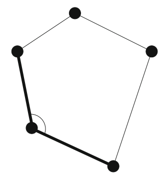
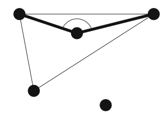
Every large point set has an obtuse angle

Chapter 17

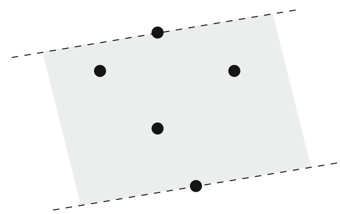
Around 1950 Paul Erdős conjectured that every set of more than 2^d points in $\mathbb{R}^d$ determines at least one *obtuse angle*, that is, an angle that is strictly greater than $\frac{\pi}{2}$. In other words, any set of points in $\mathbb{R}^d$ which only has acute angles (including right angles) has size at most 2^d. This problem was posed as a "prize question" by the Dutch Mathematical Society — but solutions were received only for $d = 2$ and for $d = 3$.

For $d = 2$ the problem is easy: The five points may determine a convex pentagon, which always has an obtuse angle (in fact, at least one angle of at least $108°$). Otherwise we have one point contained in the convex hull of three others that form a triangle. But this point "sees" the three edges of the triangle in three angles that sum to $360°$, so one of the angles is at least $120°$. (The second case also includes situations where we have three points on a line, and thus a $180°$ angle.)





Unrelated to this, Victor Klee asked a few years later — and Erdős spread the question — how large a point set in $\mathbb{R}^d$ could be and still have the following "antipodality property": For *any* two points in the set there is a strip (bounded by two parallel hyperplanes) that contains the point set, and that has the two chosen points on different sides on the boundary.



Then, in 1962, Ludwig Danzer and Branko Grünbaum solved both problems in one stroke: They sandwiched both maximal sizes into a chain of inequalities, which starts and ends in 2^d. Thus the answer is 2^d both for Erdős' and for Klee's problem.

In the following, we consider (finite) sets $S \subseteq \mathbb{R}^d$ of points, their convex hulls $\mathrm{conv}(S)$, and general convex polytopes $Q \subseteq \mathbb{R}^d$. (See the appendix on polytopes on page 73 for the basic concepts.) We assume that these sets have the full dimension d, that is, they are not contained in a hyperplane. Two convex sets *touch* if they have at least one boundary point in common, while their interiors do not intersect. For any set $Q \subseteq \mathbb{R}^d$ and any vector $\boldsymbol{s} \in \mathbb{R}^d$ we denote by $Q + \boldsymbol{s}$ the image of Q under the translation that moves $\mathbf{0}$ to $\boldsymbol{s}$. Similarly, $Q - \boldsymbol{s}$ is the translate obtained by the map that moves $\boldsymbol{s}$ to the origin.

Don't be intimidated: This chapter is an excursion into d-dimensional geometry, but the arguments in the following do not require any "high-dimensional intuition," since they all can be followed, visualized (and thus *understood*) in three dimensions, or even in the plane. Hence, our figures will illustrate the proof for $d = 2$ (where a "hyperplane" is just a line), and you could create your own pictures for $d = 3$ (where a "hyperplane" is a plane).

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_17

Theorem 1. *For every d, one has the following chain of inequalities:*

$$2^d \overset{(1)}{\leq} \max\left\{\#S \,\middle|\, S\subseteq\mathbb{R}^d,\ \sphericalangle(\boldsymbol{s}_i,\boldsymbol{s}_j,\boldsymbol{s}_k)\leq\tfrac{\pi}{2}\ \textit{for every}\ \{\boldsymbol{s}_i,\boldsymbol{s}_j,\boldsymbol{s}_k\}\subseteq S\right\}$$

$$\overset{(2)}{\leq} \max\left\{\#S \,\middle|\, \begin{array}{l} S\subseteq\mathbb{R}^d \textit{ such that for any two points } \{\boldsymbol{s}_i,\boldsymbol{s}_j\}\subseteq S\\ \textit{there is a strip } \mathcal{S}(i,j) \textit{ that contains } S\textit{, with } \boldsymbol{s}_i \textit{ and } \boldsymbol{s}_j\\ \textit{lying in the parallel boundary hyperplanes of } \mathcal{S}(i,j)\end{array}\right\}$$

