In praise of inequalities

Chapter 20

Analysis abounds with inequalities, as witnessed for example by the famous book "Inequalities" by Hardy, Littlewood and Pólya. Let us single out two of the most basic inequalities with two applications each, and let us listen in to George Pólya, who was himself a champion of the Book Proof, about what he considers the most appropriate proofs.

Our first inequality is variously attributed to Cauchy, Schwarz and/or to Buniakowski:

Theorem I (Cauchy–Schwarz inequality)
Let $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ *be an inner product on a real vector space* V *(with the norm* $|\boldsymbol{a}|^2 := \langle \boldsymbol{a}, \boldsymbol{a} \rangle$*). Then*

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle^2 \;\le\; |\boldsymbol{a}|^2 |\boldsymbol{b}|^2$$

holds for all vectors $\boldsymbol{a}, \boldsymbol{b} \in V$*, with equality if and only if* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are linearly dependent.*

■ **Proof.** The following (folklore) proof is probably the shortest. Consider the quadratic function

$$|x\boldsymbol{a} + \boldsymbol{b}|^2 \;=\; x^2|\boldsymbol{a}|^2 + 2x\langle \boldsymbol{a}, \boldsymbol{b} \rangle + |\boldsymbol{b}|^2$$

in the variable x. We may assume $\boldsymbol{a} \neq \boldsymbol{0}$. If $\boldsymbol{b} = \lambda \boldsymbol{a}$, then clearly $\langle \boldsymbol{a}, \boldsymbol{b} \rangle^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2$. If, on the other hand, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then $|x\boldsymbol{a} + \boldsymbol{b}|^2 > 0$ for all x, and thus the discriminant $\langle \boldsymbol{a}, \boldsymbol{b} \rangle^2 - |\boldsymbol{a}|^2 |\boldsymbol{b}|^2$ is less than 0. □

Our second example is the *inequality of the harmonic, geometric and arithmetic mean*:

Theorem II (Harmonic, geometric and arithmetic mean)
Let $a_1, \ldots, a_n$ *be positive real numbers, then*

$$\frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \;\le\; \sqrt[n]{a_1 a_2 \cdots a_n} \;\le\; \frac{a_1 + \cdots + a_n}{n}$$

with equality in both cases if and only if all a_i*'s are equal.*

■ **Proof.** The following beautiful nonstandard induction proof is attributed to Cauchy (see [8]). Let $P(n)$ be the statement of the second inequality, written in the form

$$a_1 a_2 \cdots a_n \;\le\; \Big(\frac{a_1 + \cdots + a_n}{n}\Big)^n.$$

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_20

For $n = 2$, we have $a_1 a_2 \leq (\frac{a_1+a_2}{2})^2 \iff (a_1 - a_2)^2 \geq 0$, which is true. Now we proceed in the following two steps:

(A) $P(n) \Longrightarrow P(n-1)$

(B) $P(n)$ and $P(2) \Longrightarrow P(2n)$

which will clearly imply the full result.

To prove **(A)**, set $A := \sum\limits_{k=1}^{n-1} \frac{a_k}{n-1}$, then

$$\Big(\prod_{k=1}^{n-1} a_k\Big) A \overset{P(n)}{\leq} \left(\frac{\sum\limits_{k=1}^{n-1} a_k \; + \; A}{n}\right)^n = \Big(\frac{(n-1)A + A}{n}\Big)^n = A^n$$

and hence $\prod\limits_{k=1}^{n-1} a_k \leq A^{n-1} = \left(\frac{\sum\limits_{k=1}^{n-1} a_k}{n-1}\right)^{n-1}.$

For **(B)**, we see

$$\prod_{k=1}^{2n} a_k = \Big(\prod_{k=1}^{n} a_k\Big)\Big(\prod_{k=n+1}^{2n} a_k\Big) \overset{P(n)}{\leq} \Big(\sum_{k=1}^{n} \frac{a_k}{n}\Big)^n \Big(\sum_{k=n+1}^{2n} \frac{a_k}{n}\Big)^n$$

$$\overset{P(2)}{\leq} \left(\frac{\sum\limits_{k=1}^{2n} \frac{a_k}{n}}{2}\right)^{2n} = \left(\frac{\sum\limits_{k=1}^{2n} a_k}{2n}\right)^{2n}.$$

The condition for equality is derived just as easily.

