



“How small can a set in the plane be in which you can turn a needle of length 1 completely around?”

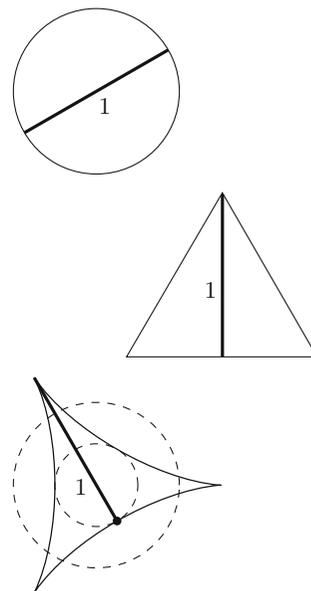
This beautiful question was posed by the Japanese mathematician Sōichi Kakeya in 1917. It gained immediate prominence and, together with its higher-dimensional analogs, helped initiate a whole new field, today called *geometric measure theory*. To be precise, by “turning around” Kakeya had a continuous motion in mind that returns the needle to the original position with its ends reversed, like a Samurai whirling his pole. Any such motion takes place in a compact subset of the plane.

Obviously, a disk of diameter 1 is such a *Kakeya needle set* (of area  $\frac{\pi}{4} \approx 0.785$ ), as is the equilateral triangle of height 1 that has area  $\frac{1}{\sqrt{3}} \approx 0.577$ . For *convex* regions Julius Pal showed that this is the minimum, but in general we can do better: The three-pointed *deltoid* in the margin is also a Kakeya needle set, as seen by moving the inner point around the small circle. The area of the deltoid is  $\frac{\pi}{8} \approx 0.393$ , and Kakeya seems to have thought that this is the minimum for connected sets.

So it was a big surprise when a few years after the question was posed Abram Samoilovitch Besicovitch produced needle sets of arbitrarily small area. His examples were rather complicated with many holes and large diameter, but in a remarkable paper Frederick Cunningham Jr. showed that one can even find simply connected needle sets of arbitrarily small area inside the circle of diameter 2.

As a matter of fact, Besicovitch was initially interested in a closely related problem, which he then applied to solve the needle problem. Call a compact set  $K \subseteq \mathbb{R}^n$  a *Kakeya set* (or, more aptly, a *Besicovitch set*) if it contains a unit line segment in every direction. Besicovitch proved the spectacular result that for every dimension there are Kakeya sets of measure 0. But how can this be? Our intuition tells us that these sets need to be somehow spread out, since they contain segments in every direction! (In contrast, one can show that all Kakeya needle sets, which not only contain a needle in every direction, but in which the needle can turn, have positive measure.)

Now these were the years when the notion of (topological) dimension came into being at the hands of Lebesgue, Menger, Hausdorff and others, which precisely captured this “spreading out” by various covering conditions; here we use the Hausdorff dimension  $\text{hd}(K)$ . We don’t need the details of the



definition: Let us just note that the Euclidean space  $\mathbb{R}^n$  has Hausdorff dimension  $n$ , and that  $\text{hd}$  is a monotone function, so every  $K \subseteq \mathbb{R}^n$  satisfies  $\text{hd}(K) \leq n$ .

**The Kakeya conjecture.** *Every Kakeya set in  $\mathbb{R}^n$  has Hausdorff dimension  $n$ .*

The conjecture is true for  $n = 1$  and  $2$ , but it is open for all  $n \geq 3$ , and it appears to get more difficult as the dimension increases. Today it is considered to be one of the major open problems in geometric measure theory.

In an inspiring paper from 1999 Thomas Wolff gave the problem a completely new twist by suggesting to look at *finite* fields  $F$ . Consider the vector space  $F^n$ . Let us call  $K \subseteq F^n$  a (*finite*) *Kakeya set* if  $K$  contains a line in every direction, meaning that to every nonzero vector  $v \in F^n$  there exists some  $w \in F^n$  such that the line  $L = \{w + tv : t \in F\}$  is in  $K$ . Wolff posed the following finite version of the Euclidean Kakeya problem:

**The finite Kakeya problem.** *Is there a constant  $c = c(n)$ , only depending on  $n$  but not on  $|F|$ , such that every Kakeya set  $K \subseteq F^n$  satisfies*

$$|K| \geq c|F|^n?$$

Clearly, this is true for  $n = 1$ , the only Kakeya set being all of  $F$ , and it is not hard to prove for  $n = 2$ , but for higher dimensions progress was again slow, until Zeev Dvir provided in his 2008 dissertation a beautiful and stunningly simple proof: All we need are two results about polynomials in  $n$  variables!

