Chapter 3
Second Order Constant Coefficient Linear Differential Equations

This chapter begins our study of second order linear differential equations, which are equations of the form

$$a(t)y'' + b(t)y' + c(t)y = f(t), \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$, called the ***coefficient functions***, and $f(t)$, known as the ***forcing function***, are all defined on a common interval I. Equation (1) is frequently made into an ***initial value problem*** by imposing ***initial conditions***: $y(t_0) = y_0$ and $y'(t_0) = y_1$, where $t_0 \in I$. Many problems in mathematics, engineering, and the sciences may be modeled by (1) so it is important to have techniques to solve these equations and to analyze the resulting solutions. This chapter will be devoted to the simplest version of (1), namely the case where the coefficient functions are constant. In Chap. 2, we introduced the Laplace transform method that codifies in a single procedure a solution method for (1), in the case where $f \in \mathcal{E}$ and initial values $y(0)$ and $y'(0)$ are given. However, it will be our approach going forward to first find the general solution to (1), without regard to initial conditions. When initial conditions are given, they determine a single function in the general solution set. The Laplace transform will still play a central role in most all that we do.

We wish to point out that our development of the Laplace transform thus far allows us to easily handle nth order constant coefficient linear differential equations for arbitrary n. Nevertheless, we will restrict our attention in this chapter to the second order case, which is the most important case for applications. Understanding this case well will provide an easy transition to the more general case to be studied in Chap. 4.

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3.1 Notation, Definitions, and Some Basic Results

For the remainder of this chapter, we will assume that the coefficient functions in (1) are constant. Thus, our focus will be the equation

$$ay'' + by' + cy = f(t), \tag{1}$$

where a, b, and c are real numbers and the forcing function $f(t)$ is a continuous function on an interval I. We assume the ***leading coefficient*** a is nonzero, otherwise (1) is first order. Equation (1) is called a ***second order constant coefficient linear differential equation***.

The left-hand side of (1) is made up of a combination of differentiations and multiplications by constants. To be specific and to introduce useful notation, let $\boldsymbol{D}$ denote the derivative operator: $\boldsymbol{D}(y) = y'$. In a similar way, let $\boldsymbol{D}^2$ denote the second derivative operator: $\boldsymbol{D}^2(y) = \boldsymbol{D}(\boldsymbol{D}y) = \boldsymbol{D}y' = y''$. If

$$\boldsymbol{L} = a\boldsymbol{D}^2 + b\boldsymbol{D} + c, \tag{2}$$

where a, b, and c are the same constants given in (1), then

$$\boldsymbol{L}(y) = ay'' + by' + cy.$$

We call $\boldsymbol{L}$ a ***(second order) constant coefficient linear differential operator***. Another useful way to describe $\boldsymbol{L}$ is in terms of the polynomial $q(s) = as^2 + bs + c$: $\boldsymbol{L}$ is obtained from q by substituting $\boldsymbol{D}$ for s. We will write $\boldsymbol{L} = q(\boldsymbol{D})$. For this reason, $\boldsymbol{L}$ is also called a ***polynomial differential operator***. Equation (1) can now be rewritten

$$\boldsymbol{L}(y) = f \quad \text{or} \quad q(\boldsymbol{D})y = f.$$

The polynomial q is referred to as the ***characteristic polynomial*** of $\boldsymbol{L}$ and will play a fundamental role in determining the solution set to (1).

The operator $\boldsymbol{L}$ can be thought of as taking a function y that has at least 2 continuous derivatives and producing a continuous function.

Example 1. Suppose $\boldsymbol{L} = \boldsymbol{D}^2 - 4\boldsymbol{D} + 3$. Find

$$\boldsymbol{L}(te^t), \qquad \boldsymbol{L}(e^t), \qquad \text{and} \quad \boldsymbol{L}(e^{3t}).$$

► **Solution.**

- $$\begin{aligned}\boldsymbol{L}(te^t) &= \boldsymbol{D}^2(te^t) - 4\boldsymbol{D}(te^t) + 3(te^t) \\ &= \boldsymbol{D}(e^t + te^t) - 4(e^t + te^t) + 3te^t \\ &= 2e^t + te^t - 4e^t - 4te^t + 3te^t \\ &= -2e^t.\end{aligned}$$

- $$\begin{aligned} \boldsymbol{L}(\mathrm{e}^t) &= \boldsymbol{D}^2(\mathrm{e}^t) - 4\boldsymbol{D}(\mathrm{e}^t) + 3(\mathrm{e}^t) \\ &= \mathrm{e}^t - 4\mathrm{e}^t + 3\mathrm{e}^t \\ &= 0. \end{aligned}$$
- $$\begin{aligned} \boldsymbol{L}(\mathrm{e}^{3t}) &= \boldsymbol{D}^2(\mathrm{e}^{3t}) - 4\boldsymbol{D}(\mathrm{e}^{3t}) + 3(\mathrm{e}^{3t}) \\ &= 9\mathrm{e}^{3t} - 12\mathrm{e}^{3t} + 3\mathrm{e}^{3t} \\ &= 0. \end{aligned}$$ ◀

The adjective "linear" describes an important property that $\boldsymbol{L}$ satisfies. To explain this, let us start with the familiar derivative operator $\boldsymbol{D}$. One learns early on in calculus the following two properties:

1. If y_1 and y_2 are continuously differentiable functions, then

$$\boldsymbol{D}(y_1 + y_2) = \boldsymbol{D}(y_1) + \boldsymbol{D}(y_2).$$

2. If y is a continuously differentiable function and c is a scalar, then

$$\boldsymbol{D}(cy) = c\boldsymbol{D}(y).$$

Simply put, $\boldsymbol{D}$ preserves addition and scalar multiplication of functions. When an operation on functions satisfies these two properties, we call it ***linear***. The second derivative operator $\boldsymbol{D}^2$ is also linear:

$$\boldsymbol{D}^2(y_1 + y_2) = \boldsymbol{D}(\boldsymbol{D}y_1 + \boldsymbol{D}y_2) = \boldsymbol{D}^2 y_1 + \boldsymbol{D}^2 y_2$$
$$\boldsymbol{D}^2(cy) = \boldsymbol{D}(\boldsymbol{D}(cy)) = \boldsymbol{D}c\boldsymbol{D}y = c\boldsymbol{D}^2 y.$$

It is easy to verify that sums and scalar products of linear operators are also linear operators, which means that any polynomial differential operator is linear. This is formalized in the following result.

Proposition 2. *The operator*

$$\boldsymbol{L} = a\boldsymbol{D}^2 + b\boldsymbol{D} + c$$

is linear. Specifically,

1. If y_1 and y_2 have sufficiently many derivatives, then

$$\boldsymbol{L}(y_1 + y_2) = \boldsymbol{L}(y_1) + \boldsymbol{L}(y_2).$$

2. If y has sufficiently many derivatives and c is a scalar, then

$$\boldsymbol{L}(cy) = c\boldsymbol{L}(y).$$

Proof. Suppose y, y_1, and y_2 are 2-times differentiable functions and c is a scalar. Then

$$\begin{aligned} \boldsymbol{L}(y_1 + y_2) &= a\boldsymbol{D}^2(y_1 + y_2) + b\boldsymbol{D}(y_1 + y_2) + c(y_1 + y_2) \\ &= a(\boldsymbol{D}^2 y_1 + \boldsymbol{D}^2 y_2) + b(\boldsymbol{D} y_1 + \boldsymbol{D} y_2) + c(y_1 + y_2) \\ &= a\boldsymbol{D}^2 y_1 + b\boldsymbol{D} y_1 + c y_1 + a\boldsymbol{D}^2 y_2 + b\boldsymbol{D} y_2 + c y_2 \\ &= \boldsymbol{L} y_1 + \boldsymbol{L} y_2. \end{aligned}$$

Thus $\boldsymbol{L}$ preserves addition. In a similar way, $\boldsymbol{L}$ preserves scalar multiplication and hence is linear. □

To illustrate the power of linearity, consider the following example.

Example 3. Let $\boldsymbol{L} = \boldsymbol{D}^2 - 4\boldsymbol{D} + 3$. Use linearity to determine

$$\boldsymbol{L}(3\mathrm{e}^t + 4t\mathrm{e}^t + 5\mathrm{e}^{3t}).$$

► **Solution.** Recall from Example 1 that

- $\boldsymbol{L}(\mathrm{e}^t) = 0$.
- $\boldsymbol{L}(t\mathrm{e}^t) = -2\mathrm{e}^t$.
- $\boldsymbol{L}(\mathrm{e}^{3t}) = 0$.

Using linearity, we obtain

$$\begin{aligned} \boldsymbol{L}(3\mathrm{e}^t + 4t\mathrm{e}^t + 5\mathrm{e}^{3t}) &= 3\boldsymbol{L}(\mathrm{e}^t) + 4\boldsymbol{L}(t\mathrm{e}^t) + 5\boldsymbol{L}(\mathrm{e}^{3t}) \\ &= 3 \cdot 0 + 4 \cdot (-2\mathrm{e}^t) + 5 \cdot 0 \\ &= -8\mathrm{e}^t. \end{aligned}$$ ◄

Solutions

An important consequence of linearity is that the set of all solutions to (1) has a particularly simple structure. We begin the description of that structure with the special case where the forcing function is identically zero. In this case, (1) becomes

$$\boldsymbol{L}(y) = 0 \tag{3}$$

and we refer to such an equation as ***homogeneous***.

Proposition 4. *Suppose* $\boldsymbol{L}$ *is a linear differential operator. Then the solution set to* $\boldsymbol{L}y = 0$ *is a linear space. Specifically, suppose* y, y_1, *and* y_2 *are solutions to* $\boldsymbol{L}y = 0$ *and* k *is a scalar. Then* $y_1 + y_2$ *and* ky *are solutions to* $\boldsymbol{L}y = 0$.

Proof. By Proposition 2, we have

$$\boldsymbol{L}(y_1 + y_2) = \boldsymbol{L}(y_1) + \boldsymbol{L}(y_2) = 0 + 0 = 0$$
$$\boldsymbol{L}(ky) = k\boldsymbol{L}(y) = k \cdot 0 = 0.$$

These equations show that the solution set is closed under addition and scalar multiplication and hence is a linear space. □

Example 5. Use Example 1 to find as many solutions as possible to the homogeneous equation $\boldsymbol{L}y = 0$, where

$$\boldsymbol{L} = \boldsymbol{D}^2 - 4\boldsymbol{D} + 3.$$

► **Solution.** In Example 1, we found that $\boldsymbol{L}(\mathrm{e}^t) = 0$ and $\boldsymbol{L}(\mathrm{e}^{3t}) = 0$. Now using Proposition 4, we have

$$c_1\mathrm{e}^t + c_2\mathrm{e}^{3t}$$

is a solution to $\boldsymbol{L}y = 0$, for all scalars c_1 and c_2. In other words,

$$\boldsymbol{L}(c_1\mathrm{e}^t + c_2\mathrm{e}^{3t}) = 0,$$

for all scalars c_1 and c_2. We will later show that *all* of the solutions to $\boldsymbol{L}y = 0$ are of this form. ◄

It is hard to overemphasize the importance of Proposition 4, since it indicates that once a few specific solutions to $\boldsymbol{L}(y) = 0$ are known, then all linear combinations are likewise solutions. This gives a strategy for describing all solutions to $\boldsymbol{L}(y) = 0$ provided we can find a few distinguished solutions. The linearity proposition will also allow for a useful way to describe all solutions to the general differential equation $\boldsymbol{L}(y) = f(t)$ by reducing it to the homogeneous differential equation $\boldsymbol{L}(y) = 0$, which we refer to as the ***associated homogeneous differential equation***. The following theorem describes this relationship.

Theorem 6. *Suppose* $\boldsymbol{L}$ *is a linear differential operator and* f *is a continuous function. If* y_{p} *is a fixed particular solution to* $\boldsymbol{L}(y) = f$ *and* y_{h} *is any solution to the associated homogeneous differential equation* $\boldsymbol{L}(y) = 0$*, then*

$$y_{\mathrm{p}} + y_{\mathrm{h}}$$

is a solution to $\boldsymbol{L}(y) = f$*. Furthermore, any solution* y *to* $\boldsymbol{L}(y) = f$ *has the form*

$$y = y_{\mathrm{p}} + y_{\mathrm{h}}.$$

Proof. Suppose y_{p} satisfies $\boldsymbol{L}(y_{\mathrm{p}}) = f$ and y_{h} satisfies $\boldsymbol{L}(y_{\mathrm{h}}) = 0$. Then by linearity,

$$\boldsymbol{L}(y_{\mathrm{p}} + y_{\mathrm{h}}) = \boldsymbol{L}(y_{\mathrm{p}}) + \boldsymbol{L}(y_{\mathrm{h}}) = f + 0 = f.$$

Thus $y_p + y_h$ is a solution to $\boldsymbol{L}(y) = f$. On the other hand, suppose $y(t)$ is any solution to $\boldsymbol{L}(y) = f$. Let $y_h = y - y_p$. Then, again by linearity,

$$\boldsymbol{L}(y_h) = \boldsymbol{L}(y - y_p) = \boldsymbol{L}(y) - \boldsymbol{L}(y_p) = f - f = 0.$$

Thus, $y_h(t)$ is a solution to $\boldsymbol{L}(y) = 0$ and $y = y_p + y_h$. □

This theorem actually provides an effective strategy for describing the solution set to a second order linear constant coefficient differential equation, which we formalize in the following algorithm. By an abuse of language, we will sometimes refer to solutions of the associated homogeneous equation $\boldsymbol{L}(y) = 0$ as ***homogeneous solutions***.

Algorithm 7. The general solution to a linear differential equation

$$\boldsymbol{L}(y) = f(t)$$

can be found as follows:

Solution Method for Second Order Linear Equations

1. Find all the solutions y_h to the associated homogeneous differential equation $\boldsymbol{L}y = 0$.
2. Find one particular solution y_p to $\boldsymbol{L}(y) = f$.
3. Add the particular solution to the homogeneous solutions:

$$y_p + y_h.$$

As y_h varies over all homogeneous solutions, we obtain all solutions to $\boldsymbol{L}(y) = f$

Example 8. Use Algorithm 7 to solve

$$y'' - 4y' + 3y = -2e^t.$$

► **Solution.** The left-hand side can be written $\boldsymbol{L}(y)$, where $\boldsymbol{L}$ is the linear differential operator

$$\boldsymbol{L} = \boldsymbol{D}^2 - 4\boldsymbol{D} + 3.$$

From Example 5, we found that

$$y_h(t) = c_1e^t + c_2e^{3t},$$

where $c_1, c_2 \in \mathbb{R}$, are all the solutions to the associated homogeneous equation $\boldsymbol{L}(y) = 0$. By Example 1, a particular solution to $\boldsymbol{L}(y) = -2e^t$ is $y_p(t) = te^t$. By Theorem 6, we have

$$y_p(t) + y_h(t) = te^t + c_1 e^t + c_2 e^{3t}$$

is a solution to $\boldsymbol{L}(y) = 2e^t$, for all scalars c_1 and c_2. ◀

The strategy outlined in Algorithm 7 is the strategy we will follow. Section 3.3 will be devoted to determining solutions to the associated homogeneous differential equation. Sections 3.4 and 3.5 will show effective methods for finding a particular solution when the forcing function $f(t) \in \mathcal{E}$ is an exponential polynomial. A more general method is found in Sect. 5.6.

Initial Value Problems

Suppose $\boldsymbol{L}$ is a constant coefficient linear differential operator, $f(t)$ is a function defined on an interval I, and $t_0 \in I$. To the equation

$$\boldsymbol{L}(y) = f$$

we can associate ***initial conditions*** of the form

$$y(t_0) = y_0, \quad \text{and} \quad y'(t_0) = y_1.$$

The differential equation $\boldsymbol{L}(y) = f$, together with the initial conditions, is called an ***initial value problem***, just as in the case of first order differential equations. After finding the general solution to $\boldsymbol{L}(y) = f$, the initial conditions are used to determine specific values for the arbitrary constants that parameterize the solution set. Here is an example.

