

The Borromean rings don't exist

Chapter 15



The “Borromean rings” — three rings arranged so that no two of them are linked, but the configuration cannot be taken apart without breaking one of the rings — form a classic artistic symbol, which appeared in the coat of arms of the aristocratic Borromeo family since the middle of the 15th century.

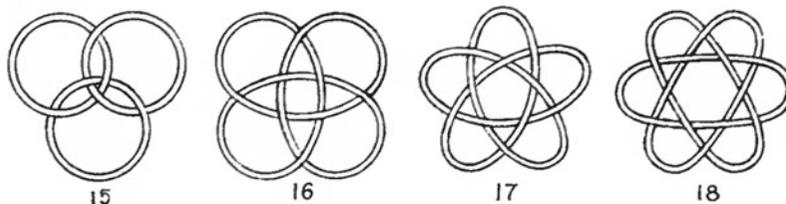
The Borromean rings are also one of the most tantalizing and enigmatic “impossible figures” of mathematics. They can easily be built as a geometric object in such a way that two of the rings are perfectly round circles of the same size; it seems, however, that then the third ring is represented by an ellipse, at best. Thus it is natural to ask:



Can the Borromean rings be built from three perfect circles?

As mathematical objects, the Borromean rings belong to the theory of knots and links, which very attractively connects geometry, topology, and combinatorics. We all have a geometric picture of what knots (closed curves in space) and links (arrangements of several such curves) look like, and we can draw them in the plane. We also have intuitive notions of when two knots or links are “the same” (equivalent), when a knot or link is “trivial,” when two circles are linked, etc.: The appendix to this chapter provides a review of the essential terms and definitions, including the fact that two diagrams present the same link or knot if and only if they can be transformed into each other by a finite sequence of “Reidemeister moves.”

Knot theory as we know it today started in 1867, when the physicist William Thomson, now known as Lord Kelvin, came up with his “vortex theory,” according to which atoms could be explained as knots in the “ether” background of the universe. Kelvin’s theory was immensely popular at the time and led to considerable efforts in the enumeration and classification of knots and links. Kelvin’s coauthor and colleague, the Scottish physicist Peter Guthrie Tait, published the first knot tables in 1876. He displayed and discussed the following links:



In this display, No. 15 shows the Borromean Rings, while No. 18 is an apparently different link that, however, shares the same characteristics: It consists of three closed curves that are pairwise not linked, whereas the whole diagram does not seem to come apart, it represents a nontrivial link. Tait indeed claimed that the links No. 15 and No. 18 were not equivalent, apparently based on the assumption that any *alternating* diagram of a link (where along any string under- and over-crossings alternate) has a minimal number of crossings among all possible diagrams. This long-standing “Tait conjecture” was proved more than 100 years later, by Thistlethwaite, Kauffman, and Murasugi in 1987. (Tait’s examples No. 16 and 17 have only one component, so they are knots. All four examples fall into a larger family that has been described and studied as the “Turk’s head links.”)

In 1892, the geometer Hermann Brunn introduced a much more general family of objects that we now call *Brunnian links*: k -component links in which any subcollection of $k - 1$ of the components is trivial. Tait’s links No. 15 (the Borromean rings) and No. 18 are examples.

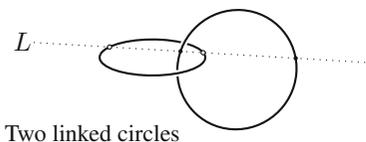
Back to the Borromean rings: Indeed they *cannot* be built from three perfect circles. The first proof for this appeared in 1987 in a long differential geometry paper by Michael F. Freedman and Richard Skora. Their beautiful geometric idea, “getting movies from spherical domes,” is very powerful: It solves the problem not only for the Borromean rings, but shows that any Brunnian link built from perfect circles is trivial. It can also be generalized to links formed by k -spheres in $(2k + 1)$ -dimensional space. Our presentation is based on a short unpublished note “Circle links” by Ian Agol.

Theorem 1. *If a link consists of disjoint perfect circles that are pairwise not linked, then the link is trivial.*

■ **Proof.** Moving each of the circles just a little bit, we may assume that they lie in planes that are distinct, no two of the planes are parallel, and none of the planes spanned by one of the circles contains the center of a second circle. (This first preparatory step is not necessary, but it simplifies some later parts of the proof quite a bit.)

