Hilbert's third problem: decomposing polyhedra

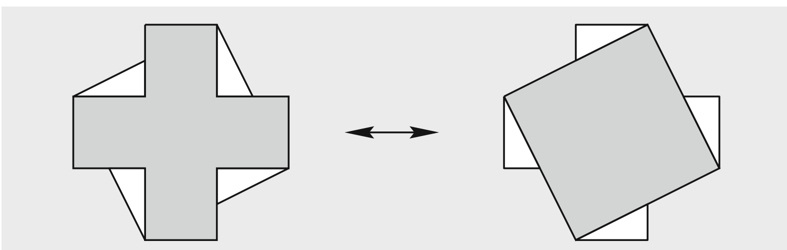
Chapter 10

In his legendary address to the International Congress of Mathematicians at Paris in 1900 David Hilbert asked — as the third of his twenty-three problems — to specify

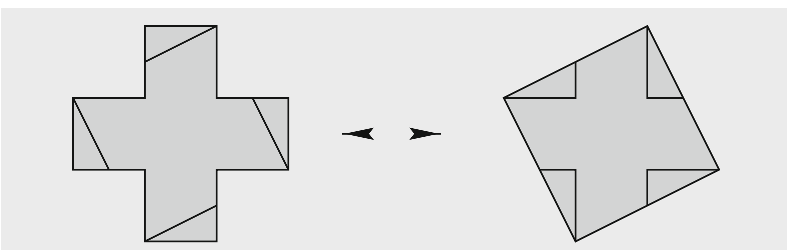
> *"two tetrahedra of equal bases and equal altitudes which can in no way be split into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra."*

David Hilbert

This problem can be traced back to two letters of Carl Friedrich Gauss from 1844 (published in Gauss' collected works in 1900). If tetrahedra of equal volume could be split into congruent pieces, then this would give one an "elementary" proof of Euclid's theorem XII.5 that pyramids with the same base and height have the same volume. It would thus provide an elementary definition of the volume for polyhedra (that would not depend on continuity arguments). A similar statement is true in plane geometry: the Bolyai–Gerwien Theorem [1, Sect. 2.7] states that planar polygons are both *equidecomposable* (can be dissected into congruent triangles) and *equicomplementable* (can be made equidecomposable by adding congruent triangles) if and only if they have the same area.



The cross is equicomplementable with a square of the same area: By adding the same four triangles we can make them equidecomposable (indeed: congruent).

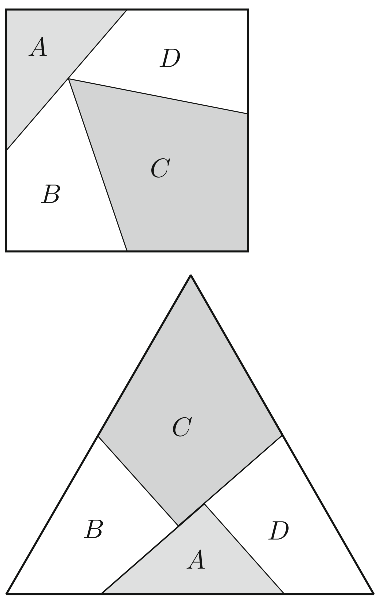


In fact, the cross and the square are even equidecomposable.

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_10

Hilbert — as we can see from his wording of the problem — did expect that there is no analogous theorem for dimension three, and he was right. In fact, the problem was completely solved by Hilbert's student Max Dehn in two papers: The first one, exhibiting non-equidecomposable tetrahedra of equal base and height, appeared already in 1900, the second one, also covering equicomplementability, appeared in 1902. However, Dehn's papers are not easy to understand, and it takes effort to see whether Dehn did not fall into a subtle trap which ensnared others: a very elegant but unfortunately wrong proof was found by Raoul Bricard (in 1896!), by Herbert Meschkowski (1960), and probably by others. However, Dehn's proof was reworked by others, clarified and redone, and after combined efforts of several authors one arrived at the "classical proof", as presented in Boltianskii's book on Hilbert's third problem and also in earlier editions of this one.