Example 9. Use Example 8 to find the solution to the following initial value problem

$$\boldsymbol{L}(y) = -2e^t, \qquad y(0) = 1,\ y'(0) = -2,$$

where $\boldsymbol{L} = \boldsymbol{D}^2 - 4\boldsymbol{D} + 3$.

► **Solution.** In Example 8, we verified that

$$y(t) = te^t + c_1 e^t + c_2 e^{3t},$$

is a solution to $\boldsymbol{L}(y) = -2e^t$ for every $c_1, c_2 \in \mathbb{R}$. Observe that $y'(t) = e^t + te^t + c_1 e^t + 3c_2 e^{3t}$. Setting $t = 0$ in both $y(t)$ and $y'(t)$ gives

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ -2 &= y'(0) = 1 + c_1 + 3c_2. \end{aligned}$$

Solving these equations gives $c_1 = 3$, and $c_2 = -2$. Thus, $y(t) = te^t + 3e^t - 2e^{3t}$ is a solution to the given initial value problem. The existence and uniqueness theorem given below implies it is the only solution. ◄

Initial Values Not Based at the Origin

You may have noticed that the initial conditions in the examples given in Chap. 2 and in Example 9 above are given at $t_0 = 0$. If the initial conditions are given elsewhere, then a simple translation can be used to shift the initial conditions back to the origin as follows. Suppose f is a function defined on an interval $[a, b]$, $t_0 \in [a, b]$, and the initial conditions are given by $y(t_0) = y_0$ and $y'(t_0) = y_1$. Let $g(t) = f(t + t_0)$ and $w(t) = y(t + t_0)$. Then $w'(t) = y'(t + t_0)$ and $w''(t) = y''(t + t_0)$. The initial value problem given by (1) in y becomes

$$aw'' + bw' + cw = g,$$

with initial conditions $w(0) = y(t_0) = y_0$ and $w'(0) = y'(t_0) = y_1$. We now solve for w; it has initial conditions at 0. The function $y(t) = w(t - t_0)$ will then be the solution to the original initial value problem on the interval $[a, b)$. Thus, it is not restrictive to give examples and base our results for initial values at $t_0 = 0$.

The Existence and Uniqueness Theorem

The existence and uniqueness theorem, as expressed by Corollary 8 of Sect. 1.4, for first order linear differential equations has an extension for second order linear differential equations. Its proof will be given in Chap. 9 in a much broader setting.

Theorem 10 (The Existence and Uniqueness Theorem). *Suppose $f(t)$ is a continuous real-valued function on an interval I. Let $t_0 \in I$. Then there is a unique real-valued function y defined on I satisfying*

$$ay'' + by' + cy = f(t), \tag{4}$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$. If $f(t)$ is of exponential type, so are the solution $y(t)$ and its derivatives $y'(t)$ and $y''(t)$. Furthermore, if $f(t)$ is in $\mathcal{E}$, then $y(t)$ is also in $\mathcal{E}$.

You will notice that the kind of solution we obtain depends on the kind of forcing function. In particular, when the forcing function is an exponential polynomial, then so is the solution. This theorem thus provides the basis for applying the Laplace transform method. Specifically, when the Laplace transform is applied to both sides of (4), we *presumed* in previous examples that the solution y and its first and second derivative have Laplace transforms. The existence and uniqueness theorem thus justifies the Laplace transform method when the forcing function is of exponential type or, more specifically, an exponential polynomial.

Exercises

1–10. Determine which of the following are second order constant coefficient linear differential equations. In those cases where it is, write the equation in the form $\boldsymbol{L}(y) = f(t)$, give the characteristic polynomial, and state whether the equation is homogeneous.

1. $y'' - yy' = 6$
2. $y'' - 3y' = \mathrm{e}^t$
3. $y''' + y' + 4y = 0$
4. $y'' + \sin(y) = 0$
5. $ty' + y = \ln t$
6. $y'' + 2y' + 3y = \mathrm{e}^{-t}$
7. $y'' - 7y' + 10y = 0$
8. $y' + 8y = t$
9. $y'' + 2 = \cos t$
10. $2y'' - 12y' + 18y = 0$

11–14. For the linear operator $\boldsymbol{L}$, determine $\boldsymbol{L}(y)$.

11. $\boldsymbol{L} = \boldsymbol{D}^2 + 3\boldsymbol{D} + 2$.

 (a) $y = \mathrm{e}^t$
 (b) $y = \mathrm{e}^{-t}$
 (c) $y = \sin t$

12. $\boldsymbol{L} = \boldsymbol{D}^2 - 2\boldsymbol{D} + 1$.

 (a) $y = 4\mathrm{e}^t$
 (b) $y = \cos t$
 (c) $y = -\mathrm{e}^{2t}$

13. $\boldsymbol{L} = \boldsymbol{D}^2 + 1$.

 (a) $y = -4\sin t$
 (b) $y = 3\cos t$
 (c) $y = 1$

14. $\boldsymbol{L} = \boldsymbol{D}^2 - 4\boldsymbol{D} + 8$.

 (a) $y = \mathrm{e}^{2t}$
 (b) $y = \mathrm{e}^{2t}\sin 2t$
 (c) $y = \mathrm{e}^{2t}\cos 2t$

15. Suppose $\boldsymbol{L}$ is a polynomial differential operator of order 2 and

 - $\boldsymbol{L}(\cos 2t) = 10\sin 2t$
 - $\boldsymbol{L}(\mathrm{e}^t) = 0$
 - $\boldsymbol{L}(\mathrm{e}^{4t}) = 0$.

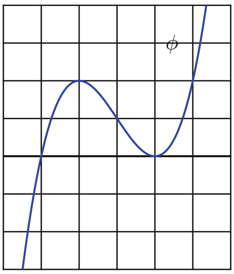
 Use this information to find other solutions to $\boldsymbol{L}(y) = 10\sin 2t$.

16. Suppose $\boldsymbol{L}$ is a polynomial differential operator of order 2 and
    - $\boldsymbol{L}(t\mathrm{e}^{3t}) = 5\mathrm{e}^{3t}$
    - $\boldsymbol{L}(\mathrm{e}^{3t}) = 0$
    - $\boldsymbol{L}(\mathrm{e}^{-2t}) = 0$.

    Use this information to find other solutions to $\boldsymbol{L}(y) = 5\mathrm{e}^{3t}$.
17. Let $\boldsymbol{L}$ be as in Exercise 15. Use the results there to solve the initial value problem
$$\boldsymbol{L}(y) = 10\sin 2t,$$
where $y(0) = 1$ and $y'(0) = -3$.
18. Let $\boldsymbol{L}$ be as in Exercise 16. Use the results there to solve the initial value problem
$$\boldsymbol{L}(y) = 5\mathrm{e}^{3t},$$
where $y(0) = -1$ and $y'(0) = 8$.
19. If $\boldsymbol{L} = a\boldsymbol{D}^2 + b\boldsymbol{D} + c$ where a, b, c are real numbers, then show that $\boldsymbol{L}(\mathrm{e}^{rt}) = (ar^2 + br + c)\mathrm{e}^{rt}$. That is, the effect of applying the operator $\boldsymbol{L}$ to the exponential function e^{rt} is to multiply e^{rt} by the number $ar^2 + br + c$.

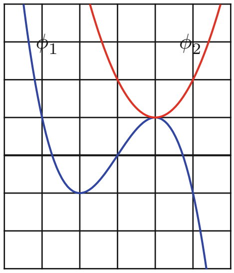
20–21. Use the existence and uniqueness theorem to establish the following.

20. Suppose $\phi(t)$ is a solution to
$$y'' + ay' + by = 0,$$
where a and b are real constants. Show that if the graph of ϕ is tangent to the t-axis, then $\phi = 0$.



21. More generally, suppose ϕ_1 and ϕ_2 are solutions to
$$y'' + ay' + by = f,$$

where a and b are real constants and f is a continuous function on an interval I. Show that if the graphs of ϕ_1 and ϕ_2 are tangent at some point, then $\phi_1 = \phi_2$.



3.2 Linear Independence

In Sects. 2.6 and 2.7, we introduced $\mathcal{B}_q$, for any polynomial q, and referred to it as the standard basis of the linear space $\mathcal{E}_q$. For a linear space, $\mathcal{F}$, of functions defined on an interval I, a ***basis*** of $\mathcal{F}$ is a subset $\mathcal{B}$ that satisfies two properties:

1. Span $\mathcal{B} = \mathcal{F}$.
2. $\mathcal{B}$ is linearly independent.

The notion of a spanning set was developed in Sect. 2.6 where we showed that $\mathcal{B}_q$ spanned the linear space $\mathcal{E}_q$. It is the purpose of this section to explain the notion of linear independence and consider some of its consequences. We will then show that $\mathcal{B}_q$ is linearly independent, thus justifying that $\mathcal{B}_q$ is a basis of $\mathcal{E}_q$ in the precise sense given above.

A set of functions $\{f_1, \ldots, f_n\}$, defined on some interval I, is said to be ***linearly independent*** if the equation

$$a_1 f_1 + \cdots + a_n f_n = 0 \tag{1}$$

implies that all the coefficients $a_1, \ldots, a_n$ are zero. Otherwise, we say that $\{f_1, \ldots, f_n\}$ is ***linearly dependent***.[1]

One must be careful about this definition. We do not try to solve equation (1) for the variable t. Rather, we are given that this equation is valid for all $t \in I$. With this information, the focus is on what this says about the coefficients $a_1, \ldots, a_n$: are they all necessarily zero or not.

We illustrate the definition with a very simple example.

Example 1. Show that the set $\{e^t, e^{-t}\}$ defined on $\mathbb{R}$ is linearly independent.

► **Solution.** Consider the equation

$$a_1 e^t + a_2 e^{-t} = 0, \tag{2}$$

for all $t \in \mathbb{R}$. In order to conclude linear independence, we need to show that a_1 and a_2 are zero. There are many ways this can be done. Below we show three approaches. Of course, only one is necessary.

Method 1: Evaluation at Specified Points

Let us evaluate (2) at two points: $t = 0$ and $t = 1$:

[1]A grammatical note: We say $f_1, \ldots, f_n$ *are* linearly independent (dependent) if the *set* $\{f_1, \ldots, f_n\}$ is linearly independent (dependent).

$$\begin{aligned} t &= 0, & a_1 + a_2 \quad &= 0, \\ t &= 1, & a_1(e) + a_2(1/e) &= 0. \end{aligned}$$

Multiply the first equation by e and subtract the result from the second equation. We get $a_2((1/e) - e) = 0$. So $a_2 = 0$ and this in turn gives $a_1 = 0$. Thus, $\{e^t, e^{-t}\}$ is linearly independent.

Method 2: Differentiation

Take the derivative of (2) to get $a_1 e^t - a_2 e^{-t} = 0$. Evaluating equation (2) and the derivative at $t = 0$ gives

$$\begin{aligned} a_1 + a_2 &= 0, \\ a_2 - a_2 &= 0. \end{aligned}$$

The only solution to these equations is $a_1 = 0$ and $a_2 = 0$. Hence, $\{e^t, e^{-t}\}$ is linearly independent. (This method will be discussed more generally when we introduce the Wronskian below.)

Method 3: The Laplace Transform

Here we take the Laplace transform of (2) to get

$$\frac{a_1}{s-1} + \frac{a_2}{s+1} = 0,$$

which is an equation valid for all $s > 1$ (since the Laplace transform of e^t is valid for $s > 1$). However, as an equation of rational functions, Corollary 7 of Appendix A.2 implies equality for all $s \neq 1, -1$. Now consider the limit as s approaches 1. If a_1 is not zero, then $\frac{a_1}{s-1}$ has an infinite limit while the second term $\frac{a_2}{s+1}$ has a finite limit. But this cannot be as the sum is 0. It must be that $a_1 = 0$ and therefore $\frac{a_2}{s+1} = 0$. This equation in turn implies $a_2 = 0$. Now it follows that $\{e^t, e^{-t}\}$ is linearly independent. (This method is a little more complicated than Methods 1 and 2 but will be the method we use to prove $\mathcal{B}_q$ is linearly independent.) ◀

Remark 2. Let us point out that when we are asked to determine whether a set of *two* functions $\{f_1, f_2\}$ is linearly dependent, it is enough to see that they are multiples of each other. For if $\{f_1, f_2\}$ is linearly dependent, then there are constants c_1 and c_2, not both zero, such that $c_1 f_1 + c_2 f_2 = 0$. By renumbering the functions if necessary, we may assume $c_1 \neq 0$. Then $f_1 = -\frac{c_2}{c_1} f_2$. Hence, f_1 and f_2 are multiples of each other. On the other hand, if one is a multiple of the other, that is, if $f_1 = mf_2$, then $f_1 - mf_2 = 0$ and this implies linear dependence. Thus, it is *immediate* that e^t and e^{-t} are linearly independent since they are not multiples of each other.

Example 3. Show that the set $\{e^t, \cos t, \sin t\}$ is linearly independent.

► **Solution.** We will use method 3 to show linear independence. Suppose $a_1 e^t + a_2 \cos t + a_3 \sin t = 0$. Take the Laplace transform to get

$$\frac{a_1}{s-1} + \frac{a_2 s + a_3}{s^2+1} = 0,$$

valid for all $s \neq 1, i, -i$, by Corollary 7 of Appendix A.2. Now consider the limit as s approaches 1. If $a_1 \neq 0$, then the first term becomes infinite while the second term is finite. Since the sum is 0, this is impossible so $a_1 = 0$. Thus $\frac{a_2 s + a_3}{s^2+1} = 0$. Now consider the limit as s approaches i. If either a_2 or a_3 is nonzero, then the quotient becomes infinite which cannot be. Thus we have $a_2 = a_3 = 0$. It now follows that $\{e^t, \cos t, \sin t\}$ is linearly independent. ◄

Let us consider an example of a set that is not linearly independent.

Example 4. Show that the set $\{e^t, e^{-t}, e^t + e^{-t}\}$ is linearly dependent.

► **Solution.** To show that a set is linearly dependent, we need only show that we can find a linear combination that adds up to 0 with coefficients not all zero. One such is

$$(1)e^t + (1)e^{-t} + (-1)(e^t + e^{-t}) = 0.$$

The coefficients, highlighted by the parentheses, are $1, 1$, and -1 and are not all zero. Thus, $\{e^t, e^{-t}, e^t + e^{-t}\}$ is linearly dependent. The dependency is clearly seen in that the third function is the sum of the first two. ◄

This example illustrates the following more general theorem.

Theorem 5. *A set of functions $\{f_1, \ldots, f_n\}$ is linearly dependent if and only if one of the functions is a linear combination of the others.*

Proof. Let us assume that one of the functions, f_1 say, is a linear combination of the other functions. Then we can write $f_1 = a_2 f_2 + \cdots + a_n f_n$, which is equivalent to

$$f_1 - a_2 f_2 - a_3 f_3 - \cdots - a_n f_n = 0.$$

Since not all of the coefficients are zero (the coefficient of f_1 is 1), it follows that $\{f_1, \ldots, f_n\}$ is linearly dependent. On the other hand, suppose $\{f_1, \ldots, f_n\}$ is linearly dependent. Then there are scalars, not all zero, such that $a_1 f_1 + \cdots + a_n f_n = 0$. By reordering if necessary, we may assume that $a_1 \neq 0$. Now we can solve for f_1 to get

$$f_1 = -\frac{a_2}{a_1} f_2 - \cdots - \frac{-a_n}{a_1} f_n,$$

Thus, one of the functions is a linear combination of the others. □

Solving Equations Involving Linear Combinations

Linear independence is precisely the requirement we need to be able to solve for coefficients involving a linear combination of functions. Consider the following example:

Example 6. Suppose we are given the following equation:

$$(c_1 + 4)e^t - (c_2 + 1)e^{-t} = 3e^t - 4c_1e^{-t},$$

valid for all $t \in \mathbb{R}$. Determine c_1 and c_2.