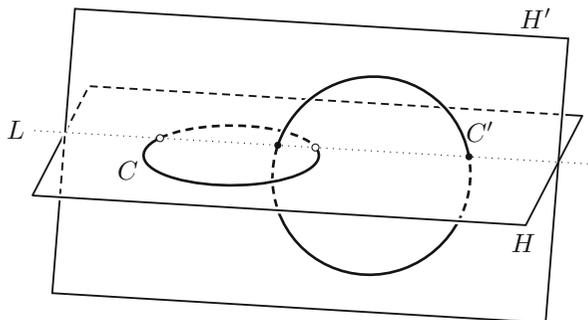
There are several different ways to define what it means that two disjoint circles in \mathbb{R}^3 are *linked*. Let us here use the following: Two circles are linked if one of them intersects (and not only touches) the disk spanned by the other one exactly once.

Let the circles be $C, C' \subseteq \mathbb{R}^3$, let D, D' be the flat disks they bound, and let H, H' be the planes they span. If C' intersects the disk D in one point, then this point lies both in $D \subseteq H$ as well as on $C' \subseteq D' \subseteq H'$, so in particular it lies in the intersection of the two planes H and H' , which is a line, $L := H \cap H'$. As this line lies in the plane H and contains a point in the interior of the disk D , it intersects C in exactly two points. The circle C' intersects the plane H once in the interior of D , so there has to be a second intersection point, which lies again on the line L , but outside D .



Two linked circles

We conclude that there are two pairs of intersection points given by $C \cap L$ and $C' \cap L$, and these two pairs alternate on the line L . In particular, we find in this situation that also C intersects the disk D' in one point.



It turns out that this “alternating property” characterizes linked circles: If two circles C, C' are not linked, then one of them misses (or only touches) the disk spanned by the other one. In that case we find fewer than four points of $C \cup C'$ on the line L , or the four points do not alternate.

For the proof of the theorem we now take a configuration of n circles in \mathbb{R}^3 that are pairwise not linked and erect *spherical domes* above the disks spanned by the circles. This entails a bold step into the fourth dimension, since we add an extra coordinate. Don't worry about how to visualize this — in the end we will look at these dome functions defined on lines, so all arguments can be visualized and verified in planar diagrams.

The spherical domes are constructed as follows: For any circle $C \subseteq \mathbb{R}^3$ with center c and radius r there is a 2-dimensional hemisphere $S \subseteq \mathbb{R}^4$, which may be obtained as the graph

$$\{(x, h(x)) \in \mathbb{R}^3 \times \mathbb{R} : x \in D\}$$

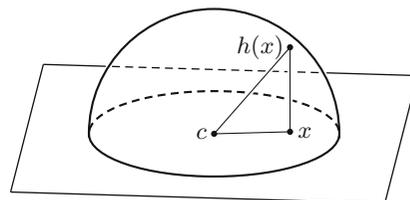
of the function

$$h : D \rightarrow \mathbb{R}, \quad h(x) := \sqrt{r^2 - |x - c|^2}$$

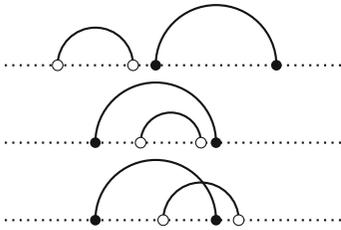
on the closed disk D spanned by the circle C . The dome S is *orthogonal above* D in the following sense: If we project it to \mathbb{R}^3 by the orthogonal projection $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3, (x, t) \mapsto x$, that “forgets the last coordinate,” then the image of the dome will be the disk D .

