Chapter 2
The Laplace Transform

2.1 Laplace Transform Method: Introduction

The method for solving a first order linear differential equation $y' + p(t)y = f(t)$ (Algorithm 3 of Sect. 5) involves multiplying the equation by an integrating factor $\mu(t) = e^{\int p(t)\,dt}$ chosen so that the left-hand side of the resulting equation becomes a perfect derivative $(\mu(t)y)'$. Then the unknown function $y(t)$ can be retrieved by integration. When $p(t) = k$ is a constant, $\mu(t) = e^{kt}$ is an exponential function. Unfortunately, for higher order linear equations, there is not a corresponding type of integrating factor. There is, however, a useful method involving multiplication by an exponential function that can be used for solving an ***nth order constant coefficient linear differential equation***, that is, an equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \tag{1}$$

in which $a_0, a_1, \ldots, a_{n-1}$ are constants. The method proceeds by multiplying (1) by the exponential term e^{-st}, where s is another variable, and then integrating the resulting equation from 0 to ∞, to obtain the following equation involving the variable s:

$$\int_0^\infty e^{-st}\left(y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y\right) dt = \int_0^\infty e^{-st} f(t)\,dt. \tag{2}$$

The integral $\int_0^\infty e^{-st} f(t)\,dt$ is called the ***Laplace transform*** of $f(t)$, and we will denote the Laplace transform of the function $f(t)$ by means of the corresponding capital letter $F(s)$ or the symbol $\mathcal{L}\{f(t)\}(s)$. Thus,

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt = F(s). \tag{3}$$

W.A. Adkins and M.G. Davidson, *Ordinary Differential Equations*,
Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-3618-8_2,

Note that $F(s)$ is a function of the new variable s, while the original function $f(t)$ is a function of the variable t. The integral involved is an *improper* integral since the domain of integration is of infinite length; at this point, we will simply assume that the integrals in question exist for all real s greater than some constant a.

Before investigating the left-hand side of (2), we will first calculate a couple of simple Laplace transforms to see what the functions $F(s)$ may look like.

Example 1. Find the Laplace transform of $f(t) = \mathrm{e}^{at}$ and $g(t) = t\mathrm{e}^{at}$.

► **Solution.** First, we find the Laplace transform of $f(t) = \mathrm{e}^{at}$.

$$
\begin{aligned}
F(s) = \mathcal{L}\{f(t)\}(s) &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{at}\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{(a-s)t}\,\mathrm{d}t = \lim_{r\to\infty}\int_0^r \mathrm{e}^{(a-s)t}\,\mathrm{d}t \\
&= \lim_{r\to\infty} \left.\frac{\mathrm{e}^{(a-s)t}}{a-s}\right|_0^r = \lim_{r\to\infty}\left(\frac{\mathrm{e}^{(a-s)r}}{a-s} - \frac{1}{a-s}\right) \\
&= \frac{1}{s-a} + \lim_{r\to\infty}\frac{\mathrm{e}^{(a-s)r}}{a-s} \\
&= \frac{1}{s-a},
\end{aligned}
$$

provided $a - s < 0$, that is, $s > a$. In this situation, the limit of the exponential term is 0. Therefore,

$$
\mathcal{L}\left\{\mathrm{e}^{at}\right\}(s) = \frac{1}{s-a} \qquad \text{for } s > a. \tag{4}
$$

A similar calculation gives the Laplace transform of $g(t) = t\mathrm{e}^{at}$, except that integration by parts will be needed in the calculation:

$$
\begin{aligned}
G(s) = \mathcal{L}\left\{t\mathrm{e}^{at}\right\}(s) &= \int_0^\infty \mathrm{e}^{-st}(t\mathrm{e}^{at})\,\mathrm{d}t = \int_0^\infty t\mathrm{e}^{(a-s)t}\,\mathrm{d}t \\
&= \lim_{r\to\infty}\left(\left.\frac{t\mathrm{e}^{(a-s)t}}{a-s}\right|_0^r - \int_0^r \frac{\mathrm{e}^{(a-s)t}}{a-s}\,\mathrm{d}t\right) \\
&= \lim_{r\to\infty}\left[\frac{t\mathrm{e}^{(a-s)t}}{a-s} - \frac{\mathrm{e}^{(a-s)t}}{(a-s)^2}\right]_0^r \\
&= \lim_{r\to\infty}\left(\frac{r\mathrm{e}^{(a-s)r}}{a-s} - \frac{\mathrm{e}^{(a-s)r}}{(a-s)^2} + \frac{1}{(a-s)^2}\right) \\
&= \frac{1}{(a-s)^2} \qquad \text{for } s > a.
\end{aligned}
$$

Therefore,

$$\mathcal{L}\left\{t\mathrm{e}^{at}\right\}(s) = \frac{1}{(s-a)^2} \qquad \text{for } s > a. \tag{5}$$

◀

In each of these formulas, the parameter a represents any real number. Thus, some specific examples of (4) and (5) are

$$\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s} && s > 0 && a = 0 \text{ in (4)},\\
\mathcal{L}\left\{\mathrm{e}^{2t}\right\} &= \frac{1}{s-2} && s > 2 && a = 2 \text{ in (4)},\\
\mathcal{L}\left\{\mathrm{e}^{-2t}\right\} &= \frac{1}{s+2} && s > -2 && a = -2 \text{ in (4)},\\
\mathcal{L}\{t\} &= \frac{1}{s^2} && s > 0 && a = 0 \text{ in (5)},\\
\mathcal{L}\left\{t\mathrm{e}^{2t}\right\} &= \frac{1}{(s-2)^2} && s > 2 && a = 2 \text{ in (5)},\\
\mathcal{L}\left\{t\mathrm{e}^{-2t}\right\} &= \frac{1}{(s+2)^2} && s > -2 && a = -2 \text{ in (5)}.
\end{aligned}$$

We now turn to the left-hand side of (2). Since the integral is additive, we can write the left-hand side as a sum of terms

$$a_j \int_0^\infty \mathrm{e}^{-st} y^{(j)}\,\mathrm{d}t = a_j \mathcal{L}\left\{y^{(j)}\right\}(s). \tag{6}$$

For now, we will not worry about whether the solution function $y(t)$ is such that the Laplace transform of $y^{(j)}$ exists. Ignoring the constant, the $j = 0$ term is the Laplace transform of $y(t)$, which we denote by $Y(s)$, and the $j = 1$ term is the Laplace transform of $y'(t)$, and this can also be expressed in terms of $Y(s) = \mathcal{L}\{y(t)\}(s)$ by use of integration by parts:

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-st} y'(t)\,\mathrm{d}t &= \mathrm{e}^{-st} y(t)\Big|_0^\infty + s\int_0^\infty \mathrm{e}^{-st} y(t)\,\mathrm{d}t\\
&= \lim_{t\to\infty} \mathrm{e}^{-st} y(t) - y(0) + sY(s).
\end{aligned}$$

We will now further restrict the type of functions that we consider by requiring that

$$\lim_{t\to\infty} \mathrm{e}^{-st} y(t) = 0.$$

We then conclude that

$$\mathcal{L}\left\{y'(t)\right\}(s) = \int_0^\infty \mathrm{e}^{-st} y'(s)\,\mathrm{d}t = sY(s) - y(0). \tag{7}$$

Table 2.1 Basic Laplace transform formulas

	f(t)		$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$af(t) + bg(t)$	$\longleftrightarrow$	$aF(s) + bG(s)$
8.	$y'(t)$	$\longleftrightarrow$	$sY(s) - y(0)$
9.	$y''(t)$	$\longleftrightarrow$	$s^2Y(s) - sy(0) - y'(0)$

By use of repeated integration by parts, it is possible to express the Laplace transforms of all of the derivatives $y^{(j)}$ in terms of $Y(s)$ and values of $y^{(k)}(t)$ at $t = 0$. The formula is

$$\begin{aligned}\mathcal{L}\left\{y^{(n)}(t)\right\}(s) &= \int_0^\infty e^{-st} y^{(n)}\,dt \\ &= s^n Y(s) - (s^{n-1}y(0) + s^{n-2}y'(0) + \cdots + sy^{n-2}(0) + y^{n-1}(0)),\end{aligned} \tag{8}$$

with the important special case for $n = 2$ being

$$\mathcal{L}\left\{y''(t)\right\}(s) = s^2Y(s) - sy(0) - y'(0). \tag{9}$$

The ***Laplace transform method*** for solving (1) is to use (8) to replace each Laplace transform of a derivative of $y(t)$ in (2) with an expression involving $Y(s)$ and initial values. This gives an *algebraic equation* in $Y(s)$. Solve for $Y(s)$, and hopefully recognize $Y(s)$ as the Laplace transform of a known function $y(t)$. This latter recognition involves having a good knowledge of Laplace transforms of a wide variety of functions, which can be manifested by means of a table of Laplace transforms. A small table of Laplace transforms, Table 2.1, is included here for use in some examples. The table will be developed fully and substantially expanded, starting in the next section. For now, we will illustrate the Laplace transform method by solving some differential equations of orders 1 and 2. The examples of order 1 could also be solved by the methods of Chap. 1.

Example 2. Solve the initial value problem

$$y' + 2y = e^{-2t}, \qquad y(0) = 0, \tag{10}$$

by the Laplace transform method.

► **Solution.** As in (2), apply the Laplace transform to both sides of the given differential equation. Thus,

$$\mathcal{L}\left\{y' + 2y\right\}(s) = \mathcal{L}\left\{e^{-2t}\right\}(s). \tag{11}$$

The right-hand side of this equation is

$$\mathcal{L}\left\{e^{-2t}\right\}(s) = \frac{1}{s+2},$$

which follows from (4) with $a = -2$. The left-hand side of (11) is

$$\mathcal{L}\left\{y' + 2y\right\}(s) = \mathcal{L}\left\{y'\right\} + 2\mathcal{L}\left\{y\right\} = sY(s) - y(0) + 2Y(s) = (s+2)Y(s),$$

where $Y(s) = \mathcal{L}\{y(t)\}(s)$. Thus, (11) becomes

$$(s+2)Y(s) = \frac{1}{s+2},$$

which can be solved for $Y(s)$ to give

$$Y(s) = \frac{1}{(s+2)^2}.$$

From (5) with $a = -2$, we see that $Y(s) = \mathcal{L}\left\{te^{-2t}\right\}(s)$, which suggests that $y(t) = te^{-2t}$. By direct substitution, we can check that $y(t) = te^{-2t}$ is, in fact, the solution of the initial value problem (10). ◄

Example 3. Solve the second order initial value problem

$$y'' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0, \tag{12}$$

by the Laplace transform method.

► **Solution.** As in the previous example, apply the Laplace transform to both sides of the given differential equation. Thus,

$$\mathcal{L}\left\{y'' + 4y\right\}(s) = \mathcal{L}\{0\} = 0. \tag{13}$$

The left-hand side is

$$\begin{aligned}\mathcal{L}\left\{y'' + 4y\right\}(s) &= \mathcal{L}\left\{y''\right\} + 4\mathcal{L}\left\{y\right\} = s^2Y(s) - sy(0) - y'(0) + 4Y(s) \\ &= -s + (s^2+4)Y(s),\end{aligned}$$

where $Y(s) = \mathcal{L}\{y(t)\}(s)$. Thus, (13) becomes

$$(s^2+4)Y(s) - s = 0,$$

which can be solved for $Y(s)$ to give

$$Y(s) = \frac{s}{s^2 + 4}.$$

Item 5, with $b = 2$, in Table 2.1, shows that $Y(s) = \mathcal{L}\{\cos 2t\}(s)$, which suggests that the solution $y(t)$ of the differential equation is $y(t) = \cos 2t$. Straightforward substitution again shows that this function satisfies the initial value problem. ◄

Here is a slightly more complicated example.

Example 4. Use the Laplace transform method to solve

$$y'' + 4y' + 4y = 2t\mathrm{e}^{-2t}, \tag{14}$$

with initial conditions $y(0) = 1$ and $y'(0) = -3$.

► **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ where, as usual, $y(t)$ is the unknown solution. Applying the Laplace transform to both sides of (14) gives

$$\mathcal{L}\left\{y'' + 4y' + 4y\right\}(s) = \mathcal{L}\left\{2t\mathrm{e}^{-2t}\right\},$$

which, after applying items 7–9 from Table 2.1 to the left-hand side, and item 4 to the right-hand side, gives

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = \frac{2}{(s+2)^2}.$$

Now, substituting the initial values gives the algebraic equation

$$s^2Y(s) - s + 3 + 4(sY(s) - 1) + 4Y(s) = \frac{2}{(s+2)^2}.$$

Collecting terms, we get

$$(s^2 + 4s + 4)Y(s) = s + 1 + \frac{2}{(s+2)^2}$$

and solving for $Y(s)$, we get

$$Y(s) = \frac{s+1}{(s+2)^2} + \frac{2}{(s+2)^4}.$$

This $Y(s)$ is not immediately recognizable as a term in the table of Laplace transforms. However, a simple partial fraction decomposition, which will be studied in more detail later in this chapter, gives

$$\begin{aligned}
Y(s) &= \frac{s+1}{(s+2)^2} + \frac{2}{(s+2)^4} \\
&= \frac{(s+2)-1}{(s+2)^2} + \frac{2}{(s+2)^4} \\
&= \frac{1}{s+2} - \frac{1}{(s+2)^2} + \frac{2}{(s+2)^4}.
\end{aligned}$$

Each of these terms can be identified as the Laplace transform of a function in Table 2.1. That is, item 4 with $a = -2$ and $n = 0$, 1, and 3 gives

$$\mathcal{L}\left\{e^{-2t}\right\}(s) = \frac{1}{s+2}, \quad \mathcal{L}\left\{te^{-2t}\right\}(s) = \frac{1}{(s+2)^2}, \quad \text{and } \mathcal{L}\left\{t^3e^{-2t}\right\} = \frac{3!}{(s+4)^4}.$$

Thus, we recognize

$$Y(s) = \mathcal{L}\left\{e^{-2t} - te^{-2t} + \frac{1}{3}t^3e^{-2t}\right\},$$

which suggests that

$$y(t) = e^{-2t} - te^{-2t} + \frac{1}{3}t^3e^{-2t},$$

is the solution to (14). As before, substitution shows that this function is in fact a solution to the initial value problem. ◀

The examples given illustrate how to use the Laplace transform to solve the nth order constant coefficient linear differential equation (1). The steps are summarized as the following algorithm.

Algorithm 5. Use the following sequence of steps to solve (1) by means of the Laplace transform.

Laplace Transform Method

1. Set the Laplace transform of the left-hand side of the equation equal to the Laplace transform of the function on the right-hand side.
2. Letting $Y(s) = \mathcal{L}\{y(t)\}(s)$, where $y(t)$ is the unknown solution to (1), use the derivative formulas for the Laplace transform to express the Laplace transform of the left-hand side of the equation as a function involving $Y(s)$, some powers of s, and the initial values $y(0)$, $y'(0)$, etc.
3. Now solve the resulting algebraic equation for $Y(s)$.
4. Identify $Y(s)$ as the Laplace transform of a known function $y(t)$. This may involve some algebraic manipulations of $Y(s)$, such as partial fractions, for example, so that you may identify individual parts of $Y(s)$ from a table of Laplace transforms, such as the short Table 2.1 reproduced above.

The Laplace transform is quite powerful for the types of differential equations to which it applies. However, steps 1 and 4 in the above summary will require a more extensive understanding of Laplace transforms than the brief introduction we have presented here. We will start to develop this understanding in the next section.

Exercises

1–14. Solve each of the following differential equations using the Laplace transform method. Determine both $Y(s) = \mathcal{L}\{y(t)\}$ and the solution $y(t)$.

1. $y' - 4y = 0, \quad y(0) = 2$
2. $y' - 4y = 1, \quad y(0) = 0$
3. $y' - 4y = e^{4t}, \quad y(0) = 0$
4. $y' + ay = e^{-at}, \quad y(0) = 1$
5. $y' + 2y = 3e^{t}, \quad y(0) = 2$
6. $y' + 2y = te^{-2t}, \quad y(0) = 0$
7. $y'' + 3y' + 2y = 0, \quad y(0) = 3, y'(0) = -6$
8. $y'' + 5y' + 6y = 0, \quad y(0) = 2, y'(0) = -6$
9. $y'' + 25y = 0, \quad y(0) = 1, y'(0) = -1$
10. $y'' + a^2 y = 0, \quad y(0) = y_0, y'(0) = y_1$
11. $y'' + 8y' + 16y = 0, \quad y(0) = 1, y'(0) = -4$
12. $y'' - 4y' + 4y = 4e^{2t}, \quad y(0) = -1, y'(0) = -4$
13. $y'' + 4y' + 4y = e^{-2t}, \quad y(0) = 0, y'(0) = 1$
14. $y'' + 4y = 8, \quad y(0) = 2, y'(0) = 1$

2.2 Definitions, Basic Formulas, and Principles

Suppose $f(t)$ is a continuous function defined for all $t \geq 0$. The ***Laplace transform*** of f is the function $F(s) = \mathcal{L}\{f(t)\}(s)$ defined by the improper integral equation

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt := \lim_{r\to\infty} \int_0^r e^{-st} f(t)\,dt \tag{1}$$

provided the limit exists at s. It can be shown that if the Laplace transform exists at $s = N$, then it exists for all $s \geq N$.[1] This means that there is a smallest number N, which will depend on the function f, so that the limit exists whenever $s > N$.

Let us consider this equation somewhat further. The function f with which we start will sometimes be called the ***input function***. Generally, "t" will denote the variable for an input function f, while the Laplace transform of f, denoted $\mathcal{L}\{f(t)\}(s)$,[2] is a new function (the ***output function*** or ***transform function***), whose variable will usually be "s". Thus, (1) is a formula for computing the value of the function $\mathcal{L}\{f\}$ at the particular point s, so that, for example, $F(2) = \mathcal{L}\{f\}(2) = \int_0^\infty e^{-2t} f(t)\,dt$ provided $s = 2$ is in the domain of $\mathcal{L}\{f(t)\}$.

When possible, we will use a lowercase letter to denote the input function and the corresponding uppercase letter to denote its Laplace transform. Thus, $F(s)$ is the Laplace transform of $f(t)$, $Y(s)$ is the Laplace transform of $y(t)$, etc. Hence, there are two distinct notations that we will be using for the Laplace transform of $f(t)$; if there is no confusion, we use $F(s)$, otherwise we will write $\mathcal{L}\{f(t)\}(s)$.

To avoid the notation becoming too heavy-handed, we will frequently write $\mathcal{L}\{f\}$ rather than $\mathcal{L}\{f\}(s)$. That is, the variable s may be suppressed when the meaning is clear. It is also worth emphasizing that, while the input function f must have a domain that includes $[0, \infty)$, the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ is only defined for all sufficiently large s, and the domain will depend on the particular input function f. In practice, this will not be an issue, and we will generally not emphasize the particular domain of $F(s)$.

Functions of Exponential Type

The fact that the Laplace transform is given by an improper integral imposes restrictions on the growth of the integrand in order to insure convergence. A function

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[1] A nice proof of this fact can be found on page 442 of the text *Advanced Calculus* (second edition) by David Widder, published by Prentice Hall (1961).

[2] Technically, f is the function while $f(t)$ is the value of the function f at t. Thus, to be correct, the notation should be $\mathcal{L}\{f\}(s)$. However, there are times when the variable t needs to be emphasized or f is given by a formula such as in $\mathcal{L}\left\{e^{2t}\right\}(s)$. Thus, we will freely use both notations: $\mathcal{L}\{f(t)\}(s)$ and $\mathcal{L}\{f\}(s)$.

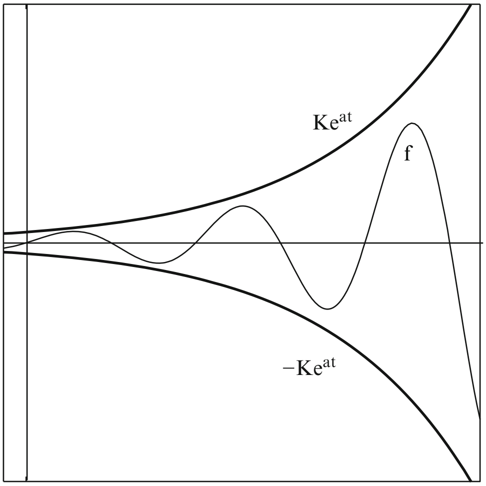
f on $[0, \infty)$ is said to be of ***exponential type with order*** a if there is a constant K such that

$$|f(t)| \leq K\mathrm{e}^{at}$$

for all $t \in [0, \infty)$. If the order is not important to the discussion, we will just say f is of ***exponential type***. The idea here is to limit the kind of growth that we allow f to have; it cannot grow faster than a multiple of an exponential function. The above inequality means

$$-K\mathrm{e}^{at} \leq f(t) \leq K\mathrm{e}^{at},$$

for all $t \in [0, \infty)$ as illustrated in the graph below. Specifically, the curve, $f(t)$, lies between the upper and lower exponential functions, $-K\mathrm{e}^{at}$ and $K\mathrm{e}^{at}$.



As we will see below, limiting growth in this way will assure us that f has a Laplace transform. If $f(t)$ is a bounded function, then there is a K so that $|f(t)| \leq K$ which implies that $f(t)$ is of exponential type of order $a = 0$. Hence, all bounded functions are of exponential type. For example, constant functions, $\cos bt$ and $\sin bt$ are of exponential type since they are bounded. Notice that if f is of exponential type of order a and $a < 0$, then $\lim_{t\to\infty} f(t) = 0$ and hence it is bounded. Since exponential type is a concept used to restrict the growth of a function, we will be interested only in exponential type of order $a \geq 0$.

The set of all functions of exponential type has the property that it is closed under addition and scalar multiplication. We will often see this property on sets of functions. A set $\mathcal{F}$ of functions (usually defined on some interval I) is a ***linear space*** (or ***vector space***) if it is closed under addition and scalar multiplication. More specifically, $\mathcal{F}$ is a linear space if

- $f_1 + f_2 \in \mathcal{F}$,
- $cf_1 \in \mathcal{F}$,

whenever $f_1, f_2 \in \mathcal{F}$ and c is a scalar. If the scalars in use are the real numbers, then $\mathcal{F}$ is referred to as a *real* linear space. If the scalars are the complex numbers, then $\mathcal{F}$ is a *complex* linear space. Unless otherwise stated, linear spaces will be real.

Proposition 1. *The set of functions of exponential type is a linear space.*

Proof. Suppose f_1 and f_2 are of exponential type and $c \in \mathbb{R}$ is a scalar. Then there are constants K_1, K_2, a_1, and a_2 so that $f_1(t) \leq K_1 e^{a_1 t}$ and $f_2(t) \leq K_2 e^{a_2 t}$. Now let $K = K_1 + K_2$ and let a be the larger of a_1 and a_2. Then

$$|f_1(t) + f_2(t)| \leq |f_2(t)| + |f_2(t)| \leq K_1 e^{a_1 t} + K_2 e^{a_2 t} \leq K_1 e^{at} + K_2 e^{at} = K e^{at}.$$

It follows that $f_1 + f_2$ is of exponential type. Further,

$$|c f_1(t)| \leq |c|\,|f_1(t)| \leq |c|\,K_1 e^{a_1 t}.$$

It follows that $c f_1$ is of exponential type. Thus, the set of all functions of exponential type is closed under addition and scalar multiplication, that is, is a linear space. $\square$

Lemma 2. *Suppose f is of exponential type of order a. Let $s > a$, then*

$$\lim_{t \to \infty} f(t) e^{-st} = 0.$$

Proof. Choose K so that $|f(t)| \leq K e^{at}$. Let $s > a$. Then

$$\left| f(t) e^{-st} \right| = \left| \frac{f(t)}{e^{at}} e^{-(s-a)t} \right| \leq K e^{-(s-a)t}.$$

Taking limits gives the result since $\lim_{t \to \infty} e^{-(s-a)t} = 0$ because $-(s-a) < 0$. $\square$

Proposition 3. *Let f be a continuous function of exponential type with order a. Then the Laplace transform $F(s) = \mathcal{L}\{f(t)\}(s)$ exists for all $s > a$ and, moreover,* $\lim_{s \to \infty} F(s) = 0$.

