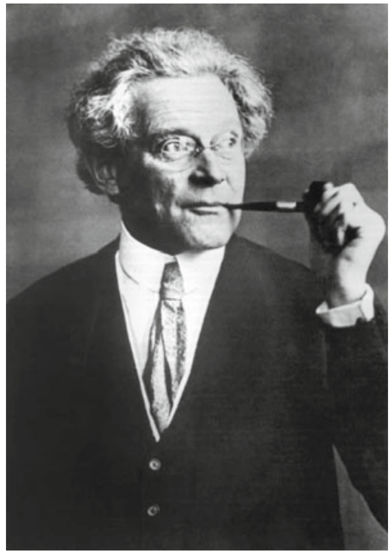
Cotangent and the Herglotz trick

Chapter 26

What is the most interesting formula involving elementary functions? In his beautiful article [2], whose exposition we closely follow, Jürgen Elstrodt nominates as a first candidate the partial fraction expansion of the cotangent function:

$$\pi \cot \pi x \quad = \quad \frac{1}{x} + \sum_{n=1}^{\infty} \Big(\frac{1}{x+n} + \frac{1}{x-n}\Big) \qquad (x \in \mathbb{R}\backslash\mathbb{Z}).$$



Gustav Herglotz

This elegant formula was proved by Euler in §178 of his *Introductio in Analysin Infinitorum* from 1748 and it certainly counts among his finest achievements. We can also write it even more elegantly as

$$\pi \cot \pi x \quad = \quad \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x+n} \tag{1}$$

but one has to note that the evaluation of the sum $\sum_{n\in\mathbb{Z}} \frac{1}{x+n}$ is a bit dangerous, since the sum is only conditionally convergent, so its value depends on the "right" order of summation.

We shall derive (1) by an argument of stunning simplicity which is attributed to Gustav Herglotz — the "Herglotz trick." To get started, set

$$f(x) \quad := \quad \pi \cot \pi x, \qquad g(x) \quad := \quad \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x+n},$$

and let us try to derive enough common properties of these functions to see in the end that they must coincide ...

(A) The functions f and g are defined for all non-integral values and are continuous there.

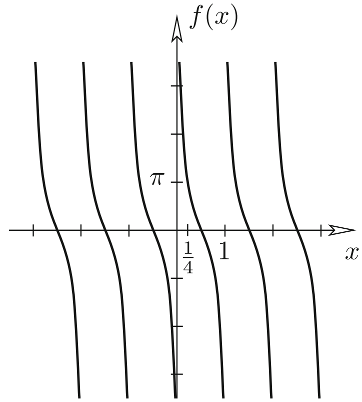
For the cotangent function $f(x) = \pi \cot \pi x = \pi \frac{\cos \pi x}{\sin \pi x}$, this is clear (see the figure). For $g(x)$, we first use the identity $\frac{1}{x+n} + \frac{1}{x-n} = -\frac{2x}{n^2-x^2}$ to rewrite Euler's formula as

$$\pi \cot \pi x \quad = \quad \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}\,. \tag{2}$$

Thus for **(A)** we have to prove that for every $x \notin \mathbb{Z}$ the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$$

converges uniformly in a neighborhood of x.



The function $f(x) = \pi \cot \pi x$

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_26

For this, we don't get any problem with the first term, for $n = 1$, or with the terms with $2n-1 \leq x^2$, since there is only a finite number of them. On the other hand, for $n \geq 2$ and $2n-1 > x^2$, that is $n^2 - x^2 > (n-1)^2 > 0$, the summands are bounded by

$$0 \;<\; \frac{1}{n^2 - x^2} \;<\; \frac{1}{(n-1)^2},$$

and this bound is not only true for x itself, but also for values in a neighborhood of x. Finally the fact that $\sum \frac{1}{(n-1)^2}$ converges (to $\frac{\pi^2}{6}$, see page 55) provides the uniform convergence needed for the proof of **(A)**.

(B) Both f and g are *periodic* of period 1, that is, $f(x+1) = f(x)$ and $g(x+1) = g(x)$ hold for all $x \in \mathbb{R}\backslash\mathbb{Z}$.

