Lattice paths and determinants

Chapter 32

The essence of mathematics is proving theorems — and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a *Lemma*, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside–Frobenius Lemma in combinatorics.

Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma — including its proof — should be beautiful!

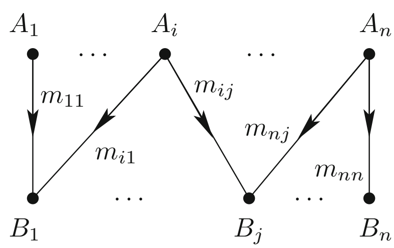
In this chapter we look at one such marvelous piece of mathematical reasoning, a counting lemma that first appeared in a paper by Bernt Lindström in 1972. Largely overlooked at the time, the result became an instant classic in 1985, when Ira Gessel and Gerard Viennot rediscovered it and demonstrated in a wonderful paper how the lemma could be successfully applied to a diversity of difficult combinatorial enumeration problems.

The starting point is the usual permutation representation of the determinant of a matrix. Let $M = (m_{ij})$ be a real $n \times n$ matrix. Then

$$\det M \;=\; \sum_{\sigma} \operatorname{sign}\sigma \; m_{1\sigma(1)}\, m_{2\sigma(2)} \cdots m_{n\sigma(n)}, \qquad (1)$$

where σ runs through all permutations of $\{1, 2, \ldots, n\}$, and the sign of σ is 1 or -1, depending on whether σ is the product of an even or an odd number of transpositions.

Now we pass to graphs, more precisely to *weighted directed bipartite* graphs. Let the vertices $A_1, \ldots, A_n$ stand for the rows of M, and $B_1, \ldots, B_n$ for the columns. For each pair of i and j draw an arrow from A_i to B_j and give it the weight m_{ij}, as in the figure.



In terms of this graph, the formula (1) has the following interpretation:

- The left-hand side is the determinant of the *path matrix* M, whose (i, j)-entry is the *weight* of the (unique) directed path from A_i to B_j.

- The right-hand side is the weighted (signed) sum over all *vertex-disjoint path systems* from $\mathcal{A} = \{A_1, \ldots, A_n\}$ to $\mathcal{B} = \{B_1, \ldots, B_n\}$. Such a system $\mathcal{P}_\sigma$ is given by paths

$$A_1 \to B_{\sigma(1)}, \;\ldots\;, A_n \to B_{\sigma(n)},$$

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_32

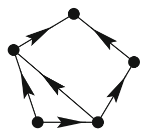
and the *weight* of the path system $\mathcal{P}_\sigma$ is the product of the weights of the individual paths:

$$w(\mathcal{P}_\sigma) \;=\; w(A_1 \to B_{\sigma(1)}) \;\cdots\; w(A_n \to B_{\sigma(n)}).$$

In this interpretation formula (1) reads

$$\det M \;=\; \sum_\sigma \operatorname{sign}\sigma\, w(\mathcal{P}_\sigma).$$

And what is the result of Gessel and Viennot? It is the natural generalization of (1) from bipartite to arbitrary graphs. It is precisely this step which makes the Lemma so widely applicable — and what's more, the proof is stupendously simple and elegant.


An acyclic directed graph

Let us first collect the necessary concepts. We are given a finite acyclic directed graph $G = (V, E)$, where *acyclic* means that there are no directed cycles in G. In particular, there are only finitely many directed paths between any two vertices A and B, where we include all trivial paths $A \to A$ of length 0. Every edge e carries a weight $w(e)$. If P is a directed path from A to B, written shortly $P : A \to B$, then we define the *weight* of P as

$$w(P) \;:=\; \prod_{e\in P} w(e),$$

which is defined to be $w(P) = 1$ if P is a path of length 0.

Now let $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ be two sets of n vertices, where $\mathcal{A}$ and $\mathcal{B}$ need not be disjoint. To $\mathcal{A}$ and $\mathcal{B}$ we associate the *path matrix* $M = (m_{ij})$ with

$$m_{ij} \;:=\; \sum_{P: A_i \to B_j} w(P).$$

A *path system $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$* consists of a permutation σ together with n paths $P_i : A_i \to B_{\sigma(i)}$, for $i = 1, \ldots, n$; we write sign $\mathcal{P} = \operatorname{sign}\sigma$. The *weight* of $\mathcal{P}$ is the product of the path weights

$$w(\mathcal{P}) \;=\; \prod_{i=1}^{n} w(P_i), \tag{2}$$

which is the product of the weights of all the edges of the path system.