In the following, however, we take advantage of a decisive simplification that was found by V. F. Kagan from Odessa already in 1903: His integrality argument, which we here present as the "cone lemma", yields a "pearl lemma" (given here in a recent version, due to Benko), and from this we derive a correct and complete proof for "Bricard's condition" (as claimed in Bricard's 1896 paper). Once we apply this to some examples we easily obtain the solution of Hilbert's third problem.
The appendix to this chapter provides some basics about polyhedra.



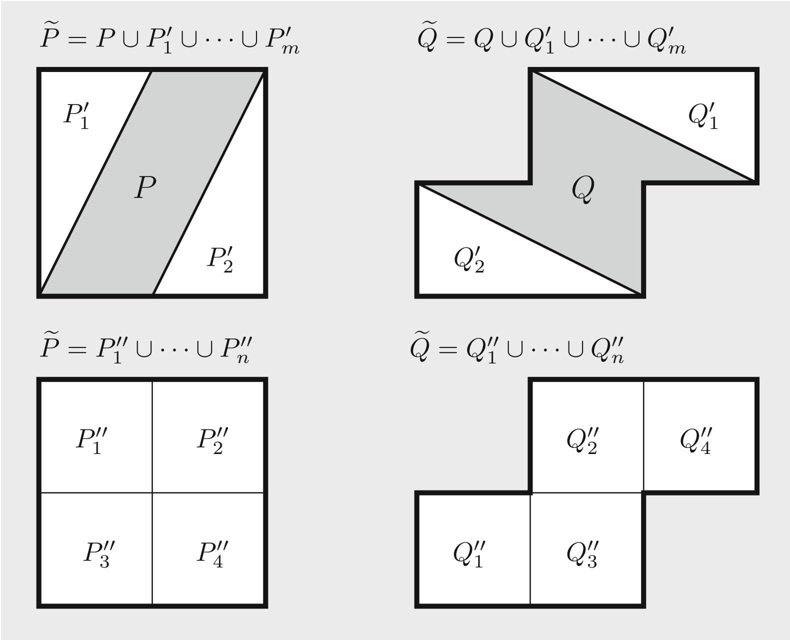
This equidecomposition of a square and an equilateral triangle into four pieces is due to Henry Dudeney (1902).
The short segment in the middle of the equilateral triangle is the intersection of pieces A and C, but it is not an edge of any one of the pieces.

As above we call two polyhedra P and Q *equidecomposable* if they can be decomposed into finite sets of polyhedra $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ such that P_i and Q_i are congruent for all i. Two polyhedra are *equicomplementable* if there are equidecomposable polyhedra $\widetilde{P} = P_1'' \cup \cdots \cup P_n''$ and $\widetilde{Q} = Q_1'' \cup \cdots \cup Q_n''$ that also have decompositions involving P and Q of the form $\widetilde{P} = P \cup P_1' \cup P_2' \cup \cdots \cup P_m'$ and $\widetilde{Q} = Q \cup Q_1' \cup Q_2' \cup \cdots \cup Q_m'$, where P_k' is congruent to Q_k' for all k. (See the large figure to the right for an illustration.) A theorem of Gerling from 1844 [1, §12] implies that for these definitions it does not matter whether we admit reflections when considering congruences, or not.

For polygons in the plane, equidecomposability and equicomplementability are defined analogously.

Clearly, equidecomposable objects are equicomplementable (this is the case $m = 0$), but the converse is far from clear. We will use "Bricard's condition" as our tool to certify — as Hilbert proposed — that certain tetrahedra of equal volume are not equicomplementable, and in particular not equidecomposable.

Before we really start to work with three-dimensional polyhedra, let us derive the pearl lemma, which is equally interesting also for planar decompositions. It refers to the *segments* in a decomposition: In any decomposition the edges of one piece may be subdivided by vertices or edges of other pieces; the pieces of this subdivision we call segments. Thus in the two-dimensional case any endpoint of a segment is given by some vertex. In the three-dimensional case the end of a segment may also be given by a crossing of two edges. However, in any case all the interior points of a segment belong to the same set of edges of pieces.



For a parallelogram P and a nonconvex hexagon Q that are equicomplementary, this figure illustrates the four decompositions we refer to.

