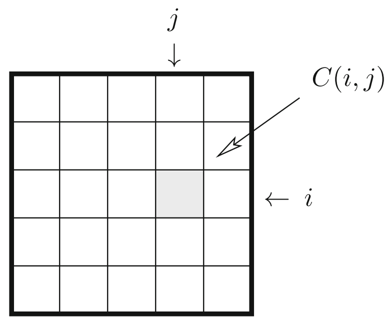
The Dinitz problem

Chapter 38

The four-color problem was a main driving force for the development of graph theory as we know it today, and coloring is still a topic that many graph theorists like best. Here is a simple-sounding coloring problem, raised by Jeff Dinitz in 1978, which defied all attacks until its astonishingly simple solution by Fred Galvin fifteen years later.



> *Consider n^2 cells arranged in an $n \times n$ square, and let (i, j) denote the cell in row i and column j. Suppose that for every cell (i, j) we are given a set $C(i, j)$ of n colors.*
> *Is it then always possible to color the whole array by picking for each cell (i, j) a color from its set $C(i, j)$ such that the colors in each row and each column are distinct?*

As a start consider the case when all color sets $C(i, j)$ are the same, say $\{1, 2, \ldots, n\}$. Then the Dinitz problem reduces to the following task: Fill the $n \times n$ square with the numbers $1, 2, \ldots, n$ in such a way that the numbers in any row and column are distinct. In other words, any such coloring corresponds to a Latin square, as discussed in the previous chapter. So, in this case, the answer to our question is “yes.”

Since this is so easy, why should it be so much harder in the general case when the set $C := \bigcup_{i,j} C(i, j)$ contains even more than n colors? The difficulty derives from the fact that not every color of C is available at each cell. For example, whereas in the Latin square case we can clearly choose an arbitrary permutation of the colors for the first row, this is not so anymore in the general problem. Already the case $n = 2$ illustrates this difficulty.

$\{1,2\}$	$\{2,3\}$
$\{1,3\}$	$\{2,3\}$

Suppose we are given the color sets that are indicated in the figure. If we choose the colors 1 and 2 for the first row, then we are in trouble since we would then have to pick color 3 for both cells in the second row.

Before we tackle the Dinitz problem, let us rephrase the situation in the language of graph theory. As usual we only consider graphs $G = (V, E)$ without loops and multiple edges. Let $\chi(G)$ denote the *chromatic number* of the graph, that is, the smallest number of colors that one can assign to the vertices such that adjacent vertices receive different colors.

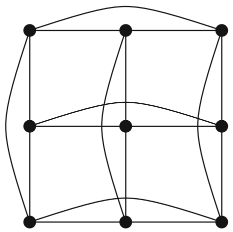
In other words, a coloring calls for a partition of V into classes (colored with the same color) such that there are no edges within a class. Calling a set $A \subseteq V$ *independent* if there are no edges within A, we infer that the chromatic number is the smallest number of independent sets which partition the vertex set V.

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_38

In 1976 Vizing, and three years later Erdős, Rubin, and Taylor, studied the following coloring variant which leads us straight to the Dinitz problem. Suppose in the graph $G = (V, E)$ we are given a set $C(v)$ of colors for each vertex v. A *list coloring* is a coloring $c : V \longrightarrow \bigcup_{v \in V} C(v)$ where $c(v) \in C(v)$ for each $v \in V$. The definition of the *list chromatic number* $\chi_\ell(G)$ should now be clear: It is the smallest number k such for *any* list of color sets $C(v)$ with $|C(v)| = k$ for all $v \in V$ there always exists a list coloring. Of course, we have $\chi_\ell(G) \leq |V|$ (we never run out of colors). Since ordinary coloring is just the special case of list coloring when all sets $C(v)$ are equal, we obtain for any graph G

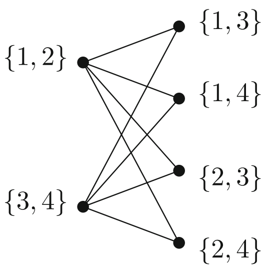
$$\chi(G) \;\leq\; \chi_\ell(G).$$

To get back to the Dinitz problem, consider the graph S_n which has as vertex set the n^2 cells of our $n \times n$ array with two cells adjacent if and only if they are in the same row or column.


The graph S_3

Since any n cells in a row are pairwise adjacent we need at least n colors. Furthermore, any coloring with n colors corresponds to a Latin square, with the cells occupied by the same number forming a color class. Since Latin squares, as we have seen, exist, we infer $\chi(S_n) = n$, and the Dinitz problem can now be succinctly stated as

$$\chi_\ell(S_n) \;=\; n?$$



One might think that perhaps $\chi(G) = \chi_\ell(G)$ holds for any graph G, but this is a long shot from the truth. Consider the graph $G = K_{2,4}$. The chromatic number is 2 since we may use one color for the two left vertices and the second color for the vertices on the right. But now suppose that we are given the color sets indicated in the figure.

