Four times $\pi^2/6$

Chapter 9

We know that the infinite series $\sum_{n\geq 1} \frac{1}{n}$ does not converge. Indeed, in Chapter 1 we have seen that even the series $\sum_{p\in\mathbb{P}} \frac{1}{p}$ diverges.

However, the sum of the reciprocals of the squares converges (although very slowly, as we will also see), and it produces an interesting value.

Euler's series

$$\sum_{n\geq 1} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}.$$

This is a classical, famous and important result by Leonhard Euler from 1734. One of its key interpretations is that it yields the first nontrivial value $\zeta(2)$ of Riemann's zeta function (see the appendix on page 62). This value is irrational, as we have seen in Chapter 8.

But not only the result has a prominent place in mathematics history, there are also a number of extremely elegant and clever proofs that have their history: For some of these the joy of discovery and rediscovery has been shared by many. In this chapter, we present four such proofs.

$$\begin{aligned}
1 &= 1.000000\\
1+\tfrac{1}{4} &= 1.250000\\
1+\tfrac{1}{4}+\tfrac{1}{9} &= 1.361111\\
1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16} &= 1.423611\\
1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25} &= 1.463611\\
1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\tfrac{1}{36} &= 1.491388\\
\pi^2/6 &= 1.644934.
\end{aligned}$$

■ **Proof.** The first proof appears as an exercise in William J. LeVeque's number theory textbook from 1956. But he says: "I haven't the slightest idea where that problem came from, but I'm pretty certain that it wasn't original with me."

The proof consists in two different evaluations of the double integral

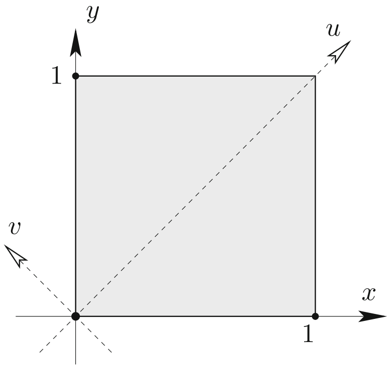
$$I \;:=\; \int_0^1\!\int_0^1 \frac{1}{1-xy}\,dx\,dy.$$

For the first one, we expand $\frac{1}{1-xy}$ as a geometric series, decompose the summands as products, and integrate effortlessly:

$$\begin{aligned}
I &= \int_0^1\!\int_0^1 \sum_{n\geq 0}(xy)^n\,dx\,dy \;=\; \sum_{n\geq 0}\int_0^1\!\int_0^1 x^n y^n\,dx\,dy\\
&= \sum_{n\geq 0}\Big(\int_0^1 x^n dx\Big)\Big(\int_0^1 y^n dy\Big) \;=\; \sum_{n\geq 0}\frac{1}{n+1}\,\frac{1}{n+1}\\
&= \sum_{n\geq 0}\frac{1}{(n+1)^2} \;=\; \sum_{n\geq 1}\frac{1}{n^2} \;=\; \zeta(2).
\end{aligned}$$

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_9

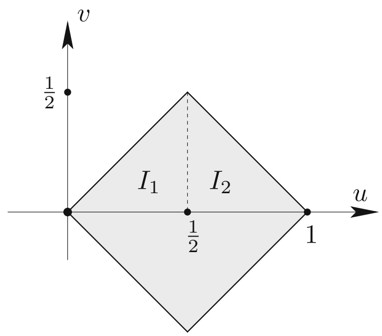
This evaluation also shows that the double integral (over a positive function with a pole at $x = y = 1$) is finite. Note that the computation is also easy and straightforward if we read it backwards — thus the evaluation of $\zeta(2)$ leads one to the double integral I.