Proof. Let f be of exponential type of order a. Then $|f(t)| \leq K e^{at}$, for some K, so $|f(t) e^{-st}| \leq K e^{-(s-a)t}$ and

$$|F(s)| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leq \int_0^\infty e^{-st}\, |f(t)|\, dt \leq K \int_0^\infty e^{-(s-a)t}\, dt = \frac{K}{s-a},$$

provided $s > a$. This shows that the integral converges absolutely, and hence the Laplace transform $F(s)$ exists for $s > a$, and in fact $|F(s)| \leq K/(s-a)$. Since $\lim_{s \to \infty} K/(s-a) = 0$, it follows that $\lim_{s \to \infty} F(s) = 0$. $\square$

It should be noted that many functions are not of exponential type. For example, in Exercises 39 and 40, you are asked to show that the function $y(t) = e^{t^2}$ is not of exponential type and does not have a Laplace transform. Proposition 3 should not be misunderstood. The restriction that f be of exponential type is a *sufficient* condition to guarantee that the Laplace transform exists. If a function is not of exponential type, it still may have a Laplace transform. See, for example, Exercise 41.

Lemma 4. *Suppose f is a continuous function defined on $[0,\infty)$ of exponential type of order $a \geq 0$. Then any antiderivative of f is also of exponential type and has order a if $a > 0$.*

Proof. Suppose $|f(t)| \leq K\mathrm{e}^{at}$, for some K and a. Let $g(t) = \int_0^t f(u)\,\mathrm{d}u$. Suppose $a > 0$. Then

$$|g(t)| \leq \int_0^t |f(u)|\,\mathrm{d}u \leq \int_0^t K\mathrm{e}^{au}\,\mathrm{d}u = \frac{K}{a}(\mathrm{e}^{at}-1) \leq \frac{K}{a}\mathrm{e}^{at}.$$

It follows that g is of exponential type of order a. If $a = 0$, then $|f| \leq K$ for some K. The antiderivative g defined above satisfies $|g| \leq Kt$. Let $b > 0$. Then since $u \leq \mathrm{e}^u$ (for u nonnegative), we have $bt \leq \mathrm{e}^{bt}$. So $|g| \leq Kt \leq (K/b)\mathrm{e}^{bt}$. It follows that g is of exponential type of order b for any positive b. Since any antiderivative of f has the form $g(t) + C$ for some constant C and constant functions are of exponential type, the lemma follows by Proposition 1. ◀

We will restrict our attention to continuous input functions in this chapter. In Chap. 6, we ease this restriction and consider Laplace transforms of discontinuous functions.

Basic Principles and Formulas

A particularly useful property of the Laplace transform, both theoretically and computationally, is that of ***linearity***. Specifically,

Theorem 5. *The Laplace transform is linear. In other words, if f and g are functions of exponential type and a and b are constants, then*

Linearity of the Laplace Transform

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}.$$

Proof. By Proposition 1, the function $af + bg$ is of exponential type and, by Proposition 3, has a Laplace transform. Since (improper) integration is linear, we have

$$\begin{aligned}\mathcal{L}\{af + bg\}(s) &= \int_0^\infty \mathrm{e}^{-st}(af(t) + bg(t))\,\mathrm{d}t \\ &= a\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t + b\int_0^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t \\ &= a\mathcal{L}\{f\}(s) + b\mathcal{L}\{g\}(s).\end{aligned}$$

□

The input derivative principles we derive below are the cornerstone principles of the Laplace transform. They are used to derive basic Laplace transform formulas and are the key to the Laplace transform method to solve differential equations.

Theorem 6. *Suppose $f(t)$ is a differentiable function on $[0, \infty)$ whose derivative $f'(t)$ is continuous and of exponential type of order $a \geq 0$. Then*

The Input Derivative Principle

The First Derivative

$$\mathcal{L}\{f'(t))\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0), \quad s > a.$$

Proof. By Lemma 4, $f(t)$ is of exponential type. By Proposition 3, both $f(t)$ and $f'(t)$ have Laplace transforms. Using integration by parts (let $u = \mathrm{e}^{-st}$, $\mathrm{d}v = f'(t)\,\mathrm{d}t$), we get

$$\begin{aligned}
\mathcal{L}\left\{f'(t)\right\}(s) &= \int_0^\infty \mathrm{e}^{-st} f'(t)\,\mathrm{d}t \\
&= \mathrm{e}^{-st} f(t)\big|_0^\infty - \int_0^\infty -s\mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= -f(0) + s\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t = s\mathcal{L}\{f(t)\}(s) - f(0).
\end{aligned}$$

For a function $g(t)$ defined of $[a, \infty)$, we use the notation $g(t)|_a^\infty$ to mean $\lim_{t\to\infty}(g(t) - g(a))$. ◀

Observe that if $f'(t)$ also satisfies the conditions of the input derivative principle, then we get

$$\begin{aligned}
\mathcal{L}\left\{f''(t)\right\}(s) &= s\mathcal{L}\left\{f'(t)\right\} - f'(0) \\
&= s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - f'(0) \\
&= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).
\end{aligned}$$

We thus get the following corollary:

Corollary 7. *Suppose $f(t)$ is a differentiable function on $[0, \infty)$ with continuous second derivative of exponential type of order $a \geq 0$. Then*

Input Derivative Principle

The Second Derivative

$$\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0), \qquad s > a.$$

Repeated applications of Theorem 6 give the following:

Corollary 8. *Suppose $f(t)$ is a differentiable function on $[0, \infty)$ with continuous nth derivative of exponential type of order $a \geq 0$. Then*

Input Derivative Principle

The nth Derivative

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\left\{f(t)\right\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0),$$

for $s > a$.

We now compute the Laplace transform of some specific input functions that will be used frequently throughout the text.

Formula 9. Verify the Laplace transform formula:

Constant Functions

$$\mathcal{L}\left\{1\right\}(s) = \frac{1}{s}, \qquad s > 0.$$

▼ *Verification.* For the constant function 1, we have

$$\begin{aligned}\mathcal{L}\left\{1\right\}(s) &= \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{r \to \infty} \left.\frac{e^{-ts}}{-s}\right|_0^r \\ &= \lim_{r \to \infty} \frac{e^{-rs} - 1}{-s} = \frac{1}{s} \qquad \text{for } s > 0.\end{aligned}$$ ▲

For the limit above, we have used the basic fact that

$$\lim_{r \to \infty} e^{rc} = \begin{cases} 0 & \text{if } c < 0 \\ \infty & \text{if } c > 0. \end{cases}$$

Formula 10. Assume n is a nonnegative integer. Verify the Laplace transform formula:

Power Functions

$$\mathcal{L}\left\{t^n\right\}(s) = \frac{n!}{s^{n+1}}, \qquad s > 0.$$

▼ *Verification.* Let $b > 0$. Since $u \leq e^u$ for $u \geq 0$, it follows that $bt \leq e^{bt}$ for all $t \geq 0$. Thus, $t \leq e^{bt}/b$ and $t^n \leq e^{bnt}/b^n$. Since b is any positive number, it follows that t^n is of exponential type of order a for all $a > 0$ and thus has a Laplace transform for $s > 0$. Let $f(t) = t^n$ and observe that $f^{(k)}(0) = 0$ for $k = 1, \ldots, n-1$ and $f^{(n)}(t) = n!$. Apply the nth order input derivative formula, Corollary 8, to get

$$\begin{aligned}\frac{n!}{s} &= \mathcal{L}\{n!\}(s) \\ &= \mathcal{L}\left\{f^{(n)}(t)\right\}(s) \\ &= s^n \mathcal{L}\{t^n\} - s^{n-1} f(0) - \cdots - f^{(n-1)}(0) \\ &= s^n \mathcal{L}\{t^n\}(s).\end{aligned}$$

It follows that

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, \qquad s > 0. \qquad \blacktriangle$$

Example 11. Find the Laplace transform of

$$f(t) = 3 - 4t + 6t^3.$$

► **Solution.** Here we use the linearity and the basic Laplace transforms determined above:

$$\begin{aligned}\mathcal{L}\left\{3 - 4t + 6t^3\right\} &= 3\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + 6\mathcal{L}\left\{t^3\right\} \\ &= 3\left(\frac{1}{s}\right) - 4\left(\frac{1}{s^2}\right) + 6\left(\frac{3!}{s^4}\right) \\ &= \frac{3}{s} - \frac{4}{s^2} + \frac{36}{s^4}.\end{aligned} \qquad \blacktriangleleft$$

The formula for the Laplace transform of t^n is actually valid even if the exponent is not an integer. Suppose that $\alpha \in \mathbb{R}$ is any real number. Then use the substitution $t = x/s$ in the Laplace transform integral to get

$$\mathcal{L}\{t^\alpha\} = \int_0^\infty t^\alpha \mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty \left(\frac{x}{s}\right)^\alpha \mathrm{e}^{-x}\,\frac{\mathrm{d}x}{s} = \frac{1}{s^{\alpha+1}} \int_0^\infty x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x \quad (s > 0).$$

The improper integral on the right converges as long as $\alpha > -1$, and it defines a function known as the ***gamma function*** or ***generalized factorial function*** evaluated at $\alpha + 1$. Thus, $\Gamma(\beta) = \int_0^\infty x^{\beta-1}\mathrm{e}^{-x}\,\mathrm{d}x$ is defined for $\beta > 0$, and the Laplace transform of the general power function is as follows:

Formula 12. If $\alpha > -1$, then

General Power Functions

$$\mathcal{L}\left\{t^{\alpha}\right\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \qquad s > 0.$$

If n is a positive integer, then $\Gamma(n+1) = n!$ (see Exercise 42) so Formula 10 is a special case of Formula 12.

Formula 13. Assume $a \in \mathbb{R}$. Verify the Laplace transform formula:

Exponential Functions

$$\mathcal{L}\left\{e^{at}\right\}(s) = \frac{1}{s-a}, \qquad s > a.$$

▼ *Verification.* If $s > a$, then

$$\mathcal{L}\left\{e^{at}\right\}(s) = \int_0^{\infty} e^{-st} e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^{\infty} = \frac{1}{s-a}. \qquad \blacktriangle$$

Formula 14. Let $b \in \mathbb{R}$. Verify the Laplace transform formulas:

Cosine Functions

$$\mathcal{L}\left\{\cos bt\right\}(s) = \frac{s}{s^2+b^2}, \qquad \text{for } s > 0.$$

and

Sine Functions

$$\mathcal{L}\left\{\sin bt\right\}(s) = \frac{b}{s^2+b^2}, \qquad \text{for } s > 0.$$

▼ *Verification.* Since both $\sin bt$ and $\cos bt$ are bounded continuous functions, they are of exponential type and hence have Laplace transforms. Let $f(t) = \cos bt$. Then $f'(t) = -b\sin bt$ and $f''(t) = -b^2\cos bt$. The input derivative principle for the second derivative, Corollary 7, implies

$$\begin{aligned}
-b^2\mathcal{L}\left\{\cos bt\right\}(s) = \mathcal{L}\left\{f''(t)\right\}(s) &= s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0) \\
&= s^2\mathcal{L}\left\{\cos bt\right\} - s(1) - (0) \\
&= s^2\mathcal{L}\left\{\cos bt\right\} - s.
\end{aligned}$$

Now subtract $s^2\mathcal{L}\{\cos bt\}$ from both sides and combine terms to get

$$-(s^2+b^2)\mathcal{L}\{\cos bt\} = -s.$$

Solving for $\mathcal{L}\{\cos bt\}$ gives

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}.$$

A similar calculation verifies the formula for $\mathcal{L}\{\sin bt\}$. ▲

Example 15. Find the Laplace transform of

$$2e^{6t} + 3\cos 2t - 4\sin 3t.$$

► **Solution.** We use the linearity of the Laplace transform together with the formulas derived above to get

$$\begin{aligned}\mathcal{L}\left\{2e^{6t} + 3\cos 2t - 4\sin 3t\right\} &= 2\mathcal{L}\left\{e^{6t}\right\} + 3\mathcal{L}\{\cos 2t\} - 4\mathcal{L}\{\sin 3t\} \\ &= \frac{2}{s-6} + 3\left(\frac{s}{s^2+2^2}\right) - 4\left(\frac{3}{s^2+3^2}\right) \\ &= \frac{2}{s-6} + \frac{3s}{s^2+4} - \frac{12}{s^2+9}.\end{aligned}$$ ◄

Formula 16. Let n be a nonnegative integer and $a \in \mathbb{R}$. Verify the following Laplace transform formula:

Power-Exponential Functions

$$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}, \quad \text{for } s > a.$$

▼ *Verification.* Notice that

$$\mathcal{L}\left\{t^n e^{at}\right\}(s) = \int_0^\infty e^{-st} t^n e^{at}\,dt = \int_0^\infty e^{-(s-a)t} t^n\,dt = \mathcal{L}\{t^n\}(s-a).$$

What this formula says is that the Laplace transform of the function $t^n e^{at}$ evaluated at the point s is the same as the Laplace transform of the function t^n evaluated at the point $s-a$. Since $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$, we conclude

$$\mathcal{L}\left\{t^n e^{at}\right\}(s) = \frac{n!}{(s-a)^{n+1}}, \quad \text{for } s > a.$$ ▲

If the function t^n in Formula 16 is replaced by an arbitrary input function $f(t)$ with a Laplace transform $F(s)$, then we obtain the following:

Theorem 17. *Suppose f has Laplace transform $F(s)$. Then*

$$\boxed{\begin{array}{c}\textbf{\textit{First Translation Principle}}\\[1ex] \mathcal{L}\{e^{at} f(t)\}(s) = F(s-a)\end{array}}$$

Proof.

$$\begin{aligned}\mathcal{L}\left\{e^{at} f(t)\right\}(s) &= \int_0^\infty e^{-st} e^{at} f(t)\, dt \\ &= \int_0^\infty e^{-(s-a)} f(t)\, dt \\ &= \mathcal{L}\{f(t)\}(s-a) = F(s-a).\end{aligned}$$

□

In words, this formula says that to compute the Laplace transform of the product of $f(t)$ and e^{at}, it is only necessary to take the Laplace transform of $f(t)$ (namely, $F(s)$) and replace the variable s by $s-a$, where a is the coefficient of t in the exponential multiplier. It is convenient to use the following notation:

$$\mathcal{L}\left\{e^{at} f(t)\right\}(s) = F(s)\big|_{s \mapsto s-a}$$

to indicate this substitution.

Formula 18. Suppose $a, b \in \mathbb{R}$. Verify the Laplace transform formulas

$$\boxed{\mathcal{L}\left\{e^{at} \cos bt\right\}(s) = \frac{s-a}{(s-a)^2 + b^2}}$$

and

$$\boxed{\mathcal{L}\left\{e^{at} \sin bt\right\}(s) = \frac{b}{(s-a)^2 + b^2}.}$$

▼ *Verification.* From Example 14, we know that

$$\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2 + b^2} \quad \text{and} \quad \mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2 + b^2}.$$

Replacing s by $s-a$ in each of these formulas gives the result. ▲

Example 19. Find the Laplace transform of

$$2e^{-t} \sin 3t \quad \text{and} \quad e^{3t} \cos \sqrt{2}t.$$

► **Solution.** Again we use linearity of the Laplace transform and the formulas derived above to get

$$\mathcal{L}\left\{2e^{-t}\sin 3t\right\} = 2\left.\frac{3}{s^2+3^2}\right|_{s\mapsto s+1} = \frac{6}{(s+1)^2+9} = \frac{6}{s^2+2s+10},$$

$$\mathcal{L}\left\{e^{3t}\cos\sqrt{2}t\right\} = \left.\frac{s}{s^2+\sqrt{2}^2}\right|_{s\mapsto s-3} = \frac{s-3}{(s-3)^2+2} = \frac{s-3}{s^2-6s+11}. \quad ◄$$

We now introduce another useful principle that can be used to compute some Laplace transforms.

Theorem 20. *Suppose $f(t)$ is an input function and $F(s) = \mathcal{L}\{f(t)\}(s)$ is the transform function. Then*

Transform Derivative Principle

$$\mathcal{L}\{-tf(t)\}(s) = \frac{d}{ds}F(s).$$

Proof. By definition, $F(s) = \int_0^\infty e^{-st} f(t)\,dt$, and thus,

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty \frac{d}{ds}(e^{-st}) f(t)\,dt \\ &= \int_0^\infty e^{-st}(-t) f(t)\,dt = \mathcal{L}\{-tf(t)\}(s). \end{aligned}$$

Interchanging the derivative and the integral can be justified. □

Repeated application of the transform derivative principle gives

Transform nth-Derivative Principle

$$(-1)^n\mathcal{L}\{t^n f(t)\}(s) = \frac{d^n}{ds^n}F(s).$$

Example 21. Use the transform derivative principle to compute

$$\mathcal{L}\{t\sin t\}(s).$$

► **Solution.** A direct application of the transform derivative principle gives

$$\begin{aligned}\mathcal{L}\{t \sin t\}(s) &= -\mathcal{L}\{-t \sin t\} \\ &= -\frac{d}{ds}\mathcal{L}\{\sin t\}(s) \\ &= -\frac{d}{ds}\frac{1}{s^2+1} \\ &= -\frac{-2s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}.\end{aligned}$$ ◄

Example 22. Compute the Laplace transform of $t^2 e^{2t}$ in two different ways: using the first translation principle and the transform derivative principle.

► **Solution.** Using the first translation principle, we get

$$\mathcal{L}\left\{t^2 e^{2t}\right\}(s) = \mathcal{L}\left\{t^2\right\}(s)\Big|_{s \mapsto s-2} = \frac{2}{(s-2)^3}.$$

Using the transform derivative principle, we get

$$\mathcal{L}\left\{t^2 e^{2t}\right\} = \frac{d^2}{ds^2}\frac{1}{s-2} = \frac{d}{ds}\frac{-1}{(s-2)^2} = \frac{2}{(s-2)^3}.$$ ◄

Suppose $f(t)$ is a function and b is a positive real number. The function $g(t) = f(bt)$ is called the ***dilation*** of f by b. If the domain of f includes $[0, \infty)$, then so does any dilation of f since b is positive. The following theorem describes the Laplace transform of a dilation.

Theorem 23. *Suppose $f(t)$ is an input function and b is a positive real number. Then*

The Dilation Principle

$$\mathcal{L}\{f(bt))\}(s) = \frac{1}{b}\mathcal{L}\{f(t)\}(s/b).$$

Proof. This result follows from a change in variable in the definition of the Laplace transform:

$$\begin{aligned}\mathcal{L}\{f(bt)\}(s) &= \int_0^\infty e^{-st} f(bt)\, dt \\ &= \int_0^\infty e^{-(s/b)r} f(r)\, \frac{dr}{b} \\ &= \frac{1}{b}\mathcal{L}\{f(t)\}(s/b).\end{aligned}$$

Table 2.2 Basic Laplace transform formulas (We are assuming n is a nonnegative integer and a and b are real)

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\frac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\frac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\frac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\frac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\frac{b}{(s-a)^2+b^2}$

Table 2.3 Basic Laplace transform principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$
Input derivative principles	$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\} - f(0)$
	$\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$
First translation principle	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
Transform derivative principle	$\mathcal{L}\{-tf(t)\}(s) = \frac{d}{ds}F(s)$
The dilation principle	$\mathcal{L}\{f(bt)\}(s) = \frac{1}{b}\mathcal{L}\{f(t)\}(s/b)$.

To get the second line, we made the change of variable $t = r/b$. Since $b > 0$, the limits of integration remain unchanged. □

Example 24. The formula $\mathcal{L}\left\{\frac{\sin t}{t}\right\}(s) = \cot^{-1}(s)$ will be derived later (in Sect. 5.4). Assuming this formula for now, determine $\mathcal{L}\left\{\frac{\sin bt}{t}\right\}(s)$.

► **Solution.** By linearity and the dilation principle, we have

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin bt}{t}\right\}(s) &= b\mathcal{L}\left\{\frac{\sin bt}{bt}\right\}(s)\\ &= b\frac{1}{b}\,\mathcal{L}\left\{\frac{\sin t}{t}\right\}\Big|_{s\mapsto s/b}\\ &= \cot^{-1}(s)\Big|_{s\mapsto s/b} = \cot^{-1}(s/b).\end{aligned}$$ ◄

We now summarize in Table 2.2 the basic Laplace transform formulas and, in Table 2.3, the basic Laplace transform principles we have thus far derived. The student should learn these well as they will be used frequently throughout the text and exercises. With the use of these tables, we can find the Laplace transform of many functions. As we continue, several new formulas will be derived. Appendix C has a complete list of Laplace transform formulas and Laplace transform principles that we derive.

Exercises

1–4. Compute the Laplace transform of each function given below directly from the integral definition given in (1).

1. $3t + 1$
2. $5t - 9e^t$
3. $e^{2t} - 3e^{-t}$
4. te^{-3t}

5–18. Use Table 2.2 and linearity to find the Laplace transform of each given function.

5. $5e^{2t}$
6. $3e^{-7t} - 7t^3$
7. $t^2 - 5t + 4$
8. $t^3 + t^2 + t + 1$
9. $e^{-3t} + 7te^{-4t}$
10. t^2e^{4t}
11. $\cos 2t + \sin 2t$
12. $e^t(t - \cos 4t)$
13. $(te^{-2t})^2$
14. $e^{-t/3}\cos\sqrt{6}t$
15. $(t + e^{2t})^2$
16. $5\cos 3t - 3\sin 3t + 4$
17. $\dfrac{t^4}{e^{4t}}$
18. $e^{5t}(8\cos 2t + 11\sin 2t)$

19–23. Use the transform derivative principle to compute the Laplace transform of the following functions.

19. te^{3t}
20. $t\cos 3t$
21. $t^2\sin 2t$
22. $te^{-t}\cos t$
23. $tf(t)$ given that $F(s) = \mathcal{L}\{f\}(s) = \ln\left(\dfrac{s^2}{s^2+1}\right)$

24–25. Use the dilation principle to find the Laplace transform of each function. The given Laplace transforms will be established later.

24. $\frac{1-\cos 5t}{t}$; given $\mathcal{L}\left\{\frac{1-\cos t}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2}{s^2+1}\right)$
25. $\mathrm{Ei}(6t)$; given $\mathcal{L}\{\mathrm{Ei}(t)\} = \frac{\ln(s+1)}{s}$

26–31. Use trigonometric or hyperbolic identities to compute the Laplace transform of the following functions.

26. $\cos^2 bt$ (*Hint:* $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$)
27. $\sin^2 bt$
28. $\sin bt \cos bt$
29. $\sin at \cos bt$
30. $\cosh bt$ (Recall that $\cosh bt = (\mathrm{e}^{bt}+\mathrm{e}^{-bt})/2$.)
31. $\sinh bt$ (Recall that $\sinh bt = (\mathrm{e}^{bt}-\mathrm{e}^{-bt})/2$.)

32–34. Use one of the input derivative formulas to compute the Laplace transform of the following functions.