► **Solution.** Subtracting the right side from both sides gives

$$(c_1 + 1)e^t + (4c_1 - c_2 - 1)e^{-t} = 0. \tag{3}$$

In Example 1, we showed that $\{e^t, e^{-t}\}$ is linear independent. Thus, we can now say that the coefficients in (3) are zero, giving us

$$\begin{aligned} c_1 \;+ \qquad\quad 1 &= 0 \\ 4c_1 - c_2 - 1 &= 0 \end{aligned}$$

Solving these equations simultaneously gives $c_1 = -1$ and $c_2 = -5$. ◄

Notice that the equations obtained by setting the coefficients equal to zero in (3) are the same as equating corresponding coefficients in the original equations: $c_1 + 4 = 3$ and $-(c_2 + 1) = -4c_1$. More generally, we have the following theorem:

Theorem 7. *Suppose $\{f_1, \ldots, f_n\}$ is a linearly independent set. If*

$$a_1 f_1 + \cdots + a_n f_n = b_1 f_1 + \cdots + b_n f_n$$

then $a_1 = b_1$, $a_2 = b_2$, …, $a_n = b_n$.

Proof. The given equation implies

$$(a_1 - b_1) f_1 + \cdots + (a_n - b_n) f_n = 0.$$

Linear independence implies that the coefficients are zero. Thus, $a_1 = b_1$, $a_2 = b_2$, …, $a_n = b_n$. □

The Linear Independence of $\mathcal{B}_q$

Let $q(s) = s^2 - 1 = (s-1)(s+1)$. Then $\mathcal{B}_q = \{e^t, e^{-t}\}$. In Example 2, we showed that $\mathcal{B}_q$ is linearly independent and used that fact in Example 6. In like manner, if $q(s) = (s-1)(s^2+1)$, then $\mathcal{B}_q = \{e^t, \cos t, \sin t\}$, and we showed in Example 3 that it too was linearly independent. The following theorem establishes the linear independence of $\mathcal{B}_q$ for any polynomial q. The method of proof is based on Method 3 in Example 2 and can be found in Appendix A.3.

Theorem 8. *Let q be a nonconstant polynomial. View $\mathcal{B}_q$ as a set of functions on $I = [0, \infty)$. Then $\mathcal{B}_q$ is linearly independent.*

One useful fact about linearly independent sets is that any subset is also linearly independent. Specifically,

Theorem 9. *Suppose $\mathcal{S}$ is a finite set of functions on an interval I which is linearly independent. Then any subset of $\mathcal{S}$ is also linearly independent.*

Proof. Suppose $\mathcal{S} = \{f_1, \ldots, f_n\}$ is linearly independent and suppose $\mathcal{S}_\circ$ is a subset of $\mathcal{S}$. We may assume by reordering if necessary that $\mathcal{S}_\circ = \{f_1, \ldots, f_k\}$, for some $k \leq n$. Suppose $c_1 f_1 + \cdots c_k f_k = 0$ for some constants $c_1, \ldots, c_k$. Let $c_{k+1} = \cdots = c_n = 0$. Then $c_1 f_1 + \cdots c_n f_n = 0$. Since $\mathcal{S}$ is linearly independent, $c_1, \ldots, c_n$ are all zero. It follows that $\mathcal{S}_\circ$ is linearly independent. □

Since $\mathcal{B}_q$ is linearly independent, it follows from Theorem 9 that any subset is also linearly independent.

Example 10. Show that the following sets are linearly independent:

1. $\left\{e^t, e^{-t}, te^{2t}\right\}$
2. $\left\{e^t, e^{-t}, te^{3t}\right\}$
3. $\{\cos t, \sin 2t\}$

► **Solution.**

1. Let $q(s) = (s-1)(s+1)(s-2)^2$. Then $\mathcal{B}_q = \left\{e^t, e^{-t}, e^{2t}, te^{2t}\right\}$ and $\left\{e^t, e^{-t}, te^{2t}\right\} \subset \mathcal{B}_q$. Theorem 9 implies linear independence.
2. Let $q(s) = (s-1)(s+1)(s-3)^2$. Then $\mathcal{B}_q = \left\{e^t, e^{-t}, e^{3t}, te^{3t}\right\}$ and $\left\{e^t, e^{-t}, te^{3t}\right\} \subset \mathcal{B}_q$. Linear independence follows from Theorem 9.
3. Let $q(s) = (s^2+1)(s^2+4)$. Then $\mathcal{B}_q = \{\cos t, \sin t, \cos 2t, \sin 2t\}$ and $\{\cos t, \sin 2t\} \subset \mathcal{B}_q$. Linear independence follows from Theorem 9. ◄

Restrictions to Subintervals

It is important to keep in mind that the common interval of definition of a set of functions plays an implicit role in the definition of linear independence. Consider the following example.

Example 11. Let $f_1(t) = |t|$ and $f_2(t) = t$. Show that $\{f_1, f_2\}$ is linearly independent if the interval of definition is $I = \mathbb{R}$ and linearly dependent if $I = [0, \infty)$.

► **Solution.** Suppose the interval of definition is $I = \mathbb{R}$. Suppose

$$c_1 f_1(t) + c_2 f_2(t) = c_1 |t| + c_2 t = 0.$$

Evaluation at $t = 1$ and $t = -1$ gives

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0 \end{aligned}$$

These equations reduce to $c_1 = 0$ and $c_2 = 0$. Thus, $\{f_1, f_2\}$ is linearly independent on $I = \mathbb{R}$. On the other hand, suppose $I = [0, \infty)$. Then $f_1(t) = t = f_2(t)$ (on I). Lemma 5 implies $\{f_1, f_2\}$ is linearly dependent. ◄

Admittedly, the previous example is rather special. However, it does teach us that we cannot presume that restricting to a smaller interval will preserve linear independence. On the other hand, if a set of functions is defined on an interval I and linearly independent when restricted to a subset of I, then the set of functions is linearly independent on I. For example, Theorem 8 says that the set $\mathcal{B}_q$ is linearly independent on $[0, \infty)$ yet defined on all of $\mathbb{R}$. Thus, $\mathcal{B}_q$ is linearly independent as functions on $\mathbb{R}$.

The Wronskian

Suppose $f_1, \ldots, f_n$ are functions on an interval I with $n - 1$ derivatives. The ***Wronskian*** of $f_1, \ldots, f_n$ is given by

$$w(f_1, \ldots, f_n)(t) = \det \begin{bmatrix} f_1(t) & f_2(t) \ldots & f_n(t) \\ f_1'(t) & f_2'(t) \ldots & f_n'(t) \\ \vdots & \vdots \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) \ldots & f_n^{(n-1)(t)} \end{bmatrix}.$$

Clearly, the Wronskian is a function on I. We sometimes refer to the $n \times n$ matrix given above as the ***Wronskian matrix*** and denote it by $W(f_1, \ldots, f_n)(t)$.

Example 12. Find the Wronskian of the following sets of functions:

1. $\{t, 1/t\}$
2. $\{e^{2t}, e^{-t}\}$
3. $\{e^t, e^{-t}, e^t + e^{-t}\}$

► **Solution.**

1. $w(t, 1/t) = \det \begin{bmatrix} t & 1/t \\ 1 & -1/t^2 \end{bmatrix} = \frac{-1}{t} - \frac{1}{t} = \frac{-2}{t}$.

2. $w(e^{2t}, e^{-t}) = \det \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix} = -e^t - 2e^t = -3e^{-t}$.

3.

$$\begin{aligned} w(e^t, e^{-t}, e^t + e^{-t}) &= \det \begin{bmatrix} e^t & e^{-t} & e^t + e^{-t} \\ e^t & -e^{-t} & e^t - e^{-t} \\ e^t & e^{-t} & e^t + e^{-t} \end{bmatrix} \\ &= e^t \det \begin{bmatrix} -e^{-t} & e^t - e^{-t} \\ e^{-t} & e^t + e^{-t} \end{bmatrix} \\ &\quad -e^t \det \begin{bmatrix} e^{-t} & e^t + e^{-t} \\ e^{-t} & e^t + e^{-t} \end{bmatrix} \\ &\quad +e^t \det \begin{bmatrix} e^{-t} & e^t + e^{-t} \\ -e^{-t} & e^t - e^{-t} \end{bmatrix} \\ &= e^t(-2) - e^t(0) + e^t(2) = 0. \end{aligned}$$

◄

Theorem 13. *Suppose $f_1, f_2, \ldots, f_n$ are functions on an interval I with $n-1$ derivatives. Suppose the Wronskian $w(f_1, f_2, \ldots, f_n)$ is nonzero for some $t_0 \in I$. Then $\{f_1, f_2, \ldots, f_n\}$ is linearly independent.*

Proof. [2]Suppose $c_1, c_2, \ldots, c_n$ are scalars such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \tag{4}$$

on I. We must show that the coefficients $c_1, \ldots, c_n$ are zero. Consider the $n-1$ derivatives of (4):

[2]We assume in this proof some familiarity with matrices and determinants. See Chap. 8 for details.

$$\begin{array}{ccccccc} c_1 f_1 & + & \cdots & + & c_2 f_2 & = & 0 \\ c_1 f_1' & + & \cdots & + & c_2 f_2' & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_1 f_1^{(n-1)} & + & \cdots & + & c_n f_n^{(n-1)} & = & 0 \end{array}$$

We can write this system in matrix form as

$$W(f_1, \ldots, f_n)\boldsymbol{c} = \boldsymbol{0},$$

where $W(f_1, \ldots, f_n)$ is the $n \times n$ Wronskian matrix,

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad \text{and} \quad \boldsymbol{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $w(f_1, \ldots, f_n)(t_0) \neq 0$, it follows that the Wronskian matrix at t_0, $W(f_1, \ldots, f_n)(t_0)$, is invertible. Thus,

$$\boldsymbol{c} = W^{-1}(f_1, \ldots, f_n)(t_0)\boldsymbol{0} = \boldsymbol{0}.$$

This means $c_1, \ldots, c_n$ are zero and $\{f_1, \ldots, f_n\}$ is linearly independent. □

In Example 12, we saw that $w(t, 1/t) = -2/t^2$ and $w(e^{2t}, e^{-t}) = -3e^{-t}$, both nonzero functions. Hence, $\{t, 1/t\}$ and $\{e^{2t}, e^{-t}\}$ are linearly independent. A frequent mistake in the application of Theorem 13 is to assume the converse is true. Specifically, if the Wronskian, $w(f_1, \ldots, f_n)$, is zero, we may not conclude that $f_1, \ldots, f_n$ are linearly dependent. In Example 12, we saw that $w(e^t, e^{-t}, e^t + e^{-t}) = 0$. We cannot conclude from Theorem 13 that $\{e^t, e^{-t}, e^t + e^{-t}\}$ is linearly dependent. Nevertheless, linear dependence was shown in Example 4. However, Exercises 26 and 27 give a simple example of two linearly independent functions with zero Wronskian. If we add an additional assumption to the hypothesis of Theorem 13, then the converse will hold. This is the content of Theorem 8 of Sect. 3.3 given in the next section for the case $n = 2$ and, generally, in Theorem 6 of Sect. 4.2.

We conclude this section with a summary of techniques that can be used to show linear independence or dependence. Suppose $\mathcal{S} = \{f_1, \ldots, f_n\}$ is a set of functions defined on an interval I.

To Show that $\mathcal{S}$ is Linearly Independent

1. Evaluate a linear combination of $\mathcal{S}$ at selected points in I to get a linear system. If the solution is trivial, that is, all coefficients are zero, then $\mathcal{S}$ is linearly independent.
2. Compute the Wronskian, $w(f_1, \ldots, f_n)$. If it is nonzero, then $\mathcal{S}$ is linearly independent.
3. If $S \subset \mathcal{B}_q$ for some q, then S is linearly independent.

To Show that $\mathcal{S}$ is Linearly Dependent

1. Show there is a linear relation among the functions $f_1, \ldots, f_n$. That is, show that one of the functions is a linear combination of the others.
2. *Warning:* If the Wronskian, $w(f_1, \ldots, f_n)$, is zero, you *cannot* conclude that $\mathcal{S}$ is linearly dependent.

Exercises

1–13. Determine whether the given set of functions is linearly independent or linearly dependent. Unless otherwise indicated, assume the interval of definition is $I = \mathbb{R}$.

1. $\{t, t^2\}$
2. $\{e^t, e^{2t}\}$
3. $\{e^t, e^{t+2}\}$
4. $\{\ln 2t, \ln 5t\}$, on $I = (0, \infty)$
5. $\{\ln t^2, \ln t^5\}$
6. $\{\cos(t+\pi), \cos(t-\pi)\}$
7. $\{t, 1/t\}$, on $I = [0, \infty)$
8. $\{1, t, t^2\}$
9. $\{1, 1/t, 1/t^2\}$ on $I = (0, \infty)$
10. $\{\cos^2 t, \sin^2 t, 1\}$
11. $\{e^t, 1, e^{-t}\}$
12. $\{e^t, e^t \sin 2t\}$
13. $\{t^2 e^t, t^3 e^t, t^4 e^t\}$

14–21. Compute the Wronskian of the following set of functions.

14. $\{e^{3t}, e^{5t}\}$
15. $\{t, t\ln t\}$, $I = (0, \infty)$
16. $\{t\cos(3\ln t), t\sin(3\ln t)\}$, $I = (0, \infty)$
17. $\{t^{10}, t^{20}\}$
18. $\{e^{2t}, e^{3t}, e^{4t}\}$
19. $\{e^{r_1 t}, e^{r_2 t}, e^{r_3 t}\}$
20. $\{1, t, t^2\}$
21. $\{1, t, t^2, t^3\}$

22–25. Solve the following equations for the unknown coefficients.

22. $(a+b)\cos 2t - 3\sin 2t = 2\cos 2t + (a-b)\sin 2t$
23. $(25c_1 + 10c_2)e^{2t} + 25c_2 t e^{2t} = 25te^{2t}$
24. $3a_1 t - a_2 t \ln t^3 = (a_2+1)t + (a_1 - a_2)t\ln t^5$
25. $a_1 + 3t - a_2 t^2 = a_2 + a_1 t - 3t^2$

26–27. In these two problems, we see an example of two linearly independent functions with zero Wronskian.

26. Verify that $y(t) = t^3$ and $y_2(t) = |t^3|$ are linearly independent on $(-\infty, \infty)$.
27. Show that the Wronskian, $w(y_1, y_2)(t) = 0$ for all $t \in \mathbb{R}$.

3.3 Linear Homogeneous Differential Equations

In this section, we focus on determining the solution set to a homogeneous second order constant coefficient linear differential equation. Recall that if $q(s)$ is a polynomial, we have defined the linear space $\mathcal{E}_q$ to be the set of all exponential polynomials whose Laplace transform is in $\mathcal{R}_q$, that is, can be written with denominator $q(s)$. (See Sects. 2.6 and 2.7.) Moreover, we have developed a very specific description of the space $\mathcal{E}_q$ by giving what we have called a standard basis $\mathcal{B}_q$ of $\mathcal{E}_q$ so that

$$\operatorname{Span} \mathcal{B}_q = \mathcal{E}_q.$$

Lemma 1. *Let $q(s) = as^2 + bs + c$. If y is a function whose second derivative is of exponential type, then*

$$\mathcal{L}\{q(\boldsymbol{D})y\} = q(s)\mathcal{L}\{y\}(s) - p(s),$$

where $p(s) = ay_0 s + (ay_1 + by_0)$ is a polynomial of degree 1.

Proof. If $y''(t)$ is of exponential type, then so are $y(t)$ and $y'(t)$ by Lemma 4 of Sect. 2.2. We let $\mathcal{L}\{y(t)\} = Y(s)$ and apply linearity and the input derivative principles to get

$$\begin{aligned}
\mathcal{L}\{q(\boldsymbol{D})y\} &= \mathcal{L}\left\{ay'' + by' + cy\right\} \\
&= a\mathcal{L}\left\{y''(t)\right\} + b\mathcal{L}\left\{y'(t)\right\} + c\mathcal{L}\{y(t)\} \\
&= as^2Y(s) - asy_0 - ay_1 + bsY(s) - by_0 + cY(s) \\
&= (as^2 + bs + c)Y(s) - ay_0s - ay_1 - by_0 \\
&= q(s)Y(s) - p(s),
\end{aligned}$$

where $p(s) = ay_0s + (ay_1 + by_0)$. □

Theorem 2. *Let $q(s)$ be a polynomial of degree* 2. *Then the solution set to*

$$q(\boldsymbol{D})y = 0$$

is $\mathcal{E}_q$.