Finally, we say that the path system $\mathcal{P} = (P_1, \ldots, P_n)$ is *vertex-disjoint* if the paths of $\mathcal{P}$ are pairwise vertex-disjoint.

Lemma. *Let $G = (V, E)$ be a finite weighted acyclic directed graph, $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ two n-sets of vertices, and M the path matrix from $\mathcal{A}$ to $\mathcal{B}$. Then*

$$\det M \;=\; \sum_{\substack{\mathcal{P}\text{ vertex-disjoint}\\ \text{path system}}} \operatorname{sign}\mathcal{P}\; w(\mathcal{P}). \tag{3}$$

■ **Proof.** A typical summand of $\det(M)$ is $\text{sign}\,\sigma\; m_{1\sigma(1)} \cdots m_{n\sigma(n)}$, which can be written as

$$\text{sign}\,\sigma\;\Big(\sum_{P_1:A_1\to B_{\sigma(1)}} w(P_1)\Big)\;\cdots\;\Big(\sum_{P_n:A_n\to B_{\sigma(n)}} w(P_n)\Big).$$

Summing over σ we immediately find from (2) that

$$\det M \;=\; \sum_{\mathcal{P}} \text{sign}\,\mathcal{P}\; w(\mathcal{P}),$$

where $\mathcal{P}$ runs through *all* path systems from $\mathcal{A}$ to $\mathcal{B}$ (vertex-disjoint or not). Hence to arrive at (3), all we have to show is

$$\sum_{\mathcal{P}\in N} \text{sign}\,\mathcal{P}\; w(\mathcal{P}) \;=\; 0\,, \tag{4}$$

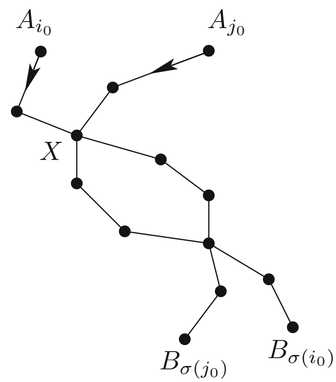
where N is the set of all path systems that are *not* vertex-disjoint. And this is accomplished by an argument of singular beauty. Namely, we exhibit an involution $\pi : N \to N$ (without fixed points) such that for $\mathcal{P}$ and $\pi\mathcal{P}$

$$w(\pi\mathcal{P}) \;=\; w(\mathcal{P}) \qquad \text{and} \qquad \text{sign}\,\pi\mathcal{P} \;=\; -\text{sign}\,\mathcal{P}\,.$$

Clearly, this will imply (4) and thus the formula (3) of the Lemma.

The involution π is defined in the most natural way. Let $\mathcal{P} \in N$ with paths $P_i : A_i \to B_{\sigma(i)}$. By definition, some pair of paths will intersect:

- Let i_0 be the minimal index such that P_{i_0} shares some vertex with another path.
- Let X be the first such common vertex on the path P_{i_0}.
- Let j_0 be the minimal index ($j_0 > i_0$) such that P_{j_0} has the vertex X in common with P_{i_0}.

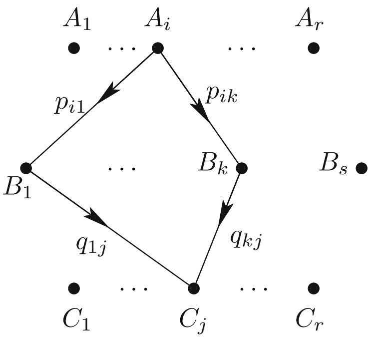


Now we construct the new system $\pi\mathcal{P} = (P_1', \ldots, P_n')$ as follows:

- Set $P_k' = P_k$ for all $k \neq i_0, j_0$.
- The new path P_{i_0}' goes from A_{i_0} to X along P_{i_0}, and then continues to $B_{\sigma(j_0)}$ along P_{j_0}. Similarly, P_{j_0}' goes from A_{j_0} to X along P_{j_0} and continues to $B_{\sigma(i_0)}$ along P_{i_0}.