The Pearl Lemma. *If P and Q are equidecomposable, then one can place a positive numbers of pearls (that is, assign positive integers) to all the segments of the decompositions $P = P_1 \cup \cdots \cup P_n$ and $Q = Q_1 \cup \cdots \cup Q_n$ in such a way that each edge of a piece P_k receives the same number of pearls as the corresponding edge of Q_k.*

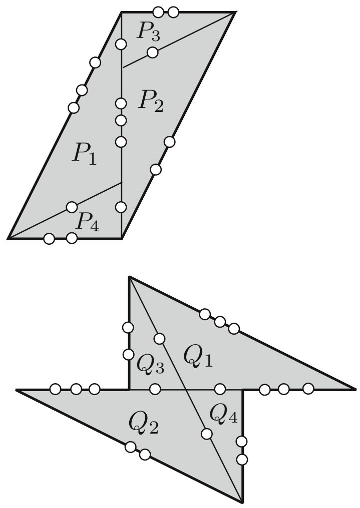
■ **Proof.** Assign a variable x_i to each segment in the decomposition of P and a variable y_j to each segment in the decomposition of Q. Now we have to find positive *integer* values for the variables x_i and y_j in such a way that the x_i-variables corresponding to the segments of any edge of some P_k yield the same sum as the y_j-variables assigned to the segments of the corresponding edge of Q_k. This yields conditions that require that "some x_i-variables have the same sum as some y_j-values", namely

$$\sum_{i:s_i \subseteq e} x_i \; - \sum_{j:s_j' \subseteq e'} y_j \; = \; 0$$

where the edge $e \subseteq P_k$ decomposes into the segments s_i, while the corresponding edge $e' \subseteq Q_k$ decomposes into the segments s_j'. This is a linear equation with integer coefficients.

We note, however, that positive *real* values satisfying all these requirements exist, namely the (real) lengths of the segments! Thus we are done, in view of the following lemma. □

The polygons P and Q considered in the figure above are, indeed, equidecomposable. The figure to the right illustrates this, and shows a possible placement of pearls.



The Cone Lemma. *If a system of homogeneous linear equations with integer coefficients has a positive* ***real*** *solution, then it also has a positive* ***integer*** *solution.*

■ **Proof.** The name of this lemma stems from the interpretation that the set

$$C \;=\; \{\mathbf{x} \in \mathbb{R}^N : A\mathbf{x} = \mathbf{0},\ \mathbf{x} > \mathbf{0}\}$$

given by an integer matrix $A \in \mathbb{Z}^{M\times N}$ describes a (relatively open) rational cone. We have to show that if this is nonempty, then it also contains integer points: $C \cap \mathbb{N}^N \neq \varnothing$.

If C is nonempty, then so is $\bar{C} := \{\mathbf{x} \in \mathbb{R}^N : A\mathbf{x} = \mathbf{0},\ \mathbf{x} \geq \mathbf{1}\}$, since for any positive vector a suitable multiple will have all coordinates equal to or larger than 1. (Here $\mathbf{1}$ denotes the vector with all coordinates equal to 1.) It suffices to verify that $\bar{C} \subseteq C$ contains a point with *rational* coordinates, since then multiplication with a common denominator for all coordinates will yield an integer point in $\bar{C} \subseteq C$.

There are many ways to prove this. We follow a well-trodden path that was first explored by Fourier and Motzkin [8, Lecture 1]: By "Fourier–Motzkin elimination" we show that the lexicographically smallest solution to the system

$$A\mathbf{x} = \mathbf{0},\ \mathbf{x} \geq \mathbf{1}$$

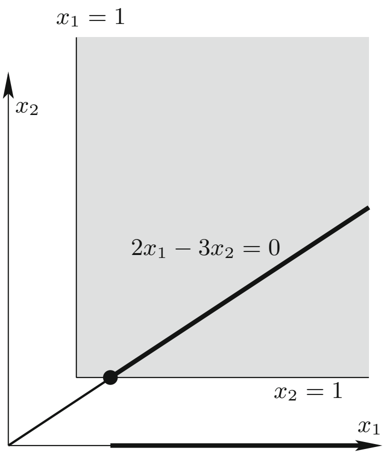
exists, and that it is rational if the matrix A is integral.