To color the left vertices we have the four possibilities $1|3$, $1|4$, $2|3$ and $2|4$, but any one of these pairs appears as a color set on the right-hand side, so a list coloring is not possible. Hence $\chi_\ell(G) \geq 3$, and the reader may find it fun to prove $\chi_\ell(G) = 3$ (there is no need to try out all possibilities!). Generalizing this example, it is not hard to find graphs G where $\chi(G) = 2$, but $\chi_\ell(G)$ is arbitrarily large! So the list coloring problem is not as easy as it looks at first glance.

Back to the Dinitz problem. A significant step towards the solution was made by Jeanette Janssen in 1992 when she proved $\chi_\ell(S_n) \leq n + 1$, and the *coup de grâce* was delivered by Fred Galvin by ingeniously combining two results, both of which had long been known. We are going to discuss these two results and show then how they imply $\chi_\ell(S_n) = n$.

First we fix some notation. Suppose v is a vertex of the graph G, then we denote as before by $d(v)$ the *degree* of v. In our square graph S_n every vertex has degree $2n - 2$, accounting for the $n - 1$ other vertices in the same row and in the same column. For a subset $A \subseteq V$ we denote by G_A the subgraph which has A as vertex set and which contains all edges of G between vertices of A. We call G_A the subgraph induced by A, and say that H is an *induced subgraph* of G if $H = G_A$ for some A.

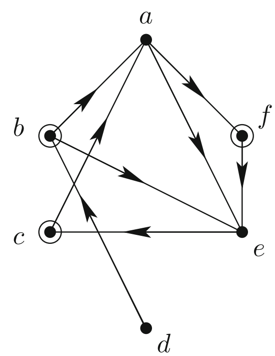
To state our first result we need *directed graphs* $\vec{G} = (V, E)$, that is, graphs where every edge e has an orientation. The notation $e = (u, v)$ means that there is an arc e, also denoted by $u \longrightarrow v$, whose initial vertex is u and whose terminal vertex is v. It then makes sense to speak of the *outdegree* $d^+(v)$ resp. the *indegree* $d^-(v)$, where $d^+(v)$ counts the number of edges with v as initial vertex, and similarly for $d^-(v)$; furthermore, $d^+(v) + d^-(v) = d(v)$. When we write G, we mean the graph $\vec{G}$ without the orientations.

The following concept originated in the analysis of games and will play a crucial role in our discussion.

Definition 1. Let $\vec{G} = (V, E)$ be a directed graph. A *kernel* $K \subseteq V$ is a subset of the vertices such that

(i) K is independent in G, and

(ii) for every $u \notin K$ there exists a vertex $v \in K$ with an edge $u \longrightarrow v$.



Let us look at the example in the figure. The set $\{b, c, f\}$ constitutes a kernel, but the subgraph induced by $\{a, c, e\}$ does not have a kernel since the three edges cycle through the vertices.

With all these preparations we are ready to state the first result.

Lemma 1. *Let $\vec{G} = (V, E)$ be a directed graph, and suppose that for each vertex $v \in V$ we have a color set $C(v)$ that is larger than the outdegree, $|C(v)| \geq d^+(v) + 1$. If every induced subgraph of $\vec{G}$ possesses a kernel, then there exists a list coloring of G with a color from $C(v)$ for each v.*

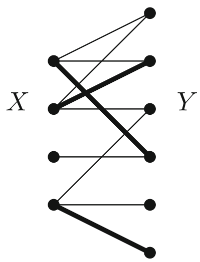
■ **Proof.** We proceed by induction on $|V|$. For $|V| = 1$ there is nothing to prove. Choose a color $c \in C = \bigcup_{v \in V} C(v)$ and set

$$A(c) := \{v \in V : c \in C(v)\}.$$

By hypothesis, the induced subgraph $G_{A(c)}$ possesses a kernel $K(c)$. Now we color all $v \in K(c)$ with the color c (this is possible since $K(c)$ is independent), and delete $K(c)$ from G and c from C. Let G' be the induced subgraph of G on $V \backslash K(c)$ with $C'(v) = C(v) \backslash c$ as the new list of color sets. Notice that for each $v \in A(c) \backslash K(c)$, the outdegree $d^+(v)$ is decreased by at least 1 (due to condition (ii) of a kernel). So $d^+(v) + 1 \leq |C'(v)|$ still holds in $\vec{G}'$. The same condition also holds for the vertices outside $A(c)$, since in this case the color sets $C(v)$ remain unchanged. The new graph G' contains fewer vertices than G, and we are done by induction. □

The method of attack for the Dinitz problem is now obvious: We have to find an orientation of the graph S_n with outdegrees $d^+(v) \leq n - 1$ for all v and which ensures the existence of a kernel for all induced subgraphs. This is accomplished by our second result.