Proof. The forcing function $f(t) = 0$ is in $\mathcal{E}$. Thus, by Theorem 10 of Sect. 3.1, any solution to $q(\boldsymbol{D})y = 0$ is in $\mathcal{E}$. Suppose y is a solution. By Lemma 1 we have $\mathcal{L}\{q(\boldsymbol{D})y\}(s) = q(s)\mathcal{L}\{y\}(s) - p(s) = 0$ where p(s) is a polynomial of degree at most 1 depending on the initial values $y(0)$ and $y'(0)$. Solving for $\mathcal{L}\{y\}$ gives

$$\mathcal{L}\{y\}(s) = \frac{p(s)}{q(s)} \in \mathcal{R}_q.$$

This implies $y \in \mathcal{E}_q$. On the other hand, suppose $y \in \mathcal{E}_q$. Then $\mathcal{L}\{y\}(s) = \frac{p(s)}{q(s)} \in \mathcal{R}_q$, and by Lemma 1, we have

$$\mathcal{L}\{q(\boldsymbol{D})y\}(s) = q(s)\frac{p(s)}{q(s)} - p_1(s) = p(s) - p_1(s),$$

where $p_1(s)$ is a polynomial that depends on the initial conditions. Note, however, that $p(s) - p_1(s)$ is a polynomial in $\mathcal{R}$, the set of *proper* rational functions, and therefore must be identically 0. Thus, $\mathcal{L}\{q(\boldsymbol{D})y\} = 0$ and this implies $q(\boldsymbol{D})y = 0$ on $[0, \infty)$. Since $q(\boldsymbol{D})y$ and 0 are exponential polynomials and equal of $[0, \infty)$, they are equal on all of $\mathbb{R}$. It follows that the solution set to $q(\boldsymbol{D})y = 0$ is $\mathcal{E}_q$. ☐

Combining Theorem 2 and the prescription for the standard basis $\mathcal{B}_q$ given in Theorem 4 of Sect. 2.6, we get the following corollary.

Corollary 3. *Suppose $q(s) = c_2s^2 + c_1s + c_0$ is a polynomial of degree 2. If*

1. $q(s) = c_2(s - r_1)(s - r_2)$, *where $r_1 \neq r_2$ are real, then* $\mathcal{B}_q = \{e^{r_1t}, e^{r_2t}\}$.
2. $q(s) = c_2(s - r)^2$ *then* $\mathcal{B}_q = \{e^{rt}, te^{rt}\}$.
3. $q(s) = c_2((s - a)^2 + b^2)$, $b > 0$, *then* $\mathcal{B}_q = \{e^{at}\cos bt, e^{at}\sin bt\}$.

In each case, the solution set to

$$q(\boldsymbol{D})y = 0$$

is given by $\mathcal{E}_q = \operatorname{Span} \mathcal{B}_q$. *That is, if* $\mathcal{B}_q = \{y_1(t), y_2(t)\}$, *then*

$$\mathcal{E}_q = \{c_1y_1(t) + c_2y_2(t) : c_1, c_2 \in \mathbb{R}\}.$$

Remark 4. Observe that the solutions to $q(\boldsymbol{D})y = 0$ in these three cases may be summarized in terms of the roots of $q(s)$ as follows:

1. If $q(s)$ has distinct real roots r_1 and r_2, then all solutions are given by

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t} : c_1, c_2 \in \mathbb{R}.$$

2. If $q(s)$ has one real root r with multiplicity 2, then all solutions are given by

$$y(t) = c_1e^{rt} + c_2te^{rt} : c_1, c_2 \in \mathbb{R}.$$

3. If $q(s)$ has complex roots $a \pm bi$, then all solutions are given by

$$y(t) = c_1e^{at}\cos bt + c_2e^{at}\sin bt : c_1, c_2 \in \mathbb{R}.$$

Example 5. Find the general solution to the following differential equations:

1. $y'' + 3y' + 2y = 0$

2. $y'' + 2y' + y = 0$
3. $y'' - 6y' + 10y = 0$

► **Solution.** 1. The characteristic polynomial for $y'' + 3y' + 2y = 0$ is

$$q(s) = s^2 + 3s + 2 = (s+1)(s+2).$$

The roots are -1 and -2. The standard basis for $\mathcal{E}_q$ is $\{e^{-t}, e^{-2t}\}$. Thus, the solutions are

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} : c_1, c_2 \in \mathbb{R}.$$

2. The characteristic polynomial of $y'' + 2y' + y = 0$ is

$$q(s) = s^2 + 2s + 1 = (s+1)^2,$$

which has root $s = -1$ with multiplicity 2. The standard basis for $\mathcal{E}_q$ is $\{e^{-t}, te^{-t}\}$. The solutions are

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} : c_1, c_2 \in \mathbb{R}.$$

3. The characteristic polynomial for $y'' - 6y' + 10y = 0$ is

$$q(s) = s^2 - 6s + 10 = (s-3)^2 + 1.$$

From this, we see that the roots of q are $3 + i$ and $3 - i$. The standard basis for $\mathcal{E}_q$ is $\{e^{3t}\cos t, e^{3t}\sin t\}$. Thus, the solutions are

$$y(t) = c_1 e^{3t}\cos t + c_2 e^{3t}\sin t : c_1, c_2 \in \mathbb{R}.$$ ◄

These examples show that it is a relatively easy process to write down the solution set to a homogeneous constant coefficient linear differential equation once the characteristic polynomial has been factored. We codify the process in the following algorithm.

Algorithm 6. Given a second order constant coefficient linear differential equation

$$q(\boldsymbol{D})y = 0,$$

the solution set is determined as follows:

Solution Method for
Second Order Homogeneous Linear Equations

1. Determine the characteristic polynomial, $q(s)$.
2. Factor $q(s)$ according to the three possibilities: distinct real roots, a double root, complex roots.
3. Construct $\mathcal{B}_q = \{y_1(t), y_2(t)\}$ as given in Sect. 2.6.
4. The solutions $y(t)$ are all linear combinations of the functions in the standard basis $\mathcal{B}_q$. In other words,

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

for $c_1,\ c_2 \in \mathbb{R}$.

Initial Value Problems

Now suppose initial conditions, $y(0) = y_0$ and $y'(0) = y_1$, are associated to a differential equation $q(\boldsymbol{D})y = 0$. To determine the unique solution guaranteed by Theorem 10 of Sect. 3.1, we first find the general solution in terms of the standard basis, $\mathcal{B}_q$. Then the undetermined scalars given in the solution can be determined by substituting the initial values into $y(t)$ and $y'(t)$. This gives an alternate approach to using the Laplace transform method which incorporates the initial conditions from the beginning.

Example 7. Solve the initial value problem

$$y'' + 2y' + y = 0,$$

with initial conditions $y(0) = 2$ and $y'(0) = -3$.

► **Solution.** We first find the general solution. Observe that $q(s) = s^2 + 2s + 1 = (s+1)^2$ is the characteristic polynomial for $y'' + 2y' + y = 0$. Thus, $\mathcal{B}_q = \{\mathrm{e}^{-t}, t\mathrm{e}^{-t}\}$ and any solution is of the form $y(t) = c_1\mathrm{e}^{-t} + c_2 t\mathrm{e}^{-t}$. Observe that $y'(t) = -c_1\mathrm{e}^{-t} + c_2(\mathrm{e}^{-t} - t\mathrm{e}^{-t})$. Now evaluate both equations at $t = 0$ to get

$$\begin{aligned} c_1 \qquad &= 2 \\ -c_1 + c_2 &= -3, \end{aligned}$$

which implies that $c_1 = 2$ and $c_2 = -1$. Thus, the unique solution is

$$y(t) = 2\mathrm{e}^{-t} - t\mathrm{e}^{-t}.$$

◄

Abel's Formula

We conclude this section with a converse to Theorem 13 of Sect. 3.2 in the second order case.

Theorem 8 (Abel's Formula). *Let $q(s) = s^2 + bs + c$ and suppose f_1 and f_2 are solutions to $q(\boldsymbol{D})y = 0$. Then the Wronskian satisfies*

$$w(f_1, f_2) = Ke^{-bt}, \tag{1}$$

for some constant K and $\{f_1, f_2\}$ is linearly independent if and only if $w(f_1, f_2)$ is nonzero.

Proof. First observe that since f_1 is a solution to $q(\boldsymbol{D})y = 0$ we have $f_1'' = -bf_1' - cf_1$ and similarly for f_2. To simplify the notation let $w = w(f_1, f_2) = f_1 f_2' - f_2 f_1'$. Then the product rule gives

$$\begin{aligned} w' &= f_1' f_2' + f_1 f_2'' - (f_2' f_1' + f_2 f_1'') \\ &= f_1 f_2'' - f_2 f_1'' \\ &= f_1(-bf_2' - cf_2) - f_2(-bf_1' - cf_1) \\ &= -b(f_1 f_2' - f_2 f_1') \\ &= -bw. \end{aligned}$$

Therefore, w satisfies the differential equation $w' + bw = 0$. By Theorem 2 of Sect. 1.4, there is a constant K so that

$$w(t) = Ke^{-bt}.$$

This gives (1). If $K \neq 0$, then $w \neq 0$ and it follows from Theorem 13 of Sect. 3.2 that $\{f_1, f_2\}$ is a linearly independent set.

Now suppose that $w = 0$. We may assume f_1 and f_2 are nonzero functions for otherwise it is automatic that $\{f_1, f_2\}$ is linearly dependent. Let t_0 be a real number where either $f_1(t_0)$ or $f_2(t_0)$ is nonzero and define $z(t) = f_2(t_0)f_1(t) - f_1(t_0)f_2(t)$. Then z is a solution to $q(\boldsymbol{D})y = 0$ and

$$\begin{aligned} z(t_0) &= f_2(t_0)f_1(t_0) - f_1(t_0)f_2(t_0) = 0 \\ z'(t_0) &= f_2(t_0)f_1'(t_0) - f_1(t_0)f_2'(t_0) = 0. \end{aligned}$$

The second line is obtained because $w = 0$. By the uniqueness and existence theorem, Theorem 10 of Sect. 3.1, it follows that $z(t) = 0$. This implies that $\{f_1, f_2\}$ is linearly dependent. □

Remark 9. Notice that we require the leading coefficient of $q(s)$ be one. If $q(s) = as^2 + bs + c$, then the Wronskian is a multiple of $e^{\frac{-bt}{a}}$. Equation (1) is known as ***Abel's formula***. Notice that the Wronskian is never zero or identically zero, depending on K. This will persist in the generalizations that you will see later.

Exercises

1–12. Determine the solution set to the following homogeneous differential equations. Write your answer as a linear combination of functions from the standard basis.

1. $y'' - y' - 2y = 0$
2. $y'' + y' - 12y = 0$
3. $y'' + 10y' + 24y = 0$
4. $y'' - 4y' - 12y = 0$
5. $y'' + 8y' + 16y = 0$
6. $y'' - 3y' - 10y = 0$
7. $y'' + 2y' + 5y = 0$
8. $2y'' - 12y' + 18y = 0$
9. $y'' + 13y' + 36y = 0$
10. $y'' + 8y' + 25y = 0$
11. $y'' + 10y' + 25y = 0$
12. $y'' - 4y' - 21y = 0$

13–16. Solve the following initial value problems.

13. $y'' - y = 0, \quad y(0) = 0, y'(0) = 1$
14. $y'' - 3y' - 10y = 0, \quad y(0) = 5, y'(0) = 4$
15. $y'' - 10y' + 25y = 0, \quad y(0) = 0, y'(0) = 1$
16. $y'' + 4y' + 13y = 0, \quad y(0) = 1, y'(0) = -5$

17–22. Determine a polynomial q so that $\mathcal{B}_q$ is the given set of functions. Compute the Wronskian and determine the constant K in Abel's formula.

17. $\{e^{3t}, e^{-7t}\}$
18. $\{e^{r_1 t}, e^{r_2 t}\}$
19. $\{e^{3t}, te^{3t}\}$
20. $\{e^{rt}, te^{rt}\}$
21. $\{e^t \cos 2t, e^t \sin 2t\}$
22. $\{e^{at} \cos bt, e^{at} \sin bt\}$

3.4 The Method of Undetermined Coefficients

In this section (and the next), we address the second part of Algorithm 7 of Sect. 3.3, namely, finding a particular solution to

$$q(\boldsymbol{D})y = f(t). \tag{1}$$

We will assume the characteristic polynomial $q(s)$ has degree 2. If we assume, $f(t) \in \mathcal{E}$ then the existence and uniqueness theorem, Theorem 10 of Sect. 3.1, implies that a solution $y(t) = y_{\mathrm{p}}(t)$ to (1) is in $\mathcal{E}$. Therefore, the form of $y_{\mathrm{p}}(t)$ is

$$y_{\mathrm{p}}(t) = a_1 y_1(t) + \cdots + a_n y_n(t),$$

where each $y_i(t)$, for $i = 1, \ldots n$, is a *simple* exponential polynomial and the coefficients $a_1, \ldots, a_n$ are to be determined. We call y_{p} a ***test function***. The ***method of undetermined coefficients*** can be broken into two parts. First, determine which simple exponential polynomials, $y_1(t), \ldots, y_n(t)$, will arise in a test function. Second, determine the coefficients, $a_1, \ldots, a_n$.

Before giving the general procedure let us consider the essence of the method in a simple example.

Example 1. Find the general solution to

$$y'' - y' - 6y = \mathrm{e}^{-t}. \tag{2}$$

► **Solution.** Let us begin by finding the solution set to the associated homogeneous equation

$$y'' - y' - 6y = 0. \tag{3}$$

Observe that the characteristic polynomial is $q(s) = s^2 - s - 6 = (s-3)(s+2)$. Thus, $\mathcal{B}_q = \left\{\mathrm{e}^{3t}, \mathrm{e}^{-2t}\right\}$ and a homogeneous solution is of the form

$$y_{\mathrm{h}} = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-2t}. \tag{4}$$

Let us now find a particular solution to $y'' - y' - 6y = \mathrm{e}^{-t}$. Since any particular solution will do, consider the case where $y(0) = 0$ and $y'(0) = 0$. Since $f(t) = \mathrm{e}^{-t} \in \mathcal{E}$, we conclude by the uniqueness and existence theorem, Theorem 10 of Sect. 3.1, that the solution $y(t)$ is in $\mathcal{E}$. We apply the Laplace transform to both sides of (2) and use Lemma 1 of Sect. 3.3 to get

$$q(s)\mathcal{L}\left\{y(t)\right\}(s) = \frac{1}{s+1},$$

Solving for $\mathcal{L}\{y(t)\}$ gives

$$\mathcal{L}\{y(t)\}(s) = \frac{1}{(s+1)q(s)} = \frac{1}{(s+1)(s-3)(s+2)}.$$

It follows that $y(t) \in \mathcal{E}_{(s+1)(s-3)(s+2)}$. It is easy to see that $\mathcal{B}_{(s+1)(s-3)(s+2)} = \left\{e^{-t}, e^{3t}, e^{-2t}\right\}$. Thus,

$$y(t) = a_1 e^{-t} + a_2 e^{3t} + a_3 e^{-2t},$$

for some a_1, a_2, and a_3. Observe now that $a_2 e^{3t} + a_3 e^{-2t}$ is a homogeneous solution. This means then that the leftover piece

$$y_p(t) = a_1 e^{-t}$$

is a particular solution for some $a_1 \in \mathbb{R}$. This is the test function. Let us now determine a_1 by plugging $y_p(t)$ into the differential equation. First, observe that $y_p'(t) = -a_1 e^{-t}$ and $y_p''(t) = a_1 e^{-t}$. Thus, (3) gives

$$\begin{aligned} e^{-t} &= y_p'' - y_p' - 6y_p \\ &= a_1 e^{-t} + a_1 e^{-t} - 6a_1 e^{-t} \\ &= -4a_1 e^{-t}. \end{aligned}$$

From this we conclude $1 = -4a_1$ or $a_1 = -1/4$. Therefore,

$$y_p(t) = \frac{-1}{4} e^{-t}$$

is a particular solution and the general solution is obtained by adding the homogeneous solution to it. Thus, the functions

$$y(t) = y_p(t) + y_h(t) = \frac{-1}{4} e^{-t} + c_1 e^{3t} + c_2 e^{-2t},$$

where c_1 and c_2 are real numbers, make up the set of all solutions. ◀

Remark 2. Let $v(s) = (s+1)$ be the denominator of $\mathcal{L}\{e^{-t}\}$. Then $v(s)q(s) = (s+1)(s-3)(s+2)$, and the standard basis for $\mathcal{E}_{vq}$ is $\left\{e^{-t}, e^{3t}, e^{-2t}\right\}$. The standard basis for $\mathcal{E}_q$ is $\left\{e^{3t}, e^{-2t}\right\}$. Observe that the test function $y_p(t)$ is made up of functions from the standard basis of $\mathcal{E}_{vq}$ that are not in the standard basis of $\mathcal{E}_q$. This will always happen. The general argument is in the proof of the following theorem.