Indeed, any linear equation $\mathbf{a}^T\mathbf{x} = 0$ can be equivalently enforced by two inequalities $\mathbf{a}^T\mathbf{x} \geq 0$, $-\mathbf{a}^T\mathbf{x} \geq 0$. (Here $\mathbf{a}$ denotes a column vector and $\mathbf{a}^T$ its transpose.) Thus it suffices to prove that any system of the type

$$A\mathbf{x} \geq \mathbf{b},\ \mathbf{x} \geq \mathbf{1}$$

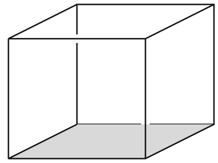
with integral A and $\mathbf{b}$ has a lexicographically smallest solution, which is rational, provided that the system has any real solution at all.

For this we argue with induction on N. The case $N = 1$ is clear. For $N > 1$ look at all the inequalities that involve x_N. If $\mathbf{x}' = (x_1, \ldots, x_{N-1})$ is fixed, these inequalities give lower bounds on x_N (among them $x_N \geq 1$) and possibly also upper bounds. So we form a new system $A'\mathbf{x}' \geq \mathbf{b},\ \mathbf{x}' \geq \mathbf{1}$ in $N-1$ variables, which contains all the inequalities from the system $A\mathbf{x} \geq \mathbf{b}$ that do not involve x_N, as well as all the inequalities obtained by requiring that all upper bounds on x_N (if there are any) are larger or equal to all the lower bounds on x_N (which include $x_N \geq 1$). This system in $N-1$ variables has a solution, and thus by induction it has a lexicographically minimal solution x'_*, which is rational. And then the smallest x_N compatible with this solution x'_* is easily found, it is determined by a linear equation or inequality with integer coefficients, and thus it is rational as well. □

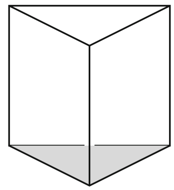


Example: Here $\bar{C}$ is given by $2x_1 - 3x_2 = 0$, $x_i \geq 1$. Eliminating x_2 yields $x_1 \geq \frac{3}{2}$. The lexicographically minimal solution to the system is $(\frac{3}{2}, 1)$.

Now we focus on decompositions of three-dimensional polyhedra. The *dihedral angles*, that is, the angles between adjacent facets, play a decisive role in the following theorem.



In a cube, all dihedral angles are $\frac{\pi}{2}$.



For a prism over an equilateral triangle, we get the dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

Theorem. ("Bricard's condition")
If three-dimensional polyhedra P and Q with dihedral angles $\alpha_1, \dots, \alpha_r$ resp. $\beta_1, \dots, \beta_s$ are equidecomposable, then there are positive integers m_i, n_j and an integer k with

$$m_1\alpha_1 + \cdots + m_r\alpha_r \;=\; n_1\beta_1 + \cdots + n_s\beta_s \;+\; k\pi.$$

The same holds more generally if P and Q are equicomplementable.

■ **Proof.** Let us first assume that P and Q are equidecomposable, with decompositions $P = P_1 \cup \cdots \cup P_n$ and $Q = Q_1 \cup \cdots \cup Q_n$, where P_i is congruent to Q_i. We assign a positive number of pearls to every segment in both decompositions, according to the pearl lemma.

Let Σ_1 be the sum of all the dihedral angles at all the pearls in the pieces of the decomposition of P. If an edge of a piece P_i contains several pearls, then the dihedral angle at this edge will appear several times in the sum Σ_1.

If a pearl is contained in several pieces, then several angles are added for this pearl, but they are all measured in the plane through the pearl that is orthogonal to the corresponding segment. If the segment is contained in an edge of P, the addition yields the (interior) dihedral angle α_j at the edge. The addition yields the angle π in the case that the segment lies in the boundary of P but not on an edge. If the pearl/the segment lies in the interior of P, then the sum of dihedral angles yields 2π or π. (The latter case occurs in case the pearl lies in the interior of a face of a piece P_i.)

Thus we get a representation

$$\Sigma_1 \;=\; m_1\alpha_1 + \cdots + m_r\alpha_r \;+\; k_1\pi$$

for positive integers m_j ($1 \le j \le r$) and nonnegative k_1. Similarly for the sum Σ_2 of all the angles at the pearls of the decomposition of Q we get

$$\Sigma_2 \;=\; n_1\beta_1 + \cdots + n_s\beta_s \;+\; k_2\pi$$

for positive integers n_j ($1 \le j \le s$) and nonnegative k_2.