$$\overset{(3)}{=} \max\left\{\#S \,\middle|\, \begin{array}{l} S\subseteq\mathbb{R}^d \textit{ such that the translates } P-\boldsymbol{s}_i,\ \boldsymbol{s}_i\in S\textit{, of}\\ \textit{the convex hull } P:=\operatorname{conv}(S) \textit{ intersect in a common}\\ \textit{point, but they only touch}\end{array}\right\}$$

$$\overset{(4)}{\leq} \max\left\{\#S \,\middle|\, \begin{array}{l} S\subseteq\mathbb{R}^d \textit{ such that the translates } Q+\boldsymbol{s}_i \textit{ of some } d\textit{-}\\ \textit{dimensional convex polytope } Q\subseteq\mathbb{R}^d \textit{ touch pairwise}\end{array}\right\}$$

$$\overset{(5)}{=} \max\left\{\#S \,\middle|\, \begin{array}{l} S\subseteq\mathbb{R}^d \textit{ such that the translates } Q^*+\boldsymbol{s}_i \textit{ of some}\\ d\textit{-dimensional centrally symmetric convex polytope}\\ Q^*\subseteq\mathbb{R}^d \textit{ touch pairwise}\end{array}\right\}$$

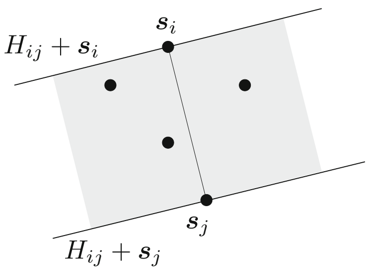
$$\overset{(6)}{\leq} 2^d.$$

■ **Proof.** We have six claims (equalities and inequalities) to verify. Let's get going.

(1) Take $S := \{0,1\}^d$ to be the vertex set of the standard unit cube in $\mathbb{R}^d$, and choose $\boldsymbol{s}_i,\boldsymbol{s}_j,\boldsymbol{s}_k\in S$. By symmetry we may assume that $\boldsymbol{s}_j=\boldsymbol{0}$ is the zero vector. Hence the angle can be computed from

$$\cos\sphericalangle(\boldsymbol{s}_i,\boldsymbol{s}_j,\boldsymbol{s}_k)=\frac{\langle\boldsymbol{s}_i,\boldsymbol{s}_k\rangle}{|\boldsymbol{s}_i||\boldsymbol{s}_k|}$$

which is clearly nonnegative. Thus S is a set with $|S|=2^d$ that has no obtuse angles.

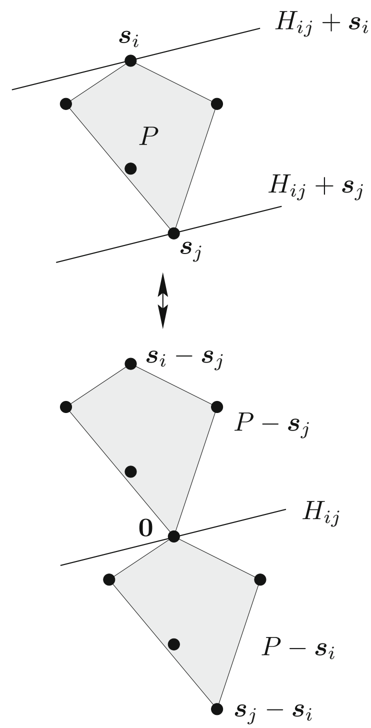


(2) If S contains no obtuse angles, then for any $\boldsymbol{s}_i,\boldsymbol{s}_j\in S$ we may define $H_{ij}+\boldsymbol{s}_i$ and $H_{ij}+\boldsymbol{s}_j$ to be the parallel hyperplanes through $\boldsymbol{s}_i$ resp. $\boldsymbol{s}_j$ that are orthogonal to the edge $[\boldsymbol{s}_i,\boldsymbol{s}_j]$. Here $H_{ij}=\{\boldsymbol{x}\in\mathbb{R}^d : \langle\boldsymbol{x},\boldsymbol{s}_i-\boldsymbol{s}_j\rangle=0\}$ is the hyperplane *through the origin* that is orthogonal to the line through $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$, and $H_{ij}+\boldsymbol{s}_j=\{\boldsymbol{x}+\boldsymbol{s}_j : \boldsymbol{x}\in H_{ij}\}$ is the translate of H_{ij} that passes through $\boldsymbol{s}_j$, etc. Hence the strip between $H_{ij}+\boldsymbol{s}_i$ and $H_{ij}+\boldsymbol{s}_j$ consists, besides $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$, exactly of all the points $\boldsymbol{x}\in\mathbb{R}^d$ such that the angles $\sphericalangle(\boldsymbol{s}_i,\boldsymbol{s}_j,\boldsymbol{x})$ and $\sphericalangle(\boldsymbol{s}_j,\boldsymbol{s}_i,\boldsymbol{x})$ are nonobtuse. Thus the strip contains all of S.

