A theorem of Pólya on polynomials

Chapter 23

Among the many contributions of George Pólya to analysis, the following has always been Erdős' favorite, both for the surprising result and for the beauty of its proof. Suppose that

$$f(z) \;=\; z^n + b_{n-1}z^{n-1} + \cdots + b_0$$

is a complex polynomial of degree $n \geq 1$ with leading coefficient 1. Associate with $f(z)$ the set

$$\mathcal{C} \;:=\; \{z \in \mathbb{C} : |f(z)| \leq 2\},$$

that is, $\mathcal{C}$ is the set of points which are mapped under f into the circle of radius 2 around the origin in the complex plane. So for $n = 1$ the domain $\mathcal{C}$ is just a circular disk of diameter 4.

George Pólya

By an astoundingly simple argument, Pólya revealed the following beautiful property of this set $\mathcal{C}$:

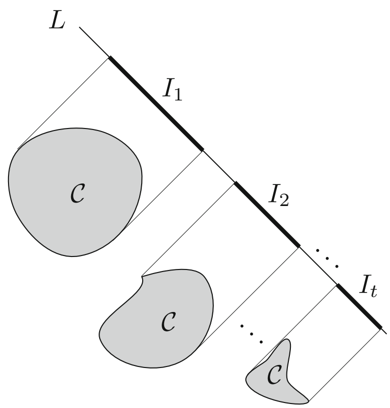
> *Take any line L in the complex plane and consider the orthogonal projection $\mathcal{C}_L$ of the set $\mathcal{C}$ onto L. Then the total length of any such projection never exceeds 4.*

What do we mean by the total length of the projection $\mathcal{C}_L$ being at most 4? We will see that $\mathcal{C}_L$ *is* a finite union of disjoint intervals $I_1, \ldots, I_t$, and the condition means that $\ell(I_1) + \cdots + \ell(I_t) \leq 4$, where $\ell(I_j)$ is the usual length of an interval.

By rotating the plane we see that it suffices to consider the case when L is the real axis of the complex plane. With these comments in mind, let us state Pólya's result.

Theorem 1. *Let $f(z)$ be a complex polynomial of degree at least 1 and leading coefficient 1. Set $\mathcal{C} = \{z \in \mathbb{C} : |f(z)| \leq 2\}$ and let $\mathcal{R}$ be the orthogonal projection of $\mathcal{C}$ onto the real axis. Then there are intervals $I_1, \ldots, I_t$ on the real line which together cover $\mathcal{R}$ and satisfy*

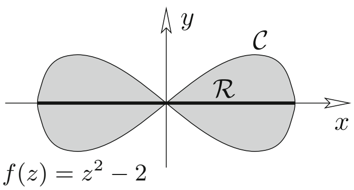
$$\ell(I_1) + \cdots + \ell(I_t) \;\leq\; 4.$$

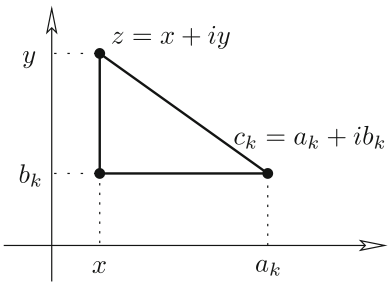


Clearly the bound of 4 in the theorem is attained for $n = 1$. To get more of a feeling for the problem let us look at the polynomial $f(z) = z^2 - 2$, which also attains the bound of 4. If $z = x + iy$ is a complex number, then x is its orthogonal projection onto the real line. Hence

$$\mathcal{R} = \{x \in \mathbb{R} : x + iy \in \mathcal{C} \text{ for some } y\}.$$

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_23





The reader can easily prove that for $f(z) = z^2 - 2$ we have $x + iy \in \mathcal{C}$ if and only if

$$(x^2 + y^2)^2 \ \leq \ 4(x^2 - y^2).$$

It follows that $x^4 \leq (x^2 + y^2)^2 \leq 4x^2$, and thus $x^2 \leq 4$, that is, $|x| \leq 2$. On the other hand, any $z = x \in \mathbb{R}$ with $|x| \leq 2$ satisfies $|z^2 - 2| \leq 2$, and we find that $\mathcal{R}$ is precisely the interval $[-2, 2]$ of length 4.