However, we can also obtain the sums Σ_1 and Σ_2 by adding all the contributions in the individual pieces P_i and Q_i. Since P_i and Q_i are congruent, we measure the same dihedral angles at the corresponding edges, and the Pearl Lemma guarantees that we get the same number of pearls from the decompositions of P resp. Q at the corresponding edges. Thus we get $\Sigma_1 = \Sigma_2$, which yields Bricard's condition (with $k = k_2 - k_1 \in \mathbb{Z}$) for the case of equidecomposability.

Now let us assume that P and Q are equicomplementable, that is, that we have decompositions

$$\widetilde{P} \;=\; P \cup P'_1 \cup \cdots \cup P'_m \qquad \text{and} \qquad \widetilde{Q} \;=\; Q \cup Q'_1 \cup \cdots \cup Q'_m,$$

where P_i' and Q_i' are congruent, and such that $\widetilde{P}$ and $\widetilde{Q}$ are equidecomposable, as

$$\widetilde{P} \;=\; P_1'' \cup \cdots \cup P_n'' \qquad \text{and} \qquad \widetilde{Q} \;=\; Q_1'' \cup \cdots \cup Q_n'',$$

where P_i'' and Q_i'' are congruent (as in the figure on page 69). Again, using the pearl lemma, we place pearls to all the segments in all four decompositions, where we impose the extra condition that each edge of $\widetilde{P}$ gets the same total number of pearls in both decompositions, and similarly for $\widetilde{Q}$. (The proof of the pearl lemma via the cone lemma allows for such extra restrictions!) We also compute the sums of angles at pearls Σ_1' and Σ_2' as well as Σ_1'' and Σ_2''.

The angle sums Σ_1'' and Σ_2'' refer to decompositions of different polyhedra, $\widetilde{P}$ and $\widetilde{Q}$, into *the same set of pieces*, hence we get $\Sigma_1'' = \Sigma_2''$ as above.

The angle sums Σ_1' and Σ_1'' refer to different decompositions *of the same polyhedron*, $\widetilde{P}$. Since we have put the same number of pearls onto the edges in both decompositions, the argument above yields $\Sigma_1' = \Sigma_1'' + \ell_1\pi$ for an integer $\ell_1 \in \mathbb{Z}$. The same way we also get $\Sigma_2' = \Sigma_2'' + \ell_2\pi$ for an integer $\ell_2 \in \mathbb{Z}$. Thus we conclude that

$$\Sigma_2' \;=\; \Sigma_1' + \ell\pi \quad \text{for } \ell = \ell_2 - \ell_1 \in \mathbb{Z}.$$

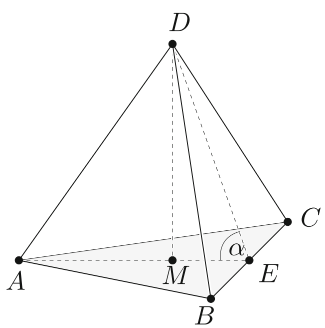
However, Σ_1' and Σ_2' refer to decompositions of $\widetilde{P}$ resp. $\widetilde{Q}$ into the same pieces, *except* that the first one uses P as a piece, while the second uses Q. Thus subtracting the contributions of P_i' resp. Q_i' from both sides, we obtain the desired conclusion: the contributions of P and Q to the respective angle sums,

$$m_1\alpha_1 + \cdots + m_r\alpha_r \qquad \text{and} \qquad n_1\beta_1 + \cdots + n_s\beta_s,$$

where m_j counts the pearls on edges with dihedral angle α_j in P and n_j counts the pearls on edges with dihedral angle β_j in Q, differ by an integer multiple of π, namely by $\ell\pi$. □

From Bricard's condition we now get a complete solution for Hilbert's third problem: We just have to compute the dihedral angles for some examples.