The second way to evaluate I comes from a change of coordinates: in the new coordinates given by $u := \frac{y+x}{2}$ and $v := \frac{y-x}{2}$ the domain of integration is a square of side length $\frac{1}{2}\sqrt{2}$, which we get from the old domain by first rotating it by $45°$ and then shrinking it by a factor of $\sqrt{2}$. Substitution of $x = u - v$ and $y = u + v$ yields

$$\frac{1}{1-xy} \;=\; \frac{1}{1-u^2+v^2}.$$

To transform the integral, we have to replace $dx\,dy$ by $2\,du\,dv$, to compensate for the fact that our coordinate transformation reduces areas by a constant factor of 2 (which is the Jacobi determinant of the transformation; see the box on the next page). The new domain of integration, and the function to be integrated, are symmetric with respect to the u-axis, so we just need to compute two times (another factor of 2 arises here!) the integral over the upper half domain, which we split into two parts in the most natural way:



$$I \;=\; 4\int_0^{1/2}\Bigg(\int_0^{u}\frac{dv}{1-u^2+v^2}\Bigg)du \;+\; 4\int_{1/2}^{1}\Bigg(\int_0^{1-u}\frac{dv}{1-u^2+v^2}\Bigg)du.$$

Using $\displaystyle\int\frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$, this becomes

$$\begin{aligned} I \;=\; & \;4\int_0^{1/2}\frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{u}{\sqrt{1-u^2}}\right)du \\ & + \;4\int_{1/2}^{1}\frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right)du. \end{aligned}$$

These integrals can be simplified and finally evaluated by substituting $u = \sin\theta$ resp. $u = \cos\theta$. But we proceed more directly, by computing that the derivative of $g(u) := \arctan\left(\frac{u}{\sqrt{1-u^2}}\right)$ is $g'(u) = \frac{1}{\sqrt{1-u^2}}$, while the derivative of $h(u) := \arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right) = \arctan\left(\sqrt{\frac{1-u}{1+u}}\right)$ is $h'(u) = -\frac{1}{2}\frac{1}{\sqrt{1-u^2}}$. So we may use $\int_a^b f'(x)f(x)dx = \left[\frac{1}{2}f(x)^2\right]_a^b = \frac{1}{2}f(b)^2 - \frac{1}{2}f(a)^2$ and get

$$\begin{aligned} I \;&=\; 4\int_0^{1/2} g'(u)g(u)\,du \;+\; 4\int_{1/2}^{1} -2h'(u)h(u)\,du \\ &=\; 2\Big[g(u)^2\Big]_0^{1/2} \;-\; 4\Big[h(u)^2\Big]_{1/2}^{1} \\ &=\; 2g(\tfrac{1}{2})^2 \;-\; 2g(0)^2 \;-\; 4h(1)^2 \;+\; 4h(\tfrac{1}{2})^2 \\ &=\; 2\big(\tfrac{\pi}{6}\big)^2 - 0 - 0 + 4\big(\tfrac{\pi}{6}\big)^2 \;=\; \tfrac{\pi^2}{6}. \end{aligned}$$

□

This proof extracted the value of Euler's series from an integral via a rather simple coordinate transformation. An ingenious proof of this type — with an entirely nontrivial coordinate transformation — was later discovered by Beukers, Calabi and Kolk. The point of departure for that proof is to split the sum $\sum_{n\geq 1}\frac{1}{n^2}$ into the even terms and the odd terms. Clearly the even terms $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots=\sum_{k\geq 1}\frac{1}{(2k)^2}$ sum to $\frac{1}{4}\zeta(2)$, so the odd terms $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\sum_{k\geq 0}\frac{1}{(2k+1)^2}$ make up three quarters of the total sum $\zeta(2)$. Thus Euler's series is equivalent to

$$\sum_{k\geq 0}\frac{1}{(2k+1)^2} \;=\; \frac{\pi^2}{8}.$$

The Substitution Formula

To compute a double integral

$$I \;=\; \int_S f(x,y)\,dx\,dy.$$

we may perform a substitution of variables

$$x = x(u,v) \quad y = y(u,v),$$

if the correspondence of $(u,v)\in T$ to $(x,y)\in S$ is bijective and continuously differentiable. Then I equals

$$\int_T f(x(u,v),y(u,v))\left|\frac{d(x,y)}{d(u,v)}\right|du\,dv,$$

where $\frac{d(x,y)}{d(u,v)}$ is the Jacobi determinant:

$$\frac{d(x,y)}{d(u,v)} \;=\; \det\begin{pmatrix}\frac{dx}{du} & \frac{dx}{dv}\\ \frac{dy}{du} & \frac{dy}{dv}\end{pmatrix}.$$