Since the cotangent has period π, we find that f has period 1 (see again the figure above). For g we argue as follows. Let

$$g_N(x) \;:=\; \sum_{n=-N}^{N} \frac{1}{x+n},$$

then

$$\begin{aligned} g_N(x+1) \;&=\; \sum_{n=-N}^{N} \frac{1}{x+1+n} \;=\; \sum_{n=-N+1}^{N+1} \frac{1}{x+n} \\ &=\; g_{N-1}(x) + \frac{1}{x+N} + \frac{1}{x+N+1}. \end{aligned}$$

Hence $g(x+1) = \lim_{N\to\infty} g_N(x+1) = \lim_{N\to\infty} g_{N-1}(x) = g(x)$.

(C) Both f and g are *odd* functions, that is, we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}\backslash\mathbb{Z}$.

The function f obviously has this property, and for g we just have to observe that $g_N(-x) = -g_N(x)$.

The final two facts constitute the Herglotz trick: First we show that f and g satisfy the same functional equation, and secondly that $h := f - g$ can be continuously extended to all of $\mathbb{R}$.

(D) The two functions f and g satisfy the same functional equation: $f(\frac{x}{2}) + f(\frac{x+1}{2}) = 2f(x)$ and $g(\frac{x}{2}) + g(\frac{x+1}{2}) = 2g(x)$.

Addition theorems:
$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$$\begin{aligned} \Longrightarrow\quad \sin(x + \tfrac{\pi}{2}) &= \cos x \\ \cos(x + \tfrac{\pi}{2}) &= -\sin x \\ \sin x &= 2 \sin\tfrac{x}{2} \cos\tfrac{x}{2} \\ \cos x &= \cos^2\tfrac{x}{2} - \sin^2\tfrac{x}{2}. \end{aligned}$$

For $f(x)$ this results from the addition theorems for the sine and cosine functions:

$$\begin{aligned} f(\tfrac{x}{2}) + f(\tfrac{x+1}{2}) \;&=\; \pi\left[\frac{\cos\frac{\pi x}{2}}{\sin\frac{\pi x}{2}} - \frac{\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}}\right] \\ &=\; 2\pi\frac{\cos(\frac{\pi x}{2} + \frac{\pi x}{2})}{\sin(\frac{\pi x}{2} + \frac{\pi x}{2})} \;=\; 2\,f(x). \end{aligned}$$

The functional equation for g follows from

$$g_N\left(\tfrac{x}{2}\right) + g_N\left(\tfrac{x+1}{2}\right) = 2\,g_{2N}(x) + \frac{2}{x+2N+1}.$$

which in turn follows from

$$\frac{1}{\frac{x}{2}+n} + \frac{1}{\frac{x+1}{2}+n} = 2\Big(\frac{1}{x+2n} + \frac{1}{x+2n+1}\Big).$$

Now let us look at

$$h(x) = f(x) - g(x) = \pi\cot\pi x - \Big(\frac{1}{x} - \sum_{n=1}^{\infty}\frac{2x}{n^2-x^2}\Big). \qquad (3)$$

We know by now that h is a continuous function on $\mathbb{R}\backslash\mathbb{Z}$ that satisfies the properties **(B)**, **(C)**, **(D)**. What happens at the integral values? From the sine and cosine series expansions, or by applying de l'Hospital's rule twice, we find

$$\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x} = 0,$$

and hence also

$$\lim_{x\to 0}\left(\pi\cot\pi x - \frac{1}{x}\right) = 0.$$

But since the last sum $\sum_{n=1}^{\infty}\frac{2x}{n^2-x^2}$ in (3) converges to 0 with $x \longrightarrow 0$, we have in fact $\lim_{x\to 0} h(x) = 0$, and thus by periodicity

$$\lim_{x\to n} h(x) = 0 \qquad \text{for all } n\in\mathbb{Z}.$$

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} \pm \cdots$$

$$\sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} \pm \cdots$$

In summary, we have shown the following:

(E) By setting $h(x) := 0$ for $x\in\mathbb{Z}$, h becomes a continuous function on all of $\mathbb{R}$ that shares the properties given in **(B)**, **(C)** and **(D)**.

