

Chapter 5

Second Order Linear Differential Equations

In this chapter, we consider the broader class of second order linear differential equations that includes the constant coefficient case. In particular, we will consider differential equations of the following form:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t). \quad (1)$$

Notice that the coefficients $a_0(t)$, $a_1(t)$, and $a_2(t)$ are functions of the independent variable t and not necessarily constants. This difference has many important consequences, the main one being that there is no general solution method as in the constant coefficient case. Nevertheless, it is still linear and, as we shall see, this implies that the solution set has a structure similar to the constant coefficient case.

In order to find solution methods, one must put some rather strong restrictions on the **coefficient functions** $a_0(t)$, $a_1(t)$ and $a_2(t)$. For example, in the following list, the coefficient functions are polynomial of a specific form. The equations in this list are classical and have important uses in the physical and engineering sciences.

$t^2y'' + ty' + (t^2 - \nu^2)y = 0$	Bessel's equation of index ν
$ty'' + (1 - t)y' + \lambda y = 0$	Laguerre's equation of index λ
$(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0$	Legendre's equation of index α
$(1 - t^2)y'' - ty' + \alpha^2y = 0$	Chebyshev's equation of index α
$y'' - 2ty' + 2\lambda y = 0$	Hermite's equation of index λ .

Unlike the constant coefficient case, the solutions to these equations are not, in general, expressible in terms of algebraic combinations of polynomials, trigonometric, or exponential functions, nor their inverses. Nevertheless, the general theory implies that solutions exist and traditionally have been loosely categorized as **special functions**. In addition to satisfying the differential equation for the given index, there are other interesting and important functional relations as the index varies. We will explore some of these relations.

5.1 The Existence and Uniqueness Theorem

In this chapter, we will assume that the coefficient functions $a_0(t)$, $a_1(t)$, and $a_2(t)$ and the **forcing function** $f(t)$ are continuous functions on some common interval I . We also assume that $a_2(t) \neq 0$ for all $t \in I$. By dividing by $a_2(t)$, when convenient, we may assume that the leading coefficient function is 1. In this case, we say that the differential equation is in **standard form**. We will adopt much of the notation that we used in Sects. 3.1 and 4.1. In particular, let \mathbf{D} denote the derivative operator and let

$$L = a_2(t)\mathbf{D}^2 + a_1(t)\mathbf{D} + a_0(t). \tag{1}$$

Then (1) in the introductory paragraph can be written as $Ly = f$. If $f = 0$, then the equation $Ly = f = 0$ is called **homogeneous**. Otherwise, it is **nonhomogeneous**. We can think of L as an operation on functions. If $y \in C^2(I)$, in other words if y is a function on an interval I having a second order continuous derivative, then Ly produces a continuous function.

Example 1. Suppose $L = t\mathbf{D}^2 + 2\mathbf{D} + t$. Find

$$L\left(\frac{\cos t}{t}\right), \quad L\left(\frac{\sin t}{t}\right), \quad L(\sin t), \quad \text{and} \quad L(e^{-t}).$$

► **Solution.** The following table gives the first and second derivatives for each function:

$y = \frac{\cos t}{t}$	$y' = \frac{-t \sin t - \cos t}{t^2}$	$y'' = \frac{-t^2 \cos t + 2t \sin t + 2 \cos t}{t^3}$
$y = \frac{\sin t}{t}$	$y' = \frac{t \cos t - \sin t}{t^2}$	$y'' = \frac{-t^2 \sin t - 2t \cos t + 2 \sin t}{t^3}$
$y = \sin t$	$y' = \cos t$	$y'' = -\sin t$
$y = e^{-t}$	$y' = -e^{-t}$	$y'' = e^{-t}$.

It now follows that

- $$\bullet \quad L\left(\frac{\cos t}{t}\right) = t \frac{-t^2 \cos t + 2t \sin t + 2 \cos t}{t^3} + 2 \frac{-t \sin t - \cos t}{t^2} + t \frac{\cos t}{t}$$

$$= \frac{-t^3 \cos t + 2t^2 \sin t + 2t \cos t - 2t^2 \sin t - 2t \cos t + t^3 \cos t}{t^3}$$

$$= 0$$
- $$\bullet \quad L\left(\frac{\sin t}{t}\right) = t \frac{-t^2 \sin t - 2t \cos t + 2 \sin t}{t^3} + 2 \frac{t \cos t - \sin t}{t^2} + t \frac{\sin t}{t}$$

$$= \frac{-t^3 \sin t - 2t^2 \cos t + 2t \sin t + 2t^2 \cos t - 2t \sin t + t^3 \sin t}{t^3}$$

$$= 0$$

- $L(\sin t) = t(-\sin t) + 2(\cos t) + t \sin t = 2 \cos t$
- $L(e^{-t}) = te^{-t} + 2(-e^{-t}) + te^{-t} = (2t - 2)e^{-t}$ ◀

The most important general property that we can say about L is that it is linear.

Proposition 2. *The operator*

$$L = a_2(t)D^2 + a_1(t)D + a_0(t)$$

given by (1) is linear. Specifically,

1. If y_1 and y_2 have second order continuous derivatives, then

$$L(y_1 + y_2) = L(y_1) + L(y_2).$$

2. If y has a second order continuous derivative and c is a scalar, then

$$L(cy) = cL(y).$$

Proof. The proof of this proposition is essentially the same as the proof of Proposition 2 of Sect. 3.3. We only need to remark that multiplication by a function $a_k(t)$ preserves addition and scalar multiplication in the same way as multiplication by a constant. \square

We call L a **second order linear differential operator**. Proposition 4 of Sect. 3.3 and Theorem 6 of Sect. 3.3 are two important consequences of linearity for the constant coefficient case. The statement and proof are essentially the same. We consolidate these results and Algorithm 7 of Sect. 3.3 into the following theorem:

Theorem 3. *Suppose L is a second order linear differential operator and y_1 and y_2 are solutions to $Ly = 0$. Then $c_1y_1 + c_2y_2$ is a solution to $Ly = 0$, for all scalars c_1 and c_2 . Suppose f is a continuous function. If y_p is a fixed particular solution to $Ly = f$ and y_h is any solution to the associated homogeneous differential equation $Ly = 0$, then*

$$y = y_p + y_h$$

is a solution to $Ly = f$. Furthermore, any solution to $Ly = f$ has this same form. Thus, to solve $Ly = f$, we proceed as follows:

1. Find all the solutions to the associated homogeneous differential equation $Ly = 0$.
2. Find one particular solution y_p .
3. Add the particular solution to the homogeneous solutions.

As an application of Theorem 3, consider the following example.

Example 4. Let $L = tD^2 + 2D + t$. Use Theorem 3 and the results of Example 1 to write the most general solution to

$$Ly = 2 \cos t.$$

► **Solution.** In Example 1, we showed that $y_p = \sin t$ was a particular solution and $y_1 = \cos t/t$ and $y_2 = \sin t/t$ are homogeneous solutions. By Theorem 3, we have that $y_h = c_1 \left(\frac{\cos t}{t}\right) + c_2 \left(\frac{\sin t}{t}\right)$ is also a homogeneous solution and

$$y = y_p + y_h = \sin t + c_1 \left(\frac{\cos t}{t}\right) + c_2 \left(\frac{\sin t}{t}\right)$$

are solutions to $Ly = 2 \cos t$. We will soon see it is the general solution. ◀

Again, suppose L is a second order linear differential operator and f is a function defined on an interval I . Let $t_0 \in I$. To the equation

$$Ly = f$$

we can associate *initial conditions* of the form

$$y(t_0) = y_0, \text{ and } y'(t_0) = y_1.$$

We refer to the initial conditions and the differential equation $Ly = f$ as an *initial value problem*.

Example 5. Let $L = tD^2 + 2D + t$. Solve the initial value problem

$$Ly = 2 \cos t, \quad y(\pi) = 1, \quad y'(\pi) = -1.$$

► **Solution.** By Example 4, all functions of the form

$$y = \sin t + c_1 \frac{\cos t}{t} + c_2 \frac{\sin t}{t}$$

are solutions. Thus, we only need to find constants c_1 and c_2 that satisfy the initial conditions. Since

$$y' = \cos t + c_1 \left(\frac{-t \sin t - \cos t}{t^2}\right) + c_2 \left(\frac{t \cos t - \sin t}{t^2}\right),$$

we have

$$\begin{aligned} 0 + c_1 \left(\frac{-1}{\pi}\right) + c_2(0) &= y(\pi) = 1 \\ -1 + c_1 \left(\frac{1}{\pi^2}\right) + c_2 \left(\frac{-\pi}{\pi^2}\right) &= y'(\pi) = -1, \end{aligned}$$

which imply $c_1 = -\pi$ and $c_2 = -1$. The solution to the initial value problem is

$$y = \sin t - \pi \frac{\cos t}{t} - \frac{\sin t}{t}. \quad \blacktriangleleft$$

In the case where the coefficient functions of L are constant, we have an existence and uniqueness theorem. (See Theorem 10 of Sect. 3.1.) In the present case, we still have the existence and uniqueness theorem; however, its proof is beyond the scope of this book.¹

Theorem 6 (Uniqueness and Existence). *Suppose $a_0(t)$, $a_1(t)$, $a_2(t)$, and f are continuous functions on an open interval I and $a_2(t) \neq 0$ for all $t \in I$. Suppose $t_0 \in I$ and y_0 and y_1 are fixed real numbers. Let $L = a_2(t)D^2 + a_1(t)D + a_0(t)$. Then there is one and only one solution to the initial value problem*

$$Ly = f, \quad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

Theorem 6 does not tell us how to find any solution. We must develop procedures for this. Let us explain in more detail what this theorem does say. Under the conditions stated, the existence and uniqueness theorem says that there always is a solution to the given initial value problem. The solution is at least twice differentiable on I and there is no other solution. In Example 5, we found $y = \sin t - \pi \frac{\cos t}{t} - \frac{\cos t}{t}$ is a solution to $ty'' + 2y' + ty = 2 \cos t$ with initial conditions $y(\pi) = 1$ and $y'(\pi) = -1$. Notice, in this case, that y is, in fact, infinitely differentiable on any interval not containing 0. The uniqueness part of Theorem 6 implies that there are no other solutions. In other words, there are no potentially hidden solutions, so that if we can find enough solutions to take care of all possible initial values, then Theorem 6 provides the theoretical underpinnings to know that we have found *all* possible solutions and need look no further. Compare this theorem with the discussion in Sect. 1.7 where we saw examples (in the nonlinear case) of initial value problems which had infinitely many distinct solutions.

