The fundamental theorem of algebra

Chapter 21

> *Every nonconstant polynomial with complex coefficients has at least one root in the field of complex numbers.*

Gauss called this theorem, for which he gave four different proofs, the "fundamental theorem of algebraic equations." It is without doubt one of the milestones in the history of mathematics. As Reinhold Remmert writes in his pertinent survey: "It was the possibility of proving this theorem in the complex domain that, more than anything else, paved the way for a general recognition of complex numbers."

Some of the greatest names have contributed to the subject, from Gauss and Cauchy to Liouville and Laplace. An article of Netto and Le Vavasseur lists nearly a hundred proofs. The proof that we present is one of the most elegant and certainly the shortest. It follows an argument of d'Alembert and Argand and uses only some elementary properties of polynomials and complex numbers. We are indebted to France Dacar and to Tord Sjödin for a polished version of the proof. Essentially the same argument appears also in the papers of Fefferman [3] and Redheffer [5], and doubtlessly in some others.

We need three facts that one learns in a first-year calculus course.

(A) Polynomial functions are continuous.

(B) Any complex number of absolute value 1 has an m-th root for any $m \geq 1$.

(C) Cauchy's minimum principle: A continuous real-valued function f on a compact set S assumes a minimum in S.

It has been commented upon that the "Fundamental theorem of algebra" is not really fundamental, that it is not necessarily a theorem since sometimes it serves as a definition, and that in its classical form it is not a result from algebra, but rather from analysis.

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Now let $p(z) = \sum_{k=0}^{n} c_k z^k$ be a complex polynomial of degree $n \geq 1$. As the first and decisive step we prove what is variously called d'Alembert's lemma or Argand's inequality.

Lemma. *If $p(a) \neq 0$, then every disk D around a contains an interior point b with $|p(b)| < |p(a)|$.*

■ **Proof.** We first claim that without loss of generality we may assume that $a = 0$ and $p(a) = 1$. Indeed, if this is not the case, then we define another polynomial $q(z) := \frac{p(z+a)}{p(a)}$, which satisfies $q(0) = 1$. Now assume

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that every disk D of radius R around the origin contains a point b with $|q(b)| < 1$. Then the disk D_a of radius R around the point a contains the point $a + b$ such that $|p(a + b)| < |p(a)|$ as claimed.

We may thus assume that $p(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n$, and letting $m \geq 1$ be the smallest index with $c_m \neq 0$ we may write $p(z)$ in the form

$$p(z) = 1 + c_m z^m + z^{m+1}(c_{m+1} + \cdots + c_n z^{n-m-1}) = 1 + c_m z^m + r(z).$$

In the first step we find $0 < \rho < 1$ such that

$$|r(z)| < |c_m z^m| < 1 \qquad \text{for all } 0 < |z| \leq \rho. \tag{1}$$

To get the first inequality we note that for $|z| < 1$

$$|r(z)| \leq |z|^{m+1}(|c_{m+1}| + \cdots + |c_n|) < |c_m||z^m| = |c_m z^m|,$$

provided that

$$0 < |z| < \frac{|c_m|}{|c_{m+1}| + \cdots + |c_n|} =: \rho_1.$$

The second inequality holds if $|z| < |c_m|^{-\frac{1}{m}} =: \rho_2$; hence we conclude that (1) is valid for every ρ with $0 < \rho < \min\{\rho_1, \rho_2, 1\}$.

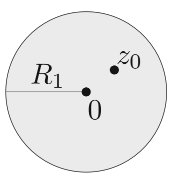
We come to our second ingredient, m-th roots of unity. Fix a constant ρ as in (1) with $\rho < R$, where R is the radius of the disk D around $a = 0$. Let ζ be an m-th root of $\frac{-\bar{c}_m}{|c_m|}$, where $\bar{c}_m$ is the complex conjugate of c_m, and set $b := \rho\zeta$. We claim that b is a desired point in D with $|p(b)| < 1$. First of all, b is in D since $|b| = \rho < R$, and further by $|c_m|^2 = c_m\bar{c}_m$ we have

$$c_m b^m = -c_m \rho^m \frac{\bar{c}_m}{|c_m|} = -|c_m|\rho^m.$$

Looking at (1) we have $|r(b)| < |c_m b^m| = |c_m|\rho^m < 1$, and hence

$$|p(b)| \leq |1 + c_m b^m| + |r(b)| = 1 - |c_m|\rho^m + |r(b)| < 1,$$

and we are done. □



The rest is easy. Clearly, $p(z)z^{-n}$ approaches the leading coefficient c_n of $p(z)$ as $|z|$ goes to infinity. Hence $|p(z)|$ goes to infinity as well with $|z| \to \infty$. Consequently, there exists $R_1 > 0$ such that $|p(z)| > |p(0)|$ for all points z on the circle $\{z : |z| = R_1\}$. Furthermore, our third fact (C) tells us that in the compact set $D_1 = \{z : |z| \leq R_1\}$ the continuous real-valued function $|p(z)|$ attains the minimum value at some point z_0. Because of $|p(z)| > |p(0)|$ for z on the boundary of D_1, z_0 must lie in the interior. But by d'Alembert's lemma this minimum value $|p(z_0)|$ must be 0 — and this is the whole proof.

References

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[4] E. NETTO & R. LE VAVASSEUR: *Les fonctions rationelles,* Enc. Sciences Math. Pures Appl. I **2** (1907), 1-232.

[5] R. M. REDHEFFER: *What! Another note just on the fundamental theorem of algebra?* Amer. Math. Monthly **71** (1964), 180-185.

[6] R. REMMERT: *The fundamental theorem of algebra,* Chapter 4 in: "Numbers" (H. D. Ebbinghaus et al., eds.), Graduate Texts in Mathematics **123**, Springer, New York 1991.

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