■ **Proof.** As above, we may express this as a double integral, namely

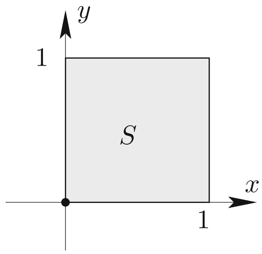
$$J \;=\; \int_0^1\!\!\int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy \;=\; \sum_{k\geq 0}\frac{1}{(2k+1)^2}.$$

So we have to compute this integral J. And for this Beukers, Calabi and Kolk proposed the new coordinates

$$u \;:=\; \arccos\sqrt{\frac{1-x^2}{1-x^2y^2}} \qquad v \;:=\; \arccos\sqrt{\frac{1-y^2}{1-x^2y^2}}.$$

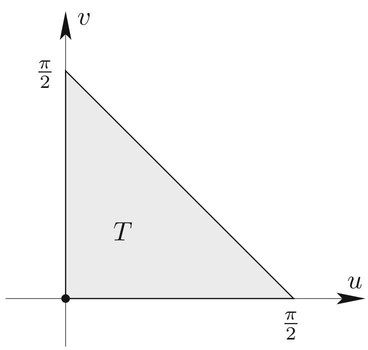
To compute the double integral, we may ignore the boundary of the domain, and consider x, y in the range $0<x<1$ and $0<y<1$. Then u, v will lie in the triangle $u>0$, $v>0$, $u+v<\pi/2$. The coordinate transformation can be inverted explicitly, which leads one to the substitution

$$x \;=\; \frac{\sin u}{\cos v} \qquad \text{and} \qquad y \;=\; \frac{\sin v}{\cos u}.$$



It is easy to check that these formulas define a bijective coordinate transformation between the interior of the unit square $S=\{(x,y): 0\leq x,y\leq 1\}$ and the interior of the triangle $T=\{(u,v): u,v\geq 0,\ u+v\leq \pi/2\}$. Now we have to compute the Jacobi determinant of the coordinate transformation, and magically it turns out to be

$$\det\begin{pmatrix}\frac{\cos u}{\cos v} & \frac{\sin u\sin v}{\cos^2 v}\\ \frac{\sin u\sin v}{\cos^2 u} & \frac{\cos v}{\cos u}\end{pmatrix} \;=\; 1-\frac{\sin^2 u\sin^2 v}{\cos^2 u\cos^2 v} \;=\; 1-x^2y^2.$$



But this means that the integral that we want to compute is transformed into

$$J \;=\; \int_0^{\pi/2}\int_0^{\pi/2-u} 1\,du\,dv,$$

which is just the area $\frac{1}{2}(\frac{\pi}{2})^2=\frac{\pi^2}{8}$ of the triangle T. □

Beautiful — even more so, as the same method of proof extends to the computation of $\zeta(2k)$ in terms of a $2k$-dimensional integral, for all $k \geq 1$. We refer to the original paper of Beuker, Calabi and Kolk [2], and to Chapter 26, where we'll achieve this on a different path, using the Herglotz trick and Euler's original approach.

After these two proofs via coordinate transformation we can't resist the temptation to present another, entirely different and completely elementary proof for $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. It appears in a sequence of exercises in the problem book by the twin brothers Akiva and Isaak Yaglom, whose Russian original edition appeared in 1954. Versions of this beautiful proof were rediscovered and presented by F. Holme (1970), I. Papadimitriou (1973), and by Ransford (1982) who attributed it to John Scholes.