The left-hand inequality, between the harmonic and the geometric mean, follows now by considering $\frac{1}{a_1}, \ldots, \frac{1}{a_n}$. □

■ **Another Proof.** Of the many other proofs of the arithmetic-geometric mean inequality (the monograph [2] lists more than fifty), let us single out a particularly striking one by Horst Alzer, with some shortenings due to France Dacar. As a matter of fact, this proof yields the stronger inequality

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \leq p_1 a_1 + p_2 a_2 + \cdots + p_n a_n$$

for any positive numbers $a_1, \ldots, a_n, p_1, \ldots, p_n$ with $\sum_{i=1}^{n} p_i = 1$. Let us denote the expression on the left side by G, and on the right side by A. Fix $c > 0$ and define the function $f(t) := \frac{1}{c} - \frac{1}{t}$ on $\mathbb{R}_{>0}$. Since $f(t) < 0$ for $t < c$ and $f(t) > 0$ for $t > c$, we get the inequality

$$\int_c^x f(t)\, dt \geq 0$$

for *every* $x > 0$, with equality if and only if $x = c$.

Now

$$0 \leq \int_c^x f(t)\, dt = \Big[\frac{t}{c} - \log t\Big]_c^x = \frac{x}{c} - 1 - \log \frac{x}{c},$$

Note that we have proved the inequality $x \geq 1 + \log x$ for $x > 0$ on the side.

and setting $c = G$ and $x = a_i$ we conclude that

$$\frac{a_i}{G} - 1 \;\geq\; \log a_i - \log G \qquad \text{for } \; i = 1, 2, \ldots, n. \tag{1}$$

Multiplying this inequality by p_i and summing over all i gives

$$\sum_{i=1}^{n} p_i \frac{a_i}{G} - \sum_{i=1}^{n} p_i \;\geq\; \sum_{i=1}^{n} p_i \log a_i - \sum_{i=1}^{n} p_i \log G.$$

With $\sum_{i=1}^{n} p_i = 1$, the left side equals $\frac{A}{G} - 1$, while the right side is

$$\log\Big(\prod_{i=1}^{n} a_i^p\Big) - \log G \;=\; \log G - \log G \;=\; 0.$$

We conclude $\frac{A}{G} - 1 \geq 0$, which is $A \geq G$. In the case of equality, all inequalities in (1) must be equalities, which implies $a_1 = \cdots = a_n = G$. □

■ **Still another Proof.** There is another nice proof, due to Michael D. Hirschhorn. It uses Bernoulli's inequality, which says

$$(1+t)^{n+1} \;\;\geq\;\; 1 + (n+1)t \qquad \text{for real } \; t \geq -1.$$

Suppose $a_1, a_2, \ldots, a_{n+1} > 0$ and set

$$t \;=\; \frac{\dfrac{a_1 + \cdots + a_{n+1}}{n+1}}{\dfrac{a_1 + \cdots + a_n}{n}} - 1.$$

By Bernoulli,

$$\begin{aligned}\left(\frac{\dfrac{a_1 + \cdots + a_{n+1}}{n+1}}{\dfrac{a_1 + \cdots + a_n}{n}}\right)^{n+1} &\geq 1 + (n+1)\left(\frac{\dfrac{a_1 + \cdots + a_{n+1}}{n+1}}{\dfrac{a_1 + \cdots + a_n}{n}} - 1\right) \\ &= 1 + n\frac{a_1 + \cdots + a_{n+1}}{a_1 + \cdots + a_n} - (n+1) \\ &= \frac{n\, a_{n+1}}{a_1 + \cdots + a_n},\end{aligned}$$

which translates into

$$\Big(\frac{a_1 + \cdots + a_{n+1}}{n+1}\Big)^{n+1} \;\;\geq\;\; a_{n+1}\Big(\frac{a_1 + \cdots + a_n}{n}\Big)^{n},$$

and the arithmetic-geometric mean inequality follows by induction. □

Our first application is a beautiful result of Laguerre (see [8]) concerning the location of roots of polynomials.