(3) P is contained in the halfspace of $H_{ij}+\boldsymbol{s}_j$ that contains $\boldsymbol{s}_i$ if and only if $P-\boldsymbol{s}_j$ is contained in the halfspace of H_{ij} that contains $\boldsymbol{s}_i-\boldsymbol{s}_j$: A property "an object is contained in a halfspace" is not destroyed if we translate both the object and the halfspace by the same amount (namely by $-\boldsymbol{s}_j$). Similarly, P is contained in the halfspace of $H_{ij}+\boldsymbol{s}_i$ that contains $\boldsymbol{s}_j$ if and only if $P-\boldsymbol{s}_i$ is contained in the halfspace of H_{ij} that contains $\boldsymbol{s}_j-\boldsymbol{s}_i$.

Putting both statements together, we find that the polytope P is contained in the strip between $H_{ij}+\boldsymbol{s}_i$ and $H_{ij}+\boldsymbol{s}_j$ if and only if $P-\boldsymbol{s}_i$ and $P-\boldsymbol{s}_j$ lie in different halfspaces with respect to the hyperplane H_{ij}.

This correspondence is illustrated by the sketch in the margin.



Furthermore, from $\boldsymbol{s}_i \in P = \text{conv}(S)$ we get that the origin $\mathbf{0}$ is contained in all the translates $P - \boldsymbol{s}_i$ ($\boldsymbol{s}_i \in S$). Thus we see that the sets $P - \boldsymbol{s}_i$ all intersect in $\mathbf{0}$, but they only touch: their interiors are pairwise disjoint, since they lie on opposite sides of the corresponding hyperplanes H_{ij}.

(4) This we get for free: "the translates must touch pairwise" is a weaker condition than "they intersect in a common point, but only touch." Similarly, we can relax the conditions by letting P be an arbitrary convex d-polytope in $\mathbb{R}^d$. Furthermore, we may replace S by $-S$.

(5) Here "$\geq$" is trivial, but that is not the interesting direction for us. We have to start with a configuration $S \subseteq \mathbb{R}^d$ and an arbitrary d-polytope $Q \subseteq \mathbb{R}^d$ such that the translates $Q + \boldsymbol{s}_i$ ($\boldsymbol{s}_i \in S$) touch pairwise. The claim is that in this situation we can use

$$Q^* := \left\{ \tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{y}) \in \mathbb{R}^d : \boldsymbol{x}, \boldsymbol{y} \in Q \right\}$$

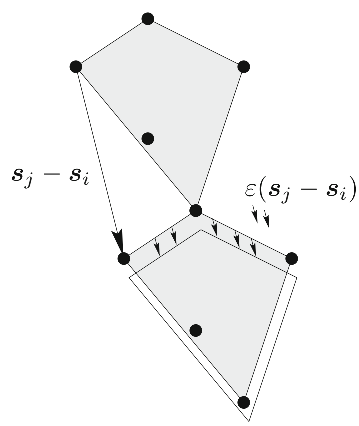
instead of Q. But this is not hard to see: First, Q^* is d-dimensional, convex, and centrally symmetric. One can check that Q^* is a polytope (its vertices are of the form $\frac{1}{2}(\boldsymbol{q}_i - \boldsymbol{q}_j)$, for vertices $\boldsymbol{q}_i, \boldsymbol{q}_j$ of Q), but this is not important for us.