Clearly $\pi(\pi\mathcal{P}) = \mathcal{P}$, since the index i_0, the vertex X, and the index j_0 are the same as before. In other words, applying π twice we switch back to the old paths P_i. Next, since $\pi\mathcal{P}$ and $\mathcal{P}$ use precisely the same edges, we certainly have $w(\pi\mathcal{P}) = w(\mathcal{P})$. And finally, since the new permutation σ' is obtained by multiplying σ with the transposition (i_0, j_0), we find that $\text{sign}\,\pi\mathcal{P} \;= -\text{sign}\,\mathcal{P}$, and that's it. □

The Gessel–Viennot Lemma can be used to derive all basic properties of determinants, just by looking at appropriate graphs. Let us consider one particularly striking example, the formula of Binet–Cauchy, which gives a very useful generalization of the product rule for determinants.



Theorem. *If P is an $r \times s$ matrix and Q an $s \times r$ matrix, $r \leq s$, then*

$$\det(PQ) \;=\; \sum_{\mathcal{Z}} (\det P_{\mathcal{Z}})(\det Q_{\mathcal{Z}}),$$

where $P_{\mathcal{Z}}$ is the $r \times r$ submatrix of P with column-set $\mathcal{Z}$, and $Q_{\mathcal{Z}}$ the $r \times r$ submatrix of Q with the corresponding rows $\mathcal{Z}$.

■ **Proof.** Let the bipartite graph on $\mathcal{A}$ and $\mathcal{B}$ correspond to P as before, and similarly the bipartite graph on $\mathcal{B}$ and $\mathcal{C}$ to Q. Consider now the concatenated graph as indicated in the figure on the left, and observe that the (i, j)-entry m_{ij} of the path matrix M from $\mathcal{A}$ to $\mathcal{C}$ is precisely $m_{ij} = \sum_k p_{ik} q_{kj}$, thus $M = PQ$.

Since the vertex-disjoint path systems from $\mathcal{A}$ to $\mathcal{C}$ in the concatenated graph correspond to pairs of systems from $\mathcal{A}$ to $\mathcal{Z}$ resp. from $\mathcal{Z}$ to $\mathcal{C}$, the result follows immediately from the Lemma, by noting that sign $(\sigma\tau)$ = (sign σ) (sign τ). □

The Lemma of Gessel–Viennot is also the source of a great number of results that relate determinants to enumerative properties. The recipe is always the same: Interpret the matrix M as a path matrix, and try to compute the right-hand side of (3). As an illustration we will consider the original problem studied by Gessel and Viennot, which led them to their Lemma:

> *Suppose that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ are two sets of natural numbers. We wish to compute the determinant of the matrix $M = (m_{ij})$, where m_{ij} is the binomial coefficient $\binom{a_i}{b_j}$.*

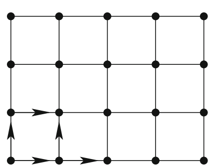
1							
1	1						
1	2	1					
1	**3**	3	**1**				
1	**4**	6	**4**	**1**			
1	5	10	10	5	1		
1	**6**	15	**20**	**15**	6	1	
1	7	21	35	35	21	7	1

In other words, Gessel and Viennot were looking at the determinants of arbitrary square matrices of Pascal's triangle, such as the matrix

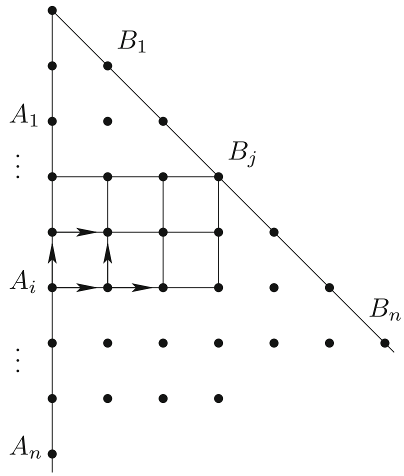
$$\det\begin{pmatrix} \binom{3}{1} & \binom{3}{3} & \binom{3}{4} \\ \binom{4}{1} & \binom{4}{3} & \binom{4}{4} \\ \binom{6}{1} & \binom{6}{3} & \binom{6}{4} \end{pmatrix} \;=\; \det\begin{pmatrix} 3 & 1 & 0 \\ 4 & 4 & 1 \\ 6 & 20 & 15 \end{pmatrix}$$

given by the bold entries of Pascal's triangle, as displayed in the margin.