Theorem 1. *Suppose all roots of the polynomial* $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ *are real. Then the roots are contained in the interval with the endpoints*

$$-\frac{a_{n-1}}{n} \pm \frac{n-1}{n}\sqrt{a_{n-1}^2 - \frac{2n}{n-1}a_{n-2}}\,.$$

■ **Proof.** Let y be one of the roots and $y_1, \ldots, y_{n-1}$ the others. Then the polynomial is $(x-y)(x-y_1)\cdots(x-y_{n-1})$. Thus by comparing coefficients

$$\begin{aligned} -a_{n-1} &= y + y_1 + \cdots + y_{n-1}, \\ a_{n-2} &= y(y_1 + \cdots + y_{n-1}) + \sum_{i<j} y_i y_j, \end{aligned}$$

and so

$$a_{n-1}^2 - 2a_{n-2} - y^2 = \sum_{i=1}^{n-1} y_i^2.$$

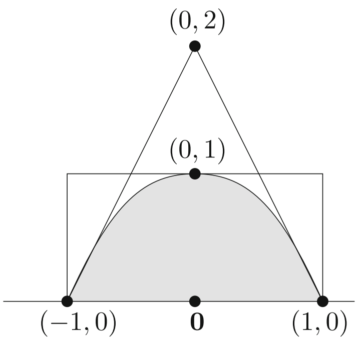
By Cauchy's inequality applied to $(y_1, \ldots, y_{n-1})$ and $(1, \ldots, 1)$,

$$\begin{aligned} (a_{n-1} + y)^2 &= (y_1 + y_2 + \cdots + y_{n-1})^2 \\ &\le (n-1)\sum_{i=1}^{n-1} y_i^2 \\ &= (n-1)(a_{n-1}^2 - 2a_{n-2} - y^2), \end{aligned}$$

or

$$y^2 + \frac{2a_{n-1}}{n}y + \frac{2(n-1)}{n}a_{n-2} - \frac{n-2}{n}a_{n-1}^2 \le 0.$$

Thus y (and hence all y_i) lie between the two roots of the quadratic function, and these roots are our bounds. □



For our second application we start from a well-known elementary property of a parabola. Consider the parabola described by $f(x) = 1 - x^2$ between $x = -1$ and $x = 1$. We associate to $f(x)$ the *tangential triangle* and the *tangential rectangle* as in the figure.

We find that the shaded area

$$A = \int_{-1}^{1} (1 - x^2)dx$$

is equal to $\frac{4}{3}$, and the areas T and R of the triangle and rectangle are both equal to 2. Thus $\frac{T}{A} = \frac{3}{2}$ and $\frac{R}{A} = \frac{3}{2}$.

In a beautiful paper, Paul Erdős and Tibor Gallai asked what happens when $f(x)$ is an arbitrary n-th degree real polynomial with $f(x) > 0$ for $-1 < x < 1$, and $f(-1) = f(1) = 0$. The area A is then $\int_{-1}^{1} f(x)dx$.

Suppose that $f(x)$ assumes in $(-1,1)$ its maximum value at b, then $R = 2f(b)$. Computing the tangents at -1 and at 1, it is readily seen (see the box below) that

$$T \;=\; \frac{2f'(1)f'(-1)}{f'(1)-f'(-1)}, \tag{2}$$

respectively $T = 0$ for $f'(1) = f'(-1) = 0$.

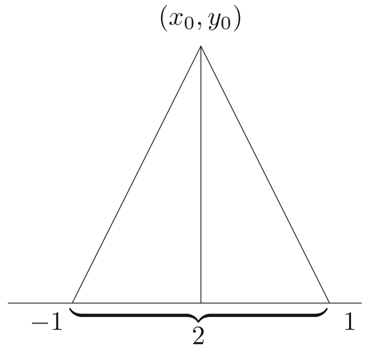
The tangential triangle

The area T of the tangential triangle is precisely y_0, where (x_0, y_0) is the point of intersection of the two tangents. The equation of these tangents are $y = f'(-1)(x+1)$ and $y = f'(1)(x-1)$, hence

$$x_0 \;=\; \frac{f'(1)+f'(-1)}{f'(1)-f'(-1)},$$

and thus

$$y_0 \;=\; f'(1)\left(\frac{f'(1)+f'(-1)}{f'(1)-f'(-1)} - 1\right) \;=\; 2\frac{f'(1)f'(-1)}{f'(1)-f'(-1)}.$$