Theorem 3. *Suppose $\boldsymbol{L} = q(\boldsymbol{D})$ is a polynomial differential operator and $f \in \mathcal{E}$. If $\mathcal{L}\{f\} = u/v$, and $\mathcal{B}_q$ is the standard basis for $\mathcal{E}_q$, then there is a particular solution $y_p(t)$ to*

$$\boldsymbol{L}(y) = f(t)$$

which is a linear combination of terms that are in $\mathcal{B}_{qv}$ but not in $\mathcal{B}_q$.

Proof. By Theorem 10 of Sect. 3.1, any solution to $\boldsymbol{L}(y) = f(t)$ is in $\mathcal{E}$, and hence, it and its derivatives have Laplace transforms. Thus,

$$\begin{aligned}
\mathcal{L}\{q(\boldsymbol{D})y\} &= \mathcal{L}\{f\} = \frac{u(s)}{v(s)} \\
\Longrightarrow q(s)\mathcal{L}\{y\} - p(s) &= \frac{u(s)}{v(s)} \quad \text{by Lemma 1 of Sect. 3.3} \\
\Longrightarrow \qquad \mathcal{L}\{y(t)\} &= \frac{p(s)}{q(s)} + \frac{u(s)}{q(s)v(s)} = \frac{u(s) + p(s)v(s)}{q(s)v(s)}.
\end{aligned}$$

It follows that $y(t)$ is in $\mathcal{E}_{qv}$ and hence a linear combination of terms in $\mathcal{B}_{qv}$. Since $\mathcal{B}_q \subset \mathcal{B}_{qv}$, we can write $y(t) = y_{\mathrm{h}}(t) + y_{\mathrm{p}}(t)$, where $y_{\mathrm{h}}(t)$ is a linear combination of terms in $\mathcal{B}_q$ and $y_{\mathrm{p}}(t)$ is a linear combination of terms in $\mathcal{B}_{qv}$ but not in $\mathcal{B}_q$. Since $y_{\mathrm{h}}(t)$ is a homogeneous solution, it follows that $y_{\mathrm{p}}(t) = y(t) - y_{\mathrm{h}}(t)$ is a particular solution of $\boldsymbol{L}(y) = f(t)$ of the required form. □

Theorem 3 is the basis for the following algorithm.

Algorithm 4. A general solution to a second order constant coefficient differential equation

$$q(\boldsymbol{D})(y) = f(t)$$

can be found by the following method.

The Method of Undetermined Coefficients

1. Compute the standard basis, $\mathcal{B}_q$, for $\mathcal{E}_q$.
2. Determine the denominator v so that $\mathcal{L}\{f\} = u/v$. This means that $f(t) \in \mathcal{E}_v$.
3. Compute the standard basis, $\mathcal{B}_{vq}$, for $\mathcal{E}_{vq}$.
4. The test function, $y_{\mathrm{p}}(t)$, is the linear combination with arbitrary coefficients of functions in $\mathcal{B}_{vq}$ that are not in $\mathcal{B}_q$.
5. The coefficients in $y_{\mathrm{p}}(t)$ are determined by plugging $y_{\mathrm{p}}(t)$ into the differential equation $q(\boldsymbol{D})(y) = f(t)$.
6. The general solution is given by

$$y(t) = y_{\mathrm{p}}(t) + y_{\mathrm{h}}(t),$$

where $y_{\mathrm{h}}(t)$ is an arbitrary function in $\mathcal{E}_q$.

Example 5. Find the general solution to $y'' - 5y' + 6y = 4\mathrm{e}^{2t}$.

► **Solution.** The characteristic polynomial is

$$q(s) = s^2 - 5s + 6 = (s-2)(s-3)$$

and the standard basis for $\mathcal{E}_q$ is $\mathcal{B}_q = \{e^{2t}, e^{3t}\}$. Since $\mathcal{L}\{4e^{2t}\} = 4/(s-2)$, we have $v(s) = s - 2$ so $v(s)q(s) = (s-2)^2(s-3)$ and the standard basis for $\mathcal{E}_{vq}$ is $\mathcal{B}_{vq} = \{e^{2t}, te^{2t}, e^{3t}\}$. The only function in $\mathcal{B}_{vq}$ that is not in $\mathcal{B}_q$ is te^{2t}. Therefore, our test function is $y_p(t) = a_1 te^{2t}$. A simple calculation gives

$$\begin{aligned} y_p &= a_1 te^{2t} \\ y_p' &= a_1 e^{2t} + 2a_1 te^{2t} \\ y_p'' &= 4a_1 e^{2t} + 4a_1 te^{2t}. \end{aligned}$$

Substitution into $y'' - 5y' + 6y = 4e^{2t}$ gives

$$\begin{aligned} 4e^{2t} &= y_p'' - 5y_p' + 6y_p \\ &= (4a_1e^{2t} + 4a_1te^{2t}) - 5(a_1e^{2t} + 2a_1te^{2t}) + 6(a_1te^{2t}) \\ &= -a_1e^{2t}. \end{aligned}$$

From this, it follows that $a_1 = -4$ and $y_p = -4te^{2t}$. The general solution is thus given by

$$y(t) = y_p(t) + y_h(t) = -4te^{2t} + c_1e^{2t} + c_2e^{3t},$$

where c_1 and c_2 are arbitrary real constants. ◄

Remark 6. Based on Example 1, one might have expected that the test function in Example 5 would be $y_p = c_1e^{2t}$. But this cannot be since e^{2t} is a homogeneous solution, that is, $\boldsymbol{L}(e^{2t}) = 0$, so it cannot possibly be true that $\boldsymbol{L}(a_1e^{2t}) = 4e^{2t}$. Observe that $v(s) = s-2$ and $q(s) = (s-2)(s+3)$ share a common root, namely, $s = 2$, so that the product has root $s = 2$ with multiplicity 2. This produces te^{2t} in the standard basis for $\mathcal{E}_{vq}$ that does not appear in the standard basis for $\mathcal{E}_q$. In Example 1, all the roots of vq are distinct so this phenomenon does not occur. There is thus a qualitative difference between the cases when $v(s)$ and $q(s)$ have common roots and the cases where they do not. However, Algorithm 4 does not distinguish this difference. It will always produce a test function that leads to a particular solution.

Example 7. Find the general solution to

$$y'' - 3y' + 2y = 2te^t.$$

► **Solution.** The characteristic polynomial is

$$q(s) = s^2 - 3s + 2 = (s-1)(s-2).$$

Hence, the standard basis for $\mathcal{E}_q$ is $\mathcal{B}_q = \{e^t, e^{2t}\}$. The Laplace transform of $2te^t$ is $2/(s-1)^2$ so $v(s) = (s-1)^2$, and thus, $v(s)q(s) = (s-1)^3(s-2)$. Therefore, $y(t) \in \mathcal{E}_{vq}$, which has standard basis $\mathcal{B}_{vq} = \{e^t, te^t, t^2e^t, e^{2t}\}$. Since te^t and t^2e^t are the only functions in $\mathcal{B}_{vq}$ but not in $\mathcal{B}_q$, it follows that our test function has the form $y_p(t) = a_1te^t + a_2t^2e^t$ for unknown constants a_1 and a_2. We determine a_1 and a_2 by plugging $y_p(t)$ into the differential equation. A calculation of derivatives gives

$$
\begin{aligned}
y_p &= a_1te^t + a_2t^2e^t \\
y_p' &= a_1e^t + (a_1 + 2a_2)te^t + a_2t^2e^t \\
y_p'' &= (2a_1 + 2a_2)e^t + (a_1 + 4a_2)te^t + a_2t^2e^t.
\end{aligned}
$$

Substitution into $y'' - 3y' + 2y = 2te^t$ gives

$$
\begin{aligned}
2te^t &= y_p'' - 3y_p' + 2y_p \\
&= (-a_1 + 2a_2)e^t - 2a_2te^t.
\end{aligned}
$$

Here is an example where we invoke the linear independence of $\mathcal{B}_{vq}$. By Theorems 7 and 8 of Sect. 3.2, we have that the coefficients a_1 and a_2 satisfy

$$
\begin{aligned}
-a_1 + 2a_2 &= 0 \\
-2a_2 &= 2.
\end{aligned}
$$

From this, we find $a_2 = -1$ and $a_1 = 2a_2 = -2$. Hence, a particular solution is

$$y_p(t) = -2te^{-t} - t^2e^{-t}$$

and the general solution is

$$y(t) = y_p(t) + y_h(t) = -2te^{-t} - t^2e^{-t} + c_1e^t + c_2e^{2t},$$

where c_1 and c_2 are arbitrary real constants. ◀

Linearity is particularly useful when the forcing function $f(t)$ is a sum of terms. Here is the general principle, sometimes referred to as the ***superposition principle***.

Theorem 8 (Superposition Principle). *Suppose y_{p_1} is a solution to $\boldsymbol{L}(y) = f_1$ and y_{p_2} is a solution to $\boldsymbol{L}(y) = f_2$, where $\boldsymbol{L}$ is a linear differential operator. Then $a_1y_{p_1} + a_2y_{p_2}$ is a solution to $\boldsymbol{L}y = a_1f_1 + a_2f_2$.*

Proof. By linearity,

$$\boldsymbol{L}(a_1y_{p_1} + a_2y_{p_2}) = a_1\boldsymbol{L}(y_{p_1}) + a_2\boldsymbol{L}(y_{p_2}) = a_1f_1 + a_2f_2. \qquad \square$$

Example 9. Find the general solution to

$$y'' - 5y' + 6y = 12 + 4e^{2t}. \tag{5}$$

► **Solution.** Theorem 8 allows us to find a particular solution by adding together the particular solutions to

$$y'' - 5y' + 6y = 12 \tag{6}$$

and

$$y'' - 5y' + 6y = 4e^{2t}. \tag{7}$$

In both cases, the characteristic polynomial is $q(s) = s^2 - 5s + 6 = (s-2)(s-3)$. In (6), the Laplace transform of 12 is $12/s$. Thus, $v(s) = s$, $q(s)v(s) = s(s-2)(s-3)$, and $\mathcal{B}_{qv} = \{1, e^{2t}, e^{3t}\}$. The test function is $y_{p_1} = a$. It is easy to see that $y_{p_1} = 2$. In Example 5, we found that $y_{p_2} = -4te^{2t}$ is a particular solution to (7). By Theorem 8, $y_\text{p} = 2 - 4te^{2t}$ is a particular solution to (5). Thus,

$$y = 2 - 4te^{2t} + c_1e^{2t} + c_2e^{3t}$$

is the general solution. ◄

Exercises

1–9. Given $q(s)$ and $v(s)$ below, determine the test function y_p for the differential equation $q(\boldsymbol{D})y = f$, where $\mathcal{L}f = u/v$.

1. $q(s) = s^2 - s - 2 \quad v(s) = s - 3$
2. $q(s) = s^2 + 6s + 8, \quad v(s) = s + 3$
3. $q(s) = s^2 - 5s + 6, \quad v(s) = s - 2$
4. $q(s) = s^2 - 7s + 12, \quad v(s) = (s - 4)^2$
5. $q(s) = (s - 5)^2, \quad v(s) = s^2 + 25$
6. $q(s) = s^2 + 1, \quad v(s) = s^2 + 4$
7. $q(s) = s^2 + 4, \quad v(s) = s^2 + 4$
8. $q(s) = s^2 + 4s + 5, \quad v(s) = (s - 1)^2$
9. $q(s) = (s - 1)^2, \quad v(s) = s^2 + 4s + 5$

10–24. Find the general solution for each of the differential equations given below.

10. $y'' + 3y' - 4y = e^{2t}$
11. $y'' - 3y' - 10y = 7e^{-2t}$
12. $y'' + 2y' + y = e^{t}$
13. $y'' + 2y' + y = e^{-t}$
14. $y'' + 3y' + 2y = 4$
15. $y'' + 4y' + 5y = e^{-3t}$
16. $y'' + 4y = 1 + e^{t}$
17. $y'' - y = t^2$
18. $y'' - 4y' + 4y = e^{t}$
19. $y'' - 4y' + 4y = e^{2t}$
20. $y'' + y = 2\sin t$
21. $y'' + 6y' + 9y = 25te^{2t}$
22. $y'' + 6y' + 9y = 25te^{-3t}$
23. $y'' + 6y' + 13y = e^{-3t}\cos 2t$
24. $y'' - 8y' + 25y = 104\sin 3t$

25–28. Solve each of the following initial value problems.

25. $y'' - 5y' - 6y = e^{3t}, \quad y(0) = 2,\ y'(0) = 1$
26. $y'' + 2y' + 5y = 8e^{-t}, \quad y(0) = 0,\ y'(0) = 8$
27. $y'' + y = 10e^{2t}, \quad y(0) = 0,\ y'(0) = 0$
28. $y'' - 4y = 2 - 8t, \quad y(0) = 0,\ y'(0) = 5$

3.5 The Incomplete Partial Fraction Method

In this section, we provide an alternate method for finding the solution set to the nonhomogeneous differential equation

$$q(\boldsymbol{D})(y) = f(t), \tag{1}$$

where $f \in \mathcal{E}$. This alternate method begins with the Laplace transform method and exploits the efficiency of the partial fraction decomposition algorithm developed in Sects. 2.3 and 2.4. However, the partial fraction decomposition applied to $Y(s) = \mathcal{L}\{y\}(s)$ is not needed to its completion. We therefore refer to this method as the ***incomplete partial fraction method***. For purposes of illustration and comparison to the method of undetermined coefficients let us reconsider Example 5 of Sect. 3.4.

Example 1. Find the general solution to

$$y'' - 5y' + 6y = 4\mathrm{e}^{2t}.$$

► **Solution.** Our goal is to find a particular solution $y_{\mathrm{p}}(t)$ to which we will add the homogeneous solutions $y_{\mathrm{h}}(t)$. Since any particular solution will do, we begin by applying the Laplace transform with initial conditions $y(0) = 0$ and $y'(0) = 0$. The characteristic polynomial is $q(s) = s^2 - 5s + 6 = (s-2)(s-3)$ and $\mathcal{L}\left\{4\mathrm{e}^{2t}\right\} = 4/(s-2)$. If, as usual, we let $Y(s) = \mathcal{L}\{y\}(s)$ and take the Laplace transform of the differential equation with the given initial conditions, then $q(s)Y(s) = 4/(s-2)$ so that

$$Y(s) = \frac{4}{(s-2)q(s)} = \frac{4}{(s-2)^2(s-3)}.$$

In the table below, we compute the $(s-2)$-chain for $Y(s)$ but stop when the denominator reduces to $q(s) = (s-2)(s+3)$.