Now we will show that $Q + \boldsymbol{s}_i$ and $Q + \boldsymbol{s}_j$ touch *if and only if* $Q^* + \boldsymbol{s}_i$ and $Q^* + \boldsymbol{s}_j$ touch. For this we note, in the footsteps of Minkowski, that

$$\begin{aligned}
&(Q^* + \boldsymbol{s}_i) \cap (Q^* + \boldsymbol{s}_j) \neq \varnothing \\
&\quad\Longleftrightarrow\ \exists\, \boldsymbol{q}_i', \boldsymbol{q}_i'', \boldsymbol{q}_j', \boldsymbol{q}_j'' \in Q : \tfrac{1}{2}(\boldsymbol{q}_i' - \boldsymbol{q}_i'') + \boldsymbol{s}_i = \tfrac{1}{2}(\boldsymbol{q}_j' - \boldsymbol{q}_j'') + \boldsymbol{s}_j \\
&\quad\Longleftrightarrow\ \exists\, \boldsymbol{q}_i', \boldsymbol{q}_i'', \boldsymbol{q}_j', \boldsymbol{q}_j'' \in Q : \tfrac{1}{2}(\boldsymbol{q}_i' + \boldsymbol{q}_j'') + \boldsymbol{s}_i = \tfrac{1}{2}(\boldsymbol{q}_j' + \boldsymbol{q}_i'') + \boldsymbol{s}_j \\
&\quad\Longleftrightarrow\ \exists\, \boldsymbol{q}_i, \boldsymbol{q}_j \in Q : \boldsymbol{q}_i + \boldsymbol{s}_i = \boldsymbol{q}_j + \boldsymbol{s}_j \\
&\quad\Longleftrightarrow\ (Q + \boldsymbol{s}_i) \cap (Q + \boldsymbol{s}_j) \neq \varnothing,
\end{aligned}$$

where in the third (and crucial) equivalence "$\Longleftrightarrow$" we use that every $\boldsymbol{q} \in Q$ can be written as $\boldsymbol{q} = \frac{1}{2}(\boldsymbol{q} + \boldsymbol{q})$ to get "$\Leftarrow$", and that Q is convex and thus $\frac{1}{2}(\boldsymbol{q}_i' + \boldsymbol{q}_j''), \frac{1}{2}(\boldsymbol{q}_j' + \boldsymbol{q}_i'') \in Q$ to see "$\Rightarrow$".

Thus the passage from Q to Q^* (known as *Minkowski symmetrization*) preserves the property that two translates $Q + \boldsymbol{s}_i$ and $Q + \boldsymbol{s}_j$ intersect. That is, we have shown that for any convex set Q, two translates $Q + \boldsymbol{s}_i$ and $Q + \boldsymbol{s}_j$ intersect if and only if the translates $Q^* + \boldsymbol{s}_i$ and $Q^* + \boldsymbol{s}_j$ intersect.



The following characterization shows that Minkowski symmetrization also preserves the property that two translates touch:

> *$Q + \boldsymbol{s}_i$ and $Q + \boldsymbol{s}_j$ touch if and only if they intersect, while $Q + \boldsymbol{s}_i$ and $Q + \boldsymbol{s}_j + \varepsilon(\boldsymbol{s}_j - \boldsymbol{s}_i)$ do not intersect for any $\varepsilon > 0$.*

(6) Assume that $Q^* + \boldsymbol{s}_i$ and $Q^* + \boldsymbol{s}_j$ touch. For every intersection point

$$\boldsymbol{x} \in (Q^* + \boldsymbol{s}_i) \cap (Q^* + \boldsymbol{s}_j)$$

we have

$$\boldsymbol{x} - \boldsymbol{s}_i \in Q^* \quad \text{and} \quad \boldsymbol{x} - \boldsymbol{s}_j \in Q^*,$$

thus, since Q^* is centrally symmetric,

$$\boldsymbol{s}_i - \boldsymbol{x} = -(\boldsymbol{x} - \boldsymbol{s}_i) \in Q^*,$$

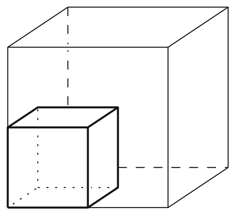
and hence, since Q^* is convex,

$$\tfrac{1}{2}(\boldsymbol{s}_i - \boldsymbol{s}_j) = \tfrac{1}{2}\left((\boldsymbol{x} - \boldsymbol{s}_j) + (\boldsymbol{s}_i - \boldsymbol{x})\right) \in Q^*.$$