¹For a proof, see Theorems 1 and 3 on pages 104–105 of the text *An Introduction to Ordinary Differential Equations* by Earl Coddington, published by Prentice Hall, (1961).

Exercises

1–11. For each of the following differential equations, determine if it is linear (yes/no). For each of those which are linear, further determine if the equation is homogeneous (homogeneous/nonhomogeneous) and constant coefficient (yes/no). Do *not* solve the equations.

1. $y'' + y'y = 0$
2. $y'' + y' + y = 0$
3. $y'' + y' + y = t^2$
4. $y'' + ty' + (1 + t^2)y^2 = 0$
5. $3t^2y'' + 2ty' + y = e^{2t}$
6. $y'' + \sqrt{y'} + y = t$
7. $y'' + \sqrt{t}y' + y = \sqrt{t}$
8. $y'' - 2y = ty$
9. $y'' + 2y + t \sin y = 0$
10. $y'' + 2y' + (\sin t)y = 0$
11. $t^2y'' + ty' + (t^2 - 5)y = 0$

12–13. For the given differential operator L , compute $L(y)$ for each given y .

12. $L = tD^2 + 1$

1. $y(t) = 1$
2. $y(t) = t$
3. $y(t) = e^{-t}$
4. $y(t) = \cos 2t$

13. $L = t^2D^2 + tD - 1$

1. $y(t) = \frac{1}{t}$
2. $y(t) = 1$
3. $y(t) = t$
4. $y(t) = t^r$

14. The differential equation $t^2y'' + ty' - y = t^{\frac{1}{2}}$, $t > 0$ has a solution of the form $y_p(t) = Ct^{\frac{1}{2}}$. Find C .

15. The differential equation $ty'' + (t-1)y' - y = t^2e^{-t}$ has a solution of the form $y_p(t) = Ct^2e^{-t}$. Find C .

16. Let $L(y) = (1 + t^2)y'' - 4ty' + 6y$

1. Check that $y(t) = t$ is a solution to the differential equation $L(y) = 2t$.
2. Check that $y_1(t) = 1 - 3t^2$ and $y_2(t) = t - \frac{t^3}{3}$ are two solutions to the differential equation $L(y) = 0$.

3. Using the results of Parts (1) and (2), find a solution to each of the following initial value problems:

a. $(1 + t^2)y'' - 4ty' + 6y = 2t, \quad y(0) = 1, \quad y'(0) = 0.$

b. $(1 + t^2)y'' - 4ty' + 6y = 2t, \quad y(0) = 0, \quad y'(0) = 1.$

c. $(1 + t^2)y'' - 4ty' + 6y = 2t, \quad y(0) = -1, \quad y'(0) = 4.$

d. $(1 + t^2)y'' - 4ty' + 6y = 2t, \quad y(0) = a, \quad y'(0) = b, \text{ where } a, b \in \mathbb{R}.$

17. Let $L(y) = (t - 1)y'' - ty' + y$:

1. Check that $y(t) = e^{-t}$ is a solution to the differential equation $L(y) = 2te^{-t}$.

2. Check that $y_1(t) = e^t$ and $y_2(t) = t$ are two solutions to the differential equation $L(y) = 0$.

3. Using the results of Parts (1) and (2), find a solution to each of the following initial value problems:

a. $(t - 1)y'' - ty' + y = 2te^{-t} \quad y(0) = 0, \quad y'(0) = 0.$

b. $(t - 1)y'' - ty' + y = 2te^{-t} \quad y(0) = 1, \quad y'(0) = 0.$

c. $(t - 1)y'' - ty' + y = 2te^{-t} \quad y(0) = 0, \quad y'(0) = 1.$

d. $(t - 1)y'' - ty' + y = 2te^{-t}, \quad y(0) = a, \quad y'(0) = b, \text{ where } a, b \in \mathbb{R}.$

18. Let $L(y) = t^2y'' - 4ty' + 6y$:

1. Check that $y(t) = \frac{1}{6}t^5$ is a solution to the differential equation $L(y) = t^5$.

2. Check that $y_1(t) = t^2$ and $y_2(t) = t^3$ are two solutions to the differential equation $L(y) = 0$.

3. Using the results of Parts (1) and (2), find a solution to each of the following initial value problems:

a. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = 1, \quad y'(1) = 0.$

b. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = 0, \quad y'(1) = 1.$

c. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = -1, \quad y'(1) = 3.$

d. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = a, \quad y'(1) = b, \text{ where } a, b \in \mathbb{R}.$

19–24. For each of the following differential equations, find the largest interval on which a unique solution of the initial value problem

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

is guaranteed by Theorem 6, if initial conditions, $y(t_0) = y_1, \quad y'(t_0) = y_2$, are given at t_0 .

19. $t^2y'' + 3ty' - y = t^4 \quad (t_0 = -1)$

20. $y'' - 2y' - 2y = \frac{1 + t^2}{1 - t^2} \quad (t_0 = 2)$

21. $(\sin t)y'' + y = \cos t \quad (t_0 = \frac{\pi}{2})$
 22. $(1 + t^2)y'' - ty' + t^2y = \cos t \quad (t_0 = 0)$
 23. $y'' + \sqrt{t}y' - \sqrt{t-3}y = 0 \quad (t_0 = 10)$
 24. $t(t^2 - 4)y'' + y = e^t \quad (t_0 = 1)$
25. The functions $y_1(t) = t^2$ and $y_2(t) = t^3$ are two distinct solutions of the initial value problem

$$t^2y'' - 4ty' + 6y = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

Why does this not violate the uniqueness part of Theorem 6?

26. Let $y(t)$ be a solution of the differential equation

$$y'' + a_1(t)y' + a_0(t)y = 0.$$

We assume that $a_1(t)$ and $a_0(t)$ are continuous functions on an interval I , so that Theorem 6 implies that a solution y is defined on I . Show that if the graph of $y(t)$ is tangent to the t -axis at some point t_0 of I , then $y(t) = 0$ for all $t \in I$.
Hint: If the graph of $y(t)$ is tangent to the t -axis at $(t_0, 0)$, what does this say about $y(t_0)$ and $y'(t_0)$?

27. More generally, let $y_1(t)$ and $y_2(t)$ be two solutions of the differential equation

$$y'' + a_1(t)y' + a_0(t)y = f(t),$$

where, as usual, we assume that $a_1(t)$, $a_0(t)$, and $f(t)$ are continuous functions on an interval I , so that Theorem 6 implies that y_1 and y_2 are defined on I . Show that if the graphs of $y_1(t)$ and $y_2(t)$ are tangent at some point t_0 of I , then $y_1(t) = y_2(t)$ for all $t \in I$.

5.2 The Homogeneous Case

In this section, we are concerned with a concise description of the solution set of the homogeneous linear differential equation

$$L(y) = a_2(t)y'' + a_1(t)y' + a_0(t)y = 0 \quad (1)$$

The main result, Theorem 2 given below, shows that we will in principle be able to find two linearly independent functions y_1 and y_2 such that all solutions to (1) are of the form $c_1y_1 + c_2y_2$, for some constants c_1 and c_2 . This is just like the second order constant coefficient case. In fact, if $q(s)$ is a characteristic polynomial of degree 2, then $\mathcal{B}_q = \{y_1, y_2\}$ is a set of two linearly independent functions that span the solution set of the corresponding homogeneous constant coefficient differential equation. In Sect. 3.2, we introduced the concept of linear independence for a set of n functions. Let us recall this important concept in the case $n = 2$.

Linear Independence

Two functions y_1 and y_2 defined on some interval I are said to be **linearly independent** if the equation

$$c_1y_1 + c_2y_2 = 0 \quad (2)$$

implies that c_1 and c_2 are both 0. Otherwise, we call y_1 and y_2 **linearly dependent**.

Example 1. Show that the functions

$$y_1(t) = \frac{\cos t}{t} \quad \text{and} \quad y_2(t) = \frac{\sin t}{t},$$

defined on the interval $(0, \infty)$ are linearly independent.

► **Solution.** The equation $c_1 \frac{\cos t}{t} + c_2 \frac{\sin t}{t} = 0$ on $(0, \infty)$ implies $c_1 \cos t + c_2 \sin t = 0$. Evaluating at $t = \frac{\pi}{2}$ and $t = \pi$ gives

$$\begin{aligned} c_2 &= 0 \\ -c_1 &= 0. \end{aligned}$$

It follows that y_1 and y_2 are independent. ◀

In Sect. 3.2, we provided several other examples that we encourage the student to reconsider by way of review.

The Main Theorem for the Homogeneous Case

Theorem 2. Let $L = a_2(t)D^2 + a_1(t)D + a_0(t)$, where $a_0(t)$, $a_1(t)$, and $a_2(t)$ are continuous functions on an interval I . Assume $a_2(t) \neq 0$ for all $t \in I$.

1. There are two linearly independent solutions to $Ly = 0$.
2. If y_1 and y_2 are any two linearly independent solutions to $Ly = 0$, then any homogeneous solution y can be written $y = c_1y_1 + c_2y_2$, for some $c_1, c_2 \in \mathbb{R}$.

Proof. Let $t_0 \in I$. By Theorem 6 of Sect. 5.1, there are functions, ψ_1 and ψ_2 , that are solutions to the initial value problems $L(y) = 0$, with initial conditions $y(t_0) = 1$, $y'(t_0) = 0$, and $y(t_0) = 0$, $y'(t_0) = 1$, respectively. Suppose $c_1\psi_1 + c_2\psi_2 = 0$. Then

$$c_1\psi_1(t_0) + c_2\psi_2(t_0) = 0.$$

Since $\psi_1(t_0) = 1$ and $\psi_2(t_0) = 0$, it follows that $c_1 = 0$. Similarly, we have

$$c_1\psi_1'(t_0) + c_2\psi_2'(t_0) = 0.$$

Since $\psi_1'(t_0) = 0$ and $\psi_2'(t_0) = 1$, it follows that $c_2 = 0$. Therefore, ψ_1 and ψ_2 are linearly independent. This proves (1).