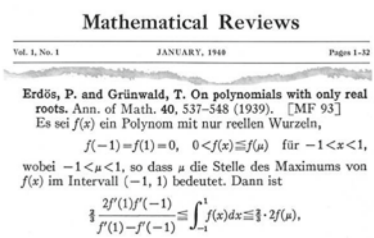
In general, there are no nontrivial bounds for $\frac{T}{A}$ and $\frac{R}{A}$. To see this, take $f(x) = 1 - x^{2n}$. Then $T = 2n$, $A = \frac{4n}{2n+1}$, and thus $\frac{T}{A} > n$. Similarly, $R = 2$ and $\frac{R}{A} = \frac{2n+1}{2n}$, which approaches 1 with n to infinity.
But, as Erdős and Gallai showed, for polynomials which have only real roots such bounds do indeed exist.

Theorem 2. *Let $f(x)$ be a real polynomial of degree $n \geq 2$ with only real roots, such that $f(x) > 0$ for $-1 < x < 1$ and $f(-1) = f(1) = 0$. Then*

$$\frac{2}{3}T \;\leq\; A \;\leq\; \frac{2}{3}R,$$

and equality holds in both cases only for $n = 2$.

Erdős and Gallai established this result with an intricate induction proof. In the review of their paper, which appeared on the first page of the first issue of the Mathematical Reviews in 1940, George Pólya explained how the first inequality can also be proved by the inequality of the arithmetic and geometric mean — a beautiful example of a conscientious review and a Book Proof at the same time.


Mathematical Reviews

Vol. 1, No. 1 JANUARY, 1940 Pages 1-32

Erdös, P. and Grünwald, T. On polynomials with only real roots. Ann. of Math. **40**, 537–548 (1939). [MF 93]
Es sei $f(x)$ ein Polynom mit nur reellen Wurzeln,

$$f(-1)=f(1)=0, \quad 0<f(x)\leqq f(\mu) \quad \text{für } -1<x<1,$$

wobei $-1<\mu<1$, so dass μ die Stelle des Maximums von $f(x)$ im Intervall $(-1, 1)$ bedeutet. Dann ist

$$\tfrac{2}{3}\frac{2f'(1)f'(-1)}{f'(1)-f'(-1)} \leqq \int_{-1}^{1} f(x)dx \leqq \tfrac{2}{3}\cdot 2f(\mu),$$

■ **Proof of $\frac{2}{3}T \le A$.** Since $f(x)$ has only real roots, and none of them in the open interval $(-1,1)$, it can be written — apart from a constant positive factor which cancels out in the end — in the form

$$f(x) \;=\; (1-x^2)\prod_i(\alpha_i - x)\prod_j(\beta_j + x) \tag{3}$$

with $\alpha_i \ge 1$, $\beta_j \ge 1$. Hence

$$A \;=\; \int_{-1}^{1}(1-x^2)\prod_i(\alpha_i - x)\prod_j(\beta_j + x)dx.$$

By making the substitution $x \longmapsto -x$, we find that also

$$A \;=\; \int_{-1}^{1}(1-x^2)\prod_i(\alpha_i + x)\prod_j(\beta_j - x)dx,$$

and hence by the inequality of the arithmetic and the geometric mean (note that all factors are ≥ 0)

$$\begin{aligned}
A \;&=\; \int_{-1}^{1}\frac{1}{2}\Big[\;(1-x^2)\prod_i(\alpha_i - x)\prod_j(\beta_j + x)\;+ \\
&\qquad\qquad (1-x^2)\prod_i(\alpha_i + x)\prod_j(\beta_j - x)\;\Big]dx \\
&\ge\; \int_{-1}^{1}(1-x^2)\Big(\prod_i(\alpha_i^2 - x^2)\prod_j(\beta_j^2 - x^2)\Big)^{1/2}dx \\
&\ge\; \int_{-1}^{1}(1-x^2)\Big(\prod_i(\alpha_i^2 - 1)\prod_j(\beta_j^2 - 1)\Big)^{1/2}dx \\
&=\; \frac{4}{3}\Big(\prod_i(\alpha_i^2 - 1)\prod_j(\beta_j^2 - 1)\Big)^{1/2}.
\end{aligned}$$