We are ready for the *coup de grâce*. Since h is a periodic continuous function, it possesses a maximum m. Let x_0 be a point in $[0,1]$ with $h(x_0) = m$. It follows from **(D)** that

$$h\left(\tfrac{x_0}{2}\right) + h\left(\tfrac{x_0+1}{2}\right) = 2m,$$

and hence that $h(\frac{x_0}{2}) = m$. Iteration gives $h(\frac{x_0}{2^n}) = m$ for all n, and hence $h(0) = m$ by continuity. But $h(0) = 0$, and so $m = 0$, that is, $h(x)\le 0$ for all $x\in\mathbb{R}$. As $h(x)$ is an *odd* function, $h(x) < 0$ is impossible, hence $h(x) = 0$ for all $x\in\mathbb{R}$, and Euler's theorem is proved. □

A great many corollaries can be derived from (1), the most famous of which concerns the values of Riemann's zeta function at even positive integers (see the appendix to Chapter 9),

$$\zeta(2k) = \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \qquad (k\in\mathbb{N}). \qquad (4)$$

So to finish our story let us see how Euler — a few years later, in 1755 — treated the series (4). We start with formula (2). Multiplying (2) by x and setting $y = \pi x$ we find for $|y| < \pi$:

$$
\begin{aligned}
y \cot y &= 1 - 2 \sum_{n=1}^{\infty} \frac{y^2}{\pi^2 n^2 - y^2} \\
&= 1 - 2 \sum_{n=1}^{\infty} \frac{y^2}{\pi^2 n^2} \frac{1}{1 - \left(\frac{y}{\pi n}\right)^2}.
\end{aligned}
$$

The last factor is the sum of a geometric series, hence

$$
\begin{aligned}
y \cot y &= 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{y}{\pi n}\right)^{2k} \\
&= 1 - 2 \sum_{k=1}^{\infty} \left(\frac{1}{\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right) y^{2k},
\end{aligned}
$$

and we have proved the remarkable result:

For all $k \in \mathbb{N}$, the coefficient of y^{2k} in the power series expansion of $y \cot y$ equals

$$
\left[y^{2k}\right] y \cot y \;=\; -\frac{2}{\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \;=\; -\frac{2}{\pi^{2k}} \zeta(2k). \tag{5}
$$

There is another, perhaps much more "canonical," way to obtain a series expansion of $y \cot y$. We know from analysis that $e^{iy} = \cos y + i \sin y$, and thus

$$
\cos y = \frac{e^{iy} + e^{-iy}}{2}, \qquad \sin y = \frac{e^{iy} - e^{-iy}}{2i},
$$

which yields

$$
y \cot y = iy \frac{e^{iy} + e^{-iy}}{e^{iy} - e^{-iy}} = iy \frac{e^{2iy} + 1}{e^{2iy} - 1}.
$$

We now substitute $z = 2iy$, and get

$$
y \cot y = \frac{z}{2} \frac{e^z + 1}{e^z - 1} = \frac{z}{2} + \frac{z}{e^z - 1}. \tag{6}
$$

Thus all we need is a power series expansion of the function $\frac{z}{e^z - 1}$; note that this function is defined and continuous on all of $\mathbb{R}$ (for $z = 0$ use the power series of the exponential function, or alternatively de l'Hospital's rule, which yields the value 1). We write

$$
\frac{z}{e^z - 1} =: \sum_{n \geq 0} B_n \frac{z^n}{n!}. \tag{7}
$$

The coefficients B_n are known as the *Bernoulli numbers*. The left-hand side of (6) is an *even* function (that is, $f(z) = f(-z)$), and thus we see that $B_n = 0$ for odd $n \geq 3$, while $B_1 = -\frac{1}{2}$ corresponds to the term of $\frac{z}{2}$ in (6).

From

$$\Big(\sum_{n\geq 0} B_n \frac{z^n}{n!}\Big)(e^z-1) \;=\; \Big(\sum_{n\geq 0} B_n \frac{z^n}{n!}\Big)\Big(\sum_{n\geq 1} \frac{z^n}{n!}\Big) \;=\; z$$

we obtain by comparing coefficients for z^n:

$$\sum_{k=0}^{n-1} \frac{B_k}{k!(n-k)!} \;=\; \begin{cases} 1 & \text{for } n=1, \\ 0 & \text{for } n\neq 1. \end{cases} \tag{8}$$

We may compute the Bernoulli numbers recursively from (8). The value $n=1$ gives $B_0=1$, $n=2$ yields $\frac{B_0}{2}+B_1=0$, that is $B_1=-\frac{1}{2}$, and so on.