Suppose y is a homogeneous solution. Let $r = y(t_0)$ and $s = y'(t_0)$. By Theorem 3 of Sect. 5.1, the function $r\psi_1 + s\psi_2$ is a solution to $Ly = 0$. Furthermore,

$$\begin{aligned} r\psi_1(t_0) + s\psi_2(t_0) &= r \\ \text{and } r\psi_1'(t_0) + s\psi_2'(t_0) &= s. \end{aligned}$$

This means the $r\psi_1 + s\psi_2$ and y satisfy the same initial conditions. By the uniqueness part of Theorem 6 of Sect. 5.1, they are equal. Thus, $y = r\psi_1 + s\psi_2$, that is, every homogeneous solution is a linear combination of ψ_1 and ψ_2 .

Now suppose y_1 and y_2 are any two linearly independent homogeneous solutions and suppose y is any other solution. From the argument above, we can write

$$\begin{aligned} y_1 &= a\psi_1 + b\psi_2, \\ y_2 &= c\psi_1 + d\psi_2, \end{aligned}$$

which in matrix form can be written

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

We multiply both sides of this matrix equation by the adjoint $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ to obtain

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = (ad - bc) \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

Suppose $ad - bc = 0$. Then

$$\begin{aligned} dy_1 - by_2 &= 0 \\ \text{and } -cy_1 + ay_2 &= 0. \end{aligned}$$

But since y_1 and y_2 are independent, this implies that a, b, c , and d are zero, which in turn implies that y_1 and y_2 are both zero. But this cannot be. We conclude that $ad - bc \neq 0$. We can now write ψ_1 and ψ_2 each as a linear combination of y_1 and y_2 . Specifically,

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Since y is a linear combination of ψ_1 and ψ_2 , it follows that y is a linear combination of y_1 and y_2 . \square

Remark 3. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that appears in the proof above appears in other contexts as well. If y_1 and y_2 are functions, recall from Sect. 3.2 that we defined the **Wronskian matrix** by

$$W(y_1, y_2)(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

and the **Wronskian** by

$$w(y_1, y_2)(t) = \det W(y_1, y_2)(t).$$

If y_1, y_2 are as in the proof above, then the relations

$$\begin{aligned} y_1 &= a\psi_1 + b\psi_2, \\ y_2 &= c\psi_1 + d\psi_2, \end{aligned}$$

in the proof, when evaluated at t_0 imply that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} = W(y_1, y_2)(t_0).$$

Since it was shown that $ad - bc \neq 0$, we have $w(y_1, y_2) \neq 0$. On the other hand, given any two differentiable functions, y_1 and y_2 (not necessarily homogeneous solutions of a linear differential equation), whose Wronskian is a nonzero function, then it is easy to see that y_1 and y_2 are independent. For suppose t_0 is chosen so that $w(y_1, y_2)(t_0) \neq 0$ and $c_1 y_1 + c_2 y_2 = 0$. Then $c_1 y_1' + c_2 y_2' = 0$, and we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 y_1(t_0) + c_2 y_2(t_0) \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) \end{bmatrix} = W(y_1, y_2) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Simple matrix algebra² gives $c_1 = 0$ and $c_2 = 0$. Hence, y_1 and y_2 are linearly independent.

We have thus shown the following proposition.

Proposition 4. *Suppose L satisfies the conditions of Theorem 2. Suppose y_1 and y_2 are solutions to $L y = 0$. Then y_1 and y_2 are linearly independent if and only if*

$$w(y_1, y_2) \neq 0.$$

The following theorem extends Abel's theorem given in Theorem 8 of Sect. 3.3. You are asked to prove this in the exercises.

Theorem 5 (Abel's Formula). *Suppose f_1 and f_2 are solutions to the second order linear differential equation*

$$y'' + a_1(t)y' + a_0(t)y = 0,$$

where a_0 and a_1 are continuous functions on an interval I . Let $t_0 \in I$. Then

$$w(f_1, f_2)(t) = K e^{-\int_{t_0}^t a_1(x) dx}, \quad (3)$$

for some constant K . Furthermore, if initial conditions $y(t_0)$ and $y'(t_0)$ are given, then

$$K = w(f_1, f_2)(t_0).$$

Remark 6. Let us now summarize what Theorems 3, 6 of Sect. 5.1, and 2 tell us. In order to solve $L(y) = f$ (satisfying the continuity hypotheses), we first need to find a particular solution y_p , which exists by the existence and uniqueness theorem. Next, Theorem 2 says that if y_1 and y_2 are any two linearly independent solutions of the associated homogeneous equation $L(y) = 0$, then all of the solutions of the associated homogeneous equation are of the form $c_1 y_1 + c_2 y_2$. Theorem 3 of Sect. 5.1 now tells us that the **general solution** to $L(y) = f$ is of the form

$$\{y_p + c_1 y_1 + c_2 y_2 : c_1, c_2 \in \mathbb{R}\}.$$

Furthermore, any set of initial conditions uniquely determines the constants c_1 and c_2 .

²cf. Chapter 8 for a discussion of matrices.

A set $\{y_1, y_2\}$ of linearly independent solutions to the homogeneous equation $L(y) = 0$ is called a **fundamental set** for the second order linear differential operator L . A fundamental set is a basis of the linear space of homogeneous solutions (cf. Sect. 3.2 for the definition of a basis). Furthermore, the standard basis, \mathcal{B}_q , in the context of constant coefficient differential equations, is a fundamental set.

In the following sections, we will develop methods, under suitable assumptions, for finding a fundamental set for L and a particular solution to the differential equation $L(y) = f$. For now, let us illustrate the main theorems with a couple of examples.

Example 7. Consider the differential equation

$$t^2 y'' + t y' + y = 0.$$

Suppose $y_1(t) = \cos(\ln t)$ and $y_2(t) = \sin(\ln t)$ are solutions. Determine the solution set.

► **Solution.** We begin by computing the Wronskian of $\{\cos \ln t, \sin \ln t\}$:

$$\begin{aligned} w(\cos \ln t, \sin \ln t) &= \det \begin{pmatrix} \cos \ln t & \sin \ln t \\ -\frac{\sin \ln t}{t} & \frac{\cos \ln t}{t} \end{pmatrix} \\ &= \frac{\cos^2 \ln t + \sin^2 \ln t}{t} \\ &= \frac{1}{t}. \end{aligned}$$

Proposition 4 implies that $\{\cos(\ln t), \sin(\ln t)\}$ is linearly independent on $(0, \infty)$ and thus a fundamental set for $L(y) = 0$. Theorem 2 now implies that

$$\{c_1 \cos(\ln t) + c_2 \sin(\ln t) : c_1, c_2 \in \mathbb{R}\}$$

is the solution set. ◀

Example 8. Consider the differential equation

$$t y'' + 2 y' + t y = 2 \cos t.$$

Determine the solution set.

► **Solution.** By Example 1 of Sect. 5.1, a particular solution is $y_p = \sin t$, and $y_1 = \frac{\cos t}{t}$ and $y_2 = \frac{\sin t}{t}$ are homogeneous solutions. By Example 1, the homogeneous

solutions y_1 and y_2 are linearly independent. By Theorem 2, all homogeneous solutions are of the form

$$y_h = c_1 y_1 + c_2 y_2 = c_1 \frac{\cos t}{t} + c_2 \frac{\sin t}{t}.$$

It follows from linearity, Theorem 3 of Sect. 5.1, that the solution set is

$$\left\{ y = y_p + y_h = \sin t + c_1 \frac{\cos t}{t} + c_2 \frac{\sin t}{t}, c_1, c_2 \in \mathbb{R} \right\}. \quad \blacktriangleleft$$

Exercises

1–6. Determine if each of the following pairs of functions are linearly independent or linearly dependent.

1. $y_1(t) = 2t$, $y_2(t) = 5t$
2. $y_1(t) = 2^t$, $y_2(t) = 5^t$
3. $y_1(t) = \ln t$, $y_2(t) = t \ln t$ on the interval $(0, \infty)$
4. $y_1(t) = e^{2t+1}$, $y_2(t) = e^{2t-3}$
5. $y_1(t) = \ln(2t)$, $y_2(t) = \ln(5t)$ on the interval $(0, \infty)$
6. $y_1(t) = \ln t^2$, $y_2(t) = \ln t^5$

7–9. For each exercise below, verify that the functions f_1 and f_2 satisfy the given differential equation. Verify Abel's formula as given in Theorem 5 of Sect. 5.2 for the given initial point t_0 . Determine the solution set.

7. $(t-1)y'' - ty' + y = 0$, $f_1(t) = e^t - t$, $f_2(t) = t$, $t_0 = 0$
8. $(1+t^2)y'' - 2ty' + 2y = 0$, $f_1(t) = 1 - t^2$, $f_2(t) = t$, $t_0 = 1$
9. $t^2y'' + ty' + 4y = 0$, $f_1(t) = \cos(2 \ln t)$, $f_2(t) = \sin(2 \ln t)$, $t_0 = 1$

10. Prove Abel's formula as stated in Theorem 5. Hint, carefully look at the proof of Abel's formula given in the second order constant coefficient case (cf. Theorem 8 of Sect. 3.3).

11. 1. Verify that $y_1(t) = t^3$ and $y_2(t) = |t^3|$ are linearly independent on $(-\infty, \infty)$.
2. Show that the Wronskian, $w(y_1, y_2)(t) = 0$, for all $t \in \mathbb{R}$.
3. Explain why Parts (a) and (b) do not contradict Proposition 4.
4. Verify that $y_1(t)$ and $y_2(t)$ are solutions to the linear differential equation $t^2y'' - 2ty' = 0$, $y(0) = 0$, $y'(0) = 0$.
5. Explain why Parts (a), (b), and (d) do not contradict the existence and uniqueness theorem, Theorem 6 of Sect. 5.1.