As a preliminary step to the solution of the problem we recall a well-known result which connects binomial coefficients to lattice paths. Consider an $a \times b$-lattice as in the margin. Then the number of paths from the lower left-hand corner to the upper right-hand corner, where the only steps that are allowed for the paths are up (North) and to the right (East), is $\binom{a+b}{a}$.



The proof of this is easy: each path consists of an arbitrary sequence of b "east" and a "north" steps, and thus it can be encoded by a sequence of the form NENEEEN, consisting of $a+b$ letters, a N's and b E's. The number of such strings is the number of ways to choose a positions of letters N from a total of $a + b$ positions, which is $\binom{a+b}{a} = \binom{a+b}{b}$.



Now look at the figure to the right, where A_i is placed at the point $(0, -a_i)$ and B_j at $(b_j, -b_j)$.

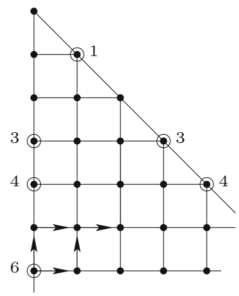
The number of paths from A_i to B_j in this grid that use only steps to the north and east is, by what we just proved, $\binom{b_j+(a_i-b_j)}{b_j} = \binom{a_i}{b_j}$. In other words, the matrix of binomials M is precisely the path matrix from $\mathcal{A}$ to $\mathcal{B}$ in the directed lattice graph for which all edges have weight 1, and all edges are directed to go north or east. Hence to compute $\det M$ we may apply the Gessel–Viennot Lemma. A moment's thought shows that every vertex-disjoint path system $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$ must consist of paths $P_i : A_i \to B_i$ for all i. Thus the only possible permutation is the identity, which has sign $= 1$, and we obtain the beautiful result

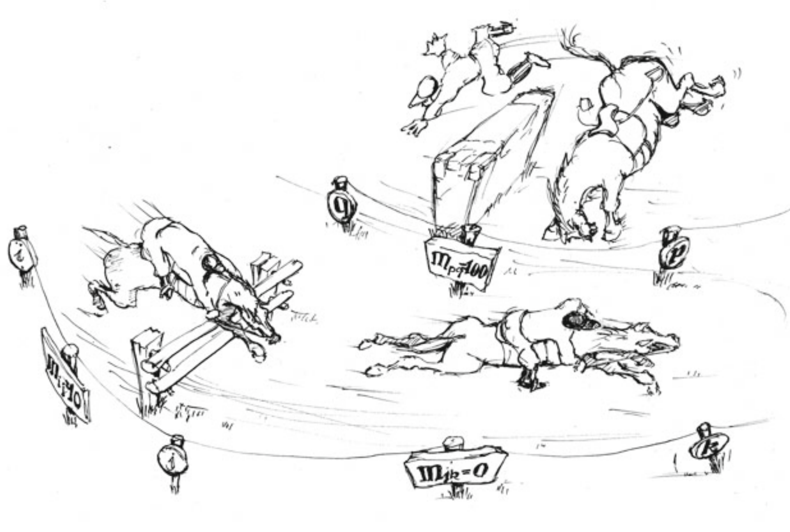
$$\det\left(\binom{a_i}{b_j}\right) \quad = \quad \#\text{ vertex-disjoint path systems from } \mathcal{A} \text{ to } \mathcal{B}.$$

In particular, this implies the far from obvious fact that $\det M$ is always nonnegative, since the right-hand side of the equality *counts* something. More precisely, one gets from the Gessel–Viennot Lemma that $\det M = 0$ if and only if $a_i < b_i$ for some i.

In our previous small example,

$$\det\begin{pmatrix} \binom{3}{1} & \binom{3}{3} & \binom{3}{4} \\ \binom{4}{1} & \binom{4}{3} & \binom{4}{4} \\ \binom{6}{1} & \binom{6}{3} & \binom{6}{4} \end{pmatrix} \quad = \quad \#$$ vertex-disjoint path systems in





"Lattice paths"

References

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[2] B. LINDSTRÖM: *On the vector representation of induced matroids,* Bulletin London Math. Soc. **5** (1973), 85-90.