32. e^{at}
33. $\sinh bt$
34. $\cosh bt$
35. Use the input derivative formula to derive the Laplace transform formula $\mathcal{L}\left\{\int_0^t f(u)\,\mathrm{d}u\right\} = F(s)/s$. *Hint:* Let $g(t) = \int_0^t f(u)\,\mathrm{d}u$ and note that $g'(t) = f(t)$. Now apply the input derivative formula to $g(t)$.

36–41. *Functions of Exponential Type:* Verify the following claims.

36. Suppose f is of exponential type of order a and $b > a$. Show f is of exponential type of order b.
37. Show that the product of two functions of exponential type is of exponential type.
38. Show that the definition given for a function of exponential type is equivalent to the following: A continuous function f on $[0,\infty)$ is of exponential type of order a if there are constants $K \geq 0$ and $N \geq 0$ such that $|f(t)| \leq K\mathrm{e}^{at}$ for all $t \geq N$ (i.e., we do not need to require that $N = 0$).
39. Show that the function $y(t) = \mathrm{e}^{t^2}$ is not of exponential type.
40. Verify that the function $f(t) = \mathrm{e}^{t^2}$ does not have a Laplace transform. That is, show that the improper integral that defines $F(s)$ does not converge for *any* value of s.
41. Let $y(t) = \sin(\mathrm{e}^{t^2})$. Why is $y(t)$ of exponential type? Compute $y'(t)$ and show that it is not of exponential type. Nevertheless, show that $y'(t)$ has a Laplace transform. *The moral*: The derivative of a function of exponential type is not necessarily of exponential type, and there are functions that are not of exponential type that have a Laplace transform.
42. Recall from the discussion for Formula 12 that the *gamma function* is defined by the improper integral

$$\Gamma(\beta) = \int_0^\infty x^{\beta-1}\mathrm{e}^{-x}\,\mathrm{d}x, \qquad (\beta > 0).$$

(a) Show that $\Gamma(1) = 1$.
(b) Show that Γ satisfies the recursion formula $\Gamma(\beta + 1) = \beta\Gamma(\beta)$.
(*Hint*: Integrate by parts.)
(c) Show that $\Gamma(n + 1) = n!$ when n is a nonnegative integer.

43. Show that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.
(*Hint*: Let I be the integral and note that

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

Then evaluate the integral using polar coordinates.)

44. Use the integral from Exercise 43 to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Then compute each of the following:

(a) $\Gamma(\frac{3}{2})$ (b) $\Gamma(\frac{5}{2})$ (c) $\mathcal{L}\left\{\sqrt{t}\right\}$ (d) $\mathcal{L}\left\{t^{3/2}\right\}$.

2.3 Partial Fractions: A Recursive Algorithm for Linear Terms

A careful look at Table 2.2 reveals that the Laplace transform of each function we considered is a rational function. Laplace inversion, which is discussed in Sect. 2.5, will involve writing rational functions as sums of those simpler ones found in the table.

All students of calculus should be familiar with the technique of obtaining the partial fraction decomposition of a rational function. Briefly, a given proper rational function[3] $p(s)/q(s)$ is a sum of ***partial fractions*** of the form

$$\frac{A_j}{(s-r)^j} \quad \text{and} \quad \frac{B_k s + C_k}{(s^2+cs+d)^k},$$

where A_j, B_k, and C_k are constants. The partial fractions are determined by the linear factors, $s-r$, and the irreducible quadratic factors, s^2+cs+d, of the denominator $q(s)$, where the powers j and k occur up to the multiplicity of the factors. After finding a common denominator and equating the numerators, we obtain a system of linear equations to solve for the undetermined coefficients A_j, B_k, C_k. Notice that the degree of the denominator determines the number of coefficients that are involved in the form of the partial fraction decomposition. Even when the degree is relatively small, this process can be very tedious and prone to simple numerical mistakes.

Our purpose in this section and the next is to provide an alternate algorithm for obtaining the partial fraction decomposition of a rational function. This algorithm has the advantage that it is constructive (assuming the factorization of the denominator), recursive (meaning that only one coefficient at a time is determined), and self-checking. This recursive method for determining partial fractions should be well practiced by the student. It is the method we will use throughout the text and is an essential technique in solving nonhomogeneous differential equations discussed in Sect. 3.5. You may wish to review Appendix A.2 where notation and results about polynomials and rational functions are given.

In this section, we will discuss the algorithm in the linear case, that is, when the denominator has a linear term as a factor. In Sect. 2.4, we discuss the case where the denominator has an irreducible quadratic factor.

Theorem 1 (Linear Partial Fraction Recursion). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-r)^n q(s)}$$

[3] A ***rational function*** is the quotient of two polynomials. A rational function is ***proper*** if the degree of the numerator is less than the degree of the denominator.

and $q(r) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-r)^n q(s)} = \frac{A_1}{(s-r)^n} + \frac{p_1(s)}{(s-r)^{n-1}q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \left.\frac{p_0(s)}{q(s)}\right|_{s=r} = \frac{p_0(r)}{q(r)} \qquad \textit{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-r}. \tag{2}$$

Proof. After finding a common denominator in (1) and equating numerators, we get the polynomial equation $p_0(s) = A_1 q(s) + (s-r)p_1(s)$. Evaluating at $s = r$ gives $p_0(r) = A_1 q(r)$, and hence $A_1 = \frac{p_0(r)}{q(r)}$. Now that A_1 is determined, we have $p_0(s) - A_1 q(s) = (s-r)p_1(s)$, and hence $p_1(s) = \frac{p_0(s)-A_1q(s)}{s-r}$. $\square$

Notice that in the calculation of p_1, it is necessary that $p_0(s) - A_1 q(s)$ have a factor of $s - r$. If such a factorization does not occur when working an example, then an error has been made. This is what is meant when we stated above that this recursive method is self-checking. In practice, we frequently factor $p_0(s) - A_1 q(s)$ and delete the $s - r$ factor. However, for large degree polynomials, it may be best to use the division algorithm for polynomials or synthetic division.

An application of Theorem 1 produces two items:

- The partial fraction of the form

$$\frac{A_1}{(s-r)^n}$$

- A remainder term of the form

$$\frac{p_1(s)}{(s-r)^{n-1}q(s)}$$

such that the original rational function $p_0(s)/(s-r)^n q(s)$ is the sum of these two pieces. We can now repeat the process on the new rational function $p_1(s)/((s-r)^{n-1}q(s))$, where the multiplicity of $(s-r)$ in the denominator has been reduced by 1, and continue in this manner until we have removed completely $(s-r)$ as a factor of the denominator. In this manner, we produce a sequence,

$$\frac{A_1}{(s-r)^n}, \ldots, \frac{A_n}{(s-r)},$$

which we will refer to as the $(s-r)$-***chain*** for $p_0(s)/((s-r)^n q(s))$. The number of terms, n, is referred to as the ***length*** of the chain. The ***chain table*** below summarizes the data obtained.

The $(s-r)$*-chain*	
$\dfrac{p_0(s)}{(s-r)^n q(s)}$	$\dfrac{A_1}{(s-r)^n}$
$\dfrac{p_1(s)}{(s-r)^{n-1}(s)q(s)}$	$\dfrac{A_2}{(s-r)^{n-1}}$
$\vdots$	$\vdots$
$\dfrac{p_{n-1}(s)}{(s-r)q(s)}$	$\dfrac{A_n}{(s-r)}$
$\dfrac{p_n(s)}{q(s)}$	

Notice that the partial fractions are placed in the second column while the remainder terms are placed in the first column under the previous remainder term. This form is conducive to the recursion algorithm. From the table, we get

$$\frac{p_0(s)}{(s-r)^n q(s)} = \frac{A_1}{(s-r)^n} + \cdots + \frac{A_n}{(s-r)} + \frac{p_n(s)}{q(s)}.$$

By factoring another linear term out of $q(s)$, the process can be repeated through all linear factors of $q(s)$. In the examples that follow, we will organize one step of the recursion process as follows:

Partial Fraction Recursion Algorithm
by a Linear Term

$$\frac{p_0(s)}{(s-r)^n q(s)} = \frac{A_1}{(s-r)^n} + \frac{p_1(s)}{(s-r)^{n-1} q(s)}$$

where $\quad A_1 = \left.\dfrac{p_0(s)}{q(s)}\right|_{s=r} = \square$

and $\quad p_1(s) = \dfrac{1}{s-r}\left(p_0(s) - A_1 q(s)\right) = \square$

The curved arrows indicate where the results of calculations are inserted. First, A_1 is calculated and inserted in two places: in the $(s-r)$-chain and in the calculation for $p_1(s)$. Afterward, $p_1(s)$ is calculated and the result inserted in the numerator of the remainder term. Now the process is repeated on $p_1(s)/((s-r)^{n-1}q(s))$ until the $(s-r)$-chain is completed.

Consider the following examples.

Example 2. Find the partial fraction decomposition for

$$\frac{s-2}{(s-3)^2(s-4)}.$$

► **Solution.** We will first compute the $(s-3)$ -chain. According to Theorem 1, we can write

$$\frac{s-2}{(s-3)^2(s-4)} = \frac{A_1}{(s-3)^2} + \frac{p_1(s)}{(s-3)(s-4)}$$

$$\text{where} \quad A_1 = \left.\frac{s-2}{s-4}\right|_{s=3} = -1$$

$$\text{and} \quad p_1(s) = \frac{1}{s-3}(s-2-(-1)(s-4)) = \frac{1}{s-3}(2s-6) = 2.$$

We thus get

$$\frac{s-2}{(s-3)^2(s-4)} = \frac{-1}{(s-3)^2} + \frac{2}{(s-3)(s-4)}.$$

We now repeat the recursion algorithm on the remainder term $\frac{2}{(s-3)(s-4)}$ to get

$$\frac{2}{(s-3)(s-4)} = \frac{A_2}{s-3} + \frac{p_2(s)}{s-4}$$

$$\text{where} \quad A_1 = \left.\frac{2}{s-4}\right|_{s=3} = -2$$

$$\text{and} \quad p_2(s) = \frac{1}{s-3}(2-(-2)(s-4)) = \frac{1}{s-3}(2s-6) = 2.$$

Putting these calculations together gives the $(s-3)$-chain

The $(s-3)$ *-chain*	
$\dfrac{s-2}{(s-3)^2(s-4)}$	$\dfrac{-1}{(s-3)^2}$
$\dfrac{2}{(s-3)(s-4)}$	$\dfrac{-2}{(s-3)}$
$\dfrac{2}{s-4}$	

The $(s-4)$-chain has length one and is given as the remainder entry in the $(s-3)$-chain; thus

$$\frac{s-2}{(s-3)^2(s-4)} = \frac{-1}{(s-3)^2} - \frac{2}{(s-3)} + \frac{2}{s-4}. \qquad \blacktriangleleft$$

A more substantial example is given next. The partial fraction recursion algorithm remains exactly the same so we will dispense with the curved arrows.

Example 3. Find the partial fraction decomposition for

$$\frac{16s}{(s+1)^3(s-1)^2}.$$

Remark 4. Before we begin with the solution, we remark that the traditional method for computing the partial fraction decomposition introduces the equation

$$\frac{16s}{(s+1)^3(s-1)^2} = \frac{A_1}{(s+1)^3} + \frac{A_2}{(s+1)^2} + \frac{A_3}{s+1} + \frac{A_4}{(s-1)^2} + \frac{A_5}{s-1}$$

and, after finding a common denominator, requires the simultaneous solution to a system of five equations in five unknowns, a doable task but one prone to simple algebraic errors.

► **Solution.** We will first compute the $(s+1)$ -chain. According to Theorem 1, we can write

$$\frac{16s}{(s+1)^3(s-1)^2} = \frac{A_1}{(s+1)^3} + \frac{p_1(s)}{(s+1)^2(s-1)^2},$$

$$\text{where} \quad A_1 = \left.\frac{16s}{(s-1)^2}\right|_{s=-1} = -\frac{16}{4} = -4$$

$$\text{and} \quad p_1(s) = \frac{1}{s+1}(16s - (-4)(s-1)^2)$$
$$= \frac{4}{s+1}(s^2 + 2s + 1) = 4(s+1).$$

We now repeat the recursion step on the remainder term $\frac{4(s+1)}{(s+1)^2(s-1)^2}$ to get

$$\frac{4(s+1)}{(s+1)^2(s-1)^2} = \frac{A_2}{(s+1)^2} + \frac{p_2(s)}{(s+1)(s-1)^2},$$

$$\text{where} \quad A_2 = \left.\frac{4(s+1)}{(s-1)^2}\right|_{s=-1} = \frac{0}{4} = 0$$
$$\text{and} \quad p_2(s) = \frac{1}{s+1}(4(s+1) - (0)(s-1)^2) = 4.$$

Notice here that we could have canceled the $(s+1)$ term at the beginning and arrived immediately at $\frac{4}{(s+1)(s-1)^2}$. Then no partial fraction with $(s+1)^2$ in the denominator would occur. We chose though to continue the recursion step to show the process. The recursion process is now repeated on $\frac{4}{(s+1)(s-1)^2}$ to get

$$\frac{4}{(s+1)(s-1)^2} = \frac{A_3}{s+1} + \frac{p_3(s)}{(s-1)^2},$$

$$\text{where} \quad A_3 = \left.\frac{4}{(s-1)^2}\right|_{s=-1} = \frac{4}{4} = 1$$
$$\text{and} \quad p_3(s) = \frac{1}{s+1}(4 - (1)(s-1)^2)$$
$$= \frac{-1}{s+1}(s^2 - 2s - 3) = \frac{-1}{s+1}(s+1)(s-3) = -(s-3).$$

Putting these calculations together gives the $(s+1)$ -chain

The $(s+1)$ *-chain*	
$\frac{16}{(s+1)^3(s-1)^2}$	$\frac{-4}{(s+1)^3}$
$\frac{4(s+1)}{(s+1)^2(s-1)^2}$	$\frac{0}{(s+1)^2}$
$\frac{4}{(s+1)(s-1)^2}$	$\frac{1}{(s+1)}$
$\frac{-(s-3)}{(s-1)^2}$	

We now compute the $(s-1)$ -chain for the remainder $\frac{-(s-3)}{(s-1)^2}$. It is implicit that $q(s) = 1$.

$$\frac{-(s-3)}{(s-1)^2} = \frac{A_4}{(s-1)^2} + \frac{p_4(s)}{s-1},$$

$$\text{where} \quad A_4 = -\left.\frac{(s-3)}{1}\right|_{s=1} = 2$$

$$\text{and} \quad p_4(s) = \frac{1}{s-1}(-(s-3)-(2)) = \frac{1}{s-1}(-s+1) = -1.$$

The $(s-1)$ -chain is thus

The $(s-1)$ *-chain*	
$\frac{-(s-3)}{(s-1)^2}$	$\frac{2}{(s-1)^2}$
$\frac{-1}{(s-1)}$	

We now have

$$\frac{-(s-3)}{(s-1)^2} = \frac{2}{(s-1)^2} + \frac{-1}{s-1},$$

and putting this chain together with the $s+1$ -chain gives

$$\frac{16s}{(s+1)^3(s-1)^2} = \frac{-4}{(s+1)^3} + \frac{0}{(s+1)^2} + \frac{1}{s+1} + \frac{2}{(s-1)^2} + \frac{-1}{s-1}. \quad \blacktriangleleft$$

Product of Distinct Linear Factors

Let $p(s)/q(s)$ be a proper rational function. Suppose $q(s)$ is the product of distinct linear factors, that is,

$$q(s) = (s-r_1)\cdots(s-r_n),$$

where $r_1, \ldots r_n$ are distinct scalars. Then each chain has length one and the partial fraction decomposition has the form

$$\frac{p(s)}{q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{(s - r_n)}.$$

The scalar A_i is the first and only entry in the $(s - r_i)$ -chain. Thus,

$$A_i = \left.\frac{p(s)}{q_i(s)}\right|_{s=r_i} = \frac{p(r_i)}{q_i(r_i)},$$

where $q_i(s) = q(s)/(s - r_i)$ is the polynomial obtained from $q(s)$ by factoring out $(s - r_i)$. If we do this for each $i = 1, \ldots, n$, it is unnecessary to calculate any remainder terms.

Example 5. Find the partial fraction decomposition of

$$\frac{-4s + 14}{(s - 1)(s + 4)(s - 2)}.$$

► **Solution.** The denominator $q(s) = (s - 1)(s + 4)(s - 2)$ is a product of distinct linear factors. Each partial fraction is determined as follows:

- For $\frac{A_1}{s-1}$: $\quad A_1 = \left.\frac{-4s + 14}{(s + 4)(s - 2)}\right|_{s=1} = \frac{10}{-5} = -2$
- For $\frac{A_2}{s+4}$: $\quad A_2 = \left.\frac{-4s + 14}{(s - 1)(s - 2)}\right|_{s=-4} = \frac{30}{30} = 1$
- For $\frac{A_3}{s-2}$: $\quad A_3 = \left.\frac{-4s + 14}{(s - 1)(s + 4)}\right|_{s=2} = \frac{6}{6} = 1$

The partial fraction decomposition is thus

$$\frac{-4s + 14}{(s - 1)(s + 4)(s - 2)} = \frac{-2}{s - 1} + \frac{1}{s + 4} + \frac{1}{s - 2}.$$ ◄

Linear Partial Fractions and the Laplace Transform Method

In the example, notice how linear partial fraction recursion facilitates the Laplace transform method.

Example 6. Use the Laplace transform method to solve the following differential equation:

$$y'' + 3y' + 2y = \mathrm{e}^{-t}, \tag{3}$$

with initial conditions $y(0) = 1$ and $y'(0) = 3$.

► **Solution.** We will use the Laplace transform to turn this differential equation in y into an algebraic equation in $Y(s) = \mathcal{L}\{y(t)\}$. Apply the Laplace transform to both sides. For the left-hand side, we get

$$\begin{aligned}\mathcal{L}\left\{y'' + 3y' + 2y\right\} &= \mathcal{L}\left\{y''\right\} + 3\mathcal{L}\left\{y'\right\} + 2\mathcal{L}\{y\}\\ &= s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s)\\ &= (s^2 + 3s + 2)Y(s) - s - 6.\end{aligned}$$

The first line uses the linearity of the Laplace transform, the second line uses the first and second input derivative principles, and the third line uses the given initial conditions and then simplifies the result. Since $\mathcal{L}\{\mathrm{e}^{-t}\} = 1/(s + 1)$, we get the algebraic equation

$$(s^2 + 3s + 2)Y(s) - s - 6 = \frac{1}{s+1},$$

from which it is easy to solve for $Y(s)$. Since $s^2 + 3s + 2 = (s + 1)(s + 2)$, we get

$$Y(s) = \frac{s+6}{(s+1)(s+2)} + \frac{1}{(s+1)^2(s+2)}.$$

We are now left with the task of finding an input function whose Laplace transform is $Y(s)$. To do this, we first compute the partial fraction decomposition of each term. The first term $\frac{s+6}{(s+1)(s+2)}$ has denominator which is a product of two distinct linear terms. Each partial fraction is determined as follows:

- For $\dfrac{A_1}{s+1}$: $\quad A_1 = \left.\dfrac{s+6}{s+2}\right|_{s=-1} = \dfrac{5}{1} = 5$
- For $\dfrac{A_2}{s+2}$: $\quad A_2 = \left.\dfrac{s+6}{s+1}\right|_{s=-2} = \dfrac{4}{-1} = -4$

The partial fraction decomposition is thus

$$\frac{s+6}{(s+1)(s+2)} = \frac{5}{s+1} - \frac{4}{s+2}.$$

For the second term $\frac{1}{(s+1)^2(s+2)}$, we compute the $(s+1)$-chain

The $(s+1)$ *-chain*	
$\frac{1}{(s+1)^2(s+2)}$	$\frac{1}{(s+1)^2}$
$\frac{-1}{(s+1)(s+2)}$	$\frac{-1}{s+1}$
$\frac{1}{s+2}$	

from which we get

$$\frac{1}{(s+1)^2(s+2)} = \frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}.$$

It follows that

$$Y(s) = \frac{4}{s+1} - \frac{3}{s+2} + \frac{1}{(s+1)^2}.$$

Now we can determine the input function $y(t)$ directly from the basic Laplace transform table, Table 2.2. We get

$$y(t) = 4e^{-t} - 3e^{-2t} + te^{-t}.$$

This is the solution to (3). ◀

Exercises

1–11. For each exercise below, compute the chain table through the indicated linear term.

1. $\dfrac{5s+10}{(s-1)(s+4)}$; $(s-1)$

2. $\dfrac{10s-2}{(s+1)(s-2)}$; $(s-2)$

3. $\dfrac{1}{(s+2)(s-5)}$; $(s-5)$

4. $\dfrac{5s+9}{(s-1)(s+3)}$; $(s+3)$

5. $\dfrac{3s+1}{(s-1)(s^2+1)}$; $(s-1)$

6. $\dfrac{3s^2-s+6}{(s+1)(s^2+4)}$; $(s+1)$

7. $\dfrac{s^2+s-3}{(s+3)^3}$; $(s+3)$

8. $\dfrac{5s^2-3s+10}{(s+1)(s+2)^2}$; $(s+2)$

9. $\dfrac{s}{(s+2)^2(s+1)^2}$; $(s+1)$

10. $\dfrac{16s}{(s-1)^3(s-3)^2}$; $(s-1)$

11. $\dfrac{1}{(s-5)^5(s-6)}$; $(s-5)$

12–32. Find the partial fraction decomposition of each proper rational function.

12. $\dfrac{5s+9}{(s-1)(s+3)}$

13. $\dfrac{8+s}{s^2-2s-15}$

14. $\dfrac{1}{s^2-3s+2}$

15. $\dfrac{5s-2}{s^2+2s-35}$

16. $\dfrac{3s+1}{s^2+s}$

17. $\dfrac{2s+11}{s^2-6s-7}$

18. $\dfrac{2s^2+7}{(s-1)(s-2)(s-3)}$

19. $\dfrac{s^2+s+1}{(s-1)(s^2+3s-10)}$

20. $\dfrac{s^2}{(s-1)^3}$

21. $\dfrac{7}{(s+4)^4}$

22. $\dfrac{s}{(s-3)^3}$

23. $\dfrac{s^2+s-3}{(s+3)^3}$

24. $\dfrac{5s^2-3s+10}{(s+1)(s+2)^2}$

25. $\dfrac{s^2-6s+7}{(s^2-4s-5)^2}$

26. $\dfrac{81}{s^3(s+9)}$

27. $\dfrac{s}{(s+2)^2(s+1)^2}$

28. $\dfrac{s^2}{(s+2)^2(s+1)^2}$

29. $\dfrac{8s}{(s-1)(s-2)(s-3)^3}$

30. $\dfrac{25}{s^2(s-5)(s+1)}$

31. $\dfrac{s}{(s-2)^2(s-3)^2}$

32. $\dfrac{16s}{(s-1)^3(s-3)^2}$

33–38. Use the *Laplace transform method* to solve the following differential equations. (Give both $Y(s)$ and $y(t)$.)

33. $y'' + 2y' + y = 9e^{2t}, \quad y(0) = 0, y'(0) = 0$
34. $y'' + 3y' + 2y = 12e^{2t}, \quad y(0) = 1, y'(0) = -1$
35. $y'' - 4y' - 5y = 150t, \quad y(0) = -1, y'(0) = 1$
36. $y'' + 4y' + 4y = 4\cos 2t, \quad y(0) = 0, y'(0) = 1$
37. $y'' - 3y' + 2y = 4, \quad y(0) = 2, y'(0) = 3$
38. $y'' - 3y' + 2y = e^{t}, \quad y(0) = -3, y'(0) = 0$

2.4 Partial Fractions: A Recursive Algorithm for Irreducible Quadratics

We continue the discussion of Sect. 2.3. Here we consider the case where a real rational function has a denominator with irreducible quadratic factors.