5.3 The Cauchy–Euler Equations

When the coefficient functions of a second order linear differential equation are nonconstant, the corresponding equation can become very difficult to solve. In order to expect to find solutions, one must put certain restrictions on the coefficient functions. A class of nonconstant coefficient linear differential equations, known as Cauchy–Euler equations, have solutions that are easy to obtain.

A *Cauchy–Euler equation* is a second order linear differential equation of the following form:

$$at^2y'' + bty' + cy = 0, \quad (1)$$

where a, b , and c are real constants and $a \neq 0$. When put in standard form, we obtain

$$y'' + \frac{b}{at}y' + \frac{c}{at^2}y = 0.$$

The functions $\frac{b}{at}$ and $\frac{c}{at^2}$ are continuous everywhere except at 0. Thus, the existence and uniqueness theorem guarantees that solutions exist in either of the intervals $(-\infty, 0)$ or $(0, \infty)$. To work in a specific interval, we will assume $t > 0$. We will refer to $L = at^2\mathbf{D}^2 + bt\mathbf{D} + c$ as a *Cauchy–Euler operator*.

The Laplace transform method does not work in any simple fashion here. However, the simple change in variable $t = e^x$ will transform equation (1) into a constant coefficient linear differential equation. To see this, let $Y(x) = y(e^x)$. Then the chain rule gives

$$\begin{aligned} Y'(x) &= e^x y'(e^x) \\ \text{and } Y''(x) &= e^x y'(e^x) + (e^x)^2 y''(e^x) \\ &= Y'(x) + (e^x)^2 y''(e^x). \end{aligned}$$

Thus,

$$\begin{aligned} a(e^x)^2 y''(e^x) &= aY''(x) - aY'(x) \\ be^x y'(e^x) &= bY'(x) \\ cy(e^x) &= cY(x). \end{aligned}$$

Addition of these terms gives

$$\begin{aligned} a(e^x)^2 y''(e^x) + be^x y'(e^x) + cy(e^x) &= aY''(x) - aY'(x) + bY'(x) + cY(x) \\ &= aY''(x) + (b - a)Y'(x) + cY(x). \end{aligned}$$

With t replaced by e^x in (1), we now obtain

$$aY''(x) + (b - a)Y'(x) + cY(x) = 0. \quad (2)$$

The polynomial

$$Q(s) = as^2 + (b - a)s + c$$

is the characteristic polynomial of (2) and known as the *indicial polynomial* of (1). Equation (2) is a second order constant coefficient differential equation and by now routine to solve. Its solutions depend on the way $Q(s)$ factors. We consider the three possibilities.

Q Has Distinct Real Roots

Suppose r_1 and r_2 are distinct roots to the indicial polynomial $Q(s)$. Then e^{r_1x} and e^{r_2x} are solutions to (2). Solutions to (1) are obtained by the substitution $x = \ln t$: we have $e^{r_1x} = e^{r_1 \ln t} = t^{r_1}$ and similarly $e^{r_2x} = t^{r_2}$. Since t^{r_1} is not a multiple of t^{r_2} , they are independent, and hence,

$$\{t^{r_1}, t^{r_2}\}$$

is a fundamental set for $L(y) = 0$.

Example 1. Find a fundamental set and general solution for the equation $t^2y'' - 2y = 0$.

► **Solution.** The indicial polynomial is $Q(s) = s^2 - s - 2 = (s - 2)(s + 1)$ and it has 2 and -1 as roots, and thus, $\{t^2, t^{-1}\}$ is a fundamental set for this Cauchy–Euler equation. The general solution is $y = c_1t^2 + c_2t^{-1}$. ◀

Q Has a Double Root

Suppose r is a double root of Q . Then e^{rx} and xe^{rx} are independent solutions to (2). The substitution $x = \ln t$ then gives t^r and $t^r \ln t$ as independent solutions to (1). Hence,

$$\{t^r, t^r \ln t\}$$

is a fundamental set for $L(y) = 0$.

Example 2. Find a fundamental set and the general solution for the equation $4t^2y'' + 8ty' + y = 0$.

► **Solution.** The indicial polynomial is $Q(s) = 4s^2 + 4s + 1 = (2s + 1)^2$ and has $-\frac{1}{2}$ as a root with multiplicity 2. Thus, $\{t^{-\frac{1}{2}}, t^{-\frac{1}{2}} \ln t\}$ is a fundamental set. The general solution is $y = c_1t^{-\frac{1}{2}} + c_2t^{-\frac{1}{2}} \ln t$. ◀

Q Has Conjugate Complex Roots

Suppose Q has complex roots $\alpha \pm i\beta$, where $\beta \neq 0$. Then $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are independent solutions to (2). The substitution $x = \ln t$ then gives

$$\{t^\alpha \cos(\beta \ln t), t^\alpha \sin(\beta \ln t)\}$$

as a fundamental set for $\mathbf{L}y = 0$.

Example 3. Find a fundamental set and the general solution for the equation $t^2 y'' + ty' + y = 0$.

► **Solution.** The indicial polynomial is $Q(s) = s^2 + 1$ which has $\pm i$ as complex roots. Theorem 4 implies that $\{\cos \ln t, \sin \ln t\}$ is a fundamental set. The general solution is $y = c_1 \cos \ln t + c_2 \sin \ln t$. ◀

We now summarize the above results into one theorem.

Theorem 4. Let $\mathbf{L} = at^2 \mathbf{D}^2 + bt \mathbf{D} + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Let $Q(s) = as^2 + (b - a)s + c$ be the indicial polynomial.

1. If r_1 and r_2 are distinct real roots of $Q(s)$, then

$$\{t^{r_1}, t^{r_2}\}$$

is a fundamental set for $\mathbf{L}(y) = 0$.

2. If r is a double root of $Q(s)$, then

$$\{t^r, t^r \ln t\}$$

is a fundamental set for $\mathbf{L}(y) = 0$.

3. If $\alpha \pm i\beta$ are complex conjugate roots of $Q(s)$, $\beta \neq 0$, then

$$\{t^\alpha \sin(\beta \ln t), t^\alpha \cos(\beta \ln t)\}$$

is a fundamental set for $\mathbf{L}(y) = 0$.

Exercises

1–11. Find the general solution of each of the following homogeneous Cauchy–Euler equations on the interval $(0, \infty)$.

1. $t^2 y'' + 2ty' - 2y = 0$

2. $2t^2 y'' - 5ty' + 3y = 0$

3. $9t^2 y'' + 3ty' + y = 0$

4. $t^2 y'' + ty' - 2y = 0$

5. $4t^2 y'' + y = 0$

6. $t^2 y'' - 3ty' - 21y = 0$

7. $t^2 y'' + 7ty' + 9y = 0$

8. $t^2 y'' + y = 0$

9. $t^2 y'' + ty' - 4y = 0$

10. $t^2 y'' + ty' + 4y = 0$

11. $t^2 y'' - 3ty' + 13y = 0$

12–15. Solve each of the following initial value problems.

12. $t^2 y'' + 2ty' - 2y = 0,$

$y(1) = 0, y'(1) = 1$

13. $4t^2 y'' + y = 0,$

$y(1) = 2, y'(1) = 0$

14. $t^2 y'' + ty' + 4y = 0,$

$y(1) = -3, y'(1) = 4$

15. $t^2 y'' - 4ty' + 6y = 0,$

$y(0) = 1, y'(0) = -1$

5.4 Laplace Transform Methods

In this section, we will develop some further properties of the Laplace transform and use them to solve some linear differential equations with nonconstant coefficient functions. However, we will see that the use of the Laplace transform is limited. Several new Laplace transform rules and formulas are developed in this section. For quick reference Tables 5.1 and 5.2 in Sect. 5.7 summarize these results.

Let us begin by recalling an important definition we saw in Sect. 2.2. A continuous function f on $[0, \infty)$ is said to be of **exponential type with order a** if there is a constant K such that

$$|f(t)| \leq Ke^{at}$$

for all $t \in [0, \infty)$. If the order is not important to the discussion, we will just say f is of **exponential type**. A function of exponential type has limited growth; it cannot grow faster than a multiple of an exponential function. The above inequality means

$$-Ke^{at} \leq f(t) \leq Ke^{at},$$

for all $t \in [0, \infty)$ as illustrated in Fig. 5.1, where the boldfaced curve, $f(t)$, lies between the upper and lower exponential functions. If f is of exponential type, then Proposition 3 of Sect. 2.2 tells us that $F(s) = \mathcal{L}\{f\}(s)$ exists and

$$\lim_{s \rightarrow \infty} F(s) = 0.$$

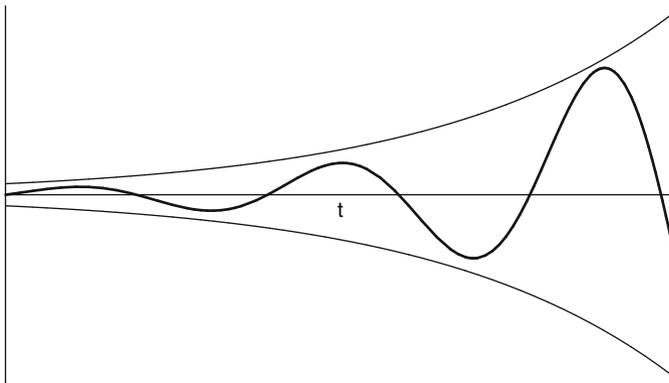


Fig. 5.1 The exponential function Ke^{at} bounds $f(t)$

Asymptotic Values

An interesting property of the Laplace transform is that certain limiting values of $f(t)$ can be deduced from its Laplace transform and vice versa.

Theorem 1 (Initial Value Theorem). *Suppose f and its derivative f' are of exponential order. Let $F(s) = \mathcal{L}\{f(t)\}(s)$. Then*

Initial Value Principle

$$f(0) = \lim_{s \rightarrow \infty} sF(s).$$

Proof. Let $H(s) = \mathcal{L}\{f'(t)\}(s)$. By Proposition 3 of Sect. 2.2, we have

$$0 = \lim_{s \rightarrow \infty} H(s) = \lim_{s \rightarrow \infty} (sF(s) - f(0)) = \lim_{s \rightarrow \infty} (sF(s) - f(0)).$$

This implies the result. □

Example 2. Verify the initial value theorem for $f(t) = \cos at$.