We conclude that $\frac{1}{2}(\boldsymbol{s}_i + \boldsymbol{s}_j)$ is contained in $Q^* + \boldsymbol{s}_j$ for all i. Consequently, for $P := \text{conv}(S)$ we get

$$P_j := \tfrac{1}{2}(P + \boldsymbol{s}_j) = \text{conv}\left\{\tfrac{1}{2}(\boldsymbol{s}_i + \boldsymbol{s}_j) : \boldsymbol{s}_i \in S\right\} \subseteq Q^* + \boldsymbol{s}_j,$$

which implies that the sets $P_j = \frac{1}{2}(P + \boldsymbol{s}_j)$ can only touch.



Scaling factor $\frac{1}{2}$, $\text{vol}(P_j) = \frac{1}{8}\text{vol}(P)$

Finally, the sets P_j are contained in P, because all the points $\boldsymbol{s}_i$, $\boldsymbol{s}_j$ and $\frac{1}{2}(\boldsymbol{s}_i + \boldsymbol{s}_j)$ are in P, since P is convex. But the P_j are just smaller, scaled, translates of P, contained in P. The scaling factor is $\frac{1}{2}$, which implies that

$$\text{vol}(P_j) \;=\; \frac{1}{2^d}\text{vol}(P),$$

since we are dealing with d-dimensional sets. This means that at most 2^d sets P_j fit into P, and hence $|S| \leq 2^d$.

This completes our proof: the chain of inequalities is closed. □

. . . but that's not the end of the story. Danzer and Grünbaum asked the following natural question:

> *What happens if one requires all angles to be* ***acute*** *rather than just nonobtuse, that is, if right angles are forbidden?*

They constructed configurations of $2d - 1$ points in $\mathbb{R}^d$ with only acute angles, conjecturing that this may be best possible. Grünbaum proved that this is indeed true for $d \leq 3$. But twenty-one years later, in 1983, Paul Erdős and Zoltan Füredi showed that the conjecture is false — quite dramatically, if the dimension is high! Their proof is a great example for the power of probabilistic arguments; see Chapter 45 for an introduction to the "probabilistic method." Our version of the proof uses a slight improvement in the choice of the parameters due to our reader David Bevan.

Theorem 2. *For every $d \geq 2$, there is a set $S \subseteq \{0,1\}^d$ of $2\lfloor \frac{\sqrt{6}}{9}\big(\frac{2}{\sqrt{3}}\big)^d \rfloor$ points in $\mathbb{R}^d$ (vertices of the unit d-cube) that determine only acute angles. In particular, in dimension $d = 34$ there is a set of $72 > 2{\cdot}34 - 1$ points with only acute angles.*

■ **Proof.** Set $m := \lfloor \frac{\sqrt{6}}{9}\big(\frac{2}{\sqrt{3}}\big)^d \rfloor$, and pick $3m$ vectors

$$\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(3m) \;\in\; \{0,1\}^d$$

by choosing all their coordinates independently and randomly, to be either 0 or 1, with probability $\frac{1}{2}$ for each alternative. (You may toss a perfect coin $3md$ times for this; however, if d is large you may get bored by this soon.)

We have seen above that all angles determined by $0/1$-vectors are nonobtuse. Three vectors $\boldsymbol{x}(i), \boldsymbol{x}(j), \boldsymbol{x}(k)$ determine a right angle with apex $\boldsymbol{x}(j)$ if and only if the scalar product $\langle \boldsymbol{x}(i) - \boldsymbol{x}(j), \boldsymbol{x}(k) - \boldsymbol{x}(j) \rangle$ vanishes, that is, if we have

$$x(i)_\ell - x(j)_\ell = 0 \quad \text{or} \quad x(k)_\ell - x(j)_\ell = 0 \quad \text{for each coordinate } \ell.$$

We call (i, j, k) a *bad triple* if this happens. (If $\boldsymbol{x}(i) = \boldsymbol{x}(j)$ or $\boldsymbol{x}(j) = \boldsymbol{x}(k)$, then the angle is not defined, but also then the triple (i, j, k) is certainly bad.)