Let us fix some notation.  $F[x_1, \dots, x_n]$  denotes the ring of polynomials  $p(x_1, \dots, x_n)$  over the finite field  $F$ . A monomial  $x_1^{s_1} \cdots x_n^{s_n}$  is sometimes written shortly as  $x^s$ , where  $\sum_{i=1}^n s_i$  is the degree of  $x^s$ . The *degree*  $\deg p$  of  $p(x) = \sum a_s x^s$  is the maximum degree of the monomials  $x^s$  with nonzero coefficient  $a_s$ . The *zero polynomial* has all  $a_s = 0$  and is said to have degree  $-1$ . The polynomial  $p(x)$  *vanishes* on  $E \subseteq F^n$  if  $p(a) = 0$  holds for all  $a \in E$ .

The two ingredients of the proof generalize the following well-known facts about polynomials in one variable:

- (1) Every polynomial of degree  $d \geq 0$  in one variable has at most  $d$  roots.
- (2) For every set  $E \subseteq F$  of size  $|E| \leq d$  there is a nonzero polynomial  $p(x)$  of degree at most  $d$  that vanishes on  $E$ .

In the following  $q = |F|$  shall denote the size of  $F$ .

Just take  $p_E(x) := \prod_{a \in E} (x - a)$ .

In particular, a nonzero polynomial can vanish on all of  $F$ .

**Lemma 1.** Every nonzero polynomial  $p(x) \in F[x_1, \dots, x_n]$  of degree  $d$  has at most  $dq^{n-1}$  roots in  $F^n$ .

■ **Proof.** We use induction on  $n$ , with fact (1) above as the starting case  $n = 1$ . Let us split  $p(x)$  into summands according to the powers of  $x_n$ ,

$$p(x) = g_0 + g_1x_n + g_2x_n^2 + \dots + g_\ell x_n^\ell,$$

where  $g_i \in F[x_1, \dots, x_{n-1}]$  for  $0 \leq i \leq \ell \leq d$ , and  $g_\ell$  is nonzero. We write every  $v \in F^n$  in the form  $v = (a, b)$  with  $a \in F^{n-1}$ ,  $b \in F$ , and estimate the number of roots  $p(a, b) = 0$ .

**Case 1.** Roots  $(a, b)$  with  $g_\ell(a) = 0$ .

Since  $g_\ell \neq 0$  and  $\deg g_\ell \leq d - \ell$ , by induction the polynomial  $g_\ell$  has at most  $(d - \ell)q^{n-2}$  roots in  $F^{n-1}$ , and for each  $a$  there are at most  $q$  different choices for  $b$ , which gives at most  $(d - \ell)q^{n-1}$  such roots for  $p(x)$  in  $F^n$ .

**Case 2.** Roots  $(a, b)$  with  $g_\ell(a) \neq 0$ .

Here  $p(a, x_n) \in F[x_n]$  is not the zero polynomial in the single variable  $x_n$ , it has degree  $\ell$ , and hence for each  $a$  by (1) there are at most  $\ell$  elements  $b$  with  $p(a, b) = 0$ . Since the number of  $a$ 's is at most  $q^{n-1}$  we get at most  $\ell q^{n-1}$  roots for  $p(x)$  in this way.

Summing the two cases gives at most

$$(d - \ell)q^{n-1} + \ell q^{n-1} = dq^{n-1}$$

roots for  $p(x)$ , as asserted. □

**Lemma 2.** For every set  $E \subseteq F^n$  of size  $|E| < \binom{n+d}{d}$  there is a nonzero polynomial  $p(x) \in F[x_1, \dots, x_n]$  of degree at most  $d$  that vanishes on  $E$ .