► **Solution.** On the one hand, $\cos at|_{t=0} = 1$. On the other hand,

$$s\mathcal{L}\{\cos at\}(s) = \frac{s^2}{s^2 + a^2}$$

which has limit 1 as $s \rightarrow \infty$. ◀

Theorem 3 (Final Value Theorem). *Suppose f and f' are of exponential type and $\lim_{t \rightarrow \infty} f(t)$ exists. If $F(s) = \mathcal{L}\{f(t)\}(s)$, then*

Final Value Principle

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s).$$

Proof. Let $H(s) = \mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$. Then $sF(s) = H(s) + f(0)$ and

$$\begin{aligned} \lim_{s \rightarrow 0} sF(s) &= \lim_{s \rightarrow 0} H(s) + f(0) \\ &= \lim_{s \rightarrow 0} \lim_{M \rightarrow \infty} \int_0^M e^{-st} f'(t) dt + f(0) \\ &= \lim_{M \rightarrow \infty} \int_0^M f'(t) dt + f(0) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{M \rightarrow \infty} f(M) - f(0) + f(0) \\
 &= \lim_{M \rightarrow \infty} f(M).
 \end{aligned}$$

The interchange of the limit operations in line 2 above can be justified for functions of exponential type. □

Integration in Transform Space

The transform derivative principle, Theorem 20 of Sect. 2.2, tells us that multiplication of an input function by $-t$ induces differentiation of the Laplace transform. One might expect then that division by $-t$ will induce integration in the transform space. This idea is valid but we must be careful about assumptions. First, if $f(t)$ has a Laplace transform, it is not necessarily the case that $f(t)/t$ will likewise. For example, the constant function $f(t) = 1$ has Laplace transform $\frac{1}{s}$ but $\frac{f(t)}{t} = \frac{1}{t}$ does not have a Laplace transform. Second, integration produces an arbitrary constant of integration. What is this constant? The precise statement is as follows:

Theorem 4 (Integration in Transform Space). *Suppose f is of exponential type with order a and $\frac{f(t)}{t}$ has a continuous extension to 0, that is, $\lim_{t \rightarrow 0^+} \frac{f(t)}{t}$ exists. Then $\frac{f(t)}{t}$ is of exponential type with order a and*

Transform Integral Principle

$$\mathcal{L} \left\{ \frac{f(t)}{t} \right\} (s) = \int_s^\infty F(\sigma) d\sigma,$$

where $s > a$.

Proof. Let $L = \lim_{t \rightarrow 0^+} \frac{f(t)}{t}$ and define

$$h(t) = \begin{cases} \frac{f(t)}{t} & \text{if } t > 0 \\ L & \text{if } t = 0 \end{cases}.$$

Since f is continuous, so is h . Since f is of exponential type with order a , there is a K so that $|f(t)| \leq Ke^{at}$. Since $\frac{1}{t} \leq 1$ on $[1, \infty)$,

$$|h(t)| = \left| \frac{f(t)}{t} \right| \leq |f(t)| \leq Ke^{at},$$

for all $t \geq 1$. Since h is continuous on $[0, 1]$, it is bounded by B , say. Thus, $|h(t)| \leq B \leq Be^{at}$ for all $t \in [0, 1]$. If M is the larger of K and B , then

$$|h(t)| \leq M e^{at},$$

for all $t \in [0, \infty)$, and hence h is of exponential type. Let $H(s) = \mathcal{L}\{h(t)\}(s)$ and $F(s) = \mathcal{L}\{f(t)\}(s)$. Then, since $-th(t) = -f(t)$, we have, by Theorem 20 of Sect. 2.2, $H'(s) = -F(s)$. Thus H is an antiderivative of $-F$, and we have

$$H(s) = - \int_a^s F(\sigma) d\sigma + C.$$

Proposition 3 of Sect. 2.2 implies $0 = \lim_{s \rightarrow \infty} H(s) = - \int_a^\infty F(\sigma) d\sigma + C$, and hence, $C = \int_a^\infty F(\sigma) d\sigma$. Therefore,

$$\begin{aligned} \mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) &= H(s) \\ &= \int_s^a F(\sigma) d\sigma + \int_a^\infty F(\sigma) d\sigma \\ &= \int_s^\infty F(\sigma) d\sigma. \end{aligned} \quad \square$$

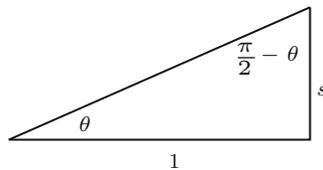
The Laplace transform of several new functions can now be deduced from this theorem. Consider an example.

Example 5. Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$.

► **Solution.** Since $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$, Theorem 4 applies to give

$$\begin{aligned} \mathcal{L}\left\{\frac{\sin t}{t}\right\}(s) &= \int_s^\infty \frac{1}{\sigma^2 + 1} d\sigma \\ &= \tan^{-1} \sigma \Big|_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}(s) \\ &= \tan^{-1} \frac{1}{s}. \end{aligned}$$

The last line can be seen by considering the right triangle



where $\theta = \tan^{-1} s$. ◀

Solving Linear Differential Equations

We now consider by example how one can use the Laplace transform method to solve some differential equations.

Example 6. Find a solution of exponential type that solves

$$ty'' - (1 + t)y' + y = 0.$$

► **Solution.** Note that the existence and uniqueness theorem implies that solutions exist on intervals that do not contain 0. We presume that such a solution has a continuous extension to $t = 0$ and is of exponential type. Let y be such a solution. Let $y(0) = y_0$, $y'(0) = y_1$, and $Y(s) = \mathcal{L}\{y(t)\}(s)$. Application of transform derivative principle, Theorem 20 of Sect. 2.2, to each component of the differential equation gives

$$\begin{aligned} \mathcal{L}\{ty''\} &= -(s^2Y(s) - sy(0) - y'(0))' \\ &= -(2sY(s) + s^2Y'(s) - y(0)) \\ \mathcal{L}\{-(1+t)y'\} &= \mathcal{L}\{-y'\} + \mathcal{L}\{-ty'\} \\ &= -sY(s) + y_0 + (sY(s) - y_0)' \\ &= sY'(s) - (s-1)Y(s) + y_0 \\ \mathcal{L}\{y\} &= Y(s). \end{aligned}$$

The sum of the left-hand terms is given to be 0. Thus, adding the right-hand terms and simplifying give

$$(s - s^2)Y'(s) + (-3s + 2)Y(s) + 2y_0 = 0,$$

which can be rewritten in the following way:

$$Y'(s) + \frac{3s-2}{s(s-1)}Y(s) = \frac{2y_0}{s(s-1)}.$$

This equation is a first order linear differential equation in $Y(s)$. Since $\frac{3s-2}{s(s-1)} = \frac{2}{s} + \frac{1}{s-1}$, it is easy to see that an integrating factor is $I = s^2(s-1)$, and hence,

$$(IY(s))' = 2y_0s.$$

Integrating and solving for Y give

$$Y(s) = \frac{y_0}{s-1} + \frac{c}{s^2(s-1)}.$$

The inverse Laplace transform is $y(t) = y_0 e^t + c(e^t - t - 1)$. For simplicity, we can write this solution in the form

$$y(t) = c_1 e^t + c_2(t + 1),$$

where $c_1 = y_0 + c$ and $c_2 = -c$. It is easy to verify that e^t and $t + 1$ are linearly independent and solutions to the given differential equation. ◀

Example 7. Find a solution of exponential type that solves

$$ty'' + 2y' + ty = 0.$$

► **Solution.** Again, assume y is a solution of exponential type. Let $Y(s) = \mathcal{L}\{y(t)\}(s)$. As in the preceding example, we apply the Laplace transform and simplify. The result is a simple linear differential equation:

$$Y'(s) = \frac{-y_0}{s^2 + 1},$$

where y_0 is the initial condition $y(0) = y_0$. Integration gives $Y(s) = y_0(-\tan^{-1} s + C)$. By Proposition 3 of Sect. 2.2, we have $0 = \lim_{s \rightarrow \infty} Y(s) = y_0(-\frac{\pi}{2} + C)$ which implies $C = \frac{\pi}{2}$ and

$$Y(s) = y_0 \left(\frac{\pi}{2} - \tan^{-1} s \right) = y_0 \tan^{-1} \frac{1}{s}.$$

By Example 5, we get

$$y(t) = y_0 \frac{\sin t}{t}. \quad \blacktriangleleft$$

Theorem 2 of Sect. 5.2 implies that there are two linearly independent solutions. The Laplace transform method has found only one, namely, $\frac{\sin t}{t}$. In Sect. 5.5, we will introduce a technique that will find another independent solution. When applied to this example, we will find $y(t) = \frac{\cos t}{t}$ is another solution. (cf. Example 2 of Sect. 5.5.) It is easy to check that the Laplace transform of $\frac{\cos t}{t}$ does not exist, and thus, the Laplace transform method cannot find it as a solution. Furthermore, the constant of integration, C , in this example cannot be arbitrary because of Proposition 3 of Sect. 2.2. It frequently happens in examples that C must be carefully chosen.

We observe that the presence of the linear factor t in Examples 6 and 7 produces a differential equation of order 1 which can be solved by techniques learned in Chapter 1. Correspondingly, the presence of higher order terms, t^n , produces differential equations of order n . For example, the Laplace transform applied to the differential equation $t^2 y'' + 6y = 0$ gives, after a short calculation, $s^2 Y''(s) + 4s Y'(s) + 8Y(s) = 0$. The resulting differential equation in $Y(s)$ is still second order and no simpler than the original. In fact, both are Cauchy–Euler. Thus,

when the coefficient functions are polynomial of order greater than one, the Laplace transform method will generally be of little use. For this reason, we will usually limit our examples to second order linear differential equations with coefficient functions that are linear terms, that is, of the form $at + b$. Even with this restriction, we still will need to solve a first order differential equation in $Y(s)$ and determine its inverse Laplace transform; not always easy problems.