Theorem 1 (Quadratic Partial Fraction Recursion). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d is an irreducible quadratic that is factored completely out of $q(s)$. Then there is a unique linear term $B_1 s + C_1$ and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^2+cs+d)^{n-1} q(s)}. \tag{1}$$

If $a+ib$ is a complex root of s^2+cs+d, then $B_1 s + C_1$ and the polynomial $p_1(s)$ are given by

$$B_1 s + C_1\big|_{s=a+bi} = \left.\frac{p_0(s)}{q(s)}\right|_{s=a+bi} \quad \textit{and} \quad p_1(s) = \frac{p_0(s) - (B_1 s + C_1)q(s)}{s^2+cs+d}. \tag{2}$$

Proof. After finding a common denominator in (1) and equating numerators, we get the polynomial equation

$$p_0(s) = (B_1 s + C_1)q(s) + (s^2+cs+d)p_1(s). \tag{3}$$

Evaluating at $s = a+ib$ gives $p_0(a+ib) = (B_1(a+ib)+C_1)(q(a+ib))$, and hence,

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)}. \tag{4}$$

Equating the real and imaginary parts of both sides of (4) gives the equations

$$B_1 a + C_1 = \mathrm{Re}\left(\frac{p_0(a+ib)}{q(a+ib)}\right),$$

$$B_1 b = \mathrm{Im}\left(\frac{p_0(a+ib)}{q(a+ib)}\right).$$

Since $b \neq 0$ because the quadratic s^2+cs+d has no real roots, these equations can be solved for B_1 and C_1, so both B_1 and C_1 are determined by (4). Now solving for $p_1(s)$ in (3) gives

$$p_1(s) = \frac{p_0(s) - (B_1 s + C_1)q(s)}{s^2 + cs + d}. \qquad \square$$

An application of Theorem 1 produces two items:

- The partial fraction of the form

$$\frac{B_1 s + C_1}{(s^2 + cs + d)^n}$$

- A remainder term of the form

$$\frac{p_1(s)}{(s^2 + cs + d)^{n-1} q(s)}$$

such that the original rational function $p_0(s)/(s^s + as + b)^n q(s)$ is the sum of these two pieces. We can now repeat the process in the same way as the linear case.

The result is called the $(s^2 + cs + d)$-***chain*** for the rational function $p_0(s)/(s^2 + cs + d)^n q(s)$. The table below summarizes the data obtained.

The $(s^2 + cs + d)$*-chain*	
$\frac{p_0(s)}{(s^2 + cs + d)^n q(s)}$	$\frac{B_1 s + C_1}{(s^2 + cs + d)^n}$
$\frac{p_1(s)}{(s^2 + cs + d)^{n-1} q(s)}$	$\frac{B_2 s + C_2}{(s^2 + cs + d)^{n-1}}$
$\vdots$	$\vdots$
$\frac{p_{n-1}(s)}{(s^2 + cs + d)\, q(s)}$	$\frac{B_n s + C_n}{(s^2 + cs + d)}$
$\frac{p_n(s)}{q(s)}$	

From this table, we can immediately read off the following decomposition:

$$\frac{p_0(s)}{(s^2 + cs + d)^n q(s)} = \frac{B_1 s + C_1}{(s^2 + cs + d)^n} + \cdots + \frac{B_n s + C_n}{(s^2 + cs + d)} + \frac{p_n(s)}{q(s)}.$$

In the examples that follow, we will organize one step of the recursion algorithm as follows:

Partial Fraction Recursion Algorithm
by a Quadratic Term

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^2+cs+d)^{n-1} q(s)}$$

$$\text{where} \quad B_1 s + C_1|_{s=a+bi} = \left.\frac{p_0(s)}{q(s)}\right|_{s=a+bi} \Rightarrow B_1 = \square \text{ and } C_1 = \square$$

$$\text{and} \quad p_1(s) = \frac{1}{s^2+cs+d}(p_0(s) - (B_1 s + C_1)q(s)) = \square$$

As in the linear case, the curved arrows indicate where the results of calculations are inserted. First, B_1 and C_1 are calculated and inserted in two places: in the $(s^2 + cs+d)$-chain and in the calculation for $p_1(s)$. Afterward, $p_1(s)$ is calculated and the result inserted in the numerator of the remainder term. Now the process is repeated on $p_1(s)/((s^2+cs+d)^{n-1}q(s))$ until the (s^2+cs+d)-chain is completed.

Here are some examples of this process in action.

Example 2. Find the partial fraction decomposition for

$$\frac{5s}{(s^2+4)(s+1)}.$$

► **Solution.** We have a choice of computing the linear chain through $s+1$ or the quadratic chain through s^2+4. It is usually easier to compute linear chains. However, to illustrate the recursive algorithm for the quadratic case, we will compute the (s^2+4)-chain. The roots of s^2+4 are $s = \pm 2\mathrm{i}$. We need only focus on one root and we will choose $s = 2\mathrm{i}$.

According to Theorem 1, we can write

$$\frac{5s}{(s^2+4)(s+1)} = \frac{B_1 s + C_1}{(s^2+4)} + \frac{p_1(s)}{s+1},$$

$$\begin{aligned} \text{where} \quad B_1(2\mathrm{i}) + C_1 &= \left.\frac{5s}{s+1}\right|_{s=2\mathrm{i}} = \frac{10\mathrm{i}}{2\mathrm{i}+1} \\ &= \frac{(10\mathrm{i})(-2\mathrm{i}+1)}{(2\mathrm{i}+1)(-2\mathrm{i}+1)} = \frac{20+10\mathrm{i}}{5} = 4+2\mathrm{i} \\ &\Rightarrow B_1 = 1 \text{ and } C_1 = 4 \end{aligned}$$

$$\text{and} \quad p_1(s) = \frac{1}{s^2+4}(5s - (s+4)(s+1))$$
$$= \frac{1}{s^2+4}(-s^2-4) = -1.$$

It follows now that

$$\frac{5s}{(s^2+4)(s+1)} = \frac{s+4}{(s^2+4)} + \frac{-1}{s+1}. \qquad ◄$$

Note that in the above calculation, B_1 and C_1 are determined by comparing the real and imaginary parts of the complex numbers $2B_1\mathrm{i} + C_1 = 2\mathrm{i} + 4$ so that the imaginary parts give $2B_1 = 2$ so $B_1 = 1$ and the real parts give $C_1 = 4$.

Example 3. Find the partial fraction decomposition for

$$\frac{30s+40}{(s^2+1)^2(s^2+2s+2)}.$$

Remark 4. We remark that since the degree of the denominator is 6, the traditional method of determining the partial fraction decomposition would involve solving a system of six equations in six unknowns.

► **Solution.** First observe that both factors in the denominator, s^2+1 and $s^2+2s+2 = (s+1)^2+1$, are irreducible quadratics. We begin by determining the s^2+1 -chain. Note that $s = i$ is a root of s^2+1.

Applying the recursive algorithm gives

$$\frac{30s+40}{(s^2+1)^2(s^2+2s+2)} = \frac{B_1 s + C_1}{(s^2+1)^2} + \frac{p_1(s)}{(s^2+1)(s^2+2s+2)},$$

$$\text{where} \quad B_1\mathrm{i} + C_1 = \left.\frac{30s+40}{(s^2+2s+2)}\right|_{s=i} = \frac{30\mathrm{i}+40}{1+2\mathrm{i}}$$
$$= \frac{(40+30\mathrm{i})(1-2\mathrm{i})}{(1+2\mathrm{i})(1-2\mathrm{i})} = \frac{100-50\mathrm{i}}{5} = 20 - 10\mathrm{i}$$
$$\Rightarrow B_1 = -10 \quad \text{and } C_1 = 20$$
$$\text{and} \quad p_1(s) = \frac{1}{s^2+1}(30s + 40 - (-10s+20)(s^2+2s+2))$$
$$= \frac{1}{s^2+1}(10s(s^2+1)) = 10s.$$

We now repeat the recursion algorithm on the remainder term

$$\frac{10s}{(s^2+1)(s^2+2s+2)}.$$

$$\frac{10s}{(s^2+1)(s^2+2s+2)} = \frac{B_2 s + C_2}{(s^2+1)} + \frac{p_2(s)}{(s^2+2s+2)},$$

$$\begin{aligned}
\text{where} \quad B_2\mathrm{i} + C_2 &= \left.\frac{10s}{(s^2+2s+2)}\right|_{s=i} = \frac{10\mathrm{i}}{1+2\mathrm{i}} \\
&= \frac{(10\mathrm{i})(1-2\mathrm{i})}{(1+2\mathrm{i})(1-2\mathrm{i})} = \frac{20+10\mathrm{i}}{5} = 4+2\mathrm{i} \\
&\Rightarrow B_2 = 2 \text{ and } C_2 = 4 \\
\text{and} \quad p_2(s) &= \frac{1}{s^2+1}(10s - (2s+4)(s^2+2s+2)) \\
&= \frac{1}{s^2+1}(-2(s+4)(s^2+1)) = -2(s+4).
\end{aligned}$$

We can now write down the (s^2+1)-chain.

The (s^2+1)*-chain*	
$\dfrac{30s+40}{(s^2+1)^2(s^2+2s+2)}$	$\dfrac{-10s+20}{(s^2+1)^2}$
$\dfrac{10s}{(s^2+1)(s^2+2s+2)}$	$\dfrac{2s+4}{(s^2+1)}$
$\dfrac{-2(s+4)}{(s^2+2s+2)}$	

Since the last remainder term is already a partial fraction, we obtain

$$\frac{30s+40}{(s^2+1)^2(s^2+2s+2)} = \frac{-10s+20}{(s^2+1)^2} + \frac{2s+4}{s^2+1} + \frac{-2s-8}{(s+1)^2+1}. \qquad \blacktriangleleft$$

Quadratic Partial Fractions and the Laplace Transform Method

In the following example, notice how quadratic partial fraction recursion facilitates the Laplace transform method.

Example 5. Use the Laplace transform method to solve

$$y'' + 4y = \cos 3t, \tag{5}$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

► **Solution.** Applying the Laplace transform to both sides of (5) and substituting the given initial conditions give

$$(s^2 + 4)Y(s) = \frac{s}{s^2 + 9}$$

and thus

$$Y(s) = \frac{s}{(s^2 + 4)(s^2 + 9)}.$$

Using quadratic partial fraction recursion, we obtain the $(s^2 + 4)$-chain

The $(s^2 + 4)$*-chain*	
$\frac{s}{(s^2 + 4)(s^2 + 9)}$	$\frac{s/5}{s^2 + 4}$
$\frac{-s/5}{s^2 + 9}$	

It follows from Table 2.2 that

$$y(t) = \frac{1}{5}(\cos 2t - \cos 3t).$$

This is the solution to (5). ◄

Exercises

1–6. For each exercise below, compute the chain table through the indicated irreducible quadratic term.

1. $\dfrac{1}{(s^2+1)^2(s^2+2)}$; (s^2+1)

2. $\dfrac{s^3}{(s^2+2)^2(s^2+3)}$; (s^2+2)

3. $\dfrac{8s+8s^2}{(s^2+3)^3(s^2+1)}$; (s^2+3)

4. $\dfrac{4s^4}{(s^2+4)^4(s^2+6)}$; (s^2+4)

5. $\dfrac{1}{(s^2+2s+2)^2(s^2+2s+3)^2}$; (s^2+2s+2)

6. $\dfrac{5s-5}{(s^2+2s+2)^2(s^2+4s+5)}$; (s^2+2s+2)

7–16. Find the decomposition of the given rational function into partial fractions over $\mathbb{R}$.

7. $\dfrac{s}{(s^2+1)(s-3)}$

8. $\dfrac{4s}{(s^2+1)^2(s+1)}$

9. $\dfrac{9s^2}{(s^2+4)^2(s^2+1)}$

10. $\dfrac{9s}{(s^2+1)^2(s^2+4)}$

11. $\dfrac{2}{(s^2-6s+10)(s-3)}$

12. $\dfrac{30}{(s^2-4s+13)(s-1)}$

13. $\dfrac{25}{(s^2-4s+8)^2(s-1)}$

14. $\dfrac{s}{(s^2+6s+10)^2(s+3)^2}$

15. $\dfrac{s+1}{(s^2+4s+5)^2(s^2+4s+6)^2}$

16. $\dfrac{s^2}{(s^2+5)^3(s^2+6)^2}$

    Hint: Let $u = s^2$.

17–20. Use the *Laplace transform method* to solve the following differential equations. (Give both $Y(s)$ and $y(t)$.)

17. $y'' + 4y' + 4y = 4\cos 2t, \quad y(0) = 0,\ y'(0) = 1$
18. $y'' + 6y' + 9y = 50\sin t, \quad y(0) = 0,\ y'(0) = 2$
19. $y'' + 4y = \sin 3t, \quad y(0) = 0,\ y'(0) = 1$
20. $y'' + 2y' + 2y = 2\cos t + \sin t, \quad y(0) = 0,\ y'(0) = 0$

2.5 Laplace Inversion

In this section, we consider Laplace inversion and the kind of input functions that can arise when the transform function is a rational function. Given a transform function $F(s)$, we call an input function $f(t)$ the ***inverse Laplace transform*** of $F(s)$ if $\mathcal{L}\{f(t)\}(s) = F(s)$. We say *the* inverse Laplace transform because in most circumstances, it can be chosen uniquely. One such circumstance is when the input function is continuous. We state this fact as a theorem. For a proof of this result, see Appendix A.1

Theorem 1. *Suppose $f_1(t)$ and $f_2(t)$ are continuous functions defined on $[0, \infty)$ with the same Laplace transform. Then $f_1(t) = f_2(t)$.*

It follows from this theorem that if a transform function has a continuous input function, then it can have only one such input function. In Chap. 6, we will consider some important classes of discontinuous input functions, but for now, we will assume that all input functions are continuous and we write $\mathcal{L}^{-1}\{F(s)\}$ for the inverse Laplace transform of F. That is, $\mathcal{L}^{-1}\{F(s)\}$ *is the unique continuous function $f(t)$ that has $F(s)$ as its Laplace transform.* Symbolically,

Defining Property of the Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \iff \mathcal{L}\{f(t)\} = F(s).$$

We can thus view $\mathcal{L}^{-1}$ as an operation on transform functions $F(s)$ that produces input functions $f(t)$. Because of the defining property of the inverse Laplace transform, each formula for the Laplace transform has a corresponding formula for the inverse Laplace transform.

Example 2. List the corresponding inverse Laplace transform formula for each formula in Table 2.2.

► **Solution.** Each line of Table 2.4 corresponds to the same line in Table 2.2. ◄

By identifying the parameters n, a, and b in specific functions $F(s)$, it is possible to read off $\mathcal{L}^{-1}\{F(s)\} = f(t)$ from Table 2.4 for some $F(s)$.

Example 3. Find the inverse Laplace transform of each of the following functions $F(s)$:

1. $\dfrac{6}{s^4}$ 2. $\dfrac{2}{(s+3)^3}$ 3. $\dfrac{5}{s^2+25}$ 4. $\dfrac{s-1}{(s-1)^2+4}$

► **Solution.**

1. $\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3$ ($n = 3$ in Formula 2)

2. $\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^3}\right\} = t^2 e^{-3t}$ ($n = 2, a = -3$ in Formula 4)
3. $\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\} = \sin 5t$ ($b = 5$ in Formula 6)
4. $\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} = e^t \cos 2t$ ($a = 1, b = 2$ in Formula 7) ◀

It is also true that each Laplace transform principle recorded in Table 2.3 results in a corresponding principle for the inverse Laplace transform. We will single out the linearity principle and the first translation principle at this time.

Theorem 4 (Linearity). *The inverse Laplace transform is linear. In other words, if $F(s)$ and $G(s)$ are transform functions with continuous inverse Laplace transforms and a and b are constants, then*

Linearity of the Inverse Laplace Transform

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.$$

Proof. Let $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$. Since the Laplace transform is linear by Theorem 5 of Sect. 2.2, we have

$$\mathcal{L}\{(af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = aF(s) + bG(s).$$

Since $af(t) + bg(t)$ is continuous, it follows that

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t) = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}. \qquad \square$$

Theorem 5. *If $F(s)$ has a continuous inverse Laplace transform, then*

Inverse First Translation Principle

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

Proof. Let $f(t) = \mathcal{L}^{-1}\{F(s)\}$. Then the first translation principle (Theorem 17 of Sect. 2.2) gives

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s-a),$$

and applying $\mathcal{L}^{-1}$ to both sides of this equation gives

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\}. \qquad \square$$

Table 2.4 Basic inverse Laplace transform formulas (We are assuming n is a nonnegative integer and a and b are real)

1.	$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
2.	$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$
3.	$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = \mathrm{e}^{at}$
4.	$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = t^n\mathrm{e}^{at}$
5.	$\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$
6.	$\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin bt$
7.	$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = \mathrm{e}^{at}\cos bt$
8.	$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = \mathrm{e}^{at}\sin bt$

Suppose $p(s)/q(s)$ is a proper rational function. Its partial fraction decomposition is a linear combination of the ***simple (real) rational functions***, by which we mean rational functions of the form

$$\frac{1}{(s-a)^k}, \quad \frac{b}{((s-a)^2+b^2)^k}, \quad \text{and} \quad \frac{s-a}{((s-a)^2+b^2)^k}, \tag{1}$$

where a, b are real, $b > 0$, and k is a positive integer. The linearity of the inverse Laplace transform implies that Laplace inversion of rational functions reduces to finding the inverse Laplace transform of the three simple rational functions given above. The inverse Laplace transforms of the first of these simple rational functions can be read off directly from Table 2.4, while the last two can be determined from this table if $k = 1$. To illustrate, consider the following example.

Example 6. Suppose

$$F(s) = \frac{5s}{(s^2+4)(s+1)}.$$

Find $\mathcal{L}^{-1}\{F(s)\}$.

► **Solution.** Since $F(s)$ does not appear in Table 2.4 (or the equivalent Table 2.2), the inverse Laplace transform is not immediately evident. However, using the recursive partial fraction method we found in Example 2 of Sect. 2.4 that

$$\begin{aligned}\frac{5s}{(s^2+4)(s+1)} &= \frac{s+4}{(s^2+4)} - \frac{1}{s+1} \\ &= \frac{s}{s^2+2^2} + 2\frac{2}{s^2+2^2} - \frac{1}{s+1}.\end{aligned}$$

By linearity of the inverse Laplace transform and perusal of Table 2.4, it is now evident that

$$\mathcal{L}^{-1}\left\{\frac{5s}{(s^2+4)(s+1)}\right\} = \cos 2t + 2\sin 2t - e^{-t}. \qquad \blacktriangleleft$$

When irreducible quadratics appear in the denominator, their inversion is best handled by completing the square and using the first translation principle as illustrated in the following example.

Example 7. Find the inverse Laplace transform of each rational function

$$1.\ \frac{4s-8}{s^2+6s+25} \qquad 2.\ \frac{2}{s^2-4s+7}.$$

► **Solution.** In each case, the denominator is an irreducible quadratic. We will complete the square and use the translation principle.

1. Completing the square of the denominator gives

$$s^2+6s+25 = s^2+6s+9+25-9 = (s+3)^2+4^2.$$

In order to apply the first translation principle with $a=-3$, the numerator must also be rewritten with s translated. Thus, $4s-8 = 4(s+3-3)-8 = 4(s+3)-20$. We now get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{4s-8}{s^2+6s+25}\right\} &= \mathcal{L}^{-1}\left\{\frac{4(s+3)-20}{(s+3)^2+4^2}\right\}\\ &= e^{-3t}\mathcal{L}^{-1}\left\{\frac{4s-20}{s^2+4^2}\right\}\\ &= e^{-3t}\left(4\mathcal{L}\left\{\frac{s}{s^2+4^2}\right\} - 5\mathcal{L}\left\{\frac{4}{s^2+4^2}\right\}\right)\\ &= e^{-3t}(4\cos 4t - 5\sin 4t).\end{aligned}$$

Notice how linearity of Laplace inversion is used here.

2. Completing the square of the denominator gives

$$s^2-4s+7 = s^2-4s+4+3 = (s-2)^2+\sqrt{3}^2.$$

We now get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^2-4s+7}\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+\sqrt{3}^2}\right\}\\ &= e^{2t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+\sqrt{3}^2}\right\}\end{aligned}$$

$$= e^{2t} \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{3}}{s^2 + \sqrt{3}^2} \right\}$$
$$= \frac{2}{\sqrt{3}} e^{2t} \sin \sqrt{3} t. \qquad \blacktriangleleft$$

In the examples above, the first translation principle reduces the calculation of the inverse Laplace transform of a simple rational function involving irreducible quadratics with a translated s variable to one without a translation. More generally, the inverse first translation principle gives

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{b}{((s-a)^2 + b^2)^k} \right\} &= e^{at} \mathcal{L}^{-1} \left\{ \frac{b}{(s^2 + b^2)^k} \right\}, \\ \mathcal{L}^{-1} \left\{ \frac{s-a}{((s-a)^2 + b^2)^k} \right\} &= e^{at} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + b^2)^k} \right\}. \end{aligned} \tag{2}$$

Table 2.4 does not contain the inverse Laplace transforms of the functions on the right unless $k = 1$. Unfortunately, explicit formulas for these inverse Laplace transforms are not very simple for a general $k \geq 1$. There is however a recursive method for computing these inverse Laplace transform formulas which we now present. This method, which we call a ***reduction of order formula***, may remind you of reduction formulas in calculus for integrating powers of trigonometric functions by expressing an integral of an nth power in terms of integrals of lower order powers.

Proposition 8 (Reduction of Order formulas). *If $b \neq 0$ is a real number and $k \geq 1$ is a positive integer, then*

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + b^2)^{k+1}} \right\} = \frac{-t}{2kb^2} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + b^2)^k} \right\} + \frac{2k-1}{2kb^2} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + b^2)^k} \right\},$$

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + b^2)^{k+1}} \right\} = \frac{t}{2k} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + b^2)^k} \right\}.$$

Proof. Let $f(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(s^2+b^2)^k} \right\}$. Then the transform derivative principle (Theorem 20 of Sect. 2.2) applies to give

$$\begin{aligned} \mathcal{L}\{t f(t)\} &= -\frac{d}{ds} \left(\mathcal{L}\{f(t)\} \right) = -\frac{d}{ds} \left(\frac{1}{(s^2 + b^2)^k} \right) \\ &= \frac{2ks}{(s^2 + b^2)^{k+1}}. \end{aligned}$$

Now divide by $2k$ and take the inverse Laplace transform of both sides to get

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^{k+1}}\right\} = \frac{t}{2k}f(t) = \frac{t}{2k}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\},$$

which is the second of the required formulas.

The first formula is done similarly. Let $g(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^k}\right\}$. Then

$$\begin{aligned}\mathcal{L}\{tg(t)\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{L}\{g(t)\}\right) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{(s^2+b^2)^k}\right)\\ &= -\frac{(s^2+b^2)^k - 2ks^2(s^2+b^2)^{k-1}}{(s^2+b^2)^{2k}}\\ &= \frac{2s^2k-(s^2+b^2)}{(s^2+b^2)^{k+1}} = \frac{(2k-1)(s^2+b^2)-2kb^2}{(s^2+b^2)^{k+1}}\\ &= \frac{2k-1}{(s^2+b^2)^k} - \frac{2kb^2}{(s^2+b^2)^{k+1}}.\end{aligned}$$

Divide by $2kb^2$, solve for the second term in the last line, and apply the inverse Laplace transform to get

$$\begin{aligned}&\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^{k+1}}\right\}\\ &\quad= \frac{-t}{2kb^2}g(t) + \frac{(2k-1)}{2kb^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\}\\ &\quad= \frac{-t}{2kb^2}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^k}\right\} + \frac{(2k-1)}{2kb^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\},\end{aligned}$$

which is the first formula. □

These equations are examples of one step recursion relations involving a pair of functions, both of which depend on a positive integer k. The kth formula for both families implies the $(k+1)^{\text{st}}$. Since we already know the formulas for

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^k}\right\}, \tag{3}$$

when $k = 1$, the reduction formulas give the formulas for the case $k = 2$, which, in turn, allow one to calculate the formulas for $k = 3$, etc. With a little work, we can calculate these inverse Laplace transforms for any $k \geq 1$.