■ **Proof.** The first step is to establish a remarkable relation between values of the (squared) cotangent function. Namely, for all $m \geq 1$ one has

For $m = 1, 2, 3$ this yields
$\cot^2 \frac{\pi}{3} = \frac{1}{3}$
$\cot^2 \frac{\pi}{5} + \cot^2 \frac{2\pi}{5} = 2$
$\cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7} = 5$

$$\cot^2\left(\tfrac{\pi}{2m+1}\right) + \cot^2\left(\tfrac{2\pi}{2m+1}\right) + \cdots + \cot^2\left(\tfrac{m\pi}{2m+1}\right) = \frac{2m(2m-1)}{6}. \quad (1)$$

To establish this, we start with the relation $e^{ix} = \cos x + i \sin x$. Taking the n-th power $e^{inx} = (e^{ix})^n$, we get

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n.$$

The imaginary part of this is

$$\sin nx = \binom{n}{1} \sin x \cos^{n-1} x - \binom{n}{3} \sin^3 x \cos^{n-3} x \pm \cdots \quad (2)$$

Now we let $n = 2m + 1$, while for x we will consider the m different values $x = \frac{r\pi}{2m+1}$, for $r = 1, 2, \ldots, m$. For each of these values we have $nx = r\pi$, and thus $\sin nx = 0$, while $0 < x < \frac{\pi}{2}$ implies that for $\sin x$ we get m distinct positive values.

In particular, we can divide (2) by $\sin^n x$, which yields

$$0 = \binom{n}{1} \cot^{n-1} x - \binom{n}{3} \cot^{n-3} x \pm \cdots,$$

that is,

$$0 = \binom{2m+1}{1} \cot^{2m} x - \binom{2m+1}{3} \cot^{2m-2} x \pm \cdots$$

for each of the m distinct values of x. Thus for the polynomial of degree m

$$p(t) := \binom{2m+1}{1} t^m - \binom{2m+1}{3} t^{m-1} \pm \cdots + (-1)^m \binom{2m+1}{2m+1}$$

we know m *distinct* roots

$$a_r = \cot^2\left(\tfrac{r\pi}{2m+1}\right) \quad \text{for} \quad r = 1, 2, \ldots, m.$$

The roots are distinct because $\cot^2 x = \cot^2 y$ implies $\sin^2 x = \sin^2 y$ and thus $x = y$ for $x, y \in \{\frac{r\pi}{2m+1} : 1 \leq r \leq m\}$.

Hence the polynomial coincides with

$$p(t) \;=\; \binom{2m+1}{1}\left(t-\cot^2\left(\tfrac{\pi}{2m+1}\right)\right)\cdots\left(t-\cot^2\left(\tfrac{m\pi}{2m+1}\right)\right).$$

Comparison of the coefficients of t^{m-1} in $p(t)$ now yields that the sum of the roots is

$$a_1+\cdots+a_r \;=\; \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} \;=\; \frac{2m(2m-1)}{6},$$

which proves (1).

Comparison of coefficients: If $p(t) = c(t-a_1)\cdots(t-a_m)$, then the coefficient of t^{m-1} is $-c(a_1+\cdots+a_m)$.

We also need a second identity, of the same type,

$$\csc^2\left(\tfrac{\pi}{2m+1}\right)+\csc^2\left(\tfrac{2\pi}{2m+1}\right)+\cdots+\csc^2\left(\tfrac{m\pi}{2m+1}\right) \;=\; \frac{2m(2m+2)}{6}, \qquad (3)$$

for the cosecant function $\csc x = \frac{1}{\sin x}$. But

$$\csc^2 x = \frac{1}{\sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \cot^2 x + 1,$$

so we can derive (3) from (1) by adding m to both sides of the equation.