As a first step towards the proof write $f(z) = (z - c_1) \cdots (z - c_n)$ with $c_k = a_k + ib_k$, and consider the *real* polynomial $p(x) = (x - a_1) \cdots (x - a_n)$. Let $z = x + iy \in \mathcal{C}$, then by the theorem of Pythagoras

$$|x - a_k|^2 + |y - b_k|^2 = |z - c_k|^2$$

and hence $|x - a_k| \leq |z - c_k|$ for all k, that is,

$$|p(x)| \ = \ |x - a_1| \cdots |x - a_n| \ \leq \ |z - c_1| \cdots |z - c_n| \ = \ |f(z)| \leq 2.$$

Thus we find that $\mathcal{R}$ is contained in the set $\mathcal{P} = \{x \in \mathbb{R} : |p(x)| \leq 2\}$, and if we can show that this latter set is covered by intervals of total length at most 4, then we are done. Accordingly, our main Theorem 1 will be a consequence of the following result.

Theorem 2. *Let $p(x)$ be a real polynomial of degree $n \geq 1$ with leading coefficient 1, and all roots real. Then the set $\mathcal{P} = \{x \in \mathbb{R} : |p(x)| \leq 2\}$ can be covered by intervals of total length at most 4.*

As Pólya shows in his paper [2], Theorem 2 is, in turn, a consequence of the following famous result due to Chebyshev. To make this chapter self-contained, we have included a proof in the appendix (following the beautiful exposition by Pólya and Szegő).

Pavnuty Chebyshev on a Soviet stamp from 1946

Chebyshev's Theorem.
Let $p(x)$ be a real polynomial of degree $n \geq 1$ with leading coefficient 1. Then

$$\max_{-1 \leq x \leq 1} |p(x)| \ \geq \ \frac{1}{2^{n-1}}.$$

Let us first note the following immediate consequence.

Corollary. *Let $p(x)$ be a real polynomial of degree $n \geq 1$ with leading coefficient 1, and suppose that $|p(x)| \leq 2$ for all x in the interval $[a, b]$. Then $b - a \leq 4$.*

■ **Proof.** Consider the substitution $y = \frac{2}{b-a}(x - a) - 1$. This maps the x-interval $[a, b]$ onto the y-interval $[-1, 1]$. The corresponding polynomial

$$q(y) \ = \ p(\tfrac{b-a}{2}(y + 1) + a)$$

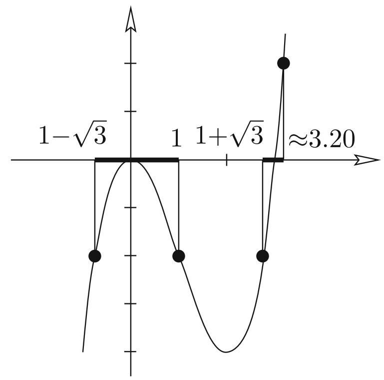
has leading coefficient $(\frac{b-a}{2})^n$ and satisfies

$$\max_{-1 \leq y \leq 1} |q(y)| \ = \ \max_{a \leq x \leq b} |p(x)|.$$

By Chebyshev's theorem we deduce

$$2 \;\geq\; \max_{a \leq x \leq b} |p(x)| \;\geq\; (\tfrac{b-a}{2})^n \tfrac{1}{2^{n-1}} \;=\; 2(\tfrac{b-a}{4})^n,$$

and thus $b - a \leq 4$, as desired. $\square$



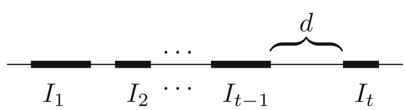
For the polynomial $p(x) = x^2(x-3)$ we get $\mathcal{P} = [1-\sqrt{3}, 1] \cup [1+\sqrt{3}, \approx 3.2]$

This corollary brings us already very close to the statement of Theorem 2. If the set $\mathcal{P} = \{x : |p(x)| \leq 2\}$ is an *interval*, then the length of $\mathcal{P}$ is at most 4. The set $\mathcal{P}$ may, however, not be an interval, as in the example depicted here, where $\mathcal{P}$ consists of two intervals.

What can we say about $\mathcal{P}$? Since $p(x)$ is a continuous function, we know at any rate that $\mathcal{P}$ is the union of disjoint closed intervals $I_1, I_2, \ldots$, and that $p(x)$ assumes the value 2 or -2 at each endpoint of an interval I_j. This implies that there are only finitely many intervals $I_1, \ldots, I_t$, since $p(x)$ can assume any value only finitely often.