Example 1. For a regular tetrahedron T_0 with edge lengths ℓ, we calculate the dihedral angle from the sketch. The midpoint M of the base triangle divides the height AE of the base triangle by 1:2, and since $|AE| = |DE|$, we find $\cos\alpha = \frac{1}{3}$, and thus



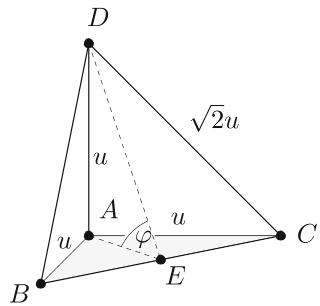
$$\alpha \;=\; \arccos\tfrac{1}{3}.$$

Thus we find that a *regular tetrahedron cannot be equidecomposable or equicomplementable with a cube*. Indeed, all the dihedral angles in a cube equal $\frac{\pi}{2}$, so Bricard's condition requires that

$$m_1 \arccos\tfrac{1}{3} \;=\; n_1\tfrac{\pi}{2} + k\pi$$

for positive integers m_1, n_1 and an integer k. But this cannot hold, since we know from Theorem 3 of Chapter 8 that $\frac{1}{\pi}\arccos\frac{1}{3}$ is irrational.

Example 2. Let T_1 be a tetrahedron spanned by three orthogonal edges AB, AC, AD of length u. This tetrahedron has three dihedral angles that are right angles, and three more dihedral angles of equal size φ, which we calculate from the sketch as



$$\cos\varphi = \frac{|AE|}{|DE|} = \frac{\frac{1}{2}\sqrt{2}u}{\frac{1}{2}\sqrt{3}\,\sqrt{2}u} = \frac{1}{\sqrt{3}}.$$

It follows that

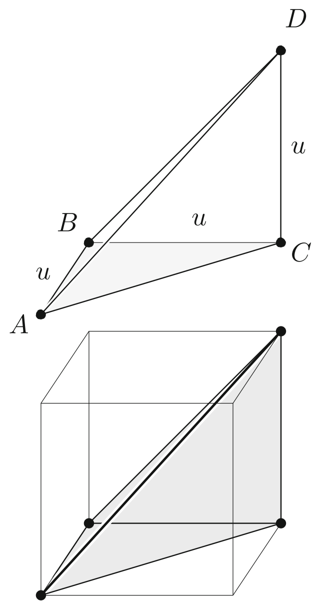
$$\varphi = \arccos\tfrac{1}{\sqrt{3}}.$$

Thus the only dihedral angles occuring in T_1 are π, $\frac{\pi}{2}$, and $\arccos\frac{1}{\sqrt{3}}$. From this Bricard's condition tells us that this tetrahedron as well is not equicomplementable with a cube of the same volume, this time using that

$$\tfrac{1}{\pi}\arccos\tfrac{1}{\sqrt{3}}$$

is irrational, as we proved in Chapter 8 (take $n = 3$ in Theorem 3).

Example 3. Finally, let T_2 be a tetrahedron with three consecutive edges AB, BC and CD that are mutually orthogonal (an "orthoscheme") and of the same length u.



It is easy to calculate the angles in such a tetrahedron (three of them equal $\frac{\pi}{2}$, two of them equal $\frac{\pi}{4}$, and one of them is $\frac{\pi}{3}$), if we use that the cube of side length u can be decomposed into six tetrahedra of this type (three congruent copies, and three mirror images). Thus all dihedral angles in T_2 are rational multiples of π, and thus with the same proofs as above (in particular, the irrationality results that we have quoted from Chapter 8) Bricard's Condition implies that T_2 is not equidecomposable, and not even equicomplementable, with T_0 or T_1.

This solves Hilbert's third problem, since T_1 and T_2 have congruent bases and the same height.

Appendix: Polytopes and polyhedra

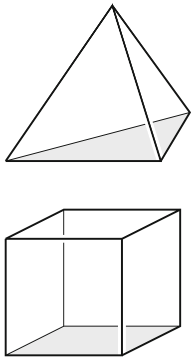
A *convex polytope* in $\mathbb{R}^d$ is the convex hull of a finite set $S = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n\}$, that is, a set of the form

$$P = \operatorname{conv}(S) := \Big\{\sum_{i=1}^{n}\lambda_i\boldsymbol{s}_i : \lambda_i \geq 0,\ \sum_{i=1}^{n}\lambda_i = 1\Big\}.$$

Polytopes are certainly familiar objects: Prime examples are given by convex *polygons* (2-dimensional convex polytopes) and by convex *polyhedra* (3-dimensional convex polytopes).