Claim. *If two disjoint circles $C, C' \subseteq \mathbb{R}^3$ are not linked, then their spherical domes $S, S' \subseteq \mathbb{R}^3 \times \mathbb{R}$ do not intersect.*

Proof of the Claim. We prove that if the domes S, S' above the discs D, D' spanned by two disjoint circles $C, C' \subseteq \mathbb{R}^3$ intersect, then the circles are linked. For this, let (x_0, t_0) be a point in the intersection $S \cap S'$. As (x_0, t_0) lies in S , we get $x_0 \in D$. Similarly, as (x_0, t_0) lies in S' , we get that $x_0 \in D'$. Hence x_0 lies in the line L , and it also lies on $D \cap D'$, where both “lifting functions” h and h' are defined.



$$|x - c|^2 + |h(x) - 0|^2 = r^2$$

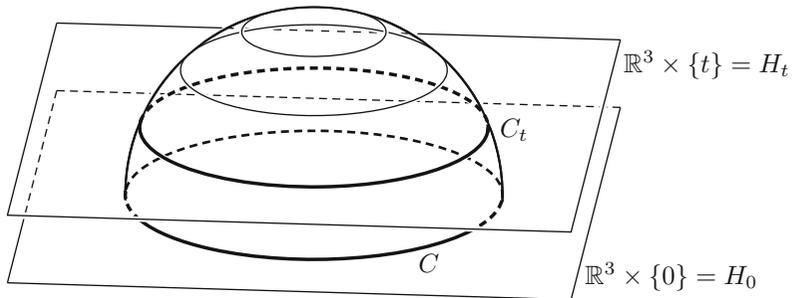


The half-circles above L intersect if and only if their end points alternate on the line L .

The lifting functions h, h' describe spherical domes defined on D resp. D' . Restricted to the line L , the functions h and h' define perfect half-circles, with domain of definition $D \cap L$ resp. $D' \cap L$. (This is the crucial point in the proof: Above an ellipse one cannot build a dome that restricts to half-circle arcs.)

Since the half-circles above $D \cap L$ resp. $D' \cap L$ intersect, their pairs of end points $S \cap L$ and $S' \cap L$ alternate on L , as illustrated in the margin. Hence the circles C and C' are linked. This finishes the proof of the claim.

Back to the configuration of disjoint perfect circles in \mathbb{R}^3 that are pairwise not linked. Freedman and Skora's brilliant idea was to use the disjoint domes guaranteed by the claim in order to construct a "movie" that *shows us* how to separate the circles in the link by a continuous motion. For this, we identify the original space \mathbb{R}^3 , which contains the link, with the slice $\mathbb{R}^3 \times \{0\}$ of the space $\mathbb{R}^3 \times \mathbb{R}$ that contains the domes; that is, the extra coordinate t is interpreted as time, and we start our movie at $t = 0$ with the original link. If we now continuously increase the fourth (time) coordinate, then what we see in time slices $\mathbb{R}^3 \times \{t\}$ is a movie in which each of the circles shrinks to a point, and then disappears.



Here is a key observation: While a circle shrinks in this movie, *the center of the circle and the plane spanned by the circle do not change*. Furthermore, the circles stay disjoint since the domes are disjoint by the claim, and thus they remain pairwise non-linked.

We can stop the shrinking for each circle at some time when the circle is so small that it does not any more intersect a plane that is spanned by any one of the other circles. Moreover, also the disk spanned by this little circle does not intersect any of the other circle planes — neither at the point of time where we stop its shrinking, nor at any later time.

Thus the movie will end with all circles shrunk so far that they have disjoint spanning disks: The circles are completely separate, and thus the link is trivial. \square

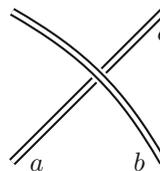
In particular, we have just proved that any Brunnian link built from perfect circles can be taken apart in a motion which maintains perfect circles along the way. It remains an open problem, however, whether each of the circles could keep its size in such a motion picture.

With Theorem 1, we have established that the Borromean rings cannot be built from perfect circles — assuming that we know that the Borromean rings form a nontrivial link. Do we know that? It is by no means easy to prove rigorously for *any* knot or link that it is nontrivial . . . However, the eminent knot theorist Ralph Fox has invented a strikingly simple method to achieve this — reportedly he designed it “in an effort to make the subject accessible to everyone” while teaching knot theory to undergraduates at Haverford College in 1956. Its first published trace can be found in an exercise of a 1963 knot theory textbook by Crowell and Fox. Thirty years later Ollie Nanyes observed that Fox’s method also solves the problem for the Borromean rings.