Now the stage is set, and everything falls into place. We use that in the range $0 < y < \frac{\pi}{2}$ we have

$$0 \;<\; \sin y \;<\; y \;<\; \tan y,$$

and thus

$$0 \;<\; \cot y \;<\; \tfrac{1}{y} \;<\; \csc y,$$

$0 < a < b < c$ implies $0 < \frac{1}{c} < \frac{1}{b} < \frac{1}{a}$

which implies

$$\cot^2 y \;<\; \tfrac{1}{y^2} \;<\; \csc^2 y.$$

Now we take this double inequality, apply it to each of the m distinct values of x, and add the results. Using (1) for the left-hand side, and (3) for the right-hand side, we obtain

$$\tfrac{2m(2m-1)}{6} \;<\; \left(\tfrac{2m+1}{\pi}\right)^2+\left(\tfrac{2m+1}{2\pi}\right)^2+\cdots+\left(\tfrac{2m+1}{m\pi}\right)^2 \;<\; \tfrac{2m(2m+2)}{6},$$

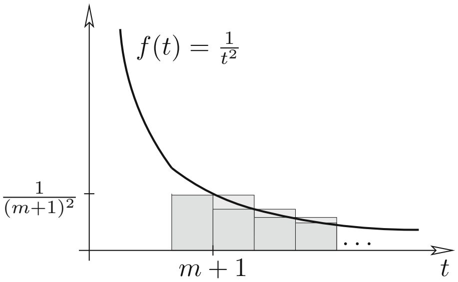
that is,

$$\tfrac{\pi^2}{6}\tfrac{2m}{2m+1}\tfrac{2m-1}{2m+1} \;<\; \tfrac{1}{1^2}+\tfrac{1}{2^2}+\cdots+\tfrac{1}{m^2} \;<\; \tfrac{\pi^2}{6}\tfrac{2m}{2m+1}\tfrac{2m+2}{2m+1}.$$

Both the left-hand and the right-hand side converge to $\frac{\pi^2}{6}$ for $m \longrightarrow \infty$: end of proof. □

So how fast does $\sum \frac{1}{n^2}$ converge to $\pi^2/6$? For this we have to estimate the difference

$$\frac{\pi^2}{6} - \sum_{n=1}^{m}\frac{1}{n^2} \;=\; \sum_{n=m+1}^{\infty}\frac{1}{n^2}.$$



This is very easy with the technique of "comparing with an integral" that we have reviewed already in the appendix to Chapter 2 (page 12). It yields

$$\sum_{n=m+1}^{\infty} \frac{1}{n^2} \;<\; \int_m^{\infty} \frac{1}{t^2} dt \;=\; \frac{1}{m}$$

for an upper bound and

$$\sum_{n=m+1}^{\infty} \frac{1}{n^2} \;>\; \int_{m+1}^{\infty} \frac{1}{t^2} dt \;=\; \frac{1}{m+1}$$

for a lower bound on the "remaining summands" — or even

$$\sum_{n=m+1}^{\infty} \frac{1}{n^2} \;>\; \int_{m+\frac{1}{2}}^{\infty} \frac{1}{t^2} dt \;=\; \frac{1}{m+\frac{1}{2}}$$

if you are willing to do a slightly more careful estimate, using that the function $f(t) = \frac{1}{t^2}$ is convex.

This means that our series does not converge too well; if we sum the first one thousand summands, then we expect an error in the third digit after the decimal point, while for the sum of the first one million summands, $m = 1000000$, we expect to get an error in the sixth decimal digit, and we do. However, then comes a big surprise: to an accuracy of 45 digits,

$$\begin{aligned} \pi^2/6 &= 1.644934066848226436472415166646025189218949901, \\ \sum_{n=1}^{10^6} \frac{1}{n^2} &= 1.644933066848726436305748499979391855885616544. \end{aligned}$$

So the sixth digit after the decimal point is wrong (too small by 1), but *the next six digits are right*! And then one digit is wrong (too large by 5), then again five are correct. This surprising discovery is quite recent, due to Roy D. North from Colorado Springs, 1988. (In 1982, Martin R. Powell, a school teacher from Amersham, Bucks, England, failed to notice the full effect due to the insufficient computing power available at the time.) It is too strange to be purely coincidental ... A look at the error term, which again to 45 digits reads

$$\sum_{n=10^6+1}^{\infty} \frac{1}{n^2} \;=\; 0.000000999999500000166666666666633333333333357,$$

reveals that clearly there is a pattern. You might try to rewrite this last number as

$$+\,10^{-6} \;-\; \tfrac{1}{2}10^{-12} \;+\; \tfrac{1}{6}10^{-18} \;-\; \tfrac{1}{30}10^{-30} \;+\; \tfrac{1}{42}10^{-42} \;+\; \cdots$$

where the coefficients $(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42})$ of 10^{-6i} form the beginning of the sequence of *Bernoulli numbers* that we'll meet again in Chapter 26. We refer our readers to the article by Borwein, Borwein & Dilcher [3] for more such surprising "coincidences" — and for proofs.