Pólya's wonderful idea was to construct another polynomial $\tilde{p}(x)$ of degree n, again with leading coefficient 1, such that $\widetilde{\mathcal{P}} = \{x : |\tilde{p}(x)| \leq 2\}$ is an *interval* of length at least $\ell(I_1) + \cdots + \ell(I_t)$. The corollary then proves $\ell(I_1) + \cdots + \ell(I_t) \leq \ell(\widetilde{\mathcal{P}}) \leq 4$, and we are done.

■ **Proof of Theorem 2.** Consider $p(x) = (x - a_1) \cdots (x - a_n)$ with $\mathcal{P} = \{x \in \mathbb{R} : |p(x)| \leq 2\} = I_1 \cup \cdots \cup I_t$, where we arrange the intervals I_j such that I_1 is the leftmost and I_t the rightmost interval. First we claim that any interval I_j contains a root of $p(x)$. We know that $p(x)$ assumes the values 2 or -2 at the endpoints of I_j. If one value is 2 and the other -2, then there is certainly a root in I_j. So assume $p(x) = 2$ at both endpoints (the case -2 being analogous). Suppose $b \in I_j$ is a point where $p(x)$ assumes its minimum in I_j. Then $p'(b) = 0$ and $p''(b) \geq 0$. If $p''(b) = 0$, then b is a multiple root of $p'(x)$, and hence a root of $p(x)$ by Fact 1 from the box on the next page. If, on the other hand, $p''(b) > 0$, then we deduce $p(b) \leq 0$ from Fact 2 from the same box. Hence either $p(b) = 0$, and we have our root, or $p(b) < 0$, and we obtain a root in the interval from b to either endpoint of I_j.



Here is the final idea of the proof. Let $I_1, \ldots, I_t$ be the intervals as before, and suppose the rightmost interval I_t contains m roots of $p(x)$, counted with their multiplicities. If $m = n$, then I_t is the only interval (by what we just proved), and we are finished. So assume $m < n$, and let d be the distance between I_{t-1} and I_t as in the figure. Let $b_1, \ldots, b_m$ be the roots of $p(x)$ which lie in I_t and $c_1, \ldots c_{n-m}$ the remaining roots. We now write $p(x) = q(x)r(x)$ where $q(x) = (x - b_1) \cdots (x - b_m)$ and $r(x) = (x - c_1) \cdots (x - c_{n-m})$, and set $p_1(x) = q(x+d)r(x)$. The polynomial $p_1(x)$ is again of degree n with leading coefficient 1. For $x \in I_1 \cup \cdots \cup I_{t-1}$ we have $|x + d - b_i| < |x - b_i|$ for all i, and hence $|q(x+d)| < |q(x)|$. It follows that

$$|p_1(x)| \;\leq\; |p(x)| \;\leq\; 2 \qquad \text{for } x \in I_1 \cup \cdots \cup I_{t-1}.$$

If, on the other hand, $x \in I_t$, then we find $|r(x-d)| \leq |r(x)|$ and thus

$$|p_1(x-d)| \;=\; |q(x)||r(x-d)| \;\leq\; |p(x)| \;\leq\; 2,$$

which means that $I_t - d \subseteq \mathcal{P}_1 = \{x : |p_1(x)| \le 2\}$.

In summary, we see that $\mathcal{P}_1$ contains $I_1 \cup \cdots \cup I_{t-1} \cup (I_t - d)$ and hence has total length at least as large as $\mathcal{P}$. Notice now that with the passage from $p(x)$ to $p_1(x)$ the intervals I_{t-1} and $I_t - d$ merge into a single interval. We conclude that the intervals $J_1, \ldots, J_s$ of $p_1(x)$ making up $\mathcal{P}_1$ have total length at least $\ell(I_1) + \cdots + \ell(I_t)$, and that the rightmost interval J_s contains more than m roots of $p_1(x)$. Repeating this procedure at most $t-1$ times, we finally arrive at a polynomial $\tilde{p}(x)$ with $\widetilde{\mathcal{P}} = \{x : |\tilde{p}(x)| \le 2\}$ being an interval of length $\ell(\widetilde{\mathcal{P}}) \ge \ell(I_1) + \cdots + \ell(I_t)$, and the proof is complete. □

Two facts about polynomials with real roots

Let $p(x)$ be a nonconstant polynomial with only real roots.