n	0	1	2	3	4	5	6	7	8
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$

The first few Bernoulli numbers

Now we are almost done: The combination of (6) and (7) yields

$$y\cot y \;=\; \sum_{k=0}^{\infty} B_{2k}\frac{(2iy)^{2k}}{(2k)!} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} y^{2k},$$

and out comes, with (5), Euler's formula for $\zeta(2k)$:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \;=\; \frac{(-1)^{k-1}2^{2k-1}B_{2k}}{(2k)!}\,\pi^{2k} \qquad (k\in\mathbb{N}). \tag{9}$$

Looking at our table of the Bernoulli numbers, we thus obtain once again the sum $\sum \frac{1}{n^2}=\frac{\pi^2}{6}$ from Chapter 9, and further

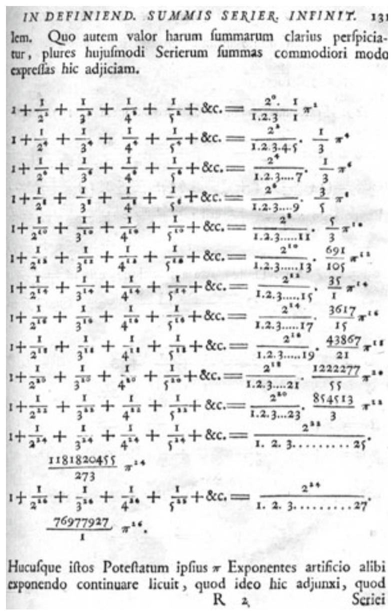
$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9450},$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{10}}=\frac{\pi^{10}}{93555}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^{12}}=\frac{691\,\pi^{12}}{638512875}, \qquad \dots$$

The Bernoulli number $B_{10}=\frac{5}{66}$ that gets us $\zeta(10)$ looks innocuous enough, but the next value $B_{12}=-\frac{691}{2730}$, needed for $\zeta(12)$, contains the large prime factor 691 in the numerator. Euler had first computed some values $\zeta(2k)$ without noticing the connection to the Bernoulli numbers. Only the appearance of the strange prime 691 put him on the right track.

Incidentally, since $\zeta(2k)$ converges to 1 for $k\longrightarrow\infty$, equation (9) tells us that the numbers $|B_{2k}|$ grow very fast — something that is not clear from the first few values.

In contrast to all this, one knows very little about the values of the Riemann zeta function at the odd integers $k\geq 3$; see page 64.



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lem. Quo autem valor harum ſummarum clarius perſpiciatur, plures hujuſmodi Serierum ſummas commodiori modo expreſſas hic adjiciam.

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c.=\frac{2^0}{1.2.3}\cdot\frac{1}{1}\pi^2$

$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.=\frac{2^2}{1.2.3.4.5}\cdot\frac{1}{3}\pi^4$

$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\&c.=\frac{2^4}{1.2.3\ldots7}\cdot\frac{1}{3}\pi^6$

$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\&c.=\frac{2^6}{1.2.3\ldots9}\cdot\frac{3}{5}\pi^8$

$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\&c.=\frac{2^8}{1.2.3\ldots11}\cdot\frac{5}{3}\pi^{10}$

$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\&c.=\frac{2^{10}}{1.2.3\ldots13}\cdot\frac{691}{105}\pi^{12}$

$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\&c.=\frac{2^{12}}{1.2.3\ldots15}\cdot\frac{35}{1}\pi^{14}$

$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\&c.=\frac{2^{14}}{1.2.3\ldots17}\cdot\frac{3617}{15}\pi^{16}$

$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\&c.=\frac{2^{16}}{1.2.3\ldots19}\cdot\frac{43867}{21}\pi^{18}$

$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\&c.=\frac{2^{18}}{1.2.3\ldots21}\cdot\frac{1222277}{55}\pi^{20}$

$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\&c.=\frac{2^{20}}{1.2.3\ldots23}\cdot\frac{854513}{3}\pi^{22}$

$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\&c.=\frac{2^{22}}{1.\,2.\,3\ldots\ldots\ldots25}\cdot\frac{1181820455}{273}\pi^{24}$

$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\&c.=\frac{2^{24}}{1.\,2.\,3\ldots\ldots\ldots27}\cdot\frac{76977927}{1}\pi^{26}.$

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