

Binomial coefficients are (almost) never powers



There is an epilogue to Bertrand's postulate which leads to a beautiful result on binomial coefficients. In 1892 Sylvester strengthened Bertrand's postulate in the following way:

If $n \geq 2k$, then at least one of the numbers $n, n-1, \dots, n-k+1$ has a prime divisor p greater than k .

Note that for $n = 2k$ we obtain precisely Bertrand's postulate. In 1934, Erdős gave a short and elementary Book Proof of Sylvester's result, running along the lines of his proof of Bertrand's postulate. There is an equivalent way of stating Sylvester's theorem:

The binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad (n \geq 2k)$$

always has a prime factor $p > k$.

With this observation in mind, we turn to another one of Erdős' jewels:

When is $\binom{n}{k}$ equal to a power m^ℓ ?

The case $k = \ell = 2$ leads to a classical topic. Multiplying $\binom{n}{2} = m^2$ by 8 and rearranging terms gives $(2n-1)^2 - 2(2m)^2 = 1$, which is a special case of *Pell's equation*, $x^2 - 2y^2 = 1$. One learns in number theory that this equation has infinitely many positive solutions (x_k, y_k) , which are given by $x_k + y_k\sqrt{2} = (3 + 2\sqrt{2})^k$ for $k \geq 1$. The smallest examples are $(x_1, y_1) = (3, 2)$, $(x_2, y_2) = (17, 12)$, and $(x_3, y_3) = (99, 70)$, yielding $\binom{2}{2} = 1^2$, $\binom{9}{2} = 6^2$, and $\binom{50}{2} = 35^2$.

For $k = 2$ and $\ell > 2$ there are no further solutions, and for $k = 3$ it is known that $\binom{n}{3} = m^\ell$ has the unique solution $n = 50$, $m = 140$, $\ell = 2$, see Györy [3]. But now we are at the end of the line. For $k \geq 4$ and any $\ell \geq 2$ no solutions exist, and this is what Erdős proved by an ingenious argument.

Theorem. *The equation $\binom{n}{k} = m^\ell$ has no integer solutions with $\ell \geq 2$ and $4 \leq k \leq n-4$.*

■ **Proof.** Note first that we may assume $n \geq 2k$ because of $\binom{n}{k} = \binom{n}{n-k}$. Suppose the theorem is false, and that $\binom{n}{k} = m^\ell$. The proof, by contradiction, proceeds in the following four steps.

(1) By Sylvester's theorem, there is a prime factor p of $\binom{n}{k}$ greater than k , hence p^ℓ divides $n(n-1)\cdots(n-k+1)$. Clearly, only one of the factors $n-i$ can be a multiple of any such $p > k$, and we conclude $p^\ell \mid n-i$, and therefore

$$n \geq p^\ell > k^\ell \geq k^2.$$

(2) Consider any factor $n-j$ of the numerator and write it in the form $n-j = a_j m_j^\ell$, where a_j is not divisible by any nontrivial ℓ -th power. We note by (1) that a_j has only prime divisors less than or equal to k . We want to show next that $a_i \neq a_j$ for $i \neq j$. Assume to the contrary that $a_i = a_j$ for some $i < j$. Then $m_i \geq m_j + 1$ and

$$\begin{aligned} k &> (n-i) - (n-j) = a_j(m_i^\ell - m_j^\ell) \geq a_j((m_j+1)^\ell - m_j^\ell) \\ &> a_j \ell m_j^{\ell-1} \geq \ell(a_j m_j^\ell)^{1/2} \geq \ell(n-k+1)^{1/2} \\ &\geq \ell\left(\frac{n}{2}+1\right)^{1/2} > n^{1/2}, \end{aligned}$$

which contradicts $n > k^2$ from above.

(3) Next we prove that the a_i 's are the integers $1, 2, \dots, k$ in some order. (According to Erdős, this is the crux of the proof.) Since we already know that they are all distinct, it suffices to prove that

$$a_0 a_1 \cdots a_{k-1} \text{ divides } k!.$$

Substituting $n-j = a_j m_j^\ell$ into the equation $\binom{n}{k} = m^\ell$, we obtain

$$a_0 a_1 \cdots a_{k-1} (m_0 m_1 \cdots m_{k-1})^\ell = k! m^\ell.$$

Cancelling the common factors of $m_0 \cdots m_{k-1}$ and m yields

$$a_0 a_1 \cdots a_{k-1} u^\ell = k! v^\ell$$

with $\gcd(u, v) = 1$. It remains to show that $v = 1$. If not, then v contains a prime divisor p . Since $\gcd(u, v) = 1$, p must be a prime divisor of $a_0 a_1 \cdots a_{k-1}$ and hence is less than or equal to k . By the theorem of Legendre (see page 8) we know that $k!$ contains p to the power $\sum_{i \geq 1} \lfloor \frac{k}{p^i} \rfloor$. We now estimate the exponent of p in $n(n-1)\cdots(n-k+1)$. Let i be a positive integer, and let $b_1 < b_2 < \cdots < b_s$ be the multiples of p^i among $n, n-1, \dots, n-k+1$. Then $b_s = b_1 + (s-1)p^i$ and hence

$$(s-1)p^i = b_s - b_1 \leq n - (n-k+1) = k-1,$$

which implies

$$s \leq \left\lfloor \frac{k-1}{p^i} \right\rfloor + 1 \leq \left\lfloor \frac{k}{p^i} \right\rfloor + 1.$$