Incomplete $(s-2)$-chain	
$\dfrac{4}{(s-2)^2(s-3)}$	$\dfrac{-4}{(s-2)^2}$
$\dfrac{p(s)}{(s-2)(s-3)}$	

The table tells us that

$$\frac{4}{(s-2)^2(s-3)} = \frac{-4}{(s-2)^2} + \frac{p(s)}{(s-2)(s-3)}.$$

There is no need to compute $p(s)$ and finish out the table since the inverse Laplace transform of $\frac{p(s)}{(s-2)(s-3)} = \frac{p(s)}{q(s)}$ is in $\mathcal{E}_q$ and hence a solution of the associated homogeneous equation. If $Y_p(s) = \frac{-4}{(s-2)^2}$, then $y_{\mathrm{p}}(t) = \mathcal{L}^{-1}\{Y_p\}(t) = -4t\mathrm{e}^{2t}$ is a particular solution. By linearity, we get the general solution:

$$y(t) = -4t\mathrm{e}^{2t} + c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{3t}. \quad \blacktriangleleft$$

Observe that the particular solution $y_{\mathrm{p}}(t)$ we have obtained here is exactly what we derived using the method of undetermined coefficients in Example 5 of Sect. 3.4 and is obtained by one iteration of the partial fraction decomposition algorithm.

The Incomplete Partial Fraction Method

We now proceed to describe the procedure generally. Consider the differential equation

$$q(D)(y) = f(t),$$

where $q(s)$ is a polynomial of degree 2 and $f \in \mathcal{E}$. Suppose $\mathcal{L}\{f\} = u(s)/v(s)$. Since we are only interested in finding a particular solution, we can choose initial conditions that are convenient for us. Thus, we will assume that $y(0) = 0$ and $y'(0) = 0$, and as usual, let $Y(s) = \mathcal{L}\{y\}(s)$. Then applying the Laplace transform to the differential equation $q(D)(y) = f(t)$ and solving for $Y(s)$ give

$$Y(s) = \frac{u(s)}{v(s)q(s)}.$$

Let us consider the linear case where $v(s) = (s-\gamma)^m$ and $u(s)$ has no factors of $s-\gamma$. (The quadratic case, $v(s) = (s^2+cs+d)^m$ is handled similarly.) This means $f(t) = p(t)e^{\gamma t}$, where the degree of p is $m-1$. It could be the case that γ is a root of $q(s)$ with multiplicity j, in which case we can write

$$q(s) = (s-\gamma)^j q_\gamma(s),$$

where $q_\gamma(\gamma) \neq 0$. Thus,

$$Y(s) = \frac{u(s)}{(s-\gamma)^{m+j} q_\gamma(s)}.$$

For convenience, let $p_0(s) = u(s)$. We now iterate the partial fraction decomposition algorithm until the denominator is $q(s)$. This occurs after m iterations. The incomplete $(s-\gamma)$-chain is given by the table below.

Incomplete $(s-\gamma)$**-chain**	
$\dfrac{p_0(s)}{(s-\gamma)^{m+j}q_\gamma(s)}$	$\dfrac{A_1}{(s-\gamma)^{m+j}}$
$\dfrac{p_1(s)}{(s-\gamma)^{m+j-1}q_\gamma(s)}$	$\dfrac{A_2}{(s-\gamma)^{m+j-1}}$
$\vdots$	$\vdots$
$\dfrac{p_{m-1}(s)}{(s-\gamma)^{j+1}q_\gamma(s)}$	$\dfrac{A_m}{(s-\gamma)^{j+1}}$
$\dfrac{p_m(x)}{(s-\gamma)^{j}q_\gamma(s)}$	

The last entry in the first column has denominator $q(s) = (s-\gamma)^j q_\gamma(s)$, and hence, its inverse Laplace transform is a solution of the associated homogeneous equation. It follows that if $Y_p(s) = \frac{A_1}{(s-\gamma)^{m+j}} + \cdots + \frac{A_m}{(s-\gamma)^{j+1}}$, then

$$y_{\mathrm{p}}(t) = \frac{A_1}{(m+j-1)!}t^{m+j-1}\mathrm{e}^{\gamma t} + \cdots + \frac{A_m}{(j)!}t^{j-1}\mathrm{e}^{\gamma t}$$

is a particular solution.

To illustrate this general procedure, let us consider two further examples.

Example 2. Find the general solution to

$$y'' + 4y' + 4y = 2t\mathrm{e}^{-2t}.$$

► **Solution.** The characteristic polynomial is $q(s) = s^2 + 4s + 4 = (s+2)^2$ and $\mathcal{L}\{2t\mathrm{e}^{-2t}\} = 2/(s+2)^2$. Again assume $y(0) = 0$ and $y'(0) = 0$ then

$$Y(s) = \frac{2}{(s+2)^4}.$$

But this term is a partial fraction. In other words, the incomplete $(s+2)$-chain for $Y(s)$ degenerates:

Incomplete $(s+2)$*-chain*	
$\frac{2}{(s+2)^4}$	$\frac{2}{(s+2)^4}$
0	

Let $Y_p(s) = 2/(s+2)^4$. Then a particular solution is

$$y_p(t) = \mathcal{L}^{-1}\{Y_p\}(t) = \frac{2}{3!}t^3 e^{-2t} = \frac{t^3}{3}e^{-2t}.$$

The homogeneous solution can be read off from the roots of $q(s) = (s+2)^2$. Thus, the general solution is

$$y = \frac{1}{3}t^3 e^{-2t} + c_1 e^{-2t} + c_2 t e^{-2t}. \qquad \blacktriangleleft$$

Example 3. Find the general solution to

$$y'' + 4y = 16te^{2t}$$

► **Solution.** The characteristic polynomial is $q(s) = s^2 + 4$ and

$$\mathcal{L}\left\{16te^{2t}\right\}(s) = \frac{16}{(s-2)^2}.$$

Again assume $y(0) = 0$ and $y'(0) = 0$. Then

$$Y(s) = \frac{16}{(s^2+4)(s-2)^2}.$$

The incomplete $s-2$-chain for $Y(s)$ is:

Incomplete $s-2$ **-chain**	
$\frac{16}{(s^2+4)(s-2)^2}$	$\frac{2}{(s-2)^2}$
$\frac{-2(s+2)}{(s^2+4)(s-2)}$	$\frac{-1}{s-2}$
$\frac{p(s)}{s^2+4}$	

Let $Y_p(s) = \frac{2}{(s-2)^2} - \frac{1}{s-2}$. Then $y_{\mathrm{p}}(t) = 2t\mathrm{e}^{2t} - \mathrm{e}^{2t}$. The homogeneous solutions are $y_{\mathrm{h}} = c_1 \cos 2t + c_2 \sin 2t$. Thus, the general solution is

$$y(t) = y_{\mathrm{p}}(t) + y_{\mathrm{h}}(t) = 2t\mathrm{e}^{2t} - \mathrm{e}^{2t} + c_1 \cos 2t + c_2 \sin 2t. \qquad \blacktriangleleft$$

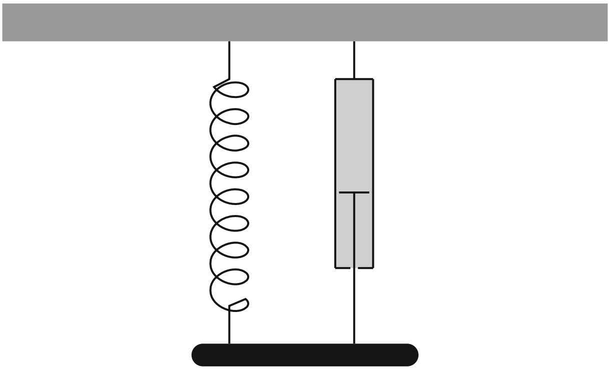
Exercises

1–12. Use the incomplete partial fraction method to solve the following differential equations.

1. $y'' - 4y = e^{-6t}$
2. $y'' + 2y' - 15y = 16e^{t}$
3. $y'' + 5y' + 6y = e^{-2t}$
4. $y'' + 3y' + 2y = 4$
5. $y'' + 2y' - 8y = 6e^{-4t}$
6. $y'' + 3y' - 10y = \sin t$
7. $y'' + 6y' + 9y = 25te^{2t}$
8. $y'' - 5y' - 6y = 10te^{4t}$
9. $y'' - 8y' + 25y = 36te^{4t}\sin(3t)$
10. $y'' - 4y' + 4y = te^{2t}$
11. $y'' + 2y' + y = \cos t$
12. $y'' + 2y' + 2y = e^{t}\cos t$

3.6 Spring Systems

In this section, we illustrate how a second order constant coefficient differential equation arises from modeling a spring-body-dashpot system. This model may arise in a simplified version of a suspension system on a vehicle or a washing machine. Consider the three main objects in the diagram below: the spring, the body, and the dashpot (shock absorber).



We assume the body only moves vertically without any twisting. Our goal is to determine the motion of the body in such a system. Various forces come into play. These include the force of gravity, the restoring force of the spring, the damping force of the dashpot, and perhaps an external force. Let us examine each of these forces and how they contribute to the overall motion of the body.

Force of Gravity

First, assume that the body has mass m. The force of gravity, F_G, acts on the body by the familiar formula

$$F_G = mg, \tag{1}$$

where g is the acceleration due to gravity. Our measurements will be positive in the downward direction so F_G is positive.

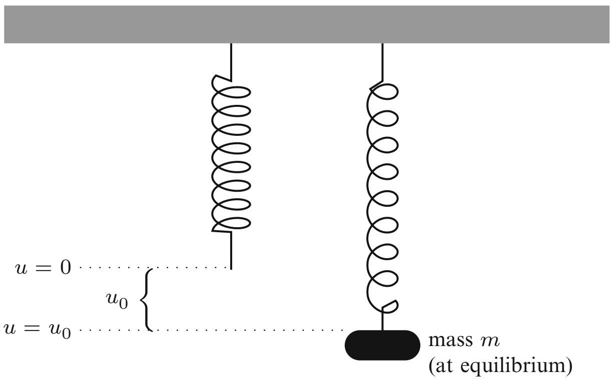
Restoring Force

When a spring is suspended with no mass attached, the end of the spring will lie at a reference point ($u = 0$). When the spring is stretched or compressed, we will denote the ***displacement*** by u. The force exerted by the spring that acts in the opposite direction to a force that stretches or compresses a spring is called the ***restoring force***. It depends on the displacement and is denoted by $F_R(u)$. Hooke's law says that the restoring force of many springs is proportional to the displacement, as long as the displacement is not too large. We will assume this. Thus, if u is the displacement we have

$$F_R(u) = -ku, \tag{2}$$

where k is a positive constant, called the ***spring constant***. When the displacement is positive (downward), the restoring force pulls the body upward, hence the negative sign.

To determine the spring constant k, consider the effect of a body of mass m attached to the spring and allowed to come to equilibrium (i.e., no movement). It will stretch the spring a certain distance, u_0, as illustrated below:



At equilibrium, the restoring force of the spring will cancel the gravitational force on the mass m. Thus, we get

$$F_R(u_0) + F_G = 0. \tag{3}$$

Combining equations (1), (2), and (3) gives us $mg - ku_0 = 0$, and hence,

$$k = \frac{mg}{u_0}. \tag{4}$$

Damping Force

In any practical situation, there will be some kind of resistance to the motion of the body. In our system, this resistance is represented by a dashpot, which in many situations is a shock absorber. The force exerted by the dashpot is called the ***damping force***, F_D. It depends on a lot of factors, but an important factor is the velocity of the body. To see that this is reasonable, imagine the difference in the forces against your body when you dive into a swimming pool off a 3-meter board and when you dive from the side of the pool. The greater the velocity when you enter the pool the greater the force that decelerates your body. We will assume that the damping force is proportional to the velocity. We thus have

$$F_{\mathrm{D}} = -\mu v = -\mu u',$$

where $v = u'$ is velocity and μ is a positive constant known as the ***damping constant***. The damping force acts in a direction opposite the velocity, hence the negative sign.

External Forces and Newton's Law of Motion

We will let $f(t)$ denote an external force acting on the body. For example, this could be the varying vertical forces acting on a suspension system of a vehicle due to driving over a bumpy road. If $a = u''$ is acceleration, then Newton's second law of motion says that the total force of a body, given by mass times acceleration, is the sum of the forces acting on that body. We thus have

$$\text{Total Force} = F_{\mathrm{G}} + F_{\mathrm{R}} + F_{\mathrm{D}} + \text{External Force},$$

which implies the equation

$$mu'' = mg - ku - \mu u' + f(t).$$

Equation (4) implies $mg = -ku_0$. Substituting and combining terms give

$$mu'' + \mu u' + k(u - u_0) = f(t).$$

If $y = u - u_0$, then y measures the displacement of the body from the equilibrium point, u_0. In this new variable, we obtain

$$my'' + \mu y' + ky = f(t). \qquad (5)$$

This second order constant coefficient differential equation is a mathematical model for the spring-body-dashpot system. The solutions that can be obtained vary dramatically depending on the constants m, μ, and k, and, of course, the external force, $f(t)$. The initial conditions,

$$y(0) = y_0 \quad \text{and} \quad y'(0) = v_0$$

represent the initial position, $y(0) = y_0$, and the initial velocity, $y'(0) = v_0$, of the given body. Once the constants, external force, and initial conditions are determined, the uniqueness and existence theorem, Theorem 10 of Sect. 3.1, guarantees a unique

Table 3.1 Units of measure in metric and English systems

System	Time	Distance	Mass	Force
Metric	seconds (s)	meters (m)	kilograms (kg)	newtons (N)
English	seconds (s)	feet (ft)	slugs (sl)	pounds (lbs)

Table 3.2 Derived quantities

Quantity	Formula
velocity (v)	distance / time
acceleration (a)	velocity /time
force (F)	mass · acceleration
spring constant (k)	force / distance
damping constant (μ)	force /velocity

solution. Of course, we should always keep in mind that (5) is a mathematical model of a real phenomenon and its solution is an approximation to what really happens. However, as long as our assumptions about the spring and damping constants are in effect, which usually require that $y(t)$ and $y'(t)$ be relatively small in magnitude, and the mass of the spring is negligible compared to the mass of the body, the solution will be a reasonably good approximation.

Units of Measurement

Before we consider specific examples, we summarize the two commonly used units of measurement: The English and metric systems. Table 3.1 summarizes the units.

The main units of the metric system are kilograms and meters while in the English system they are pounds and feet. The time unit is common to both. Table 3.2 summarizes quantities derived from these units.

In the metric system, one Newton of force (N) will accelerate a 1 kilogram mass (kg) 1 m/s^2. In the English system, a 1 pound force (lb) will accelerate a 1 slug mass (sl) 1 ft/s^2. To compute the mass of a body in the English system, one must divide the weight by the acceleration due to gravity, which is $g = 32$ ft/s^2 near the surface of the earth. Thus, a body weighing 64 lbs has a mass of 2 slugs. To compute the force a body exerts due to gravity in the metric system, one must multiply the mass by the acceleration due to gravity, which is $g = 9.8$ m/s^2. Thus, a 5 kg mass exerts a gravitational force of 49 N.

The units for the spring constant k and damping constant μ are given according to the following table.

	k	μ
Metric	N/m	N s/m
English	lbs/ft	lbs s/ft

Example 1. 1. A dashpot exerts a damping force of 10 pounds when the velocity of the mass is 2 feet per second. Find the damping constant.
2. A dashpot exerts a damping force of 6 Newtons when the velocity is 40 centimeters per second. Find the damping constant.
3. A body weighing 4 pounds stretches a spring 2 inches. Find the spring constant.
4. A mass of 8 kilograms stretches a spring 20 centimeters. Find the spring constant.