Fact 1. *If b is a multiple root of $p'(x)$, then b is also a root of $p(x)$.*

■ **Proof.** Let $b_1 < \cdots < b_r$ be the roots of $p(x)$ with multiplicities $s_1, \ldots, s_r$, $\sum_{j=1}^{r} s_j = n$. From $p(x) = (x - b_j)^{s_j} h(x)$ we infer that b_j is a root of $p'(x)$ if $s_j \ge 2$, and the multiplicity of b_j in $p'(x)$ is $s_j - 1$. Furthermore, there is a root of $p'(x)$ between b_1 and b_2, another root between b_2 and b_3, …, and one between b_{r-1} and b_r, and all these roots must be *single* roots, since $\sum_{j=1}^{r}(s_j - 1) + (r-1)$ counts already up to the degree $n-1$ of $p'(x)$. Consequently, the *multiple* roots of $p'(x)$ can only occur among the roots of $p(x)$. □

Fact 2. *We have* $p'(x)^2 \ge p(x)p''(x)$ *for all* $x \in \mathbb{R}$.

■ **Proof.** If $x = a_i$ is a root of $p(x)$, then there is nothing to show. Assume then x is not a root. The product rule of differentiation yields

$$p'(x) = \sum_{k=1}^{n} \frac{p(x)}{x - a_k}, \quad \text{that is,} \quad \frac{p'(x)}{p(x)} = \sum_{k=1}^{n} \frac{1}{x - a_k}.$$

Differentiating this again we have

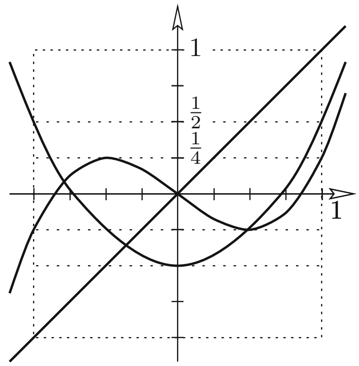
$$\frac{p''(x)p(x) - p'(x)^2}{p(x)^2} = -\sum_{k=1}^{n} \frac{1}{(x - a_k)^2} < 0. \qquad \square$$

Appendix: Chebyshev's theorem

Theorem. *Let $p(x)$ be a real polynomial of degree $n \geq 1$ with leading coefficient* 1. *Then*

$$\max_{-1 \leq x \leq 1} |p(x)| \;\geq\; \frac{1}{2^{n-1}}.$$

Before we start, let us look at some examples where we have equality. The margin depicts the graphs of polynomials of degrees 1, 2 and 3, where we have equality in each case. Indeed, we will see that for every degree there is precisely one polynomial with equality in Chebyshev's theorem.



The polynomials $p_1(x) = x$, $p_2(x) = x^2 - \frac{1}{2}$ and $p_3(x) = x^3 - \frac{3}{4}x$ achieve equality in Chebyshev's theorem.

■ **Proof.** Consider a real polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with leading coefficient 1. Since we are interested in the range $-1 \leq x \leq 1$, we set $x = \cos\vartheta$ and denote by $g(\vartheta) := p(\cos\vartheta)$ the resulting polynomial in $\cos\vartheta$,

$$g(\vartheta) = (\cos\vartheta)^n + a_{n-1}(\cos\vartheta)^{n-1} + \cdots + a_0. \tag{1}$$

The proof proceeds now in the following two steps which are both classical results and interesting in their own right.

(A) We express $g(\vartheta)$ as a so-called *cosine polynomial*, that is, a polynomial of the form

$$g(\vartheta) \;=\; b_n \cos n\vartheta + b_{n-1}\cos(n-1)\vartheta + \cdots + b_1\cos\vartheta + b_0 \tag{2}$$

with $b_k \in \mathbb{R}$, and show that its leading coefficient is $b_n = \frac{1}{2^{n-1}}$.

(B) Given any cosine polynomial $h(\vartheta)$ of order n (meaning that λ_n is the highest nonvanishing coefficient)

$$h(\vartheta) \;=\; \lambda_n \cos n\vartheta + \lambda_{n-1}\cos(n-1)\vartheta + \cdots + \lambda_0, \tag{3}$$

we show $|\lambda_n| \leq \max|h(\vartheta)|$, which when applied to $g(\vartheta)$ will then prove the theorem.