To see how to use these formulas, we will evaluate the inverse Laplace transforms in (3) for $k = 2$.

Formula 9. Use the formulas derived above to verify the following formulas:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^2}\right\} = \frac{1}{2b^3}(-bt\cos bt + \sin bt),$$
$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\} = \frac{t}{2b}\sin bt.$$

▼ *Verification.* Here we use the reduction of order formulas for $k = 1$ to get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^2}\right\} &= \frac{-t}{2b^2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} + \frac{1}{2b^2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} \\ &= \frac{-t}{2b^2}\cos bt + \frac{1}{2b^3}\sin bt \\ &= \frac{1}{2b^3}(-bt\cos bt + \sin bt),\end{aligned}$$

$$\text{and} \quad \mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\} = \frac{t}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} = \frac{t}{2b}\sin bt. \qquad \blacktriangle$$

Using the calculations just done and applying the reduction formulas for $k = 2$ will then give the inverse Laplace transforms of (3) for $k = 3$. The process can then be continued to get formulas for higher values of k. In Table 2.5, we provide the inverse Laplace transform for the powers $k = 1, \ldots, 4$. You will be asked to verify them in the exercises. In Chap. 7, we will derive a closed formula for each value k.

By writing

$$\frac{cs+d}{(s^2+b^2)^{k+1}} = c\frac{s}{(s^2+b^2)^{k+1}} + d\frac{1}{(s^2+b^2)^{k+1}},$$

the two formulas in Proposition 8 can be combined using linearity to give a single formula:

Corollary 10. *Let b, c, and d be real numbers and assume $b \neq 0$. If $k \geq 1$ is a positive integer, then*

$$\mathcal{L}^{-1}\left\{\frac{cs+d}{(s^2+b^2)^{k+1}}\right\} = \frac{t}{2kb^2}\mathcal{L}^{-1}\left\{\frac{-ds+cb^2}{(s^2+b^2)^k}\right\} + \frac{2k-1}{2kb^2}\mathcal{L}^{-1}\left\{\frac{d}{(s^2+b^2)^k}\right\}.$$

As an application of this formula, we note the following result that expresses the *form* of $\mathcal{L}^{-1}\left\{\frac{cs+d}{(s^2+b^2)^k}\right\}$ in terms of polynomials, sines, and cosines.

Table 2.5 Inversion formulas involving irreducible quadratics

$\mathcal{L}^{-1}\left\{\dfrac{b}{(s^2+b^2)^k}\right\}$	$\longleftrightarrow$	$\dfrac{b}{(s^2+b^2)^k}$
$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s^2+b^2)}$
$\frac{1}{2b^2}(\sin bt - bt\cos bt)$	$\longleftrightarrow$	$\dfrac{b}{(s^2+b^2)^2}$
$\frac{1}{8b^4}\left((3-(bt)^2)\sin bt - 3bt\cos bt\right)$	$\longleftrightarrow$	$\dfrac{b}{(s^2+b^2)^3}$
$\frac{1}{48b^6}\left((15-6(bt)^2)\sin bt - (15bt-(bt)^3)\cos bt\right)$	$\longleftrightarrow$	$\dfrac{b}{(s^2+b^2)^4}$
$\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+b^2)^k}\right\}$	$\longleftrightarrow$	$\dfrac{s}{(s^2+b^2)^k}$
$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{(s^2+b^2)}$
$\frac{1}{2b^2}bt\sin bt$	$\longleftrightarrow$	$\dfrac{s}{(s^2+b^2)^2}$
$\frac{1}{8b^4}\left(bt\sin bt - (bt)^2\cos bt\right)$	$\longleftrightarrow$	$\dfrac{s}{(s^2+b^2)^3}$
$\frac{1}{48b^6}\left((3bt-(bt)^3)\sin bt - 3(bt)^2\cos bt\right)$	$\longleftrightarrow$	$\dfrac{s}{(s^2+b^2)^4}$

Corollary 11. *Let b, c, and d be real numbers and assume $b \neq 0$. If $k \geq 1$ is a positive integer, then there are polynomials $p_1(t)$ and $p_2(t)$ of degree at most $k-1$ such that*

$$\mathcal{L}^{-1}\left\{\frac{cs+d}{(s^2+b^2)^k}\right\} = p_1(t)\sin bt + p_2(t)\cos bt. \tag{4}$$

Proof. We prove this by induction. If $k = 1$, this is certainly true since

$$\mathcal{L}^{-1}\left\{\frac{cs+d}{s^2+b^2}\right\} = \frac{d}{b}\sin bt + c\cos bt,$$

and thus, $p_1(t) = d/b$ and $p_2(t) = c$ are constants and hence polynomials of degree $0 = 1-1$. Now suppose for our induction hypothesis that $k \geq 1$ and that (4) is true for k. We need to show that this implies that it is also true for $k+1$. By this assumption, we can find polynomials $p_1(t)$, $p_2(t)$, $q_1(t)$, and $q_2(t)$ of degree at most $k-1$ so that

$$\mathcal{L}^{-1}\left\{\frac{-ds+cb^2}{(s^2+b^2)^k}\right\} = p_1(t)\sin bt + p_2(t)\cos bt$$

$$\text{and}\quad \mathcal{L}^{-1}\left\{\frac{d}{(s^2+b^2)^k}\right\} = q_1(t)\sin bt + q_2(t)\cos bt.$$

By Corollary 10,

$$\mathcal{L}^{-1}\left\{\frac{cs+d}{(s^2+b^2)^{k+1}}\right\} = \frac{t}{2kb^2}\mathcal{L}^{-1}\left\{\frac{-ds+cb^2}{(s^2+b^2)^k}\right\} + \frac{2k-1}{2kb^2}\mathcal{L}^{-1}\left\{\frac{d}{(s^2+b^2)^k}\right\}$$

$$= \frac{t}{2kb^2}(p_1(t)\sin bt + p_2(t)\cos bt)$$

$$+\frac{2k-1}{2kb^2}(q_1(t)\sin bt + q_2(t)\cos bt)$$

$$= P_1(t)\sin bt + P_2(t)\cos bt,$$

where

$$P_1(t) = \frac{t}{2kb^2}p_1(t) + \frac{2k-1}{2kb^2}q_1(t) \quad \text{and} \quad P_2(t) = \frac{t}{2kb^2}p_2(t) + \frac{2k-1}{2kb^2}q_2(t).$$

Observe that $P_1(t)$ and $P_2(t)$ are obtained from the polynomials $p_i(t)$, $q_i(t)$ by multiplying by a term of degree at most 1. Hence, these are polynomials whose degrees are at most k, since the $p_i(t)$, $q_i(t)$ have degree at most $k-1$. This completes the induction argument. $\square$

It follows from the discussion thus far that the inverse Laplace transform of *any* rational function can be computed by means of a partial fraction expansion followed by use of the formulas from Table 2.4. For partial fractions with irreducible quadratic denominator, the recursion formulas, as collected in Table 2.5, may be needed. Here is an example.

Example 12. Find the inverse Laplace transform of

$$F(s) = \frac{6s+6}{(s^2-4s+13)^3}.$$

► **Solution.** We first complete the square: $s^2-4s+13 = s^2-4s+4+9 = (s-2)^2+3^2$. Then

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{6s+6}{((s-2)^2+3^2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{6(s-2)+18}{((s-2)^2+3^2)^3}\right\}$$

$$= e^{2t}\left(6\mathcal{L}^{-1}\left\{\frac{s}{(s^2+3^2)^3}\right\} + 6\mathcal{L}^{-1}\left\{\frac{3}{(s^2+3^2)^3}\right\}\right)$$

$$= \frac{6e^{2t}}{8 \cdot 3^4} \left(3t \sin 3t - (3t)^2 \cos 3t + (3 - (3t)^2) \sin 3t - 3(3t) \cos 3t\right)$$

$$= \frac{e^{2t}}{36} \left((1 + t - 3t^2) \sin 3t - (3t + 3t^2) \cos 3t\right).$$

The third line is obtained from Table 2.5 with $b = 3$ and $k = 3$. ◄

Irreducible Quadratics and the Laplace Transform Method

We conclude with an example that uses the Laplace transform method, quadratic partial fraction recursion, and Table 2.5 to solve a second order differential equation.

Example 13. Use the Laplace transform method to solve the following differential equation:

$$y'' + 4y = 9t \sin t,$$

with initial conditions $y(0) = 0$, and $y'(0) = 0$.

► **Solution.** Table 2.5 gives $\mathcal{L}\{9t \sin t\} = 18s/(s^2 + 1)^2$. Now apply the Laplace transform to the differential equation to get

$$(s^2 + 4)Y(s) = \frac{18s}{(s^2 + 1)^2}$$

and hence

$$Y(s) = \frac{18s}{(s^2 + 1)^2(s^2 + 4)}.$$

Using quadratic partial fraction recursion, we obtain the $(s^2 + 1)$-chain

The $(s^2 + 1)$*-chain*	
$\frac{18s}{(s^2 + 1)^2(s^2 + 4)}$	$\frac{6s}{(s^2 + 1)^2}$
$\frac{-6s}{(s^2 + 1)(s^2 + 4)}$	$\frac{-2s}{s^2 + 1}$
$\frac{2s}{s^2 + 4}$	

Thus,

$$Y(s) = \frac{6s}{(s^2+1)^2} + \frac{-2s}{s^2+1} + \frac{2s}{s^2+4}.$$

Laplace inversion with Table 2.5 gives

$$y(t) = 3t \sin t - 2\cos t + 2\cos 2t. \qquad \blacktriangleleft$$

Exercises

1–18. Compute $\mathcal{L}^{-1}\{F(s)\}(t)$ for the given proper rational function $F(s)$.

1. $\dfrac{-5}{s}$
2. $\dfrac{3}{s-4}$
3. $\dfrac{3}{s^2} - \dfrac{4}{s^3}$
4. $\dfrac{4}{2s+3}$
5. $\dfrac{3s}{s^2+4}$
6. $\dfrac{2}{s^2+3}$
7. $\dfrac{2s-5}{s^2+6s+9}$
8. $\dfrac{2s-5}{(s+3)^3}$
9. $\dfrac{6}{s^2+2s-8}$
10. $\dfrac{s}{s^2-5s+6}$
11. $\dfrac{2s^2-5s+1}{(s-2)^4}$
12. $\dfrac{2s+6}{s^2-6s+5}$
13. $\dfrac{4s^2}{(s-1)^2(s+1)^2}$
14. $\dfrac{27}{s^3(s+3)}$
15. $\dfrac{8s+16}{(s^2+4)(s-2)^2}$
16. $\dfrac{5s+15}{(s^2+9)(s-1)}$
17. $\dfrac{12}{s^2(s+1)(s-2)}$
18. $\dfrac{2s}{(s-3)^3(s-4)^2}$

19–24. Use the first translation principle and Table 2.2 (or Table 2.4) to find the inverse Laplace transform.

19. $\dfrac{2s}{s^2+2s+5}$

20. $\dfrac{1}{s^2+6s+10}$

21. $\dfrac{s-1}{s^2-8s+17}$

22. $\dfrac{2s+4}{s^2-4s+12}$

23. $\dfrac{s-1}{s^2-2s+10}$

24. $\dfrac{s-5}{s^2-6s+13}$

25–34. Find the inverse Laplace transform of each rational function. Either the reduction formulas, Proposition 8, or the formulas in Table 2.5 can be used.

25. $\dfrac{8s}{(s^2+4)^2}$

26. $\dfrac{9}{(s^2+9)^2}$

27. $\dfrac{2s}{(s^2+4s+5)^2}$

28. $\dfrac{2s+2}{(s^2-6s+10)^2}$

29. $\dfrac{2s}{(s^2+8s+17)^2}$

30. $\dfrac{s+1}{(s^2+2s+2)^3}$

31. $\dfrac{1}{(s^2-2s+5)^3}$

32. $\dfrac{8s}{(s^2-6s+10)^3}$

33. $\dfrac{s-4}{(s^2-8s+17)^4}$

34. $\dfrac{2}{(s^2+4s+8)^3}$

35–38. Use the *Laplace transform method* to solve the following differential equations. (Give both $Y(s)$ and $y(t)$.)

35. $y'' + y = 4\sin t, \quad y(0) = 1,\ y'(0) = -1$
36. $y'' + 9y = 36t \sin 3t, \quad y(0) = 0,\ y'(0) = 3$
37. $y'' - 3y = 4t^2 \cos t, \quad y(0) = 0,\ y'(0) = 0$
38. $y'' + 4y = 32t \cos 2t, \quad y(0) = 0,\ y'(0) = 2$

39–44. Verify the following assertions. In each assertion, assume a, b, c are distinct. These are referred to as Heaviside expansion formulas of the first kind.

39. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)(s-b)}\right\} = \dfrac{e^{at}}{a-b} + \dfrac{e^{bt}}{b-a}$

40. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-a)(s-b)}\right\} = \dfrac{ae^{at}}{a-b} + \dfrac{be^{bt}}{b-a}$

41. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)(s-b)(s-c)}\right\} = \dfrac{e^{at}}{(a-b)(a-c)} + \dfrac{e^{bt}}{(b-a)(b-c)} + \dfrac{e^{ct}}{(c-a)(c-b)}$

42. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-a)(s-b)(s-c)}\right\} = \dfrac{ae^{at}}{(a-b)(a-c)} + \dfrac{be^{bt}}{(b-a)(b-c)} + \dfrac{ce^{ct}}{(c-a)(c-b)}$

43. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s-a)(s-b)(s-c)}\right\} = \dfrac{a^2e^{at}}{(a-b)(a-c)} + \dfrac{b^2e^{bt}}{(b-a)(b-c)} + \dfrac{c^2e^{ct}}{(c-a)(c-b)}$

44. $\mathcal{L}^{-1}\left\{\dfrac{s^k}{(s-r_1)\cdots(s-r_n)}\right\} = \dfrac{r_1^k e^{r_1 t}}{q'(r_1)} + \cdots + \dfrac{r_n^k e^{r_n t}}{q'(r_n)}$, where $q(s) = (s-r_1)\cdots(s-r_k)$. Assume $r_1, \ldots, r_n$ are distinct.

45–50. Verify the following assertions. These are referred to as Heaviside expansion formulas of the second kind.

45. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^2}\right\} = te^{at}$

46. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-a)^2}\right\} = (1+at)e^{at}$

47. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^3}\right\} = \dfrac{t^2}{2}e^{at}$

48. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-a)^3}\right\} = \left(t + \dfrac{at^2}{2}\right)\mathrm{e}^{at}$

49. $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s-a)^3}\right\} = \left(1 + 2at + \dfrac{a^2t^2}{2}\right)\mathrm{e}^{at}$

50. $\mathcal{L}^{-1}\left\{\dfrac{s^k}{(s-a)^n}\right\} = \left(\sum_{l=0}^{k}\binom{k}{l}a^{k-l}\dfrac{t^{n-l-1}}{(n-l-1)!}\right)\mathrm{e}^{at}$

2.6 The Linear Spaces $\mathcal{E}_q$: Special Cases

Let $q(s)$ be any fixed polynomial. It is the purpose of this section and the next to efficiently determine all input functions having Laplace transforms that are proper rational functions with $q(s)$ in the denominator. In other words, we want to describe all the input functions $y(t)$ such that

$$\mathcal{L}\{y(t)\}(s) = \frac{p(s)}{q(s)},$$

where $p(s)$ may be any polynomial whose degree is less than that of $q(s)$. It turns out that our description will involve in a simple way the roots of $q(s)$ and their multiplicities and will involve the notion of linear combinations and spanning sets, which we introduce below. To get an idea why we seek such a description, consider the following example of a second order linear differential equation. The more general theory of such differential equations will be discussed in Chaps. 3 and 4.

Example 1. Use the Laplace transform to find the solution set for the second order linear differential equation

$$y'' - 3y' - 4y = 0. \tag{1}$$

► **Solution.** Notice in this example that we are not specifying the initial conditions $y(0)$ and $y'(0)$. We may consider them arbitrary. Our solution set will be a family of solutions parameterized by two arbitrary constants (cf. the discussion in Sect. 1.1 under the subheading *The Arbitrary Constants*). We apply the Laplace transform to both sides of (1) and use linearity to get

$$\mathcal{L}\left\{y''\right\} - 3\mathcal{L}\left\{y'\right\} - 4\mathcal{L}\left\{y\right\} = 0.$$

Next use the input derivative principles:

$$\begin{aligned}\mathcal{L}\left\{y'(t)\right\}(s) &= s\mathcal{L}\{y(t)\}(s) - y(0),\\ \mathcal{L}\left\{y''(t)\right\}(s) &= s^2\mathcal{L}\{y(t)\}(s) - sy(0) - y'(0)\end{aligned}$$

to get

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) - 4Y(s) = 0,$$

where as usual we set $Y(s) = \mathcal{L}\{y(t)\}(s)$. Collect together terms involving $Y(s)$ and simplify to get

$$(s^2 - 3s - 4)Y(s) = sy(0) + y'(0) - 3y(0). \tag{2}$$

The polynomial coefficient of $Y(s)$ is $s^2 - 3s - 4$ and is called the ***characteristic polynomial*** of (1). To simplify the notation, let $q(s) = s^2 - 3s - 4$. Solving for $Y(s)$ gives

$$Y(s) = \frac{sy(0) + y'(0) - 3y(0)}{s^2 - 3s - 4} = \frac{p(s)}{q(s)}, \tag{3}$$

where $p(s) = sy(0) + y'(0) - 3y(0)$. Observe that $p(s)$ can be any polynomial of degree 1 since the initial values are unspecified and arbitrary. Our next step then is to find all the input functions whose Laplace transform has $q(s)$ in the denominator. Since $q(s) = s^2 - 3s - 4 = (s-4)(s+1)$, we see that the form of the partial fraction decomposition for $\frac{p(s)}{q(s)}$ is

$$Y(s) = \frac{p(s)}{q(s)} = c_1 \frac{1}{s-4} + c_2 \frac{1}{s+1}.$$

Laplace inversion gives

$$y(t) = c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t}, \tag{4}$$

where c_1 and c_2 are arbitrary real numbers, that depend on the initial conditions $y(0)$ and $y'(0)$. We encourage the student to verify by substitution that $y(t)$ is indeed a solution to (1). We will later show that all such solutions are of this form. We may now write the solution set as

$$\left\{c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t} : c_1, c_2 \in \mathbb{R}\right\}. \tag{5}$$

◀

Let us make a few observations about this example. First observe that the characteristic polynomial $q(s) = s^2 - 3s - 4$, which is the coefficient of $Y(s)$ in (2), is easy to read off directly from the left side of the differential equation $y'' - 3y' - 4y = 0$; the coefficient of each power of s in $q(s)$ is exactly the coefficient of the corresponding order of the derivative in the differential equation. Second, with the characteristic polynomial in hand, we can jump to (3) to get the *form* of $Y(s)$, namely, $Y(s)$ is a proper rational function with the characteristic polynomial in the denominator. The third matter to deal with in this example is to compute $y(t)$ knowing that its Laplace transform $Y(s)$ has the special form $\frac{p(s)}{q(s)}$. It is this third matter that we address here. In particular, we find an efficient method to write down the solution set as given by (5) directly from any characteristic polynomial $q(s)$. The roots of $q(s)$ and their multiplicity play a decisive role in the description we give.

For any polynomial $q(s)$, we let $\mathcal{R}_q$ denote all the proper rational functions that *may* be written as $\frac{p(s)}{q(s)}$ for some polynomial $p(s)$. We let $\mathcal{E}_q$ denote the set of all input functions whose Laplace transform is in $\mathcal{R}_q$. In Example 1, we found the $\mathcal{E}_q = \left\{c_1 \mathrm{e}^{4t} + c_2 \mathrm{e}^{-t} : c_1, c_2 \in \mathbb{R}\right\}$, where $q(s) = s^2 - 3s - 4$. If $c_1 = 1$ and $c_2 = 0$ then the function $\mathrm{e}^{4t} = 1\mathrm{e}^{4t} + 0\mathrm{e}^{-t} \in \mathcal{R}_q$. Observe though that $\mathcal{L}\left\{\mathrm{e}^{4t}\right\} = \frac{1}{s-4}$. At first glance, it appears that $\mathcal{L}\left\{\mathrm{e}^{4t}\right\}$ is not in $\mathcal{E}_q$. However, we *may* write

$$\mathcal{L}\left\{\mathrm{e}^{4t}\right\}(s) = \frac{1}{s-4} = \frac{1\ (s+1)}{(s-4)(s+1)} = \frac{s+1}{q(s)}.$$

Thus, $\mathcal{L}\left\{\mathrm{e}^{4t}\right\}(s) \in \mathcal{R}_q$ and indeed e^{4t} is in $\mathcal{E}_q$. In a similar way, $\mathrm{e}^{-t} \in \mathcal{E}_q$.

Recall from Sect. 2.2 the notion of a linear space of functions, namely, closure under addition and scalar multiplication.

Proposition 2. *Both $\mathcal{E}_q$ and $\mathcal{R}_q$ are linear spaces.*

Proof. Suppose $\frac{p_1(s)}{q(s)}$ and $\frac{p_2(s)}{q(s)}$ are in $\mathcal{R}_q$ and $c \in \mathbb{R}$. Then $\deg p_1(s)$ and $\deg p_2(s)$ are less than $\deg q(s)$. Further,

- $\frac{p_1(s)}{q(s)} + \frac{p_2(s)}{q(s)} = \frac{p_1(s)+p_2(s)}{q(s)}$. Since addition of polynomials does not increase the degree, we have $\deg(p_1(s) + p_2(s)) < \deg q(s)$. It follows that $\frac{p_1(s)+p_2(s)}{q(s)}$ is in $\mathcal{R}_q$.
- $c\dfrac{p_1(s)}{q(s)} = \frac{cp_1(s)}{q(s)}$ is proper and has denominator $q(s)$; hence, it is in $\mathcal{R}_q$.

It follows that $\mathcal{R}_q$ is closed under addition and scalar multiplication, and hence, $\mathcal{R}_q$ is a linear space. Now suppose f_1 and f_2 are in $\mathcal{E}_q$ and $c \in \mathbb{R}$. Then $\mathcal{L}\{f_1\} \in \mathcal{R}_q$ and $\mathcal{L}\{f_2\} \in \mathcal{R}_q$. Further,

- $\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\} \in \mathcal{R}_q$. From this, it follows that $f_1 + f_2 \in \mathcal{E}_q$.
- $\mathcal{L}\{cf_1\} = c\mathcal{L}\{f_1\} \in \mathcal{R}_q$. From this it follows that $cf_1 \in \mathcal{E}_q$.

It follows that $\mathcal{E}_q$ is closed under addition and scalar multiplication, and hence, $\mathcal{E}_q$ is a linear space. □

Description of $\mathcal{E}_q$ for $q(s)$ of Degree 2

The roots of a real polynomial of degree 2 occur in one of three ways:

1. Two distinct real roots as in Example 1
2. A real root with multiplicity two
3. Two complex roots

Let us consider an example of each type.