Let us compute $f'(1)$ and $f'(-1)$. (We may assume $f'(-1), f'(1) \ne 0$, since otherwise $T = 0$ and the inequality $\frac{2}{3}T \le A$ becomes trivial.) By (3) we see

$$f'(1) \;=\; -2\prod_i(\alpha_i - 1)\prod_j(\beta_j + 1)\,,$$

and similarly

$$f'(-1) \;=\; 2\prod_i(\alpha_i + 1)\prod_j(\beta_j - 1)\,.$$

Hence we conclude

$$A \;\ge\; \frac{2}{3}(-f'(1)f'(-1))^{1/2}.$$

Applying now the inequality of the harmonic and the geometric mean to $-f'(1)$ and $f'(1)$, we arrive by (2) at the conclusion

$$A \;\geq\; \frac{2}{3}\,\frac{2}{\frac{1}{-f'(1)}+\frac{1}{f'(-1)}} \;=\; \frac{4}{3}\,\frac{f'(1)f'(-1)}{f'(1)-f'(-1)} \;=\; \frac{2}{3}T,$$

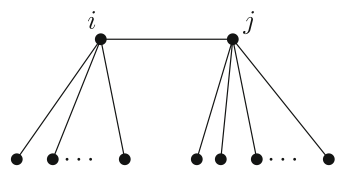
which is what we wanted to show. By analyzing the case of equality in all our inequalities the reader can easily supply the last statement of the theorem. □

The reader is invited to search for an equally inspired proof of the second inequality in Theorem 2.

Well, analysis is inequalities after all, but here is an example from graph theory where the use of inequalities comes in quite unexpected. In Chapter 41 we will discuss Turán's theorem. In the simplest case it takes on the following form.

Theorem 3. *Suppose G is a graph on n vertices without triangles. Then G has at most $\frac{n^2}{4}$ edges, and equality holds only when n is even and G is the complete bipartite graph $K_{n/2,n/2}$.*

■ **First proof.** This proof, using Cauchy's inequality, is due to Mantel. Let $V=\{1,\ldots,n\}$ be the vertex set and E the edge set of G. By d_i we denote the degree of i, hence $\sum_{i\in V} d_i = 2|E|$ (see page 199 in the chapter on double counting). Suppose ij is an edge. Since G has no triangles, we find $d_i+d_j\leq n$ since no vertex is a neighbor of both i and j.



It follows that

$$\sum_{ij\in E}(d_i+d_j) \;\leq\; n|E|.$$

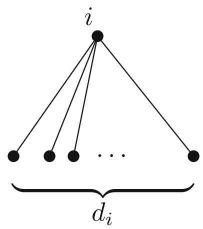
Note that d_i appears exactly d_i times in the sum, so we get

$$n|E| \;\geq\; \sum_{ij\in E}(d_i+d_j) \;=\; \sum_{i\in V} d_i^2,$$

and hence with Cauchy's inequality applied to the vectors $(d_1,\ldots,d_n)$ and $(1,\ldots,1)$,

$$n|E| \;\geq\; \sum_{i\in V} d_i^2 \;\geq\; \frac{(\sum d_i)^2}{n} \;=\; \frac{4|E|^2}{n},$$

and the result follows. In the case of equality we find $d_i = d_j$ for all i,j, and further $d_i=\frac{n}{2}$ (since $d_i+d_j=n$). Since G is triangle-free, $G=K_{n/2,n/2}$ is immediately seen from this. □



■ **Second proof.** The following proof of Theorem 3, using the inequality of the arithmetic and the geometric mean, is a folklore Book Proof. Let α be the size of a largest independent set A, and set $\beta = n - \alpha$. Since G is triangle-free, the neighbors of a vertex i form an independent set, and we infer $d_i \leq \alpha$ for all i.

The set $B = V \backslash A$ of size β meets every edge of G. Counting the edges of G according to their endvertices in B, we obtain $|E| \leq \sum_{i \in B} d_i$. The inequality of the arithmetic and geometric mean now yields

$$|E| \;\leq\; \sum_{i \in B} d_i \;\leq\; \alpha\beta \;\leq\; \left(\frac{\alpha+\beta}{2}\right)^2 \;=\; \frac{n^2}{4},$$

and again the case of equality is easily dealt with. □

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