► **Solution.** 1. The force is 10 pounds and the velocity is 2 feet per second. The damping constant is given by $\mu = \text{force}/\text{velocity} = 10/2 = 5$ lbs s/ft.
2. The force is 6 Newtons and the velocity is .4 meters per second The damping constant is given by $\mu = \text{force}/\text{velocity} = 6/.4 = 15$ N s/m.
3. The force is 4 pounds. A length of 2 inches is $1/6$ foot. The spring constant is $k = \text{force}/\text{distance} = 4/(1/6) = 24$ lbs/ft.
4. The force exerted by a mass of 8 kilograms is $8 \cdot 9.8 = 78.4$ Newtons. A length of 20 centimeters is .2 meters. The spring constant is given by $k = \text{force}/\text{distance} = 78.4/.2 = 392$ N/m. ◄

Example 2. A spring is stretched 20 centimeters by a force of 5 Newtons. A body of mass 4 kilogram is attached to such a spring with an accompanying dashpot. At $t = 0$, the mass is pulled down from its equilibrium position a distance of 50 centimeters and released with a downward velocity of 1 meter per second. Suppose the damping force is 5 Newtons when the velocity of the body is .5 meter per second. Find a mathematical model that represents the motion of the body.

► **Solution.** We will model the motion by (5). Units are converted to the kilogram-meter units of the metric system. The mass is $m = 4$. The spring constant k is given by $k = 5/(.2) = 25$. The damping constant is given by $\mu = 5/.5 = 10$. Since no external force is mentioned, we may assume it is zero. The initial conditions are $y(0) = .5$ and $y'(0) = 1$. The following equation

$$4y''(t) + 10y'(t) + 25y = 0, \qquad y(0) = .5, \quad y'(0) = 1$$

represents the model for the motion of the body. ◄

Example 3. A body weighing 4 pounds will stretch a spring 3 in. This same body is attached to such a spring with an accompanying dashpot. At $t = 0$, the mass is pulled down from its equilibrium position a distance of 1 foot and released. Suppose the damping force is 8 pounds when the velocity of the body is 2 feet per second. Find a mathematical model that represents the motion of the body.

► **Solution.** Units are converted to the pound-foot units in the English system. The mass is $m = 4/32 = 1/8$ slugs. The spring constant k is given by $k = 4/(3/12) = 16$. The damping constant is given by $\mu = 8/2 = 4$. Since no external force is mentioned, we may assume it is zero. The initial conditions are $y(0) = 1$ and $y'(0) = 0$. The following equation

$$\frac{1}{8}y''(t) + 4y'(t) + 16y = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

models the motion of the body. ◀

Let us now turn our attention to an analysis of (5). The zero-input response models the motion of the body with no external forces. We refer to this motion as ***free motion*** ($f(t) \equiv 0$). Otherwise, we refer to the motion as ***forced motion*** ($f(t) \neq 0$). In turn, each of these are divided into ***undamped*** ($\mu = 0$) and ***damped*** ($\mu \neq 0$).

Undamped Free Motion

When the damping constant is zero and there is no externally applied force, the resulting motion of the object is called ***undamped free motion*** or ***simple harmonic motion***. This is an idealized situation, for seldom, if ever, will a system be free of any damping effects. Nevertheless, (5) becomes

$$my'' + ky = 0, \tag{6}$$

with $m > 0$ and $k > 0$. The characteristic polynomial of this equation is $q(s) = ms^2 + k = m(s^2 + k/m) = m(s^2 + \beta^2)$, where $\beta = \sqrt{k/m}$. Further, we have $\mathcal{B}_q = \{\sin \beta t, \cos \beta t\}$. Hence, (6) has the general solution

$$y = c_1 \cos \beta t + c_2 \sin \beta t. \tag{7}$$

Using the trigonometric identity $\cos(\beta t - \delta) = \cos(\beta t)\cos\delta + \sin(\beta t)\sin\delta$, (7) can be rewritten as

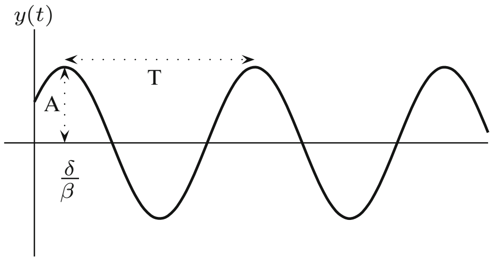
$$y = A\cos(\beta t - \delta), \tag{8}$$

where $c_1 = A\cos\delta$ and $c_2 = A\sin\delta$. Solving these equations for A and δ gives $A = \sqrt{c_1^2 + c_2^2}$ and $\tan\delta = c_2/c_1$. Therefore, the graph of $y(t)$ satisfying (6) is a pure cosine function with ***frequency*** β and with ***period***

$$T = \frac{2\pi}{\beta} = 2\pi\sqrt{\frac{m}{k}}.$$

The numbers A and δ are commonly referred to as the ***amplitude*** and ***phase angle*** of the system. Equation (8) is called the ***phase-amplitude form*** of the solution to (6). From (8), we see that A is the maximum possible value of the function $y(t)$, and hence the maximum displacement from equilibrium, and that $|y(t)| = A$ precisely when $t = (\delta + n\pi)/\beta$, where $n \in \mathbb{Z}$.

The graph of (8) is given below and well represents the oscillating motion of the body. This idealized kind of motion occurs ubiquitously in the sciences.



Example 4. Find the amplitude, phase angle, and frequency of the damped free motion of a spring-body-dashpot system with unit mass and spring constant 3. Assume the initial conditions are $y(0) = -3$ and $y'(0) = 3$.

► **Solution.** The initial value problem that models such a system is

$$y'' + 3y = 0, \qquad y(0) = -3,\ y'(0) = 3.$$

An easy calculation gives $y = -3\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t$. The amplitude is given by $A = ((-3)^2 + (\sqrt{3})^2)^{\frac{1}{2}} = 2\sqrt{3}$ and the phase angle is given implicitly by $\tan\delta = -\sqrt{3}/3$ hence $\delta = -\pi/6$. Thus,

$$y = 2\sqrt{3}\cos\left(\sqrt{3}t + \frac{\pi}{6}\right).$$ ◄

Damped Free Motion

In this case, we include the damping term $\mu\, y'$ with $\mu > 0$. In applications, the coefficient μ represents the presence of friction or resistance, which can never be completely eliminated. Thus, we want solutions to the differential equation

$$my'' + \mu y' + ky = 0. \tag{9}$$

The characteristic polynomial $q(s) = ms^2 + \mu s + k$ has roots r_1 and r_2 given by the quadratic formula

$$r_1, r_2 = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}. \tag{10}$$

The nature of the solutions of (9) are determined by whether the discriminant $D = \mu^2 - 4mk$ is negative (complex roots), zero (double root), or positive (distinct roots). We say that the system is

- ***Underdamped*** if $D < 0$.
- ***Critically damped*** if $D = 0$.
- ***Overdamped*** if $D > 0$.

Let us consider each of these cases separately.

Underdamped Systems

When μ is between 0 and $\sqrt{4mk}$ then $D < 0$, the damping is not sufficient to overcome the oscillatory behavior that we saw in the undamped case, $\mu = 0$. The resulting motion is called ***underdamped free motion***. Observe that in this case we can write

$$q(s) = m\left(s^2 + \frac{\mu}{m}s + \frac{k}{m}\right) = m((s-\alpha)^2 + \beta^2),$$

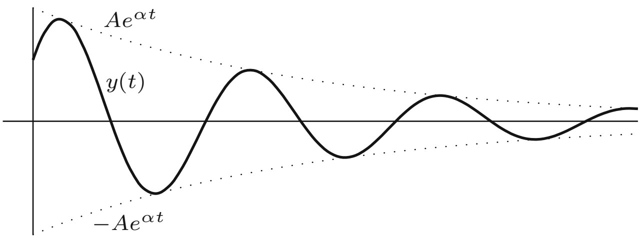
where $\alpha = -\frac{\mu}{2m}$ and $\beta = \frac{\sqrt{4mk-\mu^2}}{2m}$. Since $\mathcal{B}_q = \{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t\}$, the solution to (9) is

$$y(t) = e^{\alpha t}\left(c_1\cos\beta t + c_2\sin\beta t\right),$$

which can be rewritten as

$$y(t) = Ae^{\alpha t}\cos(\beta t - \delta), \tag{11}$$

where $A = \sqrt{c_1^2 + c_2^2}$ and $\tan\delta = c_2/c_1$, as we did earlier for the undamped case. Again we refer to (11) as the ***phase-amplitude*** form of the solution. A typical graph of (11) is given below.



Notice that y appears to be a cosine curve in which the amplitude oscillates between $Ae^{\alpha t}$ and $-Ae^{\alpha t}$. The motion of the body passes through equilibrium at regular intervals. Since $\alpha < 0$, the amplitude decreases with time. One may imagine the suspension on an automobile with rather weak shock absorbers. A push on the fender will send the vehicle oscillating as in the graph above.

Critically Damped Systems

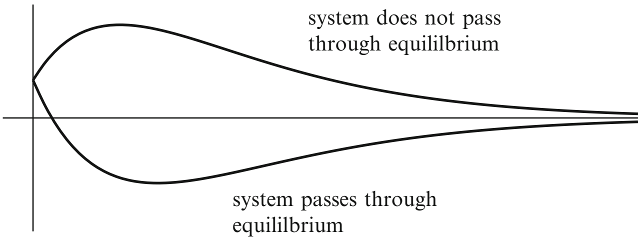
If $\mu = \sqrt{4mk}$, then the discriminant D is zero. At this critical point, the damping is just large enough to overcome oscillatory behavior. The resulting motion is called ***critically damped free motion***. Observe that the characteristic polynomial can be written

$$q(s) = m(s - r)^2,$$

where $r = -\mu/2m < 0$. Since $\mathcal{B}_q = \{e^{rt}, te^{r}t\}$ the general solution of (9) is

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} = (c_1 + c_2 t)e^{rt}. \tag{12}$$

In this case, there is no oscillatory behavior. In fact, the system will pass through equilibrium only if $t = -c_1/c_2$, and since $t > 0$, this only occurs if c_1 and c_2 have opposite signs. The following graph represents the two possibilities.



Overdamped Systems

When $\mu > \sqrt{4mk}$, then the discriminant D is positive. The resulting motion is called ***overdamped free motion***. The characteristic polynomial $q(s)$ has two distinct real roots:

$$r_1 = \frac{-\mu + \sqrt{\mu^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-\mu - \sqrt{\mu^2 - 4mk}}{2m}.$$

Both roots are negative. Since $\mathcal{B}_q = \{e^{r_1 t}, e^{r_2 t}\}$, the general solution of (9) is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{13}$$

The graphs shown for the critically damped case are representative of the possible graphs for the present case as well.

Notice that in all three cases

$$\lim_{t \to \infty} y(t) = 0$$

and thus, the motion of $y(t)$ dies out as t increases.

Undamped Forced Motion

Undamped forced motion refers to the motion of a body governed by a differential equation

$$my'' + ky = f(t),$$

where $f(t)$ is a nonzero forcing function. We will only consider the special case where the forcing function is given by $f(t) = F_0 \cos \omega t$ where F_0 is a nonzero constant. Thus, we are interested in describing the solutions of the differential equation

$$my'' + ky = F_0 \cos \omega t, \tag{14}$$

where, as usual, $m > 0$ and $k > 0$. Imagine an engine embedded within a spring-body-dashpot system. The spring system has a characteristic frequency $\beta = \sqrt{k/m}$ while the engine exerts a cyclic force with frequency ω.

To make things a little easier, we will assume the initial conditions are $y(0) = 0$ and $y'(0) = 0$. Applying the Laplace transform to (14) gives

$$Y(s) = \frac{1}{ms^2 + k} \frac{F_0 s}{s^2 + \omega^2} = \frac{F_0}{m\beta} \frac{\beta}{s^2 + \beta^2} \frac{s}{s^2 + \omega^2}. \tag{15}$$

Then the convolution theorem, Theorem 1 of Sect. 2.8, shows that

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{F_0}{m\beta} \sin \beta t * \cos \omega t. \tag{16}$$

The following convolution formula comes from Table 2.11:

$$\sin \beta t * \cos \omega t = \begin{cases} \dfrac{\beta}{\beta^2 - \omega^2} (\cos \omega t - \cos \beta t) & \text{if } \beta \neq \omega \\ \dfrac{1}{2} t \sin \omega t & \text{if } \beta = \omega. \end{cases} \tag{17}$$

Combining equations (16) and (17) gives

$$y(t) = \begin{cases} \dfrac{F_0}{m(\beta^2 - \omega^2)} (\cos \omega t - \cos \beta t) & \text{if } \beta \neq \omega \\ \dfrac{F_0}{2m\omega} t \sin \omega t & \text{if } \beta = \omega. \end{cases} \tag{18}$$

We will first consider the case $\beta \neq \omega$ in (18). Notice that, in this case, the solution $y(t)$ is the sum of two cosine functions with equal amplitude $\big(= F_0/m(\beta^2 - \omega^2)\big)$, but different frequencies β and ω. Recall the trigonometric identity

$$\cos(\theta - \phi) - \cos(\theta + \phi) = 2 \sin \theta \sin \phi.$$

If we set $\theta - \phi = \omega t$ and $\theta + \phi = \beta t$ and solve for $\theta = (\beta + \omega)t/2$ and $\phi = (\beta - \omega)t/2$, we see that we can rewrite the first part of (18) in the form

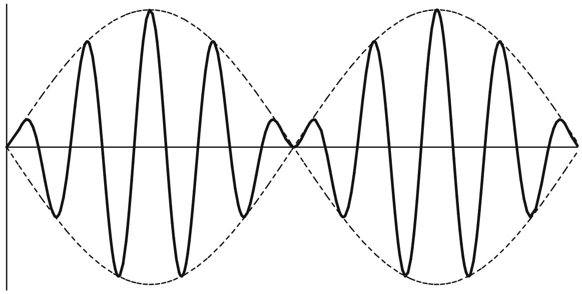
$$y(t) = \frac{2F_0}{a(\beta^2 - \omega^2)} \sin \frac{(\beta - \omega)t}{2} \sin \frac{(\beta + \omega)t}{2}. \tag{19}$$

One may think of the function $y(t)$ as a sine function, namely, $\sin((\beta + \omega)t/2)$ (with frequency $(\beta + \omega)/2$), which is multiplied by another function, namely,

$$\frac{2F_0}{a(\beta^2 - \omega^2)} \sin \frac{(\beta - \omega)t}{2},$$

which functions as a time varying amplitude function.

An interesting case is when β is close to ω so that $\beta + \omega$ is close to 2ω and $\beta - \omega$ is close to 0. In this situation, one sine function changes very rapidly, while the other, which represents the change in amplitude, changes very slowly as is illustrated below. (In order for the solution to be periodic, we must also require that β/ω be a rational number. Periodicity of the solution is further discussed in Sect. 6.8.)

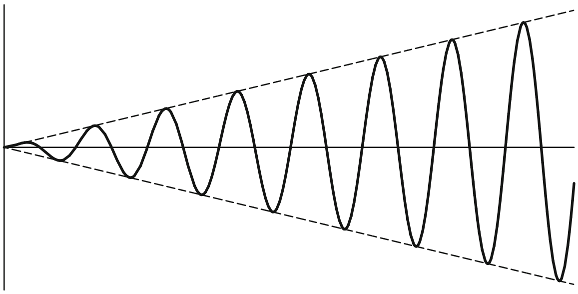


One might observe this type of motion in an unbalanced washing machine. The spinning action exerts a cyclic force, $F_0 \cos \omega t$, on the spring system which operates at a characteristic frequency β which is close to ω. The chaotic motion that results settles down momentarily only to repeat itself again. In music, this type of phenomenon, known as ***beats***, can be heard when one tries to tune a piano. When the frequency of vibration of the string is close to that of the tuning fork, one hears a pulsating beat which disappears when the two frequencies coincide. The piano is slightly out of tune.

In the case where the input frequency and characteristic frequency are equal, $\beta = \omega$, in (18), the solution

$$y(t) = \frac{F_0}{2a\omega} t \sin \omega t$$

is unbounded as $t \to \infty$ as illustrated below.



The resulting amplification of vibration eventually becomes large enough to destroy the mechanical system. This is a manifestation of ***resonance*** discussed further in Sects. 4.5 and 6.8.