Example 3. Find $\mathcal{E}_q$ for each of the following polynomials:

1. $q(s) = s^2 - 3s + 2$
2. $q(s) = s^2 - 2s + 1$
3. $q(s) = s^2 + 2s + 2$

► **Solution.** In each case, $\deg q(s) = 2$; thus

$$\mathcal{R}_q = \left\{ \frac{p(s)}{q(s)} : \deg p(s) \leq 1 \right\}.$$

1. Suppose $f(t) \in \mathcal{E}_q$. Since $q(s) = s^2 - 3s + 2 = (s-1)(s-2)$, a partial fraction decomposition of $\mathcal{L}\{f(t)\}(s) = \frac{p(s)}{q(s)}$ has the form $\frac{p(s)}{(s-1)(s-2)} = \frac{c_1}{s-1} + \frac{c_2}{s-2}$. Laplace inversion then gives $f(t) = \mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = c_1 e^t + c_2 e^{2t}$. On the other hand, we have $e^t \in \mathcal{E}_q$ since $\mathcal{L}\{e^t\} = \frac{1}{s-1} = \frac{s-2}{(s-1)(s-2)} = \frac{s-2}{q(s)} \in \mathcal{R}_q$. Similarly, $e^{2t} \in \mathcal{E}_q$. Since $\mathcal{E}_q$ is a linear space, it follows that all functions of the form $c_1 e^t + c_2 e^{2t}$ are in $\mathcal{E}_q$. From these calculations, it follows that

$$\mathcal{E}_q = \left\{c_1 e^t + c_2 e^{2t} : c_1, c_2 \in \mathbb{R}\right\}.$$

2. Suppose $f(t) \in \mathcal{E}_q$. Since $q(s) = s^2 - 2s + 1 = (s-1)^2$, a partial fraction decomposition of $\mathcal{L}\{f(t)\}(s) = \frac{p(s)}{q(s)}$ has the form $\frac{p(s)}{(s-1)^2} = \frac{c_1}{s-1} + \frac{c_2}{(s-1)^2}$. Laplace inversion then gives $f(t) = \mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = c_1 e^t + c_2 t e^t$. On the other hand, we have $e^t \in \mathcal{E}_q$ since $\mathcal{L}\{e^t\} = \frac{1}{s-1} = \frac{s-1}{(s-1)^2} = \frac{s-1}{q(s)} \in \mathcal{R}_q$. Similarly $\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2} \in \mathcal{R}_q$ so $te^t \in \mathcal{E}_q$. Since $\mathcal{E}_q$ is a linear space, it follows that all functions of the form $c_1 e^t + c_2 t e^t$ are in $\mathcal{E}_q$. From these calculations, it follows that

$$\mathcal{E}_q = \left\{c_1 e^t + c_2 t e^t : c_1, c_2 \in \mathbb{R}\right\}.$$

3. We complete the square in $q(s)$ to get $q(s) = (s+1)^2 + 1$, an irreducible quadratic. Suppose $f(t) \in \mathcal{E}_q$. A partial fraction decomposition of $\mathcal{L}\{f(t)\} = \frac{p(s)}{q(s)}$ has the form

$$\begin{aligned}\frac{p(s)}{(s+1)^2+1} &= \frac{as+b}{(s+1)^2+1} \\ &= \frac{a(s+1)+b-a}{(s+1)^2+1} \\ &= c_1\frac{s+1}{(s+1)^2+1} + c_2\frac{1}{(s+1)^2+1},\end{aligned}$$

where $c_1 = a$ and $c_2 = b - a$. Laplace inversion then gives

$$f(t) = \mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = c_1 e^{-t}\sin t + c_2 e^{-t}\cos t.$$

On the other hand, we have $e^{-t}\cos t \in \mathcal{E}_q$ since $\mathcal{L}\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1} \in \mathcal{R}_q$. Similarly, we have $e^{-t}\sin t \in \mathcal{E}_q$. Since $\mathcal{E}_q$ is a linear space, it follows that all functions of the form $c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$ are in $\mathcal{E}_q$. It follows that

$$\mathcal{E}_q = \left\{c_1 e^{-t}\sin t + c_2 e^{-t}\cos t : c_1, c_2 \in \mathbb{R}\right\}.$$ ◀

In Example 3, we observe that $\mathcal{E}_q$ takes the form

$$\mathcal{E}_q = \{c_1\phi_1 + c_2\phi_2 : c_1, c_2 \in \mathbb{R}\},$$

where

in case (1) $\phi_1(t) = \mathrm{e}^t$ and $\phi_2(t) = \mathrm{e}^{2t}$
in case (2) $\phi_1(t) = \mathrm{e}^t$ and $\phi_2(t) = t\mathrm{e}^t$
in case (3) $\phi_1(t) = \mathrm{e}^{-t}\sin t$ and $\phi_2(t) = \mathrm{e}^{-t}\cos t$

We introduce the following useful concepts and notation that will allow us to rephrase the results of Example 3 in a more convenient way and, as we will see, will generalize to arbitrary polynomials $q(s)$. Suppose $\mathcal{F}$ is a linear space of functions and $\mathcal{S} = \{\phi_1, \ldots, \phi_n\} \subset \mathcal{F}$ a subset. A ***linear combination*** of $\mathcal{S}$ is a sum of the following form:

$$c_1\phi_1 + \cdots + c_n\phi_n,$$

where $c_1, \ldots, c_n$ are scalars in $\mathbb{R}$. Since $\mathcal{F}$ is a linear space (closed under addition and scalar multiplication), all such linear combinations are back in $\mathcal{F}$. The ***span*** of $\mathcal{S}$, denoted Span $\mathcal{S}$, is the set of all such linear combinations. Symbolically, we write

$$\text{Span } \mathcal{S} = \{c_1\phi_1 + \cdots + c_n\phi_n : c_1, \ldots, c_n \in \mathbb{R}\}.$$

If every function in $\mathcal{F}$ can be written as a linear combination of $\mathcal{S}$, then we say $\mathcal{S}$ ***spans*** $\mathcal{F}$. Alternately, $\mathcal{S}$ is referred to as a ***spanning set*** for $\mathcal{F}$. Thus, there are two things that need to be checked to determine whether $\mathcal{S}$ is a spanning set for $\mathcal{F}$:

- $\mathcal{S} \subset \mathcal{F}$.
- Each function in $\mathcal{F}$ is a linear combination of functions in $\mathcal{S}$.

Returning to Example 3, we can rephrase our results in the following concise way. For each $q(s)$, define $\mathcal{B}_q$ as given below:

1. $q(s) = s^2 - 3s + 2 = (s-1)(s-2)$ $\qquad \mathcal{B}_q = \left\{\mathrm{e}^t, \mathrm{e}^{2t}\right\}$
2. $q(s) = s^2 - 2s + 1 = (s-1)^2$ $\qquad \mathcal{B}_q = \{\mathrm{e}^t, t\mathrm{e}^t\}$
3. $q(s) = (s+1)^2 + 1$ $\qquad \mathcal{B}_q = \{\mathrm{e}^{-t}\cos t, \mathrm{e}^{-t}\sin t\}$

Then, in each case,

$$\mathcal{E}_q = \text{Span } \mathcal{B}_q.$$

Notice how efficient this description is. In each case, we found two functions that make up $\mathcal{B}_q$. Once they are determined, then $\mathcal{E}_q = \text{Span } \mathcal{B}_q$ is the set of all linear combinations of the two functions in $\mathcal{B}_q$ and effectively gives all those functions whose Laplace transforms are rational functions with $q(s)$ in the denominator. The set $\mathcal{B}_q$ is called the ***standard basis*** of $\mathcal{E}_q$.

Example 3 generalizes in the following way for arbitrary polynomials of degree 2.

Theorem 4. *Suppose $q(s)$ is a polynomial of degree two. Define the standard basis $\mathcal{B}_q$ according to the way $q(s)$ factors as follows:*

$$\begin{array}{ll} 1.\ q(s) = (s - r_1)(s - r_2) & \mathcal{B}_q = \{e^{r_1 t}, e^{r_2 t}\} \\ 2.\ q(s) = (s - r)^2 & \mathcal{B}_q = \{e^{rt}, te^{rt}\} \\ 3.\ q(s) = ((s - a)^2 + b^2) & \mathcal{B}_q = \{e^{at}\cos bt, e^{at}\sin bt\} \end{array}$$

We assume r_1, r_2, r, a, and b are real, $r_1 \neq r_2$, and $b > 0$. Let $\mathcal{E}_q$ be the set of input functions whose Laplace transform is a rational function with $q(s)$ in the denominator. Then

$$\mathcal{E}_q = \text{Span } \mathcal{B}_q.$$

Remark 5. Observe that these three cases may be summarized in terms of the roots of $q(s)$ as follows:

1. If $q(s)$ has distinct real roots r_1 and r_2, then $\mathcal{B}_q = \{e^{r_1 t}, e^{r_2 t}\}$.
2. If $q(s)$ has one real root r with multiplicity 2, then $\mathcal{B}_q = \{e^{rt}, te^{rt}\}$.
3. If $q(s)$ has complex roots $a \pm bi$, then $\mathcal{B}_q = \{e^{at}\cos bt, e^{at}\sin bt\}$. Since $\sin(-bt) = -\sin bt$ and $\cos(-bt) = \cos bt$, we may assume $b > 0$.

Proof. The proof follows the pattern set forth in Example 3.

1. Suppose $f(t) \in \mathcal{E}_q$. Since $q(s) = (s - r_1)(s - r_2)$, a partial fraction decomposition of $\mathcal{L}\{f(t)\}(s) = \frac{p(s)}{q(s)}$ has the form $\frac{p(s)}{(s-1)(s-2)} = \frac{c_1}{s-r_1} + \frac{c_2}{s-r_2}$. Laplace inversion then gives $f(t) = \mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. On the other hand, we have $e^{r_1 t} \in \mathcal{E}_q$ since $\mathcal{L}\{e^{r_1 t}\} = \frac{1}{s-r_1} = \frac{s-r_2}{(s-r_1)(s-r_2)} = \frac{s-r_2}{q(s)} \in \mathcal{R}_q$. Similarly, $e^{r_2 t} \in \mathcal{E}_q$. It now follows that
$$\mathcal{E}_q = \left\{c_1 e^{r_1 t} + c_2 e^{r_2 t} : c_1, c_2 \in \mathbb{R}\right\}.$$
2. Suppose $f(t) \in \mathcal{E}_q$. Since $q(s) = (s - r)^2$, a partial fraction decomposition of $\mathcal{L}\{f(t)\}(s) = \frac{p(s)}{q(s)}$ has the form $\frac{p(s)}{(s-r)^2} = \frac{c_1}{s-r} + \frac{c_2}{(s-r)^2}$. Laplace inversion then gives $f(t) = \mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = c_1 e^{rt} + c_2 te^{rt}$. On the other hand, we have $e^{rt} \in \mathcal{E}_q$ since $\mathcal{L}\{e^{rt}\} = \frac{1}{s-r} = \frac{s-r}{(s-r)^2} = \frac{s-r}{q(s)} \in \mathcal{R}_q$. Similarly, $\mathcal{L}\{te^{rt}\}(s) = \frac{1}{(s-r)^2} \in \mathcal{R}_q$ so $te^{rt} \in \mathcal{E}_q$. It now follows that
$$\mathcal{E}_q = \left\{c_1 e^{rt} + c_2 te^{rt} : c_1, c_2 \in \mathbb{R}\right\}.$$
3. Suppose $f(t) \in \mathcal{E}_q$. A partial fraction decomposition of $\mathcal{L}\{f(t)\} = \frac{p(s)}{q(s)}$ has the form

$$\begin{aligned}\frac{p(s)}{((s-a)^2+b^2)} &= \frac{cs+d}{((s-a)^2+b^2)} \\ &= \frac{c(s-a)+d+ca}{((s-a)^2+b^2)} \\ &= c_1\frac{s-a}{(s-a)^2+b^2} + c_2\frac{b}{(s-a)^2+b^2},\end{aligned}$$

where $c_1 = c$ and $c_2 = \frac{d+ca}{b}$. Laplace inversion then gives

$$f(t) = \mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = c_1 e^{at}\cos bt + c_2 e^{at}\sin bt.$$

On the other hand, since $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \in \mathcal{R}_q$, we have $e^{at}\cos bt \in \mathcal{E}_q$. Similarly, we have $e^{at}\sin bt \in \mathcal{E}_q$. It follows that

$$\mathcal{E}_q = \left\{c_1 e^{-t}\sin t + c_2 e^{-t}\cos t : c_1, c_2 \in \mathbb{R}\right\}.$$

In each case, $\mathcal{E}_q = \text{Span } \mathcal{B}_q$, where $\mathcal{B}_q$ is prescribed as above. □

This theorem makes it very simple to find $\mathcal{B}_q$ and thus $\mathcal{E}_q$ when $\deg q(s) = 2$. The prescription boils down to finding the roots and their multiplicities.

Example 6. Find the standard basis $\mathcal{B}_q$ of $\mathcal{E}_q$ for each of the following polynomials:

1. $q(s) = s^2 + 6s + 5$
2. $q(s) = s^2 + 4s + 4$
3. $q(s) = s^2 + 4s + 13$

► **Solution.** 1. Observe that $q(s) = s^2 + 6s + 5 = (s+1)(s+5)$. The roots are $r_1 = -1$ and $r_2 = -5$. Thus, $\mathcal{B}_q = \left\{e^{-t}, e^{-5t}\right\}$ and $\mathcal{E}_q = \text{Span } \mathcal{B}_q$.
2. Observe that $q(s) = s^2+4s+4 = (s+2)^2$. The root is $r = -2$ with multiplicity 2. Thus, $\mathcal{B}_q = \left\{e^{-2t}, te^{-2t}\right\}$ and $\mathcal{E}_q = \text{Span } \mathcal{B}_q$.
3. Observe that $q(s) = s^2 + 4s + 13 = (s+2)^2 + 3^2$ is an irreducible quadratic. Its roots are $-2 \pm 3i$. Thus, $\mathcal{B}_q = \left\{e^{-2t}\cos 3t, e^{-2t}\sin 3t\right\}$ and $\mathcal{E}_q = \text{Span } \mathcal{B}_q$. ◄

As we go forward, you will see that the spanning set $\mathcal{B}_q$ for $\mathcal{E}_q$, for any polynomial $q(s)$, will be determined precisely by the roots of $q(s)$ and their multiplicities. We next consider two examples of a more general nature: when $q(s)$ is (1) a power of a single linear term and (2) a power of a single irreducible quadratic.

Power of a Linear Term

Let us now consider a polynomial which is a power of a linear term.

Proposition 7. *For $r \in \mathbb{R}$, let*

$$q(s) = (s - r)^n.$$

Then

$$\mathcal{B}_q = \left\{ e^{rt}, te^{rt}, \ldots, t^{n-1}e^{rt} \right\}$$

is a spanning set for $\mathcal{E}_q$.

Remark 8. Observe that $\mathcal{B}_q$ only depends on the single root r and its multiplicity n. Further, $\mathcal{B}_q$ has exactly n functions which is the same as the degree of $q(s)$.

Proof. Suppose $f(t) \in \mathcal{E}_q$. Then a partial fraction decomposition of $\mathcal{L}\{f(t)\}(s) = \frac{p(s)}{q(s)}$ has the form

$$\frac{p(s)}{(s-r)^n} = a_1 \frac{1}{s-r} + a_2 \frac{1}{(s-r)^2} + \cdots + a_n \frac{1}{(s-r)^n}.$$

Laplace inversion then gives

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\left\{ \frac{p(s)}{q(s)} \right\} &= a_1 e^{rt} + a_2 te^{rt} + a_3 \frac{t^2}{2!} e^{rt} + \cdots + a_n \frac{t^{n-1}}{(n-1)!} e^{rt} \\ &= c_1 e^{rt} + c_2 te^{rt} + c_3 t^2 e^{rt} + \cdots + c_n t^{n-1} e^{rt}, \end{aligned}$$

where, in the second line, we have relabeled the constants $\frac{a_k}{k!} = c_k$. Observe that

$$\mathcal{L}\left\{ t^k e^{rt} \right\} = \frac{k!}{(s-r)^{k+1}} = \frac{k!(s-r)^{n-k-1}}{(s-r)^n} \in \mathcal{E}_q.$$

If

$$\mathcal{B}_q = \left\{ e^{rt},\, te^{rt}, \ldots, t^{n-1}e^{rt} \right\}$$

then it follows that

$$\mathcal{E}_q = \text{Span}\, \mathcal{B}_q. \qquad \square$$

Example 9. Let $q(s) = (s-5)^4$. Find all functions f so that $\mathcal{L}\{f\}(s)$ has $q(s)$ in the denominator. In other words, find $\mathcal{E}_q$.

► **Solution.** We simply observe from Proposition 7 that

$$\mathcal{B}_q = \left\{ e^{5t},\, te^{5t},\, t^2e^{5t},\, t^3e^{5t} \right\}$$

and hence $\mathcal{E}_q = \text{Span}\, \mathcal{B}_q$. ◄

Power of an Irreducible Quadratic Term

Let us now consider a polynomial which is a power of an irreducible quadratic term. Recall that any irreducible quadratic $s^2 + cs + d$ may be written in the form $(s-a)^2 + b^2$, where $a \pm b$ are the complex roots.

Lemma 10. *Let n be a nonnegative integer; a, b real numbers; and $b > 0$. Let $q(s) = ((s-a)^2 + b^2)^n$. Then*

$$t^k \mathrm{e}^{at} \cos bt \in \mathcal{E}_q \quad \textit{and} \quad t^k \mathrm{e}^{at} \sin bt \in \mathcal{E}_q,$$

for all $k = 0, \ldots, n-1$.

Proof. We use the translation principle and the transform derivative principle to get

$$\begin{aligned}
\mathcal{L}\left\{t^k \mathrm{e}^{at} \cos bt\right\}(s) &= \mathcal{L}\left\{t^k \cos bt\right\}(s-a) \\
&= (-1)^k \mathcal{L}\left\{\cos bt\right\}^{(k)}\Big|_{s \mapsto (s-a)} \\
&= (-1)^k \left(\frac{s}{s^2+b^2}\right)^{(k)}\Bigg|_{s \mapsto (s-a)}.
\end{aligned}$$

An induction argument which we leave as an exercise gives that $\left(\frac{s}{s^2+b^2}\right)^{(k)}$ is a proper rational function with denominator $(s^2+b^2)^{k+1}$. Replacing s by $s-a$ gives

$$\mathcal{L}\left\{t^k \mathrm{e}^{at} \cos bt\right\}(s) = \frac{p(s)}{((s-a)^2+b^2)^{k+1}},$$

for some polynomial $p(s)$ with $\deg p(s) < 2(k+1)$. Now multiply the numerator and denominator by $((s-a)^2+b^2)^{n-k-1}$ to get

$$\mathcal{L}\left\{t^k \mathrm{e}^{at} \cos bt\right\}(s) = \frac{p(s)((s-a)^2+b^2)^{n-k-1}}{((s-a^2)^2+b^2)^n},$$

for $k = 0, \ldots, n-1$. Since the degree of the numerator is less that $2(k+1) + 2(n-k-1) = 2n$, it follows that $\mathcal{L}\left\{t^k \mathrm{e}^{at} \cos bt\right\} \in \mathcal{E}_q$. A similar calculation gives $\mathcal{L}\left\{t^k \mathrm{e}^{at} \sin bt\right\} \in \mathcal{E}_q$. □

Proposition 11. *Let*

$$q(s) = ((s-a)^2+b^2)^n$$

and assume $b > 0$. Then

$$\begin{aligned}
\mathcal{B}_q = \big\{&\mathrm{e}^{at} \cos bt, \mathrm{e}^{at} \sin bt, t\mathrm{e}^{at} \cos bt, t\mathrm{e}^{at} \sin bt, \\
&\ldots, t^{n-1}\mathrm{e}^{at} \cos bt, t^{n-1}\mathrm{e}^{at} \sin bt\big\}
\end{aligned}$$

is a spanning set for $\mathcal{E}_q$.

Remark 12. Since $\cos(-bt) = \cos bt$ and $\sin(-bt) = -\sin bt$, we may assume that $b > 0$. Observe then that $\mathcal{B}_q$ depends only on the root $a + ib$, where $b > 0$, of $q(s)$ and the multiplicity n. Also $\mathcal{B}_q$ has precisely $2n$ functions which is the degree of $q(s)$.

Proof. By Lemma 10, each term in $\mathcal{B}_q$ is in $\mathcal{E}_q$. Suppose $f(t) \in \mathcal{E}_q$ and $\mathcal{L}\{f(t)\}(s) = \frac{p(s)}{q(s)}$ for some polynomial $p(s)$. A partial fraction decomposition of $\frac{p(s)}{q(s)}$ has the form

$$\frac{p(s)}{q(s)} = \frac{a_1 s + b_1}{((s-a)^2 + b^2)} + \frac{a_2 s + b_2}{((s-a)^2 + b^2)^2} + \cdots + \frac{a_n s + b_n}{((s-a)^2 + b^2)^n}. \tag{6}$$

By Corollary 11 of Sect. 2.5 and the first translation principle, (2) of Sect. 2.5, the inverse Laplace transform of a term $\frac{a_k s + b_k}{((s-a)^2+b^2)^k}$ has the form

$$\mathcal{L}^{-1}\left\{\frac{a_k s + b_k}{((s-a)^2 + b^2)^k}\right\} = p_k(t)\mathrm{e}^{at}\cos bt + q_k(t)\mathrm{e}^{at}\sin bt,$$

where $p_k(t)$ and $q_k(t)$ are polynomials of degree at most $k-1$. We apply the inverse Laplace transform to each term in (6) and add to get

$$\mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\} = p(t)\mathrm{e}^{at}\cos bt + q(t)\mathrm{e}^{at}\sin bt,$$

where $p(t)$ and $q(t)$ are polynomials of degree at most $n-1$. This means that $f(t) = \mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\}$ is a linear combination of functions from $\mathcal{B}_q$ as defined above. It follows now that

$$\mathcal{E}_q = \text{Span}\,\mathcal{B}_q.$$ □

Example 13. Let $q(s) = ((s-3)^2 + 2^2)^3$. Find all functions f so that $\mathcal{L}\{f\}(s)$ has $q(s)$ in the denominator. In other words, find $\mathcal{E}_q$.

► **Solution.** We simply observe from Proposition 11 that

$$\mathcal{B}_q = \left\{\mathrm{e}^{3t}\cos 2t,\ \mathrm{e}^{3t}\sin 2t,\ t\mathrm{e}^{3t}\cos 2t,\ t\mathrm{e}^{3t}\sin 2t,\ t^2\mathrm{e}^{3t}\cos 2t,\ t^2\mathrm{e}^{3t}\sin 2t,\right\}$$

and hence $\mathcal{E}_q = \text{Span}\,\mathcal{B}_q$. ◄

Exercises

1–25. Find the standard basis $\mathcal{B}_q$ of $\mathcal{E}_q$ for each polynomial $q(s)$.