Laguerre Polynomials

The **Laguerre polynomial**, $\ell_n(t)$, of order n is the polynomial solution to Laguerre's differential equation

$$ty'' + (1 - t)y' + ny = 0,$$

where $y(0) = 1$ and n is a nonnegative integer.

Proposition 8. *The n th Laguerre polynomial is given by*

$$\ell_n(t) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t^k}{k!}$$

and

$$\mathcal{L}\{\ell_n(t)\}(s) = \frac{(s-1)^n}{s^{n+1}}.$$

Proof. Taking the Laplace transform of Laguerre's differential equation gives

$$(s^2 - s)Y'(s) + (s - (1 + n))Y(s) = 0$$

and hence, $Y(s) = C \frac{(s-1)^n}{s^{n+1}}$. By the initial value theorem,

$$1 = y(0) = \lim_{s \rightarrow \infty} C \frac{s(s-1)^n}{s^{n+1}} = C.$$

Now using the binomial theorem, we get $(s-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k s^{n-k}$ and hence $Y(s) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{s^{k+1}}$. It now follows by inversion that $y(t) = \ell_n(t) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t^k}{k!}$. \square

It is easy to see that the first five Laguerre polynomials are:

$$\ell_0(t) = 1$$

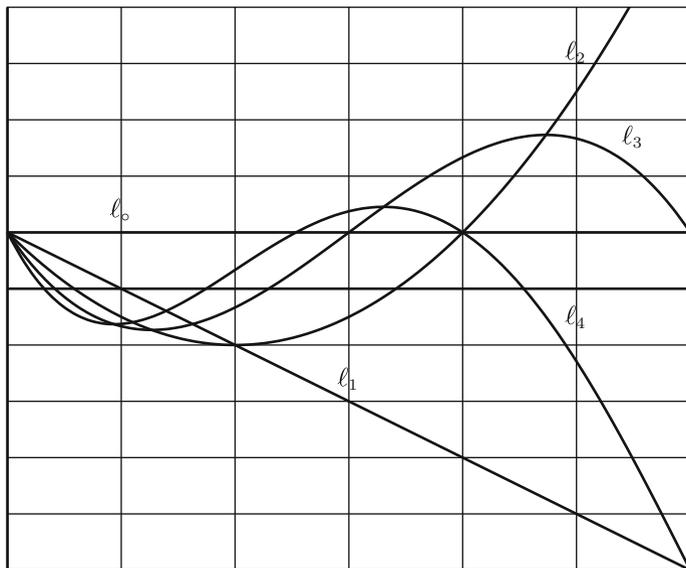
$$\ell_1(t) = 1 - t$$

$$\ell_2(t) = 1 - 2t + \frac{t^2}{2}$$

$$\ell_3(t) = 1 - 3t + \frac{3t^2}{2} - \frac{t^3}{6}$$

$$\ell_4(t) = 1 - 4t + 3t^2 - \frac{2t^3}{3} + \frac{t^4}{24}.$$

Below are their graphs on the interval $[0, 6]$.



Define the following differential operators:

$$E_{\circ} = 2t\mathbf{D}^2 + (2 - 2t)\mathbf{D} - 1,$$

$$E_{+} = t\mathbf{D}^2 + (1 - 2t)\mathbf{D} + (t - 1),$$

$$E_{-} = t\mathbf{D}^2 + \mathbf{D}.$$

Theorem 9. We have the following differential relationships among the Laguerre polynomials:

1. $E_{\circ}\ell_n = -(2n + 1)\ell_n$
2. $E_{+}\ell_n = -(n + 1)\ell_{n+1}$
3. $E_{-}\ell_n = -n\ell_{n-1}$.

Proof.

1. Let $A_n = t\mathbf{D}^2 + (1 - t)\mathbf{D} + n$ be Laguerre's differential equation. Then from the defining equation of the Laguerre polynomial ℓ_n , we have $A_n\ell_n = 0$.

Multiply this equation by 2 and add $-(1 + 2n)\ell_n$ to both sides. This gives $E_\circ \ell_n = -(2n + 1)\ell_n$

2. A simple observation gives $E_+ = A_n - t\mathbf{D} - (1 - t - n)$. Since $A_n \ell_n = 0$, it is enough to verify that $t\ell'_n + (1 - t - n)\ell_n = (n + 1)\ell_{n+1}$. This we do in transform space. Let $L_n = \mathcal{L}\{\ell_n\}$.

$$\begin{aligned} \mathcal{L}\{t\ell'_n + (1 - t - n)\ell_n\}(s) &= -(sL_n(s) - \ell_n(0))' + (1 - n)L_n(s) + L'_n(s) \\ &= -(L_n + sL'_n) + (1 - n)L_n + L'_n \\ &= -(s - 1)L'_n - nL_n \\ &= \frac{-(s - 1)^n}{s^{n+2}}(ns - (n + 1)(s - 1) - ns) \\ &= (n + 1)\mathcal{L}\{\ell_{n+1}\}(s). \end{aligned}$$

3. This equation is proved in a similar manner as above. We leave the details to the exercises. \blacktriangleleft

For a differential operator A , let $A^2y = A(Ay)$, $A^3y = A(A(Ay))$, etc. It is easy to verify by induction and the use of Theorem 9 that

$$\frac{(-1)^n}{n!} E_+^n \ell_\circ = \ell_n.$$

The operator E_+ is called a **creation operator** since successive applications to ℓ_\circ creates all the other Laguerre polynomials. In a similar way, it is easy to verify that

$$E_-^m \ell_n = 0,$$

for all $m > n$. The operator E_- is called an **annihilation operator**.

Exercises

1–4. For each of the following functions, show that Theorem 4 applies and use it to find its Laplace transform.

1. $\frac{e^{bt} - e^{at}}{t}$
2. $2 \frac{\cos bt - \cos at}{t}$
3. $2 \frac{\cos bt - \cos at}{t^2}$
4. $\frac{\sin at}{t}$

5–10. Use the Laplace transform to find solutions to each of the following differential equations. In some cases, you may find two independent solutions, and in other cases, you may only find one solution. It may be useful to have the following table for quick reference:

ty	\longleftrightarrow	$-Y'(s)$
ty'	\longleftrightarrow	$-sY'(s) - Y(s)$
ty''	\longleftrightarrow	$-s^2Y'(s) - 2sY(s) + y_0$
ty'''	\longleftrightarrow	$-s^3Y'(s) - 3s^2Y(s) + 2sy_0 + y_1$
y	\longleftrightarrow	$Y(s)$
y'	\longleftrightarrow	$sY(s) - y_0$
y''	\longleftrightarrow	$s^2Y(s) - sy_0 - y_1$

5. $ty'' + (t - 1)y' - y = 0$
6. $ty'' + (1 + t)y' + y = 0$
7. $ty'' + (2 + 4t)y' + (4 + 4t)y = 0$
8. $ty'' - 2y' + ty = 0$
9. $ty'' - 4y' + ty = 0$, assume $y(0) = 0$
10. $ty'' + (2 + 2t)y' + (2 + t)y = 0$

11–16. Use the Laplace transform to find solutions to each of the following differential equations. Use the results of Exercises 1 to 4.

11. $-ty'' + (t - 2)y' + y = 0$
12. $-ty'' - 2y' + ty = 0$
13. $ty'' + (2 - 5t)y' + (6t - 5)y = 0$
14. $ty'' + 2y' + 9ty' = 0$

15. $ty''' + 3y'' + ty' + y = 0$

16. $ty'' + (2+t)y' + y = 0$

17–25. *Laguerre Polynomials*: Each of these problems develops further properties of the Laguerre polynomials.

17. The Laguerre polynomial of order n can be defined in another way: $\ell_n(t) = \frac{1}{n!} e^t \frac{d^n}{dt^n} (e^{-t} t^n)$. Show that this definition is consistent with the definition in the text.

18. Verify (3) in Theorem 9:

$$E_- \ell_n = -n \ell_{n-1}.$$

19. The *Lie bracket* $[A, B]$ of two differential operators A and B is defined by

$$[A, B] = AB - BA.$$

Show the following:

- $[E_0, E_+] = -2E_+$.
- $[E_0, E_-] = 2E_-$.
- $[E_+, E_-] = E_0$.

20. Show that the Laplace transform of $\ell_n(at)$, $a \in \mathbb{R}$, is $\frac{(s-a)^n}{s^{n+1}}$.

21. Verify that

$$\sum_{k=0}^n \binom{n}{k} a^k \ell_k(t) (1-a)^{n-k} = \ell_n(at).$$

22. Show that

$$\int_0^t \ell_n(x) dx = \ell_n(t) - \ell_{n+1}(t).$$

23. Verify the following recursion formula:

$$\ell_{n+1}(t) = \frac{1}{n+1} ((2n+1-t)\ell_n(t) - n\ell_{n-1}(t)).$$

24. Show that

$$\int_0^t \ell_n(x) \ell_m(t-x) dx = \ell_{m+n}(t) - \ell_{m+n+1}(t).$$

25. Show that

$$\int_t^\infty e^{-x} \ell_n(x) dx = e^{-t} (\ell_n(t) - \ell_{n-1}(t)).$$

5.5 Reduction of Order

It is a remarkable feature of linear differential equations that one nonzero homogeneous solution can be used to obtain a second independent solution. Suppose $\mathbf{L} = a_2(t)\mathbf{D}^2 + a_1(t)\mathbf{D} + a_0(t)$ and suppose $y_1(t)$ is a known nonzero solution. It turns out that a second independent solution will take the form

$$y_2(t) = u(t)y_1(t), \quad (1)$$

where $u(t)$ is to be determined. By substituting y_2 into $\mathbf{L}y = 0$, we find that $u(t)$ must satisfy a second order differential equation, which, by a simple substitution, can be *reduced* to a first order separable differential equation. After $u(t)$ is found, (1) gives $y_2(t)$, a second independent solution.