Again we need a few preparations. Recall (from Chapter 11) that a *bipartite* graph $G = (X \cup Y, E)$ is a graph with the following property: The vertex set V is split into two parts X and Y such that every edge has one endvertex in X and the other in Y. In other words, the bipartite graphs are precisely those which can be colored with two colors (one for X and one for Y).



A bipartite graph with a matching

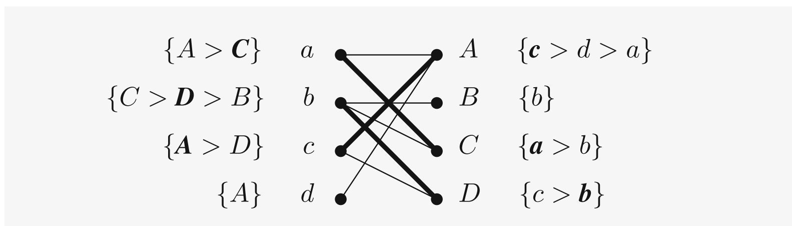
Now we come to an important concept, "stable matchings," with a down-to-earth interpretation. A *matching* M in a bipartite graph $G = (X \cup Y, E)$ is a set of edges such that no two edges in M have a common endvertex. In the displayed graph the edges drawn in bold lines constitute a matching.

Consider X to be a set of men and Y a set of women and interpret $uv \in E$ to mean that u and v might marry. A matching is then a mass-wedding with no person committing bigamy. For our purposes we need a more refined (and more realistic?) version of a matching, suggested by David Gale and Lloyd S. Shapley. Clearly, in real life every person has preferences, and this is what we add to the set-up. In $G = (X \cup Y, E)$ we assume that for every $v \in X \cup Y$ there is a ranking of the set $N(v)$ of vertices adjacent to v, $N(v) = \{z_1 > z_2 > \cdots > z_{d(v)}\}$. Thus z_1 is the top choice for v, followed by z_2, and so on.

Definition 2. A matching M of $G = (X \cup Y, E)$ is called *stable* if the following condition holds: Whenever $uv \in E \backslash M$, $u \in X$, $v \in Y$, then either $uy \in M$ with $y > v$ in $N(u)$ or $xv \in M$ with $x > u$ in $N(v)$, or both.

In our real life interpretation a set of marriages is stable if it never happens that u and v are not married but u prefers v to his partner (if he has one at all) and v prefers u to her mate (if she has one at all), which would clearly be an unstable situation.

Before proving our second result let us take a look at the following example:



The bold edges constitute a stable matching. In each priority list, the choice leading to a stable matching is printed bold.

Notice that in this example there is a unique largest matching M with four edges, $M = \{aC, bB, cD, dA\}$, but M is not stable (consider cA).

Lemma 2. *A stable matching always exists.*

■ **Proof.** Consider the following algorithm. In the first stage all men $u \in X$ propose to their top choice. If a girl receives more than one proposal she picks the one she likes best and keeps him on a string, and if she receives just one proposal she keeps that one on a string. The remaining men are rejected and form the reservoir R. In the second stage all men in R propose to their next choice. The women compare the proposals (together with the one on the string, if there is one), pick their favorite and put him on the string. The rest is rejected and forms the new set R. Now the men in R propose to their next choice, and so on. A man who has proposed to his last choice and is again rejected drops out from further consideration (as well as from the reservoir). Clearly, after some time the reservoir R is empty, and at this point the algorithm stops.

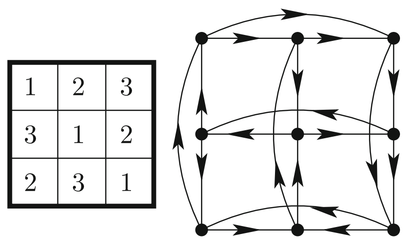
Claim. *When the algorithm stops, then the men on the strings together with the corresponding girls form a stable matching.*

Notice first that the men on the string of a particular girl move there in increasing preference (of the girl) since at each stage the girl compares the new proposals with the present mate and then picks the new favorite. Hence if $uv \in E$ but $uv \notin M$, then either u never proposed to v in which case he found a better mate before he even got around to v, implying $uy \in M$ with $y > v$ in $N(u)$, or u proposed to v but was rejected, implying $xv \in M$ with $x > u$ in $N(v)$. But this is exactly the condition of a stable matching. □

Putting Lemmas 1 and 2 together, we now get Galvin's solution of the Dinitz problem.