There are several types of polyhedra that generalize to higher dimensions in a natural way. For example, if the set S is affinely independent of cardinality $d+1$, then conv(S) is a d-dimensional *simplex* (or *d-simplex*). For $d=2$ this yields a triangle, for $d=3$ we obtain a tetrahedron. Similarly, squares and cubes are special cases of d-cubes, such as the *unit d-cube* given by

$$C_d \;=\; [0,1]^d \;\subseteq\; \mathbb{R}^d.$$



Familiar polytopes: tetrahedron and cube

General polytopes are defined as finite unions of convex polytopes. In this book nonconvex polyhedra will appear in connection with Cauchy's rigidity theorem in Chapter 14, and nonconvex polygons in connection with Pick's theorem in Chapter 13, and again when we discuss the art gallery theorem in Chapter 40.

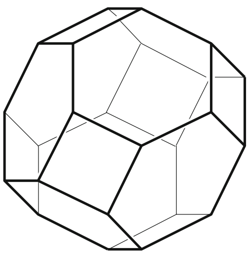
Convex polytopes can, equivalently, be defined as the bounded solution sets of finite systems of linear inequalities. Thus every convex polytope $P \subseteq \mathbb{R}^d$ has a representation of the form

$$P \;=\; \{\boldsymbol{x} \in \mathbb{R}^d : A\boldsymbol{x} \leq \boldsymbol{b}\}$$

for some matrix $A \in \mathbb{R}^{m\times d}$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$. In other words, P is the solution set of a system of m linear inequalities

$$\boldsymbol{a}_i^T\boldsymbol{x} \;\leq\; b_i,$$

where $\boldsymbol{a}_i^T$ is the i-th row of A. Conversely, every bounded such solution set is a convex polytope, and can thus be represented as the convex hull of a finite set of points.



The permutahedron has 24 vertices, 36 edges and 14 facets.

For polygons and polyhedra, we have the familiar concepts of *vertices*, *edges*, and *2-faces*. For higher-dimensional convex polytopes, we can define their faces as follows: a *face* of P is a subset $F \subseteq P$ of the form

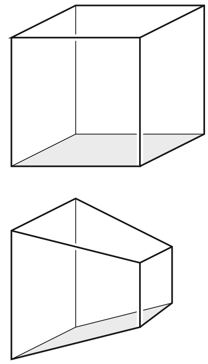
$$P \cap \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{a}^T\boldsymbol{x} = b\},$$

where $\boldsymbol{a}^T\boldsymbol{x} \leq b$ is a linear inequality that is valid for all points $\boldsymbol{x} \in P$.

All the faces of a polytope are themselves polytopes. The set V of vertices (0-dimensional faces) of a convex polytope is also the inclusion-minimal set such that conv$(V) = P$. Assuming that $P \subseteq \mathbb{R}^d$ is a d-dimensional convex polytope, the *facets* (the $(d{-}1)$-dimensional faces) determine a minimal set of hyperplanes and thus of halfspaces that contain P, and whose intersection is P. In particular, this implies the following fact that we will need later: Let F be a facet of P, denote by H_F the hyperplane it determines, and by H_F^+ and H_F^- the two closed half-spaces bounded by H_F. Then one of these two halfspaces contains P (and the other one doesn't).

The *graph* $G(P)$ of the convex polytope P is given by the set V of vertices, and by the edge set E of 1-dimensional faces. If P has dimension 3, then this graph is planar, and gives rise to the famous "Euler polyhedron formula" (see Chapter 13).

Two polytopes $P, P' \subseteq \mathbb{R}^d$ are *congruent* if there is some length-preserving affine map that takes P to P'. Such a map may reverse the orientation of space, as does the reflection of P in a hyperplane, which takes P to a *mirror image* of P. They are *combinatorially equivalent* if there is a bijection from the faces of P to the faces of P' that preserves dimension and inclusions between the faces. This notion of combinatorial equivalence is much weaker than congruence: for example, our figure shows a unit cube and a "skew" cube that are combinatorially equivalent (and thus we would call any one of them "a cube"), but they are certainly not congruent.



Combinatorially equivalent polytopes

A polytope (or a more general subset of $\mathbb{R}^d$) is called *centrally symmetric* if there is some point $\boldsymbol{x}_0 \in \mathbb{R}^d$ such that

$$\boldsymbol{x}_0 + \boldsymbol{x} \in P \quad \Longleftrightarrow \quad \boldsymbol{x}_0 - \boldsymbol{x} \in P.$$

In this situation we call $\boldsymbol{x}_0$ the *center* of P.

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