■ **Proof.** Consider the vector space  $V_d$  of all polynomials in  $F[x_1, \dots, x_n]$  of degree at most  $d$ . A basis for  $V_d$  is provided by the monomials  $x_1^{s_1} \dots x_n^{s_n}$  with  $\sum s_i \leq d$ :

$$1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_1^3, \dots, x_n^d.$$

The following pleasing argument shows that the number of monomials  $x_1^{s_1} \dots x_n^{s_n}$  of degree at most  $d$  equals the binomial coefficient  $\binom{n+d}{d}$ . What we want to count is the number of  $n$ -tuples  $(s_1, \dots, s_n)$  of nonnegative integers with  $s_1 + \dots + s_n \leq d$ . To do this, we map every  $n$ -tuple  $(s_1, \dots, s_n)$  to the increasing sequence

$$s_1 + 1 < s_1 + s_2 + 2 < \dots < s_1 + \dots + s_n + n,$$

which determines an  $n$ -subset of  $\{1, 2, \dots, d + n\}$ . The map is bijective, so the number of monomials is  $\binom{d+n}{n} = \binom{n+d}{d}$ .

Next look at the vector space  $F^E$  of all functions  $f : E \rightarrow F$ ; it has dimension  $|E|$ , which by assumption is less than  $\binom{n+d}{d} = \dim V_d$ . The evaluation map  $p(x) \mapsto (p(a))_{a \in E}$  from  $V_d$  to  $F^E$  is a linear map of vector spaces. We conclude that it has a nonzero kernel, containing as desired a nonzero polynomial that vanishes on  $E$ . □

For  $n = 2$  and  $d = 3$  we get a basis of size  $\binom{2+3}{3} = 10$ :  $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$

Now we have all things needed to give Dvir's elegant solution of the finite Kakeya problem.

**Theorem.** *Let  $K \subseteq F^n$  be a Kakeya set. Then*

$$|K| \geq \binom{|F| + n - 1}{n} \geq \frac{|F|^n}{n!}.$$

■ **Proof.** The second inequality is clear from the definition of binomial coefficients. For the first, set again  $q = |F|$  and suppose for a contradiction that

$$|K| < \binom{q + n - 1}{n} = \binom{n + q - 1}{q - 1}.$$

By Lemma 2 there exists a nonzero polynomial  $p(x) \in F[x_1, \dots, x_n]$  of degree  $d \leq q - 1$  that vanishes on  $K$ . Let us write

$$p(x) = p_0(x) + p_1(x) + \dots + p_d(x), \quad (1)$$

where  $p_i(x)$  is the sum of the monomials of degree  $i$ ; in particular,  $p_d(x)$  is nonzero. Since  $p(x)$  vanishes on the nonempty set  $K$ , we have  $d > 0$ . Take any  $v \in F^n \setminus \{0\}$ . By the Kakeya property for this  $v$  there exists a  $w \in F^n$  such that

$$p(w + tv) = 0 \quad \text{for all } t \in F.$$

Here comes the trick: Consider  $p(w + tv)$  as a polynomial in the single variable  $t$ . It has degree at most  $d \leq q - 1$  but vanishes on all  $q$  points of  $F$ , whence  $p(w + tv)$  is the zero polynomial in  $t$ . Looking at (1) above we see that the coefficient of  $t^d$  in  $p(w + tv)$  is precisely  $p_d(v)$ , which must therefore be 0. But  $v \in F^n \setminus \{0\}$  was arbitrary and  $p_d(0) = 0$  since  $d > 0$ , and we conclude that  $p_d(x)$  vanishes on all of  $F^n$ . Since

$$dq^{n-1} \leq (q-1)q^{n-1} < q^n,$$

Lemma 1, however, tells us that  $p_d(x)$  must then be the zero polynomial — contradiction and end of the proof.  $\square$

As often happens in mathematics, once a breakthrough is achieved improvements follow quickly. So it was in this case. The lower bound  $\frac{1}{n!}$  for the constant  $c(n)$  has been improved to  $\frac{1}{2^n}$ , and this is within a factor of 2 from the best possible bound. That is, there exist Kakeya sets of size roughly  $\frac{1}{2^{n-1}}|F|^n$ .

For recent developments the blog by Terence Tao at [terrytao.wordpress.com/tag/kakeya-conjecture/](http://terrytao.wordpress.com/tag/kakeya-conjecture/) is an up-to-date source.

## References

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*"Whirling a pole the Kakeya way"*