And if only to repeat the point that it pays of to look for gems hidden in exercise sections of books, in particular if they are written by brothers, here's our last proof for Euler's Theorem, as sketched in Exercise 11 of page 381 of the book "Pi and the AGM" by the brothers Jonathan and Peter Borwein. It establishes that you can get Euler's Theorem by "squaring" in an ingenious way the Gregory–Leibniz series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \dots \;=\; \frac{\pi}{4}.$$

Prove the Gregory–Leibniz identity, for example by integrating the geometric series $1 - x^2 + x^4 \pm \cdots = \frac{1}{1+x^2}$ and then evaluating at 1.

■ **Proof.** The first trick in this proof is to consider the Gregory–Leibniz series in doubly-infinite form $\sum\limits_{n=-\infty}^{\infty} \frac{(-1)^n}{2n+1}$. As for negative $n = -k < 0$ we get the same terms as for $n = k - 1 \geq 0$, since $\frac{(-1)^{-k}}{2(-k)+1} = \frac{(-1)^{k-1}}{2(k-1)+1}$, we infer that $\sum\limits_{n=-N}^{N} \frac{(-1)^n}{2n+1}$ converges to $\pi/2$ with $N \to \infty$, and thus the square of this sum converges to $\pi^2/4$. You may write this as

$$\lim_{N\to\infty} \sum_{m,n=-N}^{N} \frac{(-1)^m}{2m+1} \frac{(-1)^n}{2n+1} \;=\; \frac{\pi^2}{4}.$$

The double sum may be interpreted as the sum of all entries of a square matrix of size $(2N+1) \times (2N+1)$, and we know that for $N \to \infty$ this sum of all entries tends to $\pi^2/4$. We want to know, however, that the sum of only the *diagonal* entries, for $m = n$, also tends to $\pi^2/4$,

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{(2n+1)^2} \;=\; \frac{\pi^2}{4},$$

because then $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \pi^2/8$ will follow, and this, as we know, is equivalent to Euler's theorem. So let's show that the sum of all off-diagonal terms tends to 0! We write δ_N for this sum, and use a prime to denote that the diagonal terms with $m = n$ are deleted, so

$$\begin{aligned}
\delta_N \;&=\; \sum_{m,n=-N}^{N}{}' \frac{(-1)^{m+n}}{(2m+1)(2n+1)} \\
&=\; \sum_{m,n=-N}^{N}{}' (-1)^{m+n} \left(\frac{1}{2m-2n} \frac{1}{2m+1} - \frac{1}{2m-2n} \frac{1}{2n+1} \right) \\
&=\; \sum_{m,n=-N}^{N}{}' (-1)^{m+n} \left(\frac{1}{2m-2n} \frac{1}{2m+1} - \frac{1}{2n-2m} \frac{1}{2m+1} \right) \\
&=\; \sum_{m,n=-N}^{N}{}' (-1)^{m+n} \frac{1}{m-n} \frac{1}{2m+1} \\
&=\; \sum_{m=-N}^{N} \frac{1}{2m+1} \left(\sum_{n=-N}^{N}{}' \frac{(-1)^{m-n}}{m-n} \right).
\end{aligned}$$

Here we use that $\frac{1}{k\ell} = \frac{1}{k-\ell}(\frac{1}{\ell} - \frac{1}{k})$, for $k \neq \ell$. This replaces $\frac{1}{k\ell}$ by two summands that are not symmetric in k and ℓ.

For this we have interchanged $m \leftrightarrow n$ in the second part of the double sum.

