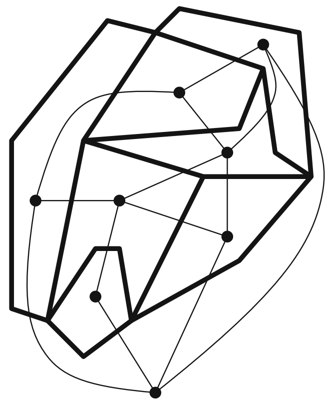
Five-coloring plane graphs

Chapter 39

Plane graphs and their colorings have been the subject of intensive research since the beginnings of graph theory because of their connection to the four-color problem. As stated originally the four-color problem asked whether it is always possible to color the regions of a plane map with four colors such that regions which share a common boundary (and not just a point) receive different colors. The figure on the right shows that coloring the regions of a map is really the same task as coloring the vertices of a plane graph. As in Chapter 13 (page 89) place a vertex in the interior of each region (including the outer region) and connect two such vertices belonging to neighboring regions by an edge through the common boundary.



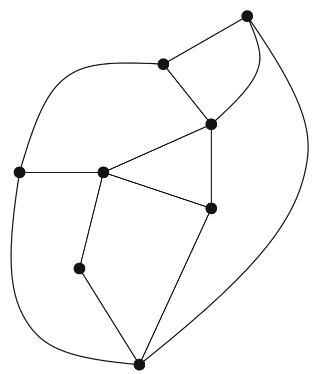
The dual graph of a map

The resulting graph G, the *dual graph* of the map M, is then a plane graph, and coloring the vertices of G in the usual sense is the same as coloring the regions of M. So we may as well concentrate on vertex-coloring plane graphs and will do so from now on. Note that we may assume that G has no loops or multiple edges, since these are irrelevant for coloring.

In the long and arduous history of attacks to prove the four-color theorem many attempts came close, but what finally succeeded in the Appel–Haken proof of 1976 and also in the more recent proof of Robertson, Sanders, Seymour and Thomas 1997 was a combination of very old ideas (dating back to the 19th century) and the very new calculating powers of modern-day computers. Twenty-five years after the original proof, the situation is still basically the same, there is even a computer-generated computer-checkable proof due to Gonthier, but no proof from The Book is in sight.

So let us be more modest and ask whether there is a neat proof that every plane graph can be 5-colored. A proof of this five-color theorem had already been given by Heawood at the turn of the century. The basic tool for his proof (and indeed also for the four-color theorem) was Euler's formula (see Chapter 13). Clearly, when coloring a graph G we may assume that G is connected since we may color the connected pieces separately. A plane graph divides the plane into a set R of regions (including the exterior region). Euler's formula states that for plane connected graphs $G = (V, E)$ we always have

$$|V| - |E| + |R| = 2.$$



This plane graph has 8 vertices, 13 edges and 7 regions.

As a warm-up, let us see how Euler's formula may be applied to prove that every plane graph G is 6-colorable. We proceed by induction on the number n of vertices. For small values of n (in particular, for $n \leq 6$) this is obvious.

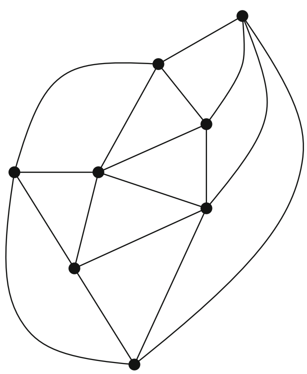
M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_39

From part (A) of the proposition on page 91 we know that G has a vertex v of degree at most 5. Delete v and all edges incident with v. The resulting graph $G' = G \backslash v$ is a plane graph on $n-1$ vertices. By induction, it can be 6-colored. Since v has at most 5 neighbors in G, at most 5 colors are used for these neighbors in the coloring of G'. So we can extend any 6-coloring of G' to a 6-coloring of G by assigning a color to v which is not used for any of its neighbors in the coloring of G'. Thus G is indeed 6-colorable.

Now let us look at the list chromatic number of plane graphs, which we have discussed in the chapter on the Dinitz problem. Clearly, our 6-coloring method works for lists of colors as well (again we never run out of colors), so $\chi_\ell(G) \le 6$ holds for any plane graph G. Erdős, Rubin and Taylor conjectured in 1979 that every plane graph has list chromatic number at most 5, and further that there are plane graphs G with $\chi_\ell(G) > 4$. They were right on both counts. Margit Voigt was the first to construct an example of a plane graph G with $\chi_\ell(G) = 5$ (her example had 238 vertices) and around the same time Carsten Thomassen gave a truly stunning proof of the 5-list coloring conjecture. His proof is a telling example of what you can do when you find the right induction hypothesis. It does not use Euler's formula at all!

Theorem. *All planar graphs G can be 5-list colored:*

$$\chi_\ell(G) \le 5.$$



A near-triangulated plane graph

■ **Proof.** First note that adding edges can only increase the chromatic number. In other words, when H is a subgraph of G, then $\chi_\ell(H) \le \chi_\ell(G)$ certainly holds. Hence we may assume that G is connected and that all the bounded faces of an embedding have triangles as boundaries. Let us call such a graph *near-triangulated*. The validity of the theorem for near-triangulated graphs will establish the statement for all plane graphs.