So for each i the number of multiples of p^i among $n, \dots, n-k+1$, and hence among the a_j 's, is bounded by $\lfloor \frac{k}{p^i} \rfloor + 1$. This implies that the exponent of p in $a_0 a_1 \cdots a_{k-1}$ is at most

$$\sum_{i=1}^{\ell-1} \left(\left\lfloor \frac{k}{p^i} \right\rfloor + 1 \right)$$

with the reasoning that we used for Legendre's theorem in Chapter 2. The only difference is that this time the sum stops at $i = \ell - 1$, since the a_j 's contain no ℓ -th powers.

Taking both counts together, we find that the exponent of p in v^ℓ is at most

$$\sum_{i=1}^{\ell-1} \left(\left\lfloor \frac{k}{p^i} \right\rfloor + 1 \right) - \sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor \leq \ell - 1,$$

and we have our desired contradiction, since v^ℓ is an ℓ -th power.

This suffices already to settle the case $\ell = 2$. Indeed, since $k \geq 4$ one of the a_i 's must be equal to 4, but the a_i 's contain no squares. So let us now assume that $\ell \geq 3$.

(4) Since $k \geq 4$, we must have $a_{i_1} = 1, a_{i_2} = 2, a_{i_3} = 4$ for some i_1, i_2, i_3 , that is,

$$n - i_1 = m_1^\ell, \quad n - i_2 = 2m_2^\ell, \quad n - i_3 = 4m_3^\ell.$$

We claim that $(n - i_2)^2 \neq (n - i_1)(n - i_3)$. If not, put $b = n - i_2$ and $n - i_1 = b - x, n - i_3 = b + y$, where $0 < |x|, |y| < k$. Hence

$$b^2 = (b - x)(b + y) \quad \text{or} \quad (y - x)b = xy,$$

where $x = y$ is plainly impossible. Now we have by part (1)

$$|xy| = b|y - x| \geq b > n - k > (k - 1)^2 \geq |xy|,$$

which is absurd.

So we have $m_2^2 \neq m_1 m_3$, where we assume $m_2^2 > m_1 m_3$ (the other case being analogous), and proceed to our last chains of inequalities. We obtain

$$\begin{aligned} 2(k-1)n &> n^2 - (n-k+1)^2 > (n-i_2)^2 - (n-i_1)(n-i_3) \\ &= 4[m_2^{2\ell} - (m_1 m_3)^\ell] \geq 4[(m_1 m_3 + 1)^\ell - (m_1 m_3)^\ell] \\ &\geq 4\ell m_1^{\ell-1} m_3^{\ell-1}. \end{aligned}$$

Since $\ell \geq 3$ and $n > k^\ell \geq k^3 > 6k$, this yields

$$\begin{aligned} 2(k-1)nm_1 m_3 &> 4\ell m_1^{\ell-1} m_3^{\ell-1} = \ell(n-i_1)(n-i_3) \\ &> \ell(n-k+1)^2 > 3\left(n - \frac{n}{6}\right)^2 > 2n^2. \end{aligned}$$

We see that our analysis so far agrees

with $\binom{50}{3} = 140^2$, as

$$50 = 2 \cdot 5^2$$

$$49 = 1 \cdot 7^2$$

$$48 = 3 \cdot 4^2$$

and $5 \cdot 7 \cdot 4 = 140$.

Now since $m_i \leq n^{1/\ell} \leq n^{1/3}$ we finally obtain

$$kn^{2/3} \geq km_1m_3 > (k-1)m_1m_3 > n,$$

or $k^3 > n$. With this contradiction, the proof is complete. \square

References

- [1] P. ERDŐS: *A theorem of Sylvester and Schur*, J. London Math. Soc. **9** (1934), 282-288.
- [2] P. ERDŐS: *On a diophantine equation*, J. London Math. Soc. **26** (1951), 176-178.
- [3] K. GYŐRY: *On the diophantine equation $\binom{n}{k} = x^l$* , Acta Arithmetica **80** (1997), 289-295.
- [4] J. J. SYLVESTER: *On arithmetical series*, Messenger of Math. **21** (1892), 1-19, 87-120; Collected Mathematical Papers Vol. 4, 1912, 687-731.