1. $q(s) = s - 4$
2. $q(s) = s + 6$
3. $q(s) = s^2 + 5s$
4. $q(s) = s^2 - 3s - 4$
5. $q(s) = s^2 - 6s + 9$
6. $q(s) = s^2 - 9s + 14$
7. $q(s) = s^2 - s - 6$
8. $q(s) = s^2 + 9s + 18$
9. $q(s) = 6s^2 - 11s + 4$
10. $q(s) = s^2 + 2s - 1$
11. $q(s) = s^2 - 4s + 1$
12. $q(s) = s^2 - 10s + 25$
13. $q(s) = 4s^2 + 12s + 9$
14. $q(s) = s^2 + 9$
15. $q(s) = 4s^2 + 25$
16. $q(s) = s^2 + 4s + 13$
17. $q(s) = s^2 - 2s + 5$
18. $q(s) = s^2 - s + 1$
19. $q(s) = (s + 3)^4$
20. $q(s) = (s - 2)^5$
21. $q(s) = s^3 - 3s^2 + 3s - 1$
22. $q(s) = (s + 1)^6$
23. $q(s) = (s^2 + 4s + 5)^2$
24. $q(s) = (s^2 - 8s + 20)^3$
25. $q(s) = (s^2 + 1)^4$

2.7 The Linear Spaces $\mathcal{E}_q$: The General Case

We continue the discussion initiated in the previous section. Let $q(s)$ be a fixed polynomial. We want to describe the linear space $\mathcal{E}_q$ of continuous functions that have Laplace transforms that are rational functions and have $q(s)$ in the denominator. In the previous section, we gave a description of $\mathcal{E}_q$ in terms of a spanning set $\mathcal{B}_q$ for polynomials of degree 2, a power of a linear term, and a power of an irreducible quadratic. We take up the general case here.

Exponential Polynomials

Let n be a nonnegative integer, and $a, b \in \mathbb{R}$, and assume $b \geq 0$. We will refer to functions of the form

$$t^n e^{at} \cos bt \quad \text{and} \quad t^n e^{at} \sin bt,$$

defined on $\mathbb{R}$, as ***simple exponential polynomials***. We introduced these functions in Lemma 10 of Sect. 2.6. Note that if $b = 0$, then $t^n e^{at} \cos bt = t^n e^{at}$, and if both $a = 0$ and $b = 0$, then $t^n e^{at} \cos bt = t^n$. Thus, the terms

$$t^n e^{at} \quad \text{and} \quad t^n$$

are simple exponential polynomials for all nonnegative integers n and real numbers a. If $n = 0$ and $a = 0$, then $t^n e^{at} \cos bt = \cos bt$ and $t^n e^{at} \sin bt = \sin bt$. Thus, the basic trigonometric functions

$$\cos bt \quad \text{and} \quad \sin bt$$

are simple exponential polynomials. For example, all of the following functions are simple exponential polynomials:

1. t^3 2. $t^5 \cos 2t$ 3. e^{-3t} 4. $t^2 e^{2t}$ 5. $t^4 e^{-8t} \sin 3t$

while *none* of the following are simple exponential polynomials:

6. $t^{\frac{1}{2}}$ 7. $\dfrac{\sin 2t}{\cos 2t}$ 8. $t e^{t^2}$ 9. $\sin(e^t)$ 10. $\dfrac{e^t}{t}$

We refer to any linear combination of simple exponential polynomials as an ***exponential polynomial***. In other words, an exponential polynomial is a function in the span of the simple exponential polynomials. We denote the set of all exponential polynomials by $\mathcal{E}$. All of the following are examples of exponential polynomials and are thus in $\mathcal{E}$:

1. $e^t + 2e^{2t} + 3e^{3t}$	2. $t^2 e^t \sin 3t + 2te^{7t}\cos 5t$	3. $1 - t + t^2 - t^3 + t^4$
4. $t - 2t\cos 3t$	5. $3e^{2t} + 4te^{2t}$	6. $2\cos 4t - 3\sin 4t$

Definition of $\mathcal{B}_q$

Recall that we defined $\mathcal{E}_q$ to be the set of input functions whose Laplace transform is in $\mathcal{R}_q$. We refine slightly our definition. We define $\mathcal{E}_q$ to be the set of *exponential polynomials* whose Laplace transform is in $\mathcal{R}_q$. That is,

$$\mathcal{E}_q = \left\{ f \in \mathcal{E} : \mathcal{L}\{f\} \in \mathcal{R}_q \right\}.$$

Thus, each function in $\mathcal{E}_q$ is defined on the real line even though the Laplace transform only uses the restriction to $[0,\infty)$.[4] We now turn our attention to describing $\mathcal{E}_q$ in terms of a spanning set $\mathcal{B}_q$. In each of the cases we considered in the previous section, $\mathcal{B}_q$ was made up of simple exponential polynomials. This will persist for the general case as well. Consider the following example.

Example 1. Let $q(s) = (s-1)^3(s^2+1)^2$. Find a set $\mathcal{B}_q$ of simple exponential polynomials that spans $\mathcal{E}_q$.

► **Solution.** Recall that $\mathcal{E}_q$ consists of those input functions $f(t)$ such that $\mathcal{L}\{f(t)\}$ is in $\mathcal{R}_q$. In other words, $\mathcal{L}\{f(t)\}(s) = \frac{p(s)}{q(s)}$, for some polynomial $p(s)$ with degree less than that of $q(s)$. A partial fraction decomposition gives the following form:

$$\begin{aligned}\frac{p(s)}{(s-1)^3(s^2+1)^2} &= \frac{a_1}{s-1} + \frac{a_2}{(s-1)^2} + \frac{a_3}{(s-1)^3} + \frac{a_4 s + a_5}{s^2+1} + \frac{a_6 s + a_7}{(s^2+1)^2} \\ &= \frac{p_1(s)}{(s-1)^3} + \frac{p_2(s)}{(s^2+1)^2},\end{aligned}$$

where $p_1(s)$ is a polynomial of degree at most 2 and $p_2(s)$ is a polynomial of degree at most 3. This decomposition allows us to treat Laplace inversion of $\frac{p(s)}{(s-1)^3(s^2+1)^2}$ in terms of the two pieces: $\frac{p_1(s)}{(s-1)^3}$ and $\frac{p_2(s)}{(s^2+1)^2}$. In the first case, the denominator is a power of a linear term, and in the second case, the denominator is a power of an irreducible quadratic. From Propositions 7 and 11 of Sect. 2.6, we get

$$\mathcal{L}^{-1}\left\{\frac{p_1(s)}{q_1(s)}\right\} = c_1 e^t + c_2 t e^t + c_3 t^2 e^t,$$

[4] In fact, any function which has a power series with infinite radius of convergence, such as an exponential polynomial, is completely determined by it values on $[0,\infty)$. This is so since $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$ and $f^{(n)}(0)$ are computed from $f(t)$ on $[0,\infty)$.

$$\mathcal{L}^{-1}\left\{\frac{p_2(s)}{q_2(s)}\right\} = c_4\cos t + c_5\sin t + c_6 t\cos t + c_7 t\sin t,$$

where $c_1, \dots, c_7$ are scalars, $q_1(s) = (s-1)^3$, and $q_2(s) = (s^2+1)^2$. It follows now by linearity of the inverse Laplace transform that

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{p(s)}{(s-1)^3(s^2+1)^2}\right\} \\ &= c_1 e^t + c_2 t e^t + c_3 t^2 e^t + c_4\cos t + c_5\sin t + c_6 t\cos t + c_7 t\sin t. \end{aligned}$$

Thus, if

$$\mathcal{B}_q = \left\{e^t, te^t, t^2e^t, \cos t, \sin t, t\cos t, t\sin t\right\}$$

then the above calculation gives $\mathcal{E}_q = \text{Span}\,\mathcal{B}_q$. Also observe that we have shown that $\mathcal{B}_q = \mathcal{B}_{q_1} \cup \mathcal{B}_{q_2}$. Further, the order of $\mathcal{B}_q$, which is 7, matches the degree of $q(s)$. ◀

From this example, we see that $\mathcal{B}_q$ is the collection of simple exponential polynomials obtained from both $\mathcal{B}_{q_1}$ and $\mathcal{B}_{q_2}$, where $q_1(s) = (s-1)^3$ and $q_2(s) = (s^2+1)^2$ are the factors of $q(s)$. More generally, suppose that $q(s) = q_1(s)\cdots q_R(s)$, where $q_i(s)$ is a power of a linear term or a power of an irreducible quadratic term. Further assume that there is no repetition among the linear or quadratic terms. Then a partial fraction decomposition can be written in the form

$$\frac{p(s)}{q(s)} = \frac{p_1(s)}{q_1(s)} + \cdots + \frac{p_R(s)}{q_R(s)}.$$

We argue as in the example above and see that $\mathcal{L}^{-1}\left\{\frac{p(s)}{q(s)}\right\}$ is a linear combination of those simple exponential polynomial gotten from $\mathcal{B}_{q_1}, \dots, \mathcal{B}_{q_R}$. If we define

$$\mathcal{B}_q = \mathcal{B}_{q_1} \cup \cdots \cup \mathcal{B}_{q_R}$$

then we get the following theorem:

Theorem 2. *Let $q(s)$ be a fixed polynomial of degree n. Suppose $q(s) = q_1(s)\cdots q_R(s)$ where $q_i(s)$ is a power of a linear or an irreducible term and there is no repetition among the terms. Define $\mathcal{B}_q = \mathcal{B}_{q_1} \cup \cdots \cup \mathcal{B}_{q_R}$. Then*

$$\text{Span}\,\mathcal{B}_q = \mathcal{E}_q.$$

Further, the degree of $p(s)$ is the same as the order of $\mathcal{B}_q$.

Proof. The essence of the proof is given in the argument in the previous paragraph. More details can be found in Appendix A.3. □

It is convenient to also express $\mathcal{B}_q$ in terms of the roots of $q(s)$ and their multiplicities. We put this forth in the following algorithm.

Algorithm 3. Let $q(s)$ be a polynomial. The following procedure is used to construct $\mathcal{B}_q$, a spanning set of $\mathcal{E}_q$.

Description of $\mathcal{B}_q$

Given a polynomial $q(s)$:

1. *Factor $q(s)$ and determine the roots and their multiplicities.*
2. *For each real root r with multiplicity m, the spanning set $\mathcal{B}_q$ will contain the simple exponential functions:*

$$e^{rt},\ te^{rt},\ldots,t^{m-1}e^{rt}.$$

3. *For each complex root $a \pm ib$ ($b > 0$) with multiplicity m, the spanning set $\mathcal{B}_q$ will contain the simple exponential functions:*

$$e^{at}\cos bt,\ e^{at}\sin bt,\ \ldots,t^{m-1}e^{at}\cos bt,\ t^{m-1}e^{at}\sin bt.$$

Example 4. Find $\mathcal{B}_q$ if

1. $q(s) = 4(s-3)^2(s-6)$
2. $q(s) = (s+1)(s^2+1)$
3. $q(s) = 7(s-1)^3(s-2)^2((s-3)^2+5^2)^2$

► **Solution.**

1. The roots are $r_1 = 3$ with multiplicity 2 and $r_2 = 6$ with multiplicity 1. Thus,

$$\mathcal{B}_q = \left\{e^{3t},\ te^{3t},\ e^{6t}\right\}.$$

2. The roots are $r = -1$ with multiplicity 1 and $a \pm ib = 0 \pm i$. Thus, $a = 0$ and $b = 1$. We now get

$$\mathcal{B}_q = \left\{e^{-t},\ \cos t,\ \sin t\right\}.$$

3. The roots are $r_1 = 1$ with multiplicity 3, $r_2 = 2$ with multiplicity 2, and $a \pm ib = 3 \pm 5i$ with multiplicity 2. We thus get

$$\mathcal{B}_q = \left\{e^{t},\ te^{t},\ t^2e^{t},\ e^{2t},\ te^{2t},\ e^{3t}\cos 5t,\ e^{3t}\sin 5t, te^{3t}\cos 5t, te^{3t}\sin 5t\right\}. \blacktriangleleft$$

Laplace Transform Correspondences

We conclude this section with two theorems. The first relates $\mathcal{E}$ and $\mathcal{R}$ by way of the Laplace transform. The second relates $\mathcal{E}_q$ and $\mathcal{R}_q$. The notion of linearity is central so we establish that $\mathcal{E}$ and $\mathcal{R}$ are linear spaces. First note the following lemma.

Lemma 5. *Suppose $\mathcal{S}$ is a set of functions on an interval I. Let $\mathcal{F} = \text{Span } \mathcal{S}$. Then $\mathcal{F}$ is a linear space.*

Proof. If f and g are in Span $\mathcal{S}$, then there are scalars $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ so that

$$f = a_1 f_1 + \cdots + a_n f_n \quad \text{and} \quad g = b_1 g_1 + \cdots + b_m g_m,$$

where $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ are in $\mathcal{S}$. The sum

$$f + g = a_1 f_1 + \cdots + a_n f_n + b_1 g_1 + \cdots + b_m g_m,$$

is again a linear combination of function in $\mathcal{S}$, and hence $f + g \in \text{Span } \mathcal{S}$. In a similar way, if c is a scalar and $f = a_1 f_1 + \cdots + a_n f_n$ is in Span $\mathcal{S}$, then $cf = ca_1 f_1 + \cdots + ca_n f_n$ is a linear combinations of functions in $\mathcal{S}$, and hence $cf \in \text{Span } \mathcal{S}$. It follows that Span $\mathcal{S}$ is closed under addition and scalar multiplication and hence is a linear space. □

Recall that we defined the set $\mathcal{E}$ of exponential polynomials as the span of the set of all simple exponential polynomials. Lemma 5 gives the following result.

Proposition 6. *The set of exponential polynomials $\mathcal{E}$ is a linear space.*

Proposition 7. *The set $\mathcal{R}$ of proper rational functions is a linear space.*

Proof. Suppose $\frac{p_1(s)}{q_1(s)}$ and $\frac{p_2(s)}{q_2(s)}$ are in $\mathcal{R}$ and $c \in \mathbb{R}$. Then

- $\frac{p_1(s)}{q_1(s)} + \frac{p_2(s)}{q_2(s)} = \frac{p_1(s)q_2(s)+p_2(s)q_1(s)}{q_1(s)q_2(s)}$ is again a proper rational function and hence in $\mathcal{R}$.
- $c\frac{p_1(s)}{q_1(s)} = \frac{cp_1(s)}{q_1(s)}$ is a again a proper rational function and hence in $\mathcal{R}$.

It follows that $\mathcal{R}$ is closed under addition and scalar multiplication, and hence $\mathcal{R}$ is a linear space. □

Theorem 8. *The Laplace transform*

$$\mathcal{L} : \mathcal{E} \to \mathcal{R}$$

establishes a linear one-to-one correspondence between the linear space of exponential polynomials, $\mathcal{E}$, and the linear space of proper rational functions, $\mathcal{R}$.

Remark 9. This theorem means the following:

1. The Laplace transform is linear, which we have already established.
2. The Laplace transform of each $f \in \mathcal{E}$ is a rational function.
3. For each proper rational function $r \in \mathcal{R}$, there is a unique exponential polynomial $f \in \mathcal{E}$ so that $\mathcal{L}\{f\} = r$.

Proof. By Lemma 10 of Sect. 2.6, the Laplace transforms of the simple exponential polynomials $t^n e^{at} \cos bt$ and $t^n e^{at} \sin bt$ are in $\mathcal{R}$. Let $\phi \in \mathcal{E}$. There are simple exponential polynomials $\phi_1, \ldots, \phi_m$ such that $\phi = c_1\phi_1 + \cdots + c_m\phi_m$, where $c_1, \ldots, c_m \in \mathbb{R}$. Since the Laplace transform is linear, we have $\mathcal{L}\{\phi\} = c_1\mathcal{L}\{f_1\} + \cdots + c_m\mathcal{L}\{\phi_m\}$. Now each term $\mathcal{L}\{\phi_i\} \in \mathcal{R}$. Since $\mathcal{R}$ is a linear space, we $\mathcal{L}\{\phi\} \in \mathcal{R}$. It follows that the Laplace transform of any exponential polynomial is a rational function.

On the other hand, a proper rational function is a linear combination of the simple rational functions given in (1) in Sect. 2.5. Observe that

$$\mathcal{L}^{-1}\{1/(s-a)^n\}(s) = \frac{t^{n-1}}{(n-1)!}e^{at}$$

is a scalar multiple of a simple exponential polynomial. Also, Corollary 11 of Sect. 2.5 and the first translation principle establish that both $1/((s-a)^2 + b^2)^k$ and $s/((s-a)^2+b^2)^k$ have inverse Laplace transforms that are linear combinations of $t^n e^{at} \sin bt$ and $t^n e^{at} \cos bt$ for $0 \leq k < n$. It now follows that the inverse Laplace transform of any rational function is an exponential polynomial. Since the Laplace transform is one-to-one by Theorem 1 of Sect. 2.5, it follows that the Laplace transform establishes a one-to-one correspondence between $\mathcal{E}$ and $\mathcal{R}$. □

We obtain by restricting the Laplace transform the following fundamental theorem.

Theorem 10. *The Laplace transform establishes a linear one-to-one correspondence between $\mathcal{E}_q$ and $\mathcal{R}_q$. In other words,*

$$\mathcal{L} : \mathcal{E}_q \to \mathcal{R}_q$$

is one-to-one and onto.

Exercises

1–11. Determine which of the following functions are in the linear space $\mathcal{E}$ of exponential polynomials.

1. $t^2 e^{-2t}$
2. $t^{-2} e^{2t}$
3. t/e^t
4. e^t/t
5. $t \sin\left(4t - \frac{\pi}{4}\right)$
6. $(t + e^t)^2$
7. $(t + e^t)^{-2}$
8. $t e^{t/2}$
9. $t^{1/2} e^t$
10. $\sin 2t / e^{2t}$
11. $e^{2t} / \sin 2t$

12–28. Find the standard basis $\mathcal{B}_q$ of $\mathcal{E}_q$ for each polynomial $q(s)$.

12. $q(s) = s^3 + s$
13. $q(s) = s^4 - 1$
14. $q(s) = s^3(s+1)^2$
15. $q(s) = (s-1)^3(s+7)^2$
16. $q(s) = (s+8)^2(s^2+9)^3$
17. $q(s) = (s+2)^3(s^2+4)^2$
18. $q(s) = (s+5)^2(s-4)^2(s+3)^2$
19. $q(s) = (s-2)^2(s+3)^2(s+3)$
20. $q(s) = (s-1)(s-2)^2(s-3)^3$
21. $q(s) = (s+4)^2(s^2+6s+13)^2$
22. $q(s) = (s+5)(s^2+4s+5)^2$
23. $q(s) = (s-3)^3(s^2+2s+10)^2$
24. $q(s) = s^3 + 8$
25. $q(s) = 2s^3 - 5s^2 + 4s - 1$
26. $q(s) = s^3 + 2s^2 - 9s - 18$
27. $q(s) = s^4 + 5s^2 + 6$
28. $q(s) = s^4 - 8s^2 + 16$

29–33. Verify the following *closure properties of the linear space of proper rational functions*.

29. *Multiplication*. Show that if $r_1(s)$ and $r_2(s)$ are in $\mathcal{R}$, then so is $r_1(s)r_2(s)$.
30. *Translation*. Show that if $r(s)$ is in $\mathcal{R}$, so is any translation of $r(s)$, that is, $r(s-a) \in \mathcal{R}$ for any a.
31. *Differentiation*. Show that if $r(s)$ is in $\mathcal{R}$, then so is the derivative $r'(s)$.
32. Let $q(s) = (s-a)^2 + b^2$. Suppose $r(s) \in \mathcal{R}_{q^n}$ but $r(s) \notin \mathcal{R}_{q^{n-1}}$. Then $r'(s) \in \mathcal{R}_{q^{n+1}}$ but $r'(s) \notin \mathcal{R}_{q^n}$.
33. Let $q(s) = (s-a)^2 + b^2$. Let $r \in \mathcal{R}_q$. Then $r^{(n)} \in \mathcal{R}_{q^{n+1}}$ but not in $\mathcal{R}_{q^n}$.

34–38. Verify the following *closure properties of the linear space of exponential polynomials*.

34. *Multiplication*. Show that if f and g are in $\mathcal{E}$, then so is fg.
35. *Translation*. Show that if f is in $\mathcal{E}$, so is any translation of f, i.e. $f(t-t_0) \in \mathcal{E}$, for any t_0.
36. *Differentiation*. Show that if f is in $\mathcal{E}$, then so is the derivative f'.
37. *Integration*. Show that if f is in $\mathcal{E}$, then so is $\int f(t)\,\mathrm{d}t$. That is, any antiderivative of f is in $\mathcal{E}$.
38. Show that $\mathcal{E}$ is not closed under inversion. That is, find a function f so that $1/f$ is not in $\mathcal{E}$.

39–41. Let $q(s)$ be a fixed polynomial. Verify the following *closure properties of the linear space $\mathcal{E}_q$*.

39. *Differentiation*. Show that if f is in $\mathcal{E}_q$, then f' is in $\mathcal{E}_q$.
40. Show that if f is in $\mathcal{E}_q$, then the nth derivative of f, $f^{(n)}$, is in $\mathcal{E}_q$.
41. Show that if $f \in \mathcal{E}_q$, then any translate is in $\mathcal{E}_q$. That is, if $t_0 \in \mathbb{R}$, then $f(t-t_0) \in \mathcal{E}_q$.

2.8 Convolution

Table 2.3 shows many examples of operations defined on the input space that induce via the Laplace transform a corresponding operation on transform space, and vice versa. For example, multiplication by $-t$ in input space corresponds to differentiation in transform space. If $F(s)$ is the Laplace transform of $f(t)$, then this correspondence can be indicated as follows:

$$-tf(t) \longleftrightarrow \frac{\mathrm{d}}{\mathrm{d}s}F(s).$$

Our goal in this section is to study another such operational identity. Specifically, we will be concentrating on the question of what is the operation in the input space that corresponds to ordinary multiplication of functions in the transform space. Put more succinctly, suppose $f(t)$ and $g(t)$ are input functions with Laplace transforms $F(s)$ and $G(s)$, respectively. What input function $h(t)$ corresponds to the product $H(s) = F(s)G(s)$ under the Laplace transform? In other words, how do we fill in the following question mark in terms of $f(t)$ and $g(t)$?

$$h(t) = ? \longleftrightarrow F(s)G(s).$$

You might guess that $h(t) = f(t)g(t)$. That is, you would be guessing that multiplication in the input space corresponds to multiplication in the transform space. This guess is *wrong* as you can quickly see by looking at almost any example. For a concrete example, let

$$F(s) = 1/s \quad \text{and} \quad G(s) = 1/s^2.$$

Then $H(s) = F(s)G(s) = 1/s^3$ and $h(t) = t^2/2$. However, $f(t) = 1$, $g(t) = t$, and, hence, $f(t)g(t) = t$. Thus $h(t) \neq f(t)g(t)$.

Suppose f and g are continuous functions on $[0, \infty)$. We define the ***convolution (product)***, $(f * g)(t)$, of f and g by the following integral:

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u. \tag{1}$$

The variable of integration we chose is u but any variable other than t can be used. Admittedly, convolution is an unusual product. It is not at all like the usual product of functions where the value (or state) at time t is determined by knowing just the value of each factor at time t. Rather, (1) tells us that the value at time t depends on knowing the values of the input function f and g for all u between 0 and t. They are then "meshed" together to give the value at t.

The following theorem, the convolution theorem, explains why convolution is so important. It is the operation of convolution in input space that corresponds under the Laplace transform to ordinary multiplication of transform functions.

Theorem 1 (The Convolution Theorem). *Let $f(t)$ and $g(t)$ be continuous functions of exponential type. Then $f * g$ is of exponential type. Further, if $F(s) = \mathcal{L}\{f(t)\}(s)$ and $G(s) = \mathcal{L}\{g(t)\}(s)$, then*

The Convolution Principle

$$\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s)$$

$$\text{or} \quad (f * g)(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s))\}(t).$$

The second formula is just the Laplace inversion of the first formula. The proof of the convolution principle will be postponed until Chap. 6, where it is proved for a broader class of functions.