The procedure is straightforward. We drop the functional dependence on t to make the notation simpler. The product rule gives

$$\begin{aligned} y_2' &= u'y_1 + uy_1' \\ \text{and } y_2'' &= u''y_1 + 2u'y_1' + uy_1''. \end{aligned}$$

Substituting these equations into $\mathbf{L}y_2$ gives

$$\begin{aligned} \mathbf{L}y_2 &= a_2y_2'' + a_1y_2' + a_0y_2 \\ &= a_2(u''y_1 + 2u'y_1' + uy_1'') + a_1(u'y_1 + uy_1') + a_0uy_1 \\ &= u''a_2y_1 + u'(2a_2y_1' + a_1y_1) + u(a_2y_1'' + a_1y_1' + a_0y_1) \\ &= u''a_2y_1 + u'(2a_2y_1' + a_1y_1). \end{aligned}$$

In the third line above, the coefficient of u is zero because y_1 is assumed to be a solution to $\mathbf{L}y = 0$. The equation $\mathbf{L}y_2 = 0$ implies

$$u''a_2y_1 + u'(2a_2y_1' + a_1y_1) = 0, \quad (2)$$

another second order differential equation in u . One obvious solution to (2) is $u(t)$ a constant, implying y_2 is a multiple of y_1 . To find another independent solution, we use the substitution $v = u'$ to get

$$v'a_2y_1 + v(2a_2y_1' + a_1y_1) = 0,$$

a first order separable differential equation in v . This substitution gives this procedure its name: The product **reduction of order**. It is now straightforward to solve for v . In fact, separating variables gives

$$\frac{v'}{v} = \frac{-2y_1'}{y_1} - \frac{a_1}{a_2}.$$

From this, we get

$$v = \frac{1}{y_1^2} e^{-\int a_1/a_2}.$$

Since $v = u'$, we integrate v to get

$$u = \int \frac{1}{y_1^2} e^{-\int a_1/a_2}, \quad (3)$$

which is independent of the constant solution. Substituting (3) into (1) then gives a new solution independent of y_1 . Admittedly, (3) is difficult to remember and not very enlightening. In the exercises, we recommend following the procedure we have outlined above. This is what we shall do in the examples to follow.

Example 1. The function $y_1(t) = e^t$ is a solution to

$$(t-1)y'' - ty' + y = 0.$$

Use reduction of order to find another independent solution and write down the general solution.

► **Solution.** Let $y_2(t) = u(t)e^t$. Then

$$\begin{aligned} y_2'(t) &= u'(t)e^t + u(t)e^t \\ y_2''(t) &= u''(t)e^t + 2u'(t)e^t + u(t)e^t. \end{aligned}$$

Substitution into the differential equation $(t-1)y'' - ty' + y = 0$ gives

$$(t-1)(u''(t)e^t + 2u'(t)e^t + u(t)e^t) - t(u'(t)e^t + u(t)e^t) + u(t)e^t = 0$$

which simplifies to

$$(t-1)u'' + (t-2)u' = 0.$$

Let $v = u'$. Then we get $(t-1)v' + (t-2)v = 0$. Separating variables gives

$$\frac{v'}{v} = \frac{-(t-2)}{t-1} = -1 + \frac{1}{t-1}$$

with solution $v = e^{-t}(t-1)$. Integration by parts gives $u(t) = \int v(t) dt = -te^{-t}$. Substitution gives

$$y_2(t) = u(t)e^t = -te^{-t}e^t = -t.$$

It is easy to verify that this is indeed a solution. Since our equation is homogeneous, we know $-y_2(t) = t$ is also a solution. Clearly t and e^t are independent. By Theorem 2 of Sect. 5.2, the general solution is

$$y(t) = c_1t + c_2e^t. \quad \blacktriangleleft$$

Example 2. In Example 7 of Sect. 5.4, we showed that $y_1 = \frac{\sin t}{t}$ is a solution to

$$ty'' + 2y' + ty = 0.$$

Use reduction of order to find a second independent solution and write down the general solution.

► **Solution.** Let $y_2(t) = u(t)\frac{\sin t}{t}$. Then

$$y_2'(t) = u'(t)\frac{\sin t}{t} + u(t)\frac{t \cos t - \sin t}{t^2}$$

$$y_2''(t) = u''(t)\frac{\sin t}{t} + 2u'(t)\frac{t \cos t - \sin t}{t^2} + u(t)\frac{-t^2 \sin t - 2t \cos t + 2 \sin t}{t^3}.$$

We next substitute y_2 into $ty'' + 2y' + ty = 0$ and simplify to get

$$u''(t) \sin t + 2u'(t) \cos t = 0.$$

Let $v = u'$. Then we get $v'(t) \sin t + 2v(t) \cos t = 0$. Separating variables gives

$$\frac{v'}{v} = \frac{-2 \cos t}{\sin t}$$

with solution

$$v(t) = \csc^2(t).$$

Integration gives $u(t) = \int v(t) dt = -\cot(t)$, and hence,

$$y_2(t) = -(\cot t)\frac{\sin t}{t} = \frac{-\cos t}{t}.$$

By Theorem 2 of Sect. 5.2, the general solution can be written as

$$c_1 \frac{\sin t}{t} + c_2 \frac{\cos t}{t}.$$

Compare this result with Examples 4 of Sect. 5.1 and 7 of Sect. 5.4. ◀

We remark that the constant of integration in the computation of u was chosen to be 0 in both examples. There is no loss in this for if a nonzero constant, c say, is chosen, then $y_2 = uy_1 + cy_1$. But cy_1 is already known to be a homogeneous solution. We gain nothing by adding a multiple of it in y_2 .

Exercises

1–16. For each differential equation and the given solution, use reduction of order to find a second independent solution and write down the general solution.

$$1. \quad t^2 y'' - 3ty' + 4y = 0, \\ y_1(t) = t^2$$

$$2. \quad t^2 y'' + 2ty' - 2y = 0, \\ y_1(t) = t$$

$$3. \quad 4t^2 y'' + y = 0, \\ y_1(t) = \sqrt{t}$$

$$4. \quad t^2 y'' + 2ty' = 0, \\ y_1(t) = \frac{1}{t}$$

$$5. \quad t^2 y'' - t(t+2)y' + (t+2)y = 0, \\ y_1(t) = t$$

$$6. \quad t^2 y'' - 4ty' + (t^2 + 6)y = 0, \\ y_1(t) = t^2 \cos t$$

$$7. \quad ty'' - y' + 4t^3 y = 0, \\ y_1(t) = \sin t^2$$

$$8. \quad ty'' - 2(t+1)y' + 4y = 0, \\ y_1(t) = e^{2t}$$

$$9. \quad y'' - 2(\sec^2 t)y = 0, \\ y_1(t) = \tan t$$

$$10. \quad ty'' + (t-1)y' - y = 0, \\ y_1(t) = e^{-t}$$

$$11. \quad y'' - (\tan t)y' - (\sec^2 t)y = 0, \\ y_1(t) = \tan t$$

$$12. \quad (1+t^2)y'' - 2ty' + 2y = 0, \\ y_1(t) = t$$

$$13. \quad (\cos 2t + 1)y'' - 4y = 0, \quad t \in (-\pi/2, \pi/2), \\ y_1(t) = \frac{\sin 2t}{1 + \cos 2t}$$

$$14. \quad t^2 y'' - 2ty' + (t^2 + 2)y = 0, \\ y_1(t) = t \cos t$$

$$15. \quad (1-t^2)y'' + 2y = 0, \quad -1 < t < 1, \\ y_1(t) = 1 - t^2$$

$$16. \quad (1-t^2)y'' - 2ty' + 2y = 0, \quad -1 < t < 1, \\ y_1(t) = t$$

5.6 Variation of Parameters

Let L be a second order linear differential operator. In this section, we address the issue of finding a particular solution to a nonhomogeneous linear differential equation $L(y) = f$, where f is continuous on some interval I . It is a pleasant feature of linear differential equations that the homogeneous solutions can be used decisively to find a particular solution. The procedure we use is called *variation of parameters* and, as you shall see, is akin to the method of reduction of order.

Suppose, in particular, that $L = D^2 + a_1(t)D + a_0(t)$, that is, we will assume that the leading coefficient function is 1, and it is important to remember that this assumption is essential for the method we develop below. Suppose $\{y_1, y_2\}$ is a fundamental set for $L(y) = 0$. We know then that all solutions of the homogeneous equation $L(y) = 0$ are of the form $c_1y_1 + c_2y_2$. To find a particular solution y_p to $L(y) = f$, the method of variation of parameters makes two assumptions. First, the parameters c_1 and c_2 are allowed to vary (hence the name). We thus replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$, and assume that the particular solution y_p , takes the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \quad (1)$$

The second assumption is

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \quad (2)$$

What is remarkable is that these two assumptions consistently lead to explicit formulas for $u_1(t)$ and $u_2(t)$ and hence a formula for y_p .

To simplify notation in the calculations that follow, we will drop the “ t ” in expressions like $u_1(t)$, etc. Before substituting y_p into $L(y) = f$, we first calculate y_p' and y_p'' :

$$\begin{aligned} y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ &= u_1y_1' + u_2y_2', \end{aligned}$$

where we used (2) to simplify. Now for the second derivative

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

We now substitute y_p into $L(y)$:

$$\begin{aligned} L(y_p) &= y_p'' + a_1y_p' + a_0y_p \\ &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + a_1(u_1y_1' + u_2y_2') + a_0(u_1y_1 + u_2y_2) \\ &= u_1'y_1' + u_2'y_2' + u_1(y_1'' + a_1y_1' + a_0y_1) + u_2(y_2'' + a_1y_2' + a_0y_2) \\ &= u_1'y_1' + u_2'y_2'. \end{aligned}$$

In the second to the last equation, the coefficients of u_1 and u_2 are zero because y_1 and y_2 are assumed to be homogeneous solutions. The second assumption, (2), and the equation $\mathbf{L}(y_p) = f$ now lead to the following system:

$$\begin{aligned}u_1' y_1 + u_2' y_2 &= 0 \\u_1' y_1' + u_2' y_2' &= f\end{aligned}$$

which can be rewritten in matrix form as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}. \quad (3)$$

The leftmost matrix in (3) is none other than the Wronskian matrix, $W(y_1, y_2)$, which has a nonzero determinant because $\{y_1, y_2\}$ is a fundamental set (cf. Proposition 4 of Sect. 5.2). By Cramer's rule, we can solve for u_1' and u_2' . We obtain

$$\begin{aligned}u_1' &= \frac{-y_2 f}{w(y_1, y_2)}, \\u_2' &= \frac{y_1 f}{w(y_1, y_2)}.\end{aligned}$$

We now obtain an explicit formula for a particular solution:

$$\begin{aligned}y_p(t) &= u_1 y_1 + u_2 y_2 \\&= \left(\int \frac{-y_2 f}{w(y_1, y_2)} \right) y_1 + \left(\int \frac{y_1 f}{w(y_1, y_2)} \right) y_2.\end{aligned}$$

The following theorem consolidates these results with Theorem 6 of Sect. 5.1.