Theorem 2. *The Borromean rings are nontrivial, and they are also not equivalent to Tait’s link No. 18.*

■ **Proof.** For every $n \geq 2$, a *Fox n -labeling* of a link diagram labels each arc of the diagram by an integer modulo n , such that at each crossing the two integers a and c of the arcs that end at the crossing and the label b of the arc of the overpass satisfy the *crossing relation*

$$a + c \equiv 2b \pmod{n}.$$

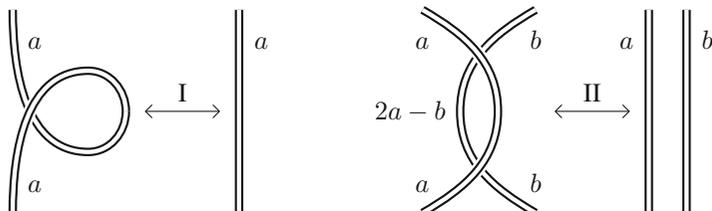


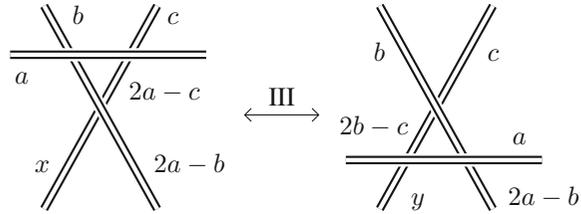
The crossing relation

Each link diagram has n *trivial* n -labelings, which use the same label for all the arcs of the diagram, so we are interested in *nontrivial* labelings, which use at least two different labels. For example, any link that consists of two disjoint “far away” parts in the plane has at least n^2 different Fox n -labelings. Now we observe a crucial fact:

Claim. *If two diagrams represent equivalent links, then they have the same number of Fox n -labelings.*

As explained in the appendix to this chapter, the diagrams for equivalent links are connected by continuous deformations and a finite sequence of Reidemeister moves of types I, II, and III; so all we have to check is that Reidemeister moves don’t change the number of Fox n -labelings. This is apparent from the following sketches, where in each of the separate drawings all the relations among the labels of different arcs are forced by the crossing relations:





In particular, for arbitrary labels $a, b,$ and $c,$ the Reidemeister moves of type III the crossing relations finally force us to put labels

$$x \equiv 2(2a - b) - (2a - c) \equiv c + 2a - 2b$$

before the move and

$$y \equiv 2a - (2b - c) \equiv c + 2a - 2b$$

after the move. This establishes the Claim!

Now we simply have to count the labelings. The interesting observations will occur for odd $n \geq 3.$

For the *Borromean rings* we claim that all Fox n -labelings are trivial if $n \geq 3$ is odd: If in the standard diagram for the Borromean rings the outer arcs get the labels $a, b,$ and c (as sketched in the margin), then the outer crossings force the inner arcs to have labels $2b - a, 2c - b,$ and $2a - c,$ and at the inner crossings of the diagram we need that

$$2(2b - a) \equiv c + (2a - c), \quad 2(2c - b) \equiv a + (2b - a), \quad 2(2a - c) \equiv b + (2c - b),$$

that is $4a \equiv 4b \equiv 4c,$ and hence $a \equiv b \equiv c \pmod{n},$ as n is odd. (For every even $n \geq 2,$ nontrivial labelings exist.) In particular, the Borromean rings have only the trivial Fox 3-labelings or 5-labelings.

For *Tait's link No. 18,* a very similar calculation, with the labels $a, b, c, d, e,$ and f assigned to the outer arcs, leads to inner labels $2a - b, 2b - c, 2c - d,$ etc., and then finally to the conditions

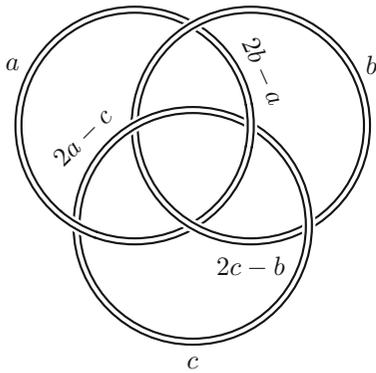
$$a - d \equiv 4(b - c), \quad b - e \equiv 4(c - d), \quad c - f \equiv 4(d - e), \quad \dots \pmod{n}.$$

For $n = 3,$ this yields $a - b + c - d \equiv 0, b - c + d - e \equiv 0,$ etc., and we quickly derive that $a \equiv b \equiv \dots \equiv f,$ so again there are only the trivial 3-labelings.