The probability that one specific triple is bad is exactly $\left(\frac{3}{4}\right)^d$: Indeed, it will be good if and only if, for one of the d coordinates ℓ, we get

$$\begin{array}{lll} \text{either} & x(i)_\ell = x(k)_\ell = 0, & x(j)_\ell = 1, \\ \text{or} & x(i)_\ell = x(k)_\ell = 1, & x(j)_\ell = 0. \end{array}$$

This leaves us with six bad options out of eight equally likely ones, and a triple will be bad if and only if one of the bad options (with probability $\frac{3}{4}$) happens for each of the d coordinates.

The number of triples we have to consider is $3\binom{3m}{3}$, since there are $\binom{3m}{3}$ sets of three vectors, and for each of them there are three choices for the apex. Of course the probabilities that the various triples are bad are not independent: but *linearity of expectation* (which is what you get by averaging over all possible selections; see the appendix) yields that the *expected* number of bad triples is exactly $3\binom{3m}{3}\left(\frac{3}{4}\right)^d$. This means — and this is the point where the probabilistic method shows its power — that there is *some* choice of the $3m$ vectors such that there are at most $3\binom{3m}{3}\left(\frac{3}{4}\right)^d$ bad triples, where

$$3\binom{3m}{3}\left(\tfrac{3}{4}\right)^d \;<\; 3\frac{(3m)^3}{6}\left(\tfrac{3}{4}\right)^d \;=\; m^3\left(\tfrac{9}{\sqrt{6}}\right)^2\left(\tfrac{3}{4}\right)^d \;\le\; m,$$

by the choice of m.

But if there are not more than m bad triples, then we can remove m of the $3m$ vectors $\boldsymbol{x}(i)$ in such a way that the remaining $2m$ vectors don't contain a bad triple, that is, they determine acute angles only. $\square$

The "probabilistic construction" of a large set of $0/1$-points without right angles can be easily implemented. David Bevan has thus constructed a set of 31 $0/1$-points in dimension $d = 15$ that determines only acute angles.

Very recently, Balázs Gerencsér and Viktor Harangi, building on ideas of an anonymous Ukrainian enthusiast, managed to construct such "acute-angled sets" of size $2^{d-1} + 1$ for all dimensions d, which however do not anymore consist of $0/1$ vectors. As we have seen above, the size $2^{d-1} + 1$ is optimal up to a factor of 2.

Appendix: Three tools from probability

Here we gather three basic tools from discrete probability theory which will come up several times: random variables, linearity of expectation and Markov's inequality.

Let (Ω, p) be a finite *probability space*, that is, Ω is a finite set and $p = \text{Prob}$ is a map from Ω into the interval $[0, 1]$ with $\sum_{\omega \in \Omega} p(\omega) = 1$. A *random variable* X on Ω is a mapping $X : \Omega \longrightarrow \mathbb{R}$. We define a probability space on the image set $X(\Omega)$ by setting $p(X = x) := \sum_{X(\omega)=x} p(\omega)$. A simple example is an unbiased dice (all $p(\omega) = \frac{1}{6}$) with $X =$ "the number on top when the dice is thrown."

The *expectation* EX of X is the average to be expected, that is,

$$EX = \sum_{\omega \in \Omega} p(\omega) X(\omega).$$

Now suppose X and Y are two random variables on Ω, then the sum $X+Y$ is again a random variable, and we obtain

$$\begin{aligned} E(X+Y) &= \sum_{\omega} p(\omega)(X(\omega) + Y(\omega)) \\ &= \sum_{\omega} p(\omega)X(\omega) + \sum_{\omega} p(\omega)Y(\omega) \;=\; EX + EY. \end{aligned}$$

Clearly, this can be extended to any finite linear combination of random variables — this is what is called the *linearity of expectation*. Note that it needs no assumption that the random variables have to be "independent" in any sense!

Our third tool concerns random variables X which take only nonnegative values, shortly denoted $X \geq 0$. Let

$$\text{Prob}(X \geq a) = \sum_{\omega : X(\omega) \geq a} p(\omega)$$

be the probability that X is at least as large as some $a > 0$. Then

$$EX = \sum_{\omega : X(\omega) \geq a} p(\omega)X(\omega) + \sum_{\omega : X(\omega) < a} p(\omega)X(\omega) \;\geq\; a \sum_{\omega : X(\omega) \geq a} p(\omega),$$

and we have proved *Markov's inequality*

$$\text{Prob}(X \geq a) \leq \frac{EX}{a}.$$

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