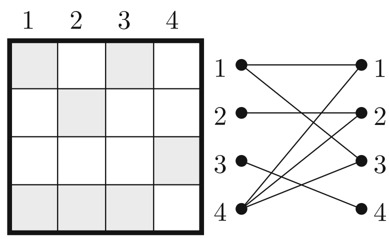
Theorem. *We have* $\chi_\ell(S_n) = n$ *for all* n.

■ **Proof.** As before we denote the vertices of S_n by (i,j), $1 \leq i,j \leq n$. Thus (i,j) and (r,s) are adjacent if and only if $i = r$ or $j = s$. Take any Latin square L with letters from $\{1,2,\dots,n\}$ and denote by $L(i,j)$ the entry in cell (i,j). Next make S_n into a directed graph $\vec{S}_n$ by orienting the horizontal edges $(i,j) \longrightarrow (i,j')$ if $L(i,j) < L(i,j')$ and the vertical edges $(i,j) \longrightarrow (i',j)$ if $L(i,j) > L(i',j)$. Thus, horizontally we orient from the smaller to the larger element, and vertically the other way around. (In the margin we have an example for $n = 3$.)



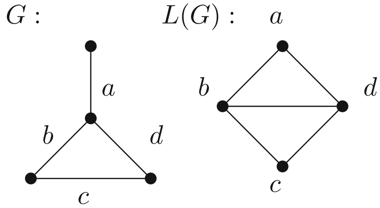
Notice that we obtain $d^+(i,j) = n-1$ for all (i,j). In fact, if $L(i,j) = k$, then $n-k$ cells in row i contain an entry larger than k, and $k-1$ cells in column j have an entry smaller than k.

By Lemma 1 it remains to show that every induced subgraph of $\vec{S}_n$ possesses a kernel. Consider a subset $A \subseteq V$, and let X be the set of rows of L, and Y the set of its columns. Associate to A the bipartite graph $G = (X \cup Y, A)$, where every $(i,j) \in A$ is represented by the edge ij with $i \in X, j \in Y$. In the example in the margin the cells of A are shaded.



The orientation on S_n naturally induces a ranking on the neighborhoods in $G = (X \cup Y, A)$ by setting $j' > j$ in $N(i)$ if $(i,j) \longrightarrow (i,j')$ in $\vec{S}_n$ respectively $i' > i$ in $N(j)$ if $(i,j) \longrightarrow (i',j)$. By Lemma 2, $G = (X \cup Y, A)$ possesses a stable matching M. This M, viewed as a subset of A, is our desired kernel! To see why, note first that M is independent in A since as edges in $G = (X \cup Y, A)$ they do not share an endvertex i or j. Secondly, if $(i,j) \in A \backslash M$, then by the definition of a stable matching there either exists $(i,j') \in M$ with $j' > j$ or $(i',j) \in M$ with $i' > i$, which for $\vec{S}_n$ means $(i,j) \longrightarrow (i,j') \in M$ or $(i,j) \longrightarrow (i',j) \in M$, and the proof is complete. □

To end the story let us go a little beyond. The reader may have noticed that the graph S_n arises from a bipartite graph by a simple construction. Take the complete bipartite graph, denoted by $K_{n,n}$, with $|X| = |Y| = n$, and *all* edges between X and Y. If we consider the edges of $K_{n,n}$ as vertices of a new graph, joining two such vertices if and only if as edges in $K_{n,n}$ they have a common endvertex, then we clearly obtain the square graph S_n. Let us say that S_n is the *line graph* of $K_{n,n}$. Now this same construction can be performed on any graph G with the resulting graph called the *line graph* $L(G)$ of G.



Construction of a line graph

In general, call H a *line graph* if $H = L(G)$ for some graph G. Of course, not every graph is a line graph, an example being the graph $K_{2,4}$ that we considered earlier, and for this graph we have seen $\chi(K_{2,4}) < \chi_\ell(K_{2,4})$. But what if H *is* a line graph? By adapting the proof of our theorem it can easily be shown that $\chi(H) = \chi_\ell(H)$ holds whenever H is the line graph of a *bipartite* graph, and the method may well go some way in verifying the supreme conjecture in this field:

Does $\chi(H) = \chi_\ell(H)$ *hold for every line graph* H*?*

Very little is known about this conjecture, and things look hard — but after all, so did the Dinitz problem twenty years ago.

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