Theorem 1. *Let $\mathbf{L} = \mathbf{D}^2 + a_1(t)\mathbf{D} + a_0(t)$, where $a_1(t)$ and $a_0(t)$ are continuous on an interval I . Suppose $\{y_1, y_2\}$ is a fundamental set of solutions for $\mathbf{L}(y) = 0$. If f is continuous on I , then a particular solution, y_p , to $\mathbf{L}(y) = f$ is given by the formula*

$$y_p = \left(\int \frac{-y_2 f}{w(y_1, y_2)} \right) y_1 + \left(\int \frac{y_1 f}{w(y_1, y_2)} \right) y_2. \quad (4)$$

Furthermore, the solution set to $\mathbf{L}(y) = f$ becomes

$$\{y_p + c_1 y_1 + c_2 y_2 : c_1, c_2 \in \mathbb{R}\}.$$

Remark 2. Equation (4), which gives an explicit formula for a particular solution, is too complicated to memorize, and we do not recommend students to do this. Rather, the point of variation of parameters is the method that leads to (4), and our recommended starting point is (3). You will see such matrix equations as we proceed in the text.

We will illustrate the method of variation of parameters with two examples.

Example 3. Find the general solution to the following equation:

$$t^2 y'' - 2y = t^2 \ln t.$$

► **Solution.** In standard form, this becomes

$$y'' - \frac{2}{t^2} y = \ln t.$$

The associated homogeneous equation is $y'' - (2/t^2)y = 0$ or, equivalently, $t^2 y'' - 2y = 0$ and is a Cauchy–Euler equation. The indicial polynomial is $Q(s) = s^2 - s - 2 = (s-2)(s+1)$, which has 2 and -1 as roots. Thus, $\{t^{-1}, t^2\}$ is a fundamental set to the homogeneous equation $y'' - (2/t^2)y = 0$, by Theorem 4 of Sect. 5.3. Let $y_p = t^{-1}u_1(t) + t^2u_2(t)$. Our starting point for determining u_1 and u_2 is the matrix equation

$$\begin{bmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \ln t \end{bmatrix}$$

which is equivalent to the system

$$\begin{aligned} t^{-1}u_1' + t^2u_2' &= 0 \\ -t^{-2}u_1' + 2tu_2' &= \ln t. \end{aligned}$$

Multiplying the bottom equation by t and then adding the equations together give $3t^2u_2' = t \ln t$, and hence,

$$u_2' = \frac{1}{3t} \ln t.$$

Substituting u_2' into the first equation and solving for u_1' give

$$u_1' = -\frac{t^2}{3} \ln t.$$

Integration by parts leads to

$$u_1 = -\frac{1}{3} \left(\frac{t^3}{3} \ln t - \frac{t^3}{9} \right)$$

and a simple substitution gives

$$u_2 = \frac{1}{6} (\ln t)^2.$$

We substitute u_1 and u_2 into (1) to get

$$y_p(t) = -\frac{1}{3} \left(\frac{t^3}{3} \ln t - \frac{t^3}{9} \right) t^{-1} + \frac{1}{6} (\ln t)^2 t^2 = \frac{t^2}{54} (9(\ln t)^2 - 6 \ln t + 2).$$

It follows that the solution set is

$$\left\{ \frac{t^2}{54} (9(\ln t)^2 - 6 \ln t + 2) + c_1 t^{-1} + c_2 t^2 : c_1, c_2 \in \mathbb{R} \right\}. \quad \blacktriangleleft$$

Example 4. Find the general solution to

$$ty'' + 2y' + ty = 1.$$

Use the results of Example 2 of Sect. 5.5.

► **Solution.** Example 2 of Sect. 5.5 showed that

$$y_1(t) = \frac{\sin t}{t} \quad \text{and} \quad y_2(t) = \frac{\cos t}{t}$$

are homogeneous solutions to $ty'' + 2y' + ty = 0$. Let $y_p = \frac{\sin t}{t} u_1(t) + \frac{\cos t}{t} u_2(t)$. Then

$$\begin{bmatrix} \frac{\sin t}{t} & \frac{\cos t}{t} \\ \frac{t \cos t - \sin t}{t^2} & \frac{-t \sin t - \cos t}{t^2} \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{t} \end{bmatrix}.$$

(We get $1/t$ in the last matrix because the differential equation in standard form is $y'' + (2/t)y' + y = 1/t$.) From the matrix equation, we get the following system:

$$\begin{aligned} \frac{\sin t}{t} u_1'(t) + \frac{\cos t}{t} u_2'(t) &= 0, \\ \frac{t \cos t - \sin t}{t^2} u_1'(t) + \frac{-t \sin t - \cos t}{t^2} u_2'(t) &= \frac{1}{t}. \end{aligned}$$

The first equation gives

$$u_1'(t) = -(\cot t) u_2'(t).$$

If we multiply the first equation by t , the second equation by t^2 , and then add, we get

$$(\cos t) u_1'(t) - (\sin t) u_2'(t) = 1.$$

Substituting in $u_1'(t)$ and solving for $u_2'(t)$ give $u_2'(t) = -\sin t$, and thus, $u_1'(t) = \cos t$. Integration gives

$$u_1(t) = \sin t,$$

$$u_2(t) = \cos t.$$

We now substitute these functions into y_p to get

$$\begin{aligned} y_p(t) &= \frac{\sin t}{t} \sin t + \frac{\cos t}{t} \cos t \\ &= \frac{\sin^2 t + \cos^2 t}{t} \\ &= \frac{1}{t}. \end{aligned}$$

The general solution is

$$y(t) = \frac{1}{t} + c_1 \frac{\sin t}{t} + c_2 \frac{\cos t}{t}.$$



Exercises

1–5. Use variation of parameters to find a particular solution and then write down the general solution. Next solve each using the method of undetermined coefficients or the incomplete partial fraction method.

1. $y'' + y = \sin t$
2. $y'' - 4y = e^{2t}$
3. $y'' - 2y' + 5y = e^t$
4. $y'' + 3y' = e^{-3t}$
5. $y'' - 3y' + 2y = e^{3t}$

6–16. Use variation of parameters to find a particular solution and then write down the general solution. In some exercises, a fundamental set $\{y_1, y_2\}$ is given.

6. $y'' + y = \tan t$
7. $y'' - 2y' + y = \frac{e^t}{t}$
8. $y'' + y = \sec t$
9. $t^2 y'' - 2t y' + 2y = t^4$
10. $t y'' - y' = 3t^2 - 1$
 $y_1(t) = 1$ and $y_2(t) = t^2$
11. $t^2 y'' - t y' + y = t$
12. $y'' - 4y' + 4y = \frac{e^{2t}}{t^2 + 1}$
13. $y'' - (\tan t)y' - (\sec^2 t)y = t$
 $y_1(t) = \tan t$ and $y_2(t) = \sec t$
14. $t y'' + (t - 1)y' - y = t^2 e^{-t}$
 $y_1(t) = t - 1$ and $y_2(t) = e^{-t}$
15. $t y'' - y' + 4t^3 y = 4t^5$
 $y_1 = \cos t^2$ and $y_2(t) = \sin t^2$
16. $y'' - y = \frac{1}{1 + e^{-t}}$

17. Show that the constants of integration in the formula for y_p in Theorem 1 can be chosen so that a particular solution can be written in the form:

$$y_p(t) = \int_a^t \frac{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} f(x) dx,$$

where a and t are in the interval I , and the absolute value signs indicate the determinant.

18–21. For each problem below, use the result of Problem 17, with $a = 0$, to obtain a particular solution to the given differential equation in the form given. Solve the differential equation using the Laplace transform method and compare.

18. $y'' + a^2y = f(t)$

19. $y'' - a^2y = f(t)$

20. $y'' - 2ay' + a^2y = f(t)$

21. $y'' - (a + b)y' + aby = f(t), a \neq b$

5.7 Summary of Laplace Transforms

Laplace transforms and rules presented in Chap. 5 are summarized in Tables 5.1 and 5.2.

Table 5.1 Laplace transform rules

Assumptions	Result	Page
<i>Initial value theorem</i>		
1. f, f' of exponential order	$f(0) = \lim_{s \rightarrow \infty} sF(s)$	356
<i>Final value theorem</i>		
2. f, f' of exponential order and $\lim_{t \rightarrow \infty} f(t)$ exists	$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$	356
<i>Transform integral formula</i>		
3. f of exponential order and $\frac{f(t)}{t}$ has a continuous extension to 0	$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma) d\sigma$	357

Table 5.2 Laplace transforms

$f(t)$	$F(s)$	Page
1. $\frac{\sin t}{t}$	$\tan^{-1} \frac{1}{s}$	358
2. $\frac{\sin at}{t}$	$\tan^{-1} \left(\frac{a}{s}\right)$	365
3. $\frac{e^{bt} - e^{at}}{t}$	$\ln \left(\frac{s-a}{s-b}\right)$	365
4. $2 \frac{\cos bt - \cos at}{t}$	$\ln \left(\frac{s^2 + a^2}{s^2 + b^2}\right)$	365
5. $2 \frac{\cos bt - \cos at}{t^2}$	$s \ln \left(\frac{s^2 + b^2}{s^2 + a^2}\right) - 2b \tan^{-1} \left(\frac{b}{s}\right) + 2a \tan^{-1} \left(\frac{a}{s}\right)$	365
<i>Laguerre polynomials</i>		
6. $\ell_n(t) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t^k}{k!}$	$\frac{(s-1)^n}{s^{n+1}}$	361
7. $\ell_n(at)$	$\frac{(s-a)^n}{s^{n+1}}$	366