The trick of the proof is to show the following stronger statement (which allows us to use induction):

Let $G = (V, E)$ be a near-triangulated graph, and let B be the cycle bounding the outer region. We make the following assumptions on the color sets $C(v)$, $v \in V$:

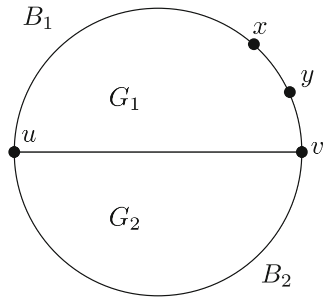
(1) *Two adjacent vertices x, y of B are already colored with (different) colors α and β.*

(2) $|C(v)| \ge 3$ *for all other vertices v of B.*

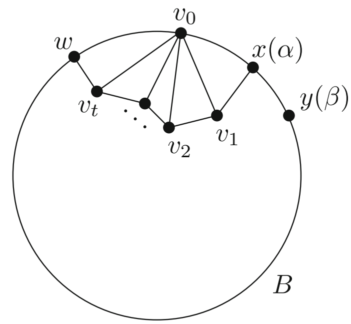
(3) $|C(v)| \ge 5$ *for all vertices v in the interior.*

Then the coloring of x, y can be extended to a proper coloring of G by choosing colors from the lists. In particular, $\chi_\ell(G) \le 5$.

For $|V| = 3$ this is obvious, since for the only uncolored vertex v we have $|C(v)| \geq 3$, so there is a color available. Now we proceed by induction.



Case 1: Suppose B has a chord, that is, an edge not in B that joins two vertices $u, v \in B$. The subgraph G_1 which is bounded by $B_1 \cup \{uv\}$ and contains x, y, u and v is near-triangulated and therefore has a 5-list coloring by induction. Suppose in this coloring the vertices u and v receive the colors γ and δ. Now we look at the bottom part G_2 bounded by B_2 and uv. Regarding u, v as pre-colored, we see that the induction hypotheses are also satisfied for G_2. Hence G_2 can be 5-list colored with the available colors, and thus the same is true for G.



Case 2: Suppose B has no chord. Let v_0 be the vertex on the other side of the α-colored vertex x on B, and let $x, v_1, \ldots, v_t, w$ be the neighbors of v_0. Since G is near-triangulated we have the situation shown in the figure.

Construct the near-triangulated graph $G' = G \backslash v_0$ by deleting from G the vertex v_0 and all edges emanating from v_0. This G' has as outer boundary $B' = (B \backslash v_0) \cup \{v_1, \ldots, v_t\}$. Since $|C(v_0)| \geq 3$ by assumption (2) there exist two colors γ, δ in $C(v_0)$ different from α. Now we replace every color set $C(v_i)$ by $C(v_i) \backslash \{\gamma, \delta\}$, keeping the original color sets for all other vertices in G'. Then G' clearly satisfies all assumptions and is thus 5-list colorable by induction. Choosing γ or δ for v_0, different from the color of w, we can extend the list coloring of G' to all of G. $\square$

So, the 5-list color theorem is proved, but the story is not quite over. A stronger conjecture claimed that the list-chromatic number of a plane graph G is at most 1 more than the ordinary chromatic number:

Is $\chi_\ell(G) \leq \chi(G) + 1$ *for every plane graph G ?*

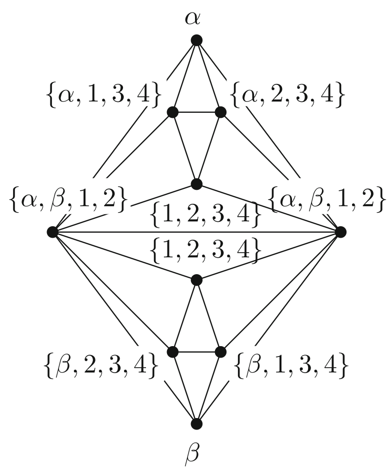
Since $\chi(G) \leq 4$ by the four-color theorem, we have three cases:

Case I: $\chi(G) = 2 \implies \chi_\ell(G) \leq 3$

Case II: $\chi(G) = 3 \implies \chi_\ell(G) \leq 4$

Case III: $\chi(G) = 4 \implies \chi_\ell(G) \leq 5$.

Thomassen's result settles Case III, and Case I was proved by an ingenious (and much more sophisticated) argument by Alon and Tarsi. Furthermore, there are plane graphs G with $\chi(G) = 2$ and $\chi_\ell(G) = 3$, for example the graph $K_{2,4}$ that we considered in the chapter on the Dinitz problem.



But what about Case II? Here the conjecture fails: This was first shown by Margit Voigt for a graph that was earlier constructed by Shai Gutner. His graph on 130 vertices can be obtained as follows. First we look at the "double octahedron" (see the figure), which is clearly 3-colorable. Let $\alpha \in \{5, 6, 7, 8\}$ and $\beta \in \{9, 10, 11, 12\}$, and consider the lists that are given in the figure. You are invited to check that with these lists a coloring is not possible. Now take 16 copies of this graph, and identify all top vertices and all bottom vertices. This yields a graph on $16 \cdot 8 + 2 = 130$ vertices which is still plane and 3-colorable. We assign $\{5, 6, 7, 8\}$ to the top vertex and $\{9, 10, 11, 12\}$ to the bottom vertex, with the inner lists corresponding to the 16 pairs (α, β), $\alpha \in \{5, 6, 7, 8\}$, $\beta \in \{9, 10, 11, 12\}$. For every choice of α and β we thus obtain a subgraph as in the figure, and so a list coloring of the big graph is not possible.

By modifying another one of Gutner's examples, Voigt and Wirth came up with an even smaller plane graph with 75 vertices and $\chi = 3$, $\chi_\ell = 5$, which in addition uses only the minimal number of 5 colors in the combined lists. The current record is 63 vertices — achieved in 1996 by a young Iranian Math Olympiad participant, Maryam Mirzakhani, who in 2014 became the first woman ever to receive a Fields Medal.

To close let us remark that Victor Campos and Frédéric Havet have recently extended Thomassen's theorem by showing that every graph that can be drawn in the plane with at most two crossings is still 5-list colorable.

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