Let us consider a few examples that confirm the convolution principle.

Example 2. Let n be a positive integer, $f(t) = t^n$, and $g(t) = 1$. Compute the convolution $f * g$ and verify the convolution principle.

► **Solution.** Observe that $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$, $\mathcal{L}\{1\}(s) = 1/s$, and

$$f * g(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u = \int_0^t u^n \cdot 1\,\mathrm{d}u = \left.\frac{u^{n+1}}{n+1}\right|_0^t = \frac{t^{n+1}}{n+1}.$$

Further,

$$\begin{aligned}\mathcal{L}\{f * g\}(s) &= \mathcal{L}\left\{\frac{t^{n+1}}{n+1}\right\}(s) = \frac{1}{n+1}\,\frac{(n+1)!}{s^{n+2}} = \frac{n!}{s^{n+1}} \cdot \frac{1}{s} \\ &= \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}\end{aligned}$$

thus verifying the convolution principle. ◄

Example 3. Compute $t^2 * t^3$ and verify the convolution principle.

► **Solution.** Here we let $f(t) = t^3$ and $g(t) = t^2$ in (1) to get

$$\begin{aligned}t^3 * t^2 &= \int_0^t f(u)g(t-u)\,\mathrm{d}u = \int_0^t u^3(t-u)^2\,\mathrm{d}u \\ &= \int_0^t t^2u^3 - 2tu^4 + u^5\,\mathrm{d}u = \frac{t^6}{4} - 2\frac{t^6}{5} + \frac{t^6}{6} = \frac{t^6}{60}.\end{aligned}$$

Additionally,

$$\mathcal{L}\{f * g\}(s) = \mathcal{L}\left\{\frac{t^6}{60}\right\}(s) = \frac{1}{60}\frac{6!}{s^7} = \frac{3!}{s^4} \cdot \frac{2}{s^3}$$
$$= \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}. \qquad \blacktriangleleft$$

Example 4. Let $f(t) = \sin t$ and $g(t) = 1$. Compute $(f * g)(t)$ and verify the convolution principle.

► **Solution.** Observe that $F(s) = \mathcal{L}\{\sin t\} = 1/(s^2 + 1)$, $G(s) = \mathcal{L}\{1\} = 1/s$, and

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u = \int_0^t \sin u\,\mathrm{d}u = -\cos u|_0^t = 1 - \cos t.$$

Further,

$$\begin{aligned}\mathcal{L}\{f * g\}(s) &= \mathcal{L}\{1 - \cos t\}(s) = \frac{1}{s} - \frac{s}{s^2+1} \\ &= \frac{s^2 + 1 - s^2}{s(s^2+1)} = \frac{1}{s(s^2+1)} = \frac{1}{s^2+1} \cdot \frac{1}{s} \\ &= \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}. \qquad \blacktriangleleft\end{aligned}$$

Properties of the Convolution Product

Convolution is sometimes called the convolution product because it behaves in many ways like an ordinary product. In fact, below are some of its properties:

$$\begin{array}{lrcl} \textit{Commutative property:} & f * g &=& g * f \\ \textit{Associative property:} & (f * g) * h &=& f * (g * h) \\ \textit{Distributive property:} & f * (g + h) &=& f * g + f * h \\ & f * 0 &=& 0 * f = 0 \end{array}$$

Indeed, these properties of convolution are easily verified from the definition given in (1). For example, the commutative property is verified by a change of variables:

$$f * g(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u$$

Let $x = t - u$ then $\mathrm{d}x = -\mathrm{d}t$ and we get

$$
\begin{aligned}
&= \int_t^0 f(t-x)g(x)\,(-1)\mathrm{d}x \\
&= \int_0^t g(x)f(t-x)\,\mathrm{d}x \\
&= g * f.
\end{aligned}
$$

It follows, for example, that

$$
t^n * 1(t) = \int_0^t u^n \,\mathrm{d}u = \int_0^t (u-t)^n \,\mathrm{d}u = t^n * 1.
$$

Both integrals are equal. The decision about which one to use depends on which you regard as the easiest to compute. You should verify the other properties listed.

There is one significant difference that convolution has from the ordinary product of functions, however. Examples 2 and 4 imply that the constant function 1 does not behave like a multiplicative identity. In fact, no such "function" exists.[5] Nevertheless, convolution by $f(t) = 1$ is worth singling out as a special case of the convolution principle.

Theorem 5. *Let $g(t)$ be a continuous function of exponential type and $G(s)$ its Laplace transform. Then $(1 * g)(t) = \int_0^t g(u)\,\mathrm{d}u$ and*

Input Integral Principle

$$
\mathcal{L}\left\{\int_0^t g(u)\,\mathrm{d}u\right\} = \frac{G(s)}{s}.
$$

Proof. Since $(1 * g)(t) = \int_0^t g(u)\,\mathrm{d}u$, the theorem follows directly from the convolution principle. However, it is noteworthy that the input integral principle follows from the input derivative principle. Here is the argument. Since g is of exponential type, so is any antiderivative by Lemma 4 of Sect. 2.2. Suppose $h(t) = \int_0^t g(u)\,\mathrm{d}u$. Then $h'(t) = g(t)$, and $h(0) = 0$ so the input derivative principle gives

$$
G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\left\{h'(t)\right\} = sH(s) - h(0) = sH(s).
$$

Hence, $H(s) = (1/s)G(s)$, and thus,

$$
\mathcal{L}\left\{\int_0^t g(u)\,\mathrm{d}u\right\} = \mathcal{L}\{h(t)\}(s) = \frac{1}{s}G(s). \qquad \square
$$

[5]In Chap. 6, we will discuss a so-called "generalized function" that will act as a multiplicative identity for convolution.

Remark 6. The requirement that g be of exponential type can be relaxed. It can be shown that if g is continuous on $[0, \infty)$ and has a Laplace transform so does any antiderivative, and the input integral principle remains valid.[6]

The input integral and convolution principles can also be used to compute the inverse Laplace transform of rational functions. Consider the following two examples

Example 7. Find the inverse Laplace transform of

$$\frac{1}{s(s^2+1)}.$$

► **Solution.** Instead of using partial fractions, we will use the input integral principle. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \int_0^t \sin u \, \mathrm{d}u \\ &= -\cos u|_0^t = 1 - \cos t.\end{aligned}$$ ◄

Example 8. Compute the inverse Laplace transform of $\frac{s}{(s-1)(s^2+9)}$.

► **Solution.** The inverse Laplace transforms of $\frac{s}{s^2+9}$ and $\frac{1}{s-1}$ are $\cos 3t$ and e^t, respectively. The convolution theorem now gives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+9)}\right\} &= \cos 3t * \mathrm{e}^t \\ &= \int_0^t \cos 3u \, \mathrm{e}^{t-u} \, \mathrm{d}u \\ &= \mathrm{e}^t \int_0^t \mathrm{e}^{-u} \cos 3u \, \mathrm{d}u \\ &= \frac{\mathrm{e}^t}{10} \left(-\mathrm{e}^{-u} \cos 3u + 3\mathrm{e}^{-u} \sin 3u\right)\Big|_0^t \\ &= \frac{1}{10}(-\cos 3t + 3 \sin 3t + \mathrm{e}^t).\end{aligned}$$

The computation of the integral involves integration by parts. We leave it to the student to verify this calculation. Of course, this calculation agrees with Laplace inversion using the method of partial fractions. ◄

[6] For a proof, see Theorem 6 and the remark that follows on page 450 of the text *Advanced Calculus* (second edition) by David Widder, published by Prentice Hall (1961).

Table 2.11 contains several general convolution formulas. The next few formulas verify some of the entries.

Formula 9. Verify the convolution product

$$\boxed{\mathrm{e}^{at} * \mathrm{e}^{bt} = \frac{\mathrm{e}^{at} - \mathrm{e}^{bt}}{a - b},} \tag{2}$$

where $a \neq b$, and verify the convolution principle.

▼ *Verification.* Use the defining equation (1) to get

$$\mathrm{e}^{at} * \mathrm{e}^{bt} = \int_0^t \mathrm{e}^{au}\mathrm{e}^{b(t-u)}\,\mathrm{d}u = \mathrm{e}^{bt}\int_0^t \mathrm{e}^{(a-b)u}\,\mathrm{d}u = \frac{\mathrm{e}^{at} - \mathrm{e}^{bt}}{a - b}.$$

Observe that

$$\begin{aligned} \mathcal{L}\left\{\frac{\mathrm{e}^{at} - \mathrm{e}^{bt}}{a - b}\right\} &= \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) \\ &= \frac{1}{(s-a)(s-b)} = \frac{1}{s-a}\cdot\frac{1}{s-b} \\ &= \mathcal{L}\left\{\mathrm{e}^{at}\right\}\cdot\mathcal{L}\left\{\mathrm{e}^{bt}\right\}, \end{aligned}$$

so this calculation is in agreement with the convolution principle. ▲

Formula 10. Verify the convolution product

$$\boxed{\mathrm{e}^{at} * \mathrm{e}^{at} = t\mathrm{e}^{at}} \tag{3}$$

and verify the convolution principle.

▼ *Verification.* Computing from the definition:

$$\mathrm{e}^{at} * \mathrm{e}^{at} = \int_0^t \mathrm{e}^{au}\mathrm{e}^{a(t-u)}\,\mathrm{d}u = \mathrm{e}^{at}\int_0^t \mathrm{d}u = t\mathrm{e}^{at}.$$

As with the previous example, note that the calculation

$$\mathcal{L}\left\{t\mathrm{e}^{at}\right\} = \frac{1}{(s-a)^2} = \mathcal{L}\left\{\mathrm{e}^{at}\right\}\mathcal{L}\left\{\mathrm{e}^{at}\right\},$$

which agrees with the convolution principle. ▲

Remark 11. Since

$$\lim_{a\to b}\frac{\mathrm{e}^{at} - \mathrm{e}^{bt}}{a - b} = \frac{\mathrm{d}}{\mathrm{d}a}\mathrm{e}^{at} = t\mathrm{e}^{at},$$

the previous two examples show that

$$\lim_{a\to b} e^{at} * e^{bt} = t e^{at} = e^{at} * e^{at},$$

so that the convolution product is, in some sense, a continuous operation.

Formula 12. Verify the following convolution product where $m, n \geq 0$:

$$\boxed{t^m * t^n = \frac{m!\,n!}{(m+n+1)!} t^{m+n+1}.}$$

▼ *Verification.* Our method for this computation is to use the convolution theorem, Theorem 1. We get

$$\mathcal{L}\{t^m * t^n\} = \mathcal{L}\{t^m\}\,\mathcal{L}\{t^n\} = \frac{m!}{s^{m+1}}\frac{n!}{s^{n+1}} = \frac{m!\,n!}{s^{m+n+2}}.$$

Now take the inverse Laplace transform to conclude

$$t^m * t^n = \mathcal{L}^{-1}\left\{\frac{m!\,n!}{s^{m+n+2}}\right\} = \frac{m!\,n!}{(m+n+1)!} t^{m+n+1}. \qquad \blacktriangle$$

As special cases of this formula, note that

$$t^2 * t^3 = \frac{1}{60} t^6 \quad \text{and} \quad t * t^4 = \frac{1}{30} t^6.$$

The first was verified directly in Example 3.

In the next example, we revisit a simple rational function whose inverse Laplace transform can be computed by the techniques of Sect. 2.5.

Example 13. Compute the inverse Laplace transform of $\frac{1}{(s^2+1)^2}$.

► **Solution.** The inverse Laplace transform of $1/(s^2+1)$ is $\sin t$. By the convolution theorem, we have

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \sin t * \sin t. \\
&= \int_0^t \sin u \sin(t-u)\,\mathrm{d}u \\
&= \int_0^t \sin u(\sin t \cos u - \sin u \cos t)\,\mathrm{d}u \\
&= \sin t \int_0^t \sin u \cos u\,\mathrm{d}u - \cos t \int_0^t \sin^2 u\,\mathrm{d}u \\
&= \left(\sin t \frac{\sin^2 t}{2} - \cos t \frac{t - \sin t \cos t}{2}\right) \\
&= \frac{\sin t - t\cos t}{2}.
\end{aligned}
$$

◀

Now, one should see how to handle $1/(s^2+1)^3$ and even higher powers: repeated applications of convolution. Let f^{*k} denote the convolution of f with itself k times. In other words,

$$f^{*k} = f * f * \cdots * f, \qquad \text{k times.}$$

Then it is easy to see that

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^{k+1}}\right\} = \sin^{*(k+1)} t \\
\text{and} \qquad &\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^{k+1}}\right\} = \cos t * \sin^{*k} t.
\end{aligned}
$$

These rational functions with powers of irreducible quadratics in the denominator were introduced in Sect. 2.5 where recursion formulas were derived.

Computing convolution products can be tedious and time consuming. In Table 2.11, we provide a list of common convolution products. Students should familiarize themselves with this list so as to know when they can be used.

Exercises

1–4. Use the definition of the convolution to compute the following convolution products.

1. $t * t$
2. $t * t^3$
3. $3 * \sin t$
4. $(3t + 1) * e^{4t}$

5–9. Compute the following convolutions using the table or the convolution principle.

5. $\sin 2t * e^{3t}$
6. $(2t + 1) * \cos 2t$
7. $t^2 * e^{-6t}$
8. $\cos t * \cos 2t$
9. $e^{2t} * e^{-4t}$

10–15. Use the convolution principle to determine the following convolutions and thus verify the entries in the convolution table.

10. $t * t^n$
11. $e^{at} * \sin bt$
12. $e^{at} * \cos bt$
13. $\sin at * \sin bt$
14. $\sin at * \cos bt$
15. $\cos at * \cos bt$

16–21. Compute the Laplace transform of each of the following functions.

16. $f(t) = \int_0^t (t - x) \cos 2x \, dx$
17. $f(t) = \int_0^t (t - x)^2 \sin 2x \, dx$
18. $f(t) = \int_0^t (t - x)^3 e^{-3x} \, dx$
19. $f(t) = \int_0^t x^3 e^{-3(t-x)} \, dx$
20. $f(t) = \int_0^t \sin 2x \cos(t - x) \, dx$
21. $f(t) = \int_0^t \sin 2x \sin 2(t - x) \, dx$

22–31. In each of the following exercises, use the convolution theorem to compute the inverse Laplace transform of the given function.

22. $\dfrac{1}{(s-2)(s+4)}$

23. $\dfrac{1}{s^2-6s+5}$

24. $\dfrac{1}{(s^2+1)^2}$

25. $\dfrac{s}{(s^2+1)^2}$

26. $\dfrac{1}{(s+6)s^3}$

27. $\dfrac{2}{(s-3)(s^2+4)}$

28. $\dfrac{s}{(s-4)(s^2+1)}$

29. $\dfrac{1}{(s-a)(s-b)} \quad a \neq b$

30. $\dfrac{G(s)}{s+2}$

31. $G(s)\dfrac{s}{s^2+2}$

32. Let f be a function with Laplace transform $F(s)$. Show that

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^{x_1} f(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1.$$

More generally, show that

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^n}\right\} = \int_0^t \int_0^{x_1} \cdots \int_0^{x_{n-1}} f(x_n)\,\mathrm{d}x_n \ldots \mathrm{d}x_2\,\mathrm{d}x_1.$$

33–38. Use the input integral principle or, more generally, the results of Problem 32 to compute the inverse Laplace transform of each function.

33. $\dfrac{1}{s^2(s^2+1)}$

34. $\dfrac{1}{s^2(s^2-4)}$

35. $\dfrac{1}{s^3(s+3)}$

36. $\dfrac{1}{s^2(s-2)^2}$

37. $\dfrac{1}{s(s^2+9)^2}$

38. $\dfrac{1}{s^3+s^2}$

2.9 Summary of Laplace Transforms and Convolutions

Laplace transforms and convolutions presented in Chap. 2 are summarized in Tables 2.6–2.11.

Table 2.6 Laplace transform rules

	$f(t)$	$F(s)$	Page
Definition of the Laplace transform			
1.	$f(t)$	$F(s)=\int_0^\infty e^{-st} f(t)\,dt$	111
Linearity			
2.	$a_1 f_1(t)+a_2 f_2(t)$	$a_1 F_1(s)+a_2 F_2(s)$	114
Dilation principle			
3.	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	122
First Translation principle			
4.	$e^{at} f(t)$	$F(s-a)$	120
Input derivative principle: first order			
5.	$f'(t)$	$sF(s)-f(0)$	115
Input derivative principle: second order			
6.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	115
Input derivative principle: nth order			
7.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$	116
Transform derivative principle: first order			
8.	$tf(t)$	$-F'(s)$	121
Transform derivative principle: second order			
9.	$t^2 f(t)$	$F''(s)$	
Transform derivative principle: nth order			
10.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	121
Convolution principle			
11.	$(f*g)(t)$ $=\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	188
Input integral principle			
12.	$\int_0^t f(v)dv$	$\frac{F(s)}{s}$	190

Table 2.7 Basic Laplace transforms

	$f(t)$	$F(s)$	Page
1.	1	$\frac{1}{s}$	116
2.	t	$\frac{1}{s^2}$	
3.	$t^n \quad (n = 0, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$	116
4.	$t^\alpha \quad (\alpha > 0)$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	118
5.	e^{at}	$\frac{1}{s-a}$	118
6.	te^{at}	$\frac{1}{(s-a)^2}$	
7.	$t^n e^{at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$	119
8.	$\sin bt$	$\frac{b}{s^2+b^2}$	118
9.	$\cos bt$	$\frac{s}{s^2+b^2}$	118
10.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	120
11.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	120

Table 2.8 Heaviside formulas

	$f(t)$	$F(s)$
1.	$\frac{r_1^k e^{r_1 t}}{q'(r_1)} + \cdots + \frac{r_n^k e^{r_n t}}{q'(r_n)},$ $q(s) = (s-r_1)\cdots(s-r_n)$	$\frac{s^k}{(s-r_1)\cdots(s-r_n)},$ $r_1, \ldots, r_n,$ distinct
2.	$\frac{e^{at}}{a-b} + \frac{e^{bt}}{b-a}$	$\frac{1}{(s-a)(s-b)}$
3.	$\frac{ae^{at}}{a-b} + \frac{be^{bt}}{b-a}$	$\frac{s}{(s-a)(s-b)}$
4.	$\frac{e^{at}}{(a-b)(a-c)} + \frac{e^{bt}}{(b-a)(b-c)} + \frac{e^{ct}}{(c-a)(c-b)}$	$\frac{1}{(s-a)(s-b)(s-c)}$
5.	$\frac{ae^{at}}{(a-b)(a-c)} + \frac{be^{bt}}{(b-a)(b-c)} + \frac{ce^{ct}}{(c-a)(c-b)}$	$\frac{s}{(s-a)(s-b)(s-c)}$

(continued)

Table 2.8 (continued)

6.	$\dfrac{a^2 e^{at}}{(a-b)(a-c)} + \dfrac{b^2 e^{bt}}{(b-a)(b-c)} + \dfrac{c^2 e^{ct}}{(c-a)(c-b)}$	$\dfrac{s^2}{(s-a)(s-b)(s-c)}$
7.	$\left(\sum_{l=0}^{k} \binom{k}{l} a^{k-l} \frac{t^{n-l-1}}{(n-l-1)!}\right) e^{at}$	$\dfrac{s^k}{(s-a)^n}$
8.	$t e^{at}$	$\dfrac{1}{(s-a)^2}$
9.	$(1+at)e^{at}$	$\dfrac{s}{(s-a)^2}$
10.	$\dfrac{t^2}{2} e^{at}$	$\dfrac{1}{(s-a)^3}$
11.	$\left(t + \dfrac{at^2}{2}\right) e^{at}$	$\dfrac{s}{(s-a)^3}$
12.	$\left(1 + 2at + \dfrac{a^2t^2}{2}\right) e^{at}$	$\dfrac{s^2}{(s-a)^3}$

In each case, a, b, and c are distinct. See Page 165.

Table 2.9 Laplace transforms involving irreducible quadratics

	$f(t)$	$F(s)$
1.	$\sin bt$	$\dfrac{b}{(s^2+b^2)}$
2.	$\dfrac{1}{2b^2}(\sin bt - bt\cos bt)$	$\dfrac{b}{(s^2+b^2)^2}$
3.	$\dfrac{1}{8b^4}\left((3-(bt)^2)\sin bt - 3bt\cos bt\right)$	$\dfrac{b}{(s^2+b^2)^3}$
4.	$\dfrac{1}{48b^6}\left((15-6(bt)^2)\sin bt - (15bt-(bt)^3)\cos bt\right)$	$\dfrac{b}{(s^2+b^2)^4}$
5.	$\cos bt$	$\dfrac{s}{(s^2+b^2)}$
6.	$\dfrac{1}{2b^2} bt\sin bt$	$\dfrac{s}{(s^2+b^2)^2}$
7.	$\dfrac{1}{8b^4}\left(bt\sin bt - (bt)^2\cos bt\right)$	$\dfrac{s}{(s^2+b^2)^3}$
8.	$\dfrac{1}{48b^6}\left((3bt-(bt)^3)\sin bt - 3(bt)^2\cos bt\right)$	$\dfrac{s}{(s^2+b^2)^4}$

Table 2.10 Reduction of order formulas

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^{k+1}}\right\} = \frac{-t}{2kb^2}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^k}\right\} + \frac{2k-1}{2kb^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^{k+1}}\right\} = \frac{t}{2k}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\}$$

See Page 155.

Table 2.11 Basic convolutions

	$f(t)$	$g(t)$	$(f * g)(t)$	Page
1.	$f(t)$	$g(t)$	$f * g(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u$	187
2.	1	$g(t)$	$\int_0^t g(\tau)\,\mathrm{d}\tau$	190
3.	t^m	t^n	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}$	193
4.	t	$\sin at$	$\dfrac{at - \sin at}{a^2}$	
5.	t^2	$\sin at$	$\dfrac{2}{a^3}\left(\cos at - (1 - \frac{a^2t^2}{2})\right)$	
6.	t	$\cos at$	$\dfrac{1 - \cos at}{a^2}$	
7.	t^2	$\cos at$	$\dfrac{2}{a^3}(at - \sin at)$	
8.	t	e^{at}	$\dfrac{\mathrm{e}^{at} - (1 + at)}{a^2}$	
9.	t^2	e^{at}	$\dfrac{2}{a^3}\left(\mathrm{e}^{at} - \left(a + at + \frac{a^2t^2}{2}\right)\right)$	
10.	e^{at}	e^{bt}	$\dfrac{1}{b-a}(\mathrm{e}^{bt} - \mathrm{e}^{at}) \quad a \neq b$	192
11.	e^{at}	e^{at}	$t\mathrm{e}^{at}$	192
12.	e^{at}	$\sin bt$	$\dfrac{1}{a^2+b^2}(b\mathrm{e}^{at} - b\cos bt - a\sin bt)$	195
13.	e^{at}	$\cos bt$	$\dfrac{1}{a^2+b^2}(a\mathrm{e}^{at} - a\cos bt + b\sin bt)$	195
14.	$\sin at$	$\sin bt$	$\dfrac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$	195
15.	$\sin at$	$\sin at$	$\dfrac{1}{2a}(\sin at - at\cos at)$	195
16.	$\sin at$	$\cos bt$	$\dfrac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$	195
17.	$\sin at$	$\cos at$	$\dfrac{1}{2}t\sin at$	195
18.	$\cos at$	$\cos bt$	$\dfrac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$	195
19.	$\cos at$	$\cos at$	$\dfrac{1}{2a}(at\cos at + \sin at)$	195