Proof of (A). To pass from (1) to the representation (2), we have to express all powers $(\cos\vartheta)^k$ as cosine polynomials. For example, the addition theorem for the cosine gives

$$\cos 2\vartheta \;=\; \cos^2\vartheta - \sin^2\vartheta \;=\; 2\cos^2\vartheta - 1,$$

so that $\cos^2\vartheta = \frac{1}{2}\cos 2\vartheta + \frac{1}{2}$. To do this for an arbitrary power $(\cos\vartheta)^k$ we go into the complex numbers, via the relation $e^{ix} = \cos x + i\sin x$. The e^{ix} are the complex numbers of absolute value 1 (see the box on complex unit roots on page 37). In particular, this yields

$$e^{in\vartheta} \;=\; \cos n\vartheta + i\sin n\vartheta. \tag{4}$$

On the other hand,

$$e^{in\vartheta} \;=\; (e^{i\vartheta})^n \;=\; (\cos\vartheta + i\sin\vartheta)^n. \tag{5}$$

Equating the real parts in (4) and (5) we obtain by $i^{4\ell+2} = -1$, $i^{4\ell} = 1$ and $\sin^2\theta = 1 - \cos^2\theta$

$$\begin{aligned}\cos n\vartheta &= \sum_{\ell\geq 0}\binom{n}{4\ell}(\cos\vartheta)^{n-4\ell}(1-\cos^2\vartheta)^{2\ell} \\ &\quad -\sum_{\ell\geq 0}\binom{n}{4\ell+2}(\cos\vartheta)^{n-4\ell-2}(1-\cos^2\vartheta)^{2\ell+1}.\end{aligned} \tag{6}$$

We conclude that $\cos n\vartheta$ is a polynomial in $\cos\vartheta$,

$$\cos n\vartheta = c_n(\cos\vartheta)^n + c_{n-1}(\cos\vartheta)^{n-1} + \cdots + c_0. \tag{7}$$

From (6) we obtain for the highest coefficient

$$c_n = \sum_{\ell\geq 0}\binom{n}{4\ell} + \sum_{\ell\geq 0}\binom{n}{4\ell+2} = 2^{n-1}.$$

$\sum_{k\geq 0}\binom{n}{2k} = 2^{n-1}$ holds for $n > 0$: Every subset of $\{1, 2, \ldots, n-1\}$ yields an even sized subset of $\{1, 2, \ldots, n\}$ if we add the element n "if needed."

Now we turn our argument around. Assuming by induction that for $k < n$, $(\cos\vartheta)^k$ can be expressed as a cosine polynomial of order k, we infer from (7) that $(\cos\vartheta)^n$ can be written as a cosine polynomial of order n with leading coefficient $b_n = \frac{1}{2^{n-1}}$.

Proof of (B). Let $h(\vartheta)$ be a cosine polynomial of order n as in (3), and assume without loss of generality $\lambda_n > 0$. Now we set $m(\vartheta) := \lambda_n \cos n\vartheta$ and find

$$m(\tfrac{k}{n}\pi) = (-1)^k\lambda_n \qquad \text{for } k = 0, 1, \ldots, n.$$

Suppose, for a proof by contradiction, that $\max|h(\vartheta)| < \lambda_n$. Then

$$m(\tfrac{k}{n}\pi) - h(\tfrac{k}{n}\pi) = (-1)^k\lambda_n - h(\tfrac{k}{n}\pi)$$

is positive for even k and negative for odd k in the range $0 \leq k \leq n$. We conclude that $m(\vartheta) - h(\vartheta)$ has at least n roots in the interval $[0, \pi]$. But this cannot be since $m(\vartheta) - h(\vartheta)$ is a cosine polynomial of order $n-1$, and thus has at most $n-1$ roots.

The proof of **(B)** and thus of Chebyshev's theorem is complete. $\square$

The energetic reader is now invited to complete the analysis, showing that $g_n(\vartheta) := \frac{1}{2^{n-1}}\cos n\vartheta$ is the *only* cosine polynomial of order n with leading coefficient 1 that achieves the equality $\max|g(\vartheta)| = \frac{1}{2^{n-1}}$.

The polynomials $T_n(x) = \cos n\vartheta$, $x = \cos\vartheta$, are called the *Chebyshev polynomials* (of the first kind); thus $\frac{1}{2^{n-1}}T_n(x)$ is the unique monic polynomial of degree n where equality holds in Chebyshev's theorem.

References

[1] P. L. CEBYCEV: *Œuvres,* Vol. I, Acad. Imperiale des Sciences, St. Petersburg 1899, pp. 387-469.

[2] G. PÓLYA: *Beitrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängenden Gebieten,* Sitzungsber. Preuss. Akad. Wiss. Berlin (1928), 228-232; Collected Papers Vol. I, MIT Press 1974, 347-351.

[3] G. PÓLYA & G. SZEGŐ: *Problems and Theorems in Analysis, Vol. II,* Springer-Verlag, Berlin Heidelberg New York 1976; Reprint 1998.