We only need to show that the terms

$$c_{m,N} \;:=\; \sum_{n=-N}^{N}{}' \;\frac{(-1)^{m-n}}{m-n}$$

are small enough in absolute value. What do we know about them? It is easy to see that $c_{-m,N} = -c_{m,N}$, so in particular $c_{0,N} = 0$. Thus we may assume that $m > 0$, and note that the summands for $n = m + k$ and $n = m - k$ cancel as long as they are in the range between $-N$ and N, that is, for $1 \le k \le N - m$. Thus $c_{m,N}$ equals the alternating sum of fractions of decreasing size given by the remaining terms, where the largest one occurs for $n = m - (N - m) - 1 = 2m - N - 1$, that is $m - n = N - m + 1$. Hence

$$c_{m,N} \;=\; (-1)^{N-m+1}\left(\frac{1}{N-m+1} - \frac{1}{N-m+2} \pm \cdots \pm \frac{1}{m+N}\right),$$

which implies that

$$|c_{m,N}| \;\le\; \frac{1}{N-m+1}.$$

This finally yields

$$\begin{aligned}
|\delta_N| \;&\le\; \sum_{m=-N}^{N}\left|\frac{1}{2m+1}\right||c_{m,N}| \;\le\; \sum_{m=-N}^{N}\frac{1}{2|m|-1}|c_{m,N}| \\
&\le\; 2\sum_{m=1}^{N}\frac{1}{m}|c_{m,N}| \;\le\; 2\sum_{m=1}^{N}\frac{1}{m}\,\frac{1}{N-m+1} \\
&=\; 2\sum_{m=1}^{N}\frac{1}{N+1}\left(\frac{1}{m} + \frac{1}{N-m+1}\right) \\
&=\; 2\,\frac{1}{N+1}(H_N + H_N) \;<\; 4\,\frac{\log N + 1}{N+1},
\end{aligned}$$

Here we use that $\frac{1}{k\ell} = \frac{1}{k+\ell}(\frac{1}{\ell} + \frac{1}{k})$ for positive k and ℓ.

We got the estimate $H_N < \log N + 1$ for the harmonic numbers on page 13.

and this goes to 0 as N goes to infinity. □

Appendix: The Riemann zeta function

The *Riemann zeta function* $\zeta(s)$ is defined for real $s > 1$ by

$$\zeta(s) \;:=\; \sum_{n\ge 1}\frac{1}{n^s}.$$

Our estimates for H_n (see page 12) imply that the series for $\zeta(1)$ diverges, but for any real $s > 1$ it does converge. The zeta function has a canonical continuation to the entire complex plane (with one simple pole at $s = 1$), which can be constructed using power series expansions. The resulting complex function is of utmost importance for the theory of prime numbers. Let us mention four diverse connections:

(1) The remarkable identity

$$\zeta(s) \;=\; \prod_p \frac{1}{1-p^{-s}}$$

is due to Euler. It encodes the basic fact that every natural number has a unique (!) decomposition into prime factors; using this, Euler's identity is a simple consequence of the geometric series expansion

$$\frac{1}{1-p^{-s}} \;=\; 1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots$$

The irrationality of $\zeta(2)=\frac{\pi^2}{6}$ together with Euler's identity implies, again, that there are infinitely many primes ...

(2) The following marvelous argument of Don Zagier computes $\zeta(4)$ from $\zeta(2)$. Consider the function

$$f(m,n) \;=\; \frac{2}{m^3n}+\frac{1}{m^2n^2}+\frac{2}{mn^3}$$

for integers $m,n\geq 1$. It is easily verified that for all m and n,

$$f(m,n)-f(m+n,n)-f(m,m+n) \;=\; \frac{2}{m^2n^2}.$$

Let us sum this equation over all $m,n\geq 1$. If $i\neq j$, then (i,j) is either of the form $(m+n,n)$ or of the form $(m,m+n)$, for $m,n\geq 1$. Thus, in the sum on the left-hand side all terms $f(i,j)$ with $i\neq j$ cancel, and so