Exercises

Assume forces are in pounds or newtons and lengths are in feet or meters.

1–4. Assume Hooke's law.

1. A body weighing 16 lbs stretches a spring 6 in. Find the spring constant.
2. The spring constant of a certain spring is $k = 20$ lbs/ft. If a body stretches the spring 9 inches, how much does it weigh?
3. A mass of 40 kilograms stretches a spring 80 centimeters. Find the spring constant.
4. The spring constant of a certain spring is $k = 784$ N/m. How far will a mass of 20 kilograms stretch the spring?

5–8. Assume the damping force of a dashpot is proportional to the velocity of the body.

5. A dashpot exerts a damping force of 4 lbs when the velocity of the mass is 6 inches per second. Find the damping constant.
6. A dashpot exerts a damping force of 40 Newtons when the velocity is 30 centimeters per second. Find the damping constant.
7. A dashpot has a damping constant $\mu = 100$ lbs s/ft and decelerates a body by 4 ft per second. What was the force exerted by the body?
8. A force of 40 N is applied on a body connected to a dashpot having a damping constant $\mu = 200$ N s/m. By how much will the dashpot decelerates the body?

9–14. For each exercise, investigate the motion of the mass. For undamped or underdamped motion, express the solution in the amplitude-phase form: that is, $y = Ae^{\alpha t}\cos(\beta t + \phi)$.

9. A spring is stretched 10 centimeters by a force of 2 Newtons. A body of mass 6 kilogram is attached to such a spring with no dashpot. At $t = 0$, the mass is pulled down from its equilibrium position a distance of 10 centimeters and released. Find a mathematical model that represents the motion of the body and solve. Determine the resulting motion. What is the amplitude, frequency, and phase shift?
10. A body of mass 4 kg will stretch a spring 80 centimeters. This same body is attached to such a spring with an accompanying dashpot. Suppose the damping force is 98 N when the velocity of the body is 2 m/s. At $t = 0$, the mass is given an initial upward velocity of 50 centimeters per second from its equilibrium position. Find a mathematical model that represents the motion of the body and solve. Determine the resulting motion. After release, does the mass every cross equilibrium? If so, when does it first cross equilibrium?
11. A body weighing 16 pounds will stretch a spring 6 inches. This same body is attached to such a spring with an accompanying dashpot. Suppose the damping force is 4 pounds when the velocity of the body is 2 feet per second. At $t = 0$,

the mass is pulled down from its equilibrium position a distance of 1 foot and released with a downward velocity of 1 foot per second. Find a mathematical model that represents the motion of the body and solve.

12. A body weighing 32 pounds will stretch a spring 2 feet. This same body is attached to such a spring with an accompanying dashpot. Suppose the damping constant is 8 lbs s/ft. At $t = 0$, the mass is pulled up from its equilibrium position a distance of 1 foot and released. Find a mathematical model that represents the motion of the body and solve. Determine the resulting motion. After release, does the mass every cross equilibrium?
13. A body weighing 2 pounds will stretch a spring 4 inches. This same body is attached to such a spring with no accompanying dashpot. At $t = 0$, the body is pushed downward from equilibrium with a velocity of 8 inches per second. Find a mathematical model that represents the motion of the body and solve. Determine the resulting motion. After release does the mass every cross equilibrium?
14. A spring is stretched 1 m by a force of 5 N. A body of mass 2 kg is attached to the spring with, accompanying dashpot. Suppose the damping force of the dashpot is 6 N when the velocity of the body is 1 m/s. At $t = 0$, the mass is pulled down from its equilibrium position a distance of 10 centimeters and given an initial downward velocity of 10 centimeters per second. Find a mathematical model that represents the motion of the body and solve. Determine the resulting motion. After release, does the mass every cross equilibrium?
15. Suppose m, μ, and k are positive. Show that the roots of the polynomial $q(s) = ms^2 + \mu s + k$

    1. Are negative if the roots are real.
    2. Have negative real parts if the roots are complex.

    Conclude that a solution to $my'' + \mu y' + ky = 0$ satisfies

$$\lim_{t\to\infty} y(t) = 0.$$

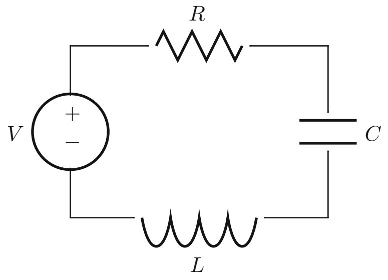
16. Prove that a solution to an overdamped or critically damped system

$$my'' + \mu y' + ky = 0$$

crosses equilibrium at most once regardless of the initial conditions.

3.7 RCL Circuits

In this section, we consider RCL series circuits. These are simple electrical circuits with a resistor, capacitor, and inductor connected to a power source in series. The diagram below gives the basic components which we discuss below.



Charge

A charge is a fundamental property of matter that exhibits electrostatic attraction or repulsion. An electron is said to have a charge of -1 and a proton has a charge of $+1$. A ***coulomb***, abbreviated C, the basic unit of measuring charge, is equivalent to the charge of about 6.242×10^{18} protons.

Current

The rate at which charged particles flow through a conductor is called ***current***. Thus, if $q(t)$ represents the charge at a cross section in the circuit at time t, then current, $I(t)$, is given by

$$I(t) = q'(t).$$

Current is measured in ***amperes***, abbreviated amp, which is coulombs per second, coulomb/s. In ***conventional flow***, $I(t)$ is positive when charge flows from the positive terminal of the supply V.

Voltage

As electrons move through a circuit, they exchange energy with its components. The standard unit of energy is the ***joule***, abbreviated J. ***Voltage*** is defined to be the quotient between energy and charge and is measured in joules per coulomb, J/C. We let $E(t)$ be the source voltage, V, in the diagram.

The charge, current, and voltage all obey basic fundamental laws with respect to the components in a RCL circuit. We discuss these laws for each component below.

Resistor

As current flows through a resistor, energy is exchanged (usually in the form of heat) resulting in a voltage drop. The voltage drop, V_R, from one side of the resistor to the other is governed by ***Ohm's law***:

$$V_{\mathrm{R}}(t) = RI(t), \tag{1}$$

where R is a positive constant called the ***resistance*** of the resistor. The unit of resistance, called the ***ohm*** and abbreviated Ω, is measured in voltage per ampere, V/A. For example, a resistor of 1 Ω will cause a voltage drop of 1 V if the current is 1 A.

Capacitor

A ***capacitor*** consists of two parallel conducting plates separated by a thin insulator. Current flows into a plate increasing the positive charge on one side and the negative charge on the other. The result is an electric field in the insulator between the plates that stores energy. That energy comes from a voltage drop across the capacitor and is governed by the ***capacitor law***:

$$V_{\mathrm{C}}(t) = \frac{1}{C} q(t),$$

where $q(t)$ is the charge on the capacitor and C is a positive constant called the ***capacitance*** of the capacitor. The unit of capacitance, called a ***farad*** and abbreviated F, is measured in charge per voltage, C/V. For example, a 1-farad capacitor will hold a charge of 1 C at a voltage drop of 1 volt. Since a coulomb is so large, capacitance is sometimes measured in millifarad (1 mF $= 10^{-3}$ F) or microfarad (1 μF $= 10^{-6}$F).

Inductor

An ***inductor*** is typically built by wrapping a conductor such as copper wire in the shape of a coil around a core of ferromagnetic material.[3] As a current flows, a

[3] Sometimes other materials are present.

Table 3.3 Standard units of measurement for RCL circuits

Quantity	Symbol	Unit of measurement	unit symbol	relation to other units
Energy		Joule	J	
Time	t	Second	s	
Charge	q	Coulomb	C	
Voltage	E or V	Volt	V	J/C
Current	I	Ampere	A	C/s
Resistance	R	Ohm	Ω	V/A
Capacitance	C	Farad	F	C/V
Inductance	L	Henry	H	V/(A/s)

magnetic field about the inductor forms which stores energy and resists any change in current. The resulting voltage drop is governed by ***Faraday's law***:

$$V_{\mathrm{L}} = LI'(t), \tag{2}$$

where L is a positive constant called the ***inductance*** of the inductor. The unit of inductance, called the ***henry*** and abbreviated H, is measured in voltage per change in ampere, V/(A/s). For example, an inductor with an inductance of 1 H produces a voltage drop of 1 V when the current through the inductor changes at a rate of 1 A/s.

Table 3.3 summarizes the standard units of measurement for RCL circuits.

Kirchoff's Laws

There are two laws that govern the behavior of the current and voltage drops in a closed circuit due to Gustaf Kirchoff.

Kirchoff's Current Law

The sum of the currents flowing into and out of a point on a closed circuit is zero.

Kirchoff's Voltage Law

The sum of the voltage drops around a closed circuit is zero.

The voltage drop across the voltage source is $-E(t)$ in conventional flow. Thus, Kirchoff's voltage law implies $V_{\mathrm{R}} + V_{\mathrm{C}} + V_{\mathrm{L}} - E = 0$. Now using Ohm's law, the capacitor law, and Faraday's law, we get

$$RI(t) + \frac{1}{C}q(t) + LI'(t) = E(t). \tag{3}$$

Table 3.4 Spring-body-mass and RCL circuit correspondence

Spring system		RCL circuit	
$my'' + \mu y' + ky = f(t)$		$Lq'' + Rq' + (1/C)q = E(t)$	
Displacement	y	Charge	q
Velocity	y'	Current	$q' = I$
Mass	m	Inductance	L
Damping constant	μ	Resistance	R
Spring constant	k	(Capacitance)$^{-1}$	$1/C$
Forcing function	$f(t)$	Applied voltage	$E(t)$

Now using the fact that $q'(t) = I(t)$, we can rewrite (3) to get

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E(t). \tag{4}$$

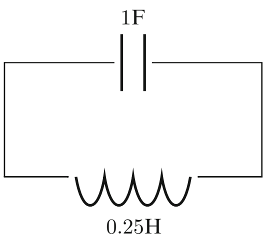
In many applications, we are interested in the current $I(t)$. If we differentiate (4) and use $I(t) = q'(t)$, we get

$$LI''(t) + RI' + \frac{1}{C}I = E'(t). \tag{5}$$

Frequently, we are given the initial charge on the capacitor $q(0)$ and the initial current $I(0)$. By evaluating (3) at $t = 0$, we obtain $I'(0)$.

The equations that model RCL circuits, (4) and (5), and the equation that models the spring-body-dashpot system, (5) of Sect. 3.6, are essentially the same. Both are second order constant coefficient linear differential equations. Table 3.4 gives the correspondence between the coefficients. The formulas in the coefficients m, μ, and k that describe concepts such as harmonic motion; underdamped, critically damped, and overdamped motion; resonance; etc. for spring systems, have a correspondence in the coefficients R, C, and L for RCL simple circuits. For example, simple harmonic motion with frequency $\beta = \sqrt{k/m}$ in a spring system occurs when there is no dashpot. Likewise, simple harmonic motion with frequency $\beta = 1/\sqrt{LC}$ in an RCL circuit occurs when there is no resistor.

Example 1. A circuit consists of a capacitor and inductor joined in series as illustrated.



There is no voltage supply. Suppose the capacitor has a capacitance of 1 F and initial charge of $q(0) = 0.2$ C. Suppose the inductor has inductance 0.25 H. If there is no initial current find the current, $I(t)$, at time t. What is the system frequency, amplitude, and phase angle? What is the charge on the capacitor at time $t = \pi/4$.

► **Solution.** We use (5) to model the current and get

$$0.25I'' + I = 0. \tag{6}$$

We have $I(0) = 0$. To determined $I'(0)$, we evaluate (3) at $t = 0$ and use $I(0) = 0$, $q(0) = 0.2$, and $E(0) = 0$ to get $0.2 + 0.25I'(0) = 0$, and hence, $I'(0) = -0.2/0.25 = -0.8$. We now multiply (6) by 4 to get the following initial value problem:

$$I'' + 4I = 0, \quad I(0) = 0, \ I'(0) = -0.8.$$

A simple calculation gives

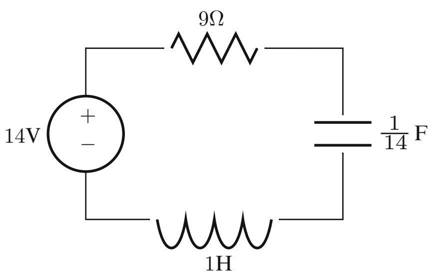
$$I(t) = -0.4 \sin 2t = 0.4 \cos\left(2t + \frac{\pi}{2}\right).$$

From the phase-amplitude form, it follows that the system frequency is 2, the amplitude is 0.4, and the phase angle is $-\pi/2$ (see the explanation after 3.6.(8) where these terms are defined). Since

$$\begin{aligned} q(t) - 0.2 &= q(t) - q(0) \\ &= \int_0^t I(\tau)\, d\tau \\ &= (-0.4) \left.\frac{-\cos 2\tau}{2}\right|_0^t = 0.2(\cos 2t - 1) \end{aligned}$$

it follows that $q(t) = 0.2 \cos 2t$. Hence, the charge on the capacitor at $t = \pi/4$ is $q(\pi/4) = 0$ coulombs. ◄

Example 2. A resistor, capacitor, and inductor, are connected in series with a voltage supply of 14 V as illustrated below.



Find the charge on the capacitor at time t if the initial charge and initial current are 0. In the long term, what will be the charge on the capacitor and the current in the circuit.

► **Solution.** We have $q(0) = 0$ and $q'(0) = I(0) = 0$. Equation (4) gives

$$q'' + 9q' + 14q = 14, \quad q(0) = 0, \ q'(0) = 0.$$

The Laplace transform method gives

$$Q(s) = \frac{14}{s(s+2)(s+7)} = \frac{1}{s} + \frac{2}{5}\frac{1}{s+7} - \frac{7}{5}\frac{1}{s+2}.$$

Hence,

$$q(t) = 1 + \frac{2}{5}\mathrm{e}^{-7t} - \frac{7}{5}\mathrm{e}^{-2t}.$$

Observe that $\lim_{t\to\infty} q(t) = 1$. This means that in the long term the charge on the capacitor will be 1C. Since $I(t) = q'(t) = \frac{14}{5}\left(\mathrm{e}^{-2t} - \mathrm{e}^{-7t}\right)$ and $\lim_{t\to\infty} I(t) = 0$, there will be no current flowing in the long term. ◄

Exercises

1–4. Find the current and charge for each RCL series circuit from the data given below. Where appropriate, express the current in phase-amplitude form: that is, $I(t) = Ae^{-\alpha t}\cos(\beta t - \delta)$.

1. $R = 10\ \Omega$, $C = 5$ mF, $L = 0.25$ H, $V = 6$ V, $q(0) = 0$ C, $I(0) = 0$ A.
2. $R = 5\ \Omega$, $C = .025$ F, $L = 0.1$ H, $V = 0$ V, $q(0) = 0.01$ C, $I(0) = 0$ A.
3. $R = 4\ \Omega$, $C = .05$ F, $L = 0.2$ H, $V = 25\sin 5t$ V, $q(0) = 0$ C, $I(0) = 2$ A.
4. $R = 11\ \Omega$, $C = 1/30$ F, $L = 1$ H, $V = 10e^{-5t}$ V, $q(0) = 1$ C, $I(0) = 2$ A.

5. An RCL circuit consists of a 0.1-H inductor and a 0.1-F capacitor. The capacitor will fail if it reaches a charge greater than 2 C. Assume there is no initial charge on the capacitor and no initial current. A voltage supply is connected to the circuit with alternating current given by $V = (1/10)\cos 5t$. Determine the charge and whether the capacitor will fail.
6. An RCL circuit consists of a 0.1-H inductor and a 0.1-F capacitor. The capacitor will fail if it reaches a charge greater than 2 C. Assume there is no initial charge on the capacitor and no initial current. A voltage supply is connected to the circuit with alternating current given by $V = (1/10)\cos 10t$. Determine the charge and whether the capacitor will fail.