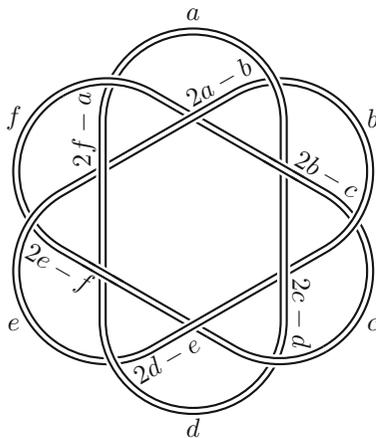
However, for $n = 5$ we find that we have to solve the equations $a + b \equiv c + d, b + c \equiv d + e,$ etc., and this leads us to the solutions with arbitrary $a \equiv c \equiv e$ and $b \equiv d \equiv f$ (and no others). Thus there are $5^2 = 25$ Fox 5-labelings for this link.

The *trivial three component link* clearly has n^3 Fox n -labelings, that is, it has 27 Fox 3-labelings and 125 Fox 5-labelings.

Thus the Borromean rings, Tait's link No. 18, and the trivial link with three components have different numbers of Fox 5-labelings (5, 25, and 125, respectively), so they are nonequivalent links. \square



Labels for the Borromean rings

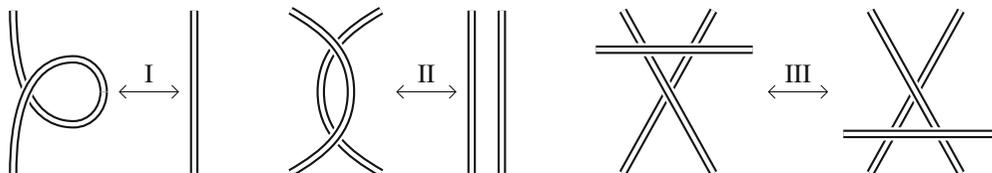


Labels for Tait's link No. 18

Appendix: Basic notions on knots and links

Topologists define a *knot* as the image of a continuous embedding of a circle in \mathbb{R}^3 ; a differential geometer might add that we are not interested in “wild” knots, but only in “tame” ones that are smooth curves. A *link* is obtained from a smooth embedding of a disjoint union of disjoint circles, known as the *components* of the link. Knots and links can also be treated as combinatorial objects, as any projection of a smooth knot or link to the plane along a sufficiently “generic” direction leads to a representation by a *diagram*, that is, a drawing of the knot or link by smooth curves in the plane with only a finite number of crossings, at which exactly two different parts of the knot or link cross — and where we indicate an over- or under-pass by a “trompe l’œil”-like fashion.

When are two knots, or two links, “the same”? Topologically, two links L and L' are defined to be *equivalent* if there is an orientation-preserving homeomorphism between (\mathbb{R}^3, L) and (\mathbb{R}^3, L') , that is, a continuous and bijective map $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with a continuous inverse such that $h(L) = L'$. Geometrically, we can describe this by a continuous deformation of space that moves L to L' . Such deformations might be hard to describe and analyze, but in 1926 Kurt Reidemeister proved a very useful combinatorial characterization: Two diagrams drawn in the plane describe equivalent knots or links if and only if one can be obtained from the other by continuous deformations and a finite number of local operations that are now known as the *Reidemeister moves* of types I, II, and III.



The “if” part of Reidemeister’s theorem is quite obvious. For the “only if” direction one studies a smooth deformation of L to L' , where also the directions and curvatures along the curves are required to change continuously. If we then maintain a “general position” projection to a plane, this will give us a continuous deformation of one diagram to the other with only a finite number of Reidemeister-type moves on the way.

A knot is *trivial* if it is equivalent to a perfect (geometric) circle in \mathbb{R}^3 , or equivalently, if it admits a spanning disk whose interior is disjoint from the knot. More generally, a link with k components is *trivial* if it is equivalent to a link formed by k “far apart” circles that have disjoint spanning disks.

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