$$\sum_{n\geq 1} f(n,n) \;=\; \sum_{n\geq 1}\frac{5}{n^4} \;=\; 5\zeta(4)$$

remains. For the right-hand side one obtains

$$\sum_{m,n\geq 1}\frac{2}{m^2n^2} \;=\; 2\sum_{m\geq 1}\frac{1}{m^2}\cdot\sum_{n\geq 1}\frac{1}{n^2} \;=\; 2\zeta(2)^2,$$

and out comes the equality

$$5\zeta(4) \;=\; 2\zeta(2)^2.$$

With $\zeta(2)=\frac{\pi^2}{6}$ we thus get $\zeta(4)=\frac{\pi^4}{90}$.
Another derivation via Bernoulli numbers appears in Chapter 26.

(3) It has been known for a long time that $\zeta(s)$ is a rational multiple of π^s, and hence irrational, if s is an *even* integer $s \geq 2$; see Chapter 26. In contrast, the irrationality of $\zeta(3)$ was proved by Roger Apéry only in 1979. Despite considerable effort the picture is rather incomplete about $\zeta(s)$ for the other odd integers, $s = 2t+1 \geq 5$. However, Keith Ball and Tanguy Rivoal proved that infinitely many of the values $\zeta(2t+1)$ are irrational. And indeed, although it is not known for any single odd value $s \geq 5$ that $\zeta(s)$ is irrational, Wadim Zudilin has proved that at least one of the four values $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ is irrational. We refer to the beautiful survey by Fischler.

(4) The location of the complex zeros of the zeta function is the subject of the "Riemann hypothesis": one of the most famous and important unresolved conjectures in all of mathematics. It claims that all the nontrivial zeros $s \in \mathbb{C}$ of the zeta function satisfy $\text{Re}(s) = \frac{1}{2}$. (The zeta function vanishes at all the negative even integers, which are referred to as the "trivial zeros.")

Surprisingly, Jeff Lagarias showed that the Riemann hypothesis is equivalent to the following elementary statement: For all $n \geq 1$,

$$\sum_{d \mid n} d \;\leq\; H_n \;+\; \exp(H_n)\log(H_n),$$

with equality only for $n = 1$, where H_n is again the n-th harmonic number.

References

[1] K. BALL & T. RIVOAL: *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs,* Inventiones math. **146** (2001), 193-207.

[2] F. BEUKERS, J. A. C. KOLK & E. CALABI: *Sums of generalized harmonic series and volumes,* Nieuw Archief voor Wiskunde (4) **11** (1993), 217-224.

[3] J. M. BORWEIN & P. B. BORWEIN: *Pi and the AGM,* Canadian Math. Soc. Series of Monographs and Advanced Texts, Wiley, New York 1987.

[4] J. M. BORWEIN, P. B. BORWEIN & K. DILCHER: *Pi, Euler numbers, and asymptotic expansions,* Amer. Math. Monthly **96** (1989), 681-687.

[5] S. FISCHLER: *Irrationalité de valeurs de zêta (d'après Apéry, Rivoal, ...),* Bourbaki Seminar, No. 910, November 2002; Astérisque **294** (2004), 27-62.

[6] J. C. LAGARIAS: *An elementary problem equivalent to the Riemann hypothesis,* Amer. Math. Monthly **109** (2002), 534-543.

[7] W. J. LEVEQUE: *Topics in Number Theory, Vol. I,* Addison-Wesley, Reading MA 1956.

[8] A. M. YAGLOM & I. M. YAGLOM: *Challenging mathematical problems with elementary solutions,* Vol. II, Holden-Day, Inc., San Francisco, CA 1967.

[9] D. ZAGIER: *Values of zeta functions and their applications,* Proc. First European Congress of Mathematics, Vol. II (Paris 1992), Progress in Math. **120**, Birkhäuser, Basel 1994, pp. 497-512.

[10] W. ZUDILIN: *Arithmetic of linear forms involving odd zeta values,* J. Théorie Nombres Bordeaux **16** (2004), 251-291.