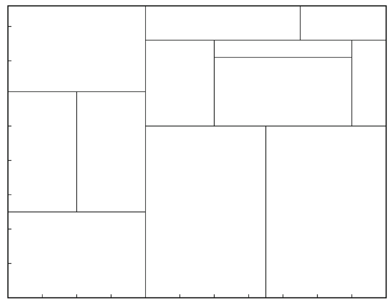
Tiling rectangles

Chapter 29

Some mathematical theorems exhibit a special feature: The statement of the theorem is elementary and easy, but to prove it can turn out to be a tantalizing task — unless you open some magic door and everything becomes clear and simple.

One such example is the following result due to Nicolaas de Bruijn:

Theorem. *Whenever a rectangle is tiled by rectangles all of which have at least one side of integer length, then the tiled rectangle has at least one side of integer length.*



The big rectangle has side lengths 11 and 8.5.

By a tiling we mean a covering of the big rectangle R with rectangles $T_1, \dots, T_m$ that have pairwise disjoint interior, as in the picture to the right.

Actually, de Bruijn proved the following result about packing copies of an $a \times b$ rectangle into a $c \times d$ rectangle: If a, b, c, d are integers, then each of a and b must divide one of c or d. This is implied by two applications of the more general theorem above to the given figure, scaled down first by a factor of $\frac{1}{a}$, and then scaled down by a factor of $\frac{1}{b}$. Each small rectangle has then one side equal to 1, and so $\frac{c}{a}$ or $\frac{d}{a}$ must be an integer.

Almost everybody's first attempt is to try induction on the number of small rectangles. Induction can be made to work, but it has to be performed very carefully, and it is not the most elegant option one can come up with. Indeed, in a delightful paper Stan Wagon surveys no less than fourteen different proofs out of which we have selected three; none of them needs induction. The first proof, essentially due to de Bruijn himself, makes use of a very clever calculus trick. The second proof by Richard Rochberg and Sherman Stein is a discrete version of the first proof, which makes it simpler still. But the champion may be the third proof suggested by Mike Paterson. It is just counting in two ways and almost one-line.

In the following we assume that the big rectangle R is placed parallel to the x, y-axes with $(0, 0)$ as the lower left-hand corner. The small rectangles T_i have then sides parallel to the axes as well.

■ **First Proof.** Let T be any rectangle in the plane, where T extends from a to b along the x-axis and from c to d along the y-axis. Here is de Bruijn's trick. Consider the double integral over T,

$$\int_c^d \int_a^b e^{2\pi i(x+y)}\,dx\,dy. \qquad (1)$$

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_29

Since

$$\int_c^d \int_a^b e^{2\pi i(x+y)} dx\,dy \;=\; \int_a^b e^{2\pi ix} dx \cdot \int_c^d e^{2\pi iy} dy,$$

it follows that the integral (1) is 0 if and only if at least one of $\int_a^b e^{2\pi ix} dx$ or $\int_c^d e^{2\pi iy} dy$ is equal to 0.

We are going to show that

$$\int_a^b e^{2\pi ix} dx = 0 \quad\Longleftrightarrow\quad b-a \text{ is an integer.} \qquad (2)$$

$$\iint_R f(x,y) \;=\; \sum_i \iint_{T_i} f(x,y)$$

Additivity of the integral

But then we will be done! Indeed, by the assumption on the tiling, each $\iint_{T_i}$ is equal to 0, and so by additivity of the integral, $\iint_R = 0$ as well, whence R has an integer side.

It remains to verify (2). From

$$\begin{aligned}\int_a^b e^{2\pi ix} dx &= \frac{1}{2\pi i} e^{2\pi ix}\Big|_a^b = \frac{1}{2\pi i}(e^{2\pi ib} - e^{2\pi ia}) \\ &= \frac{e^{2\pi ia}}{2\pi i}(e^{2\pi i(b-a)} - 1),\end{aligned}$$

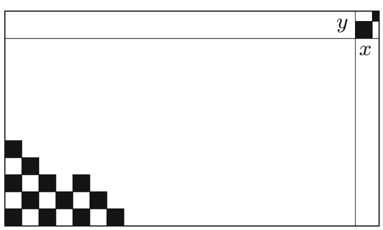
we conclude that

$$\int_a^b e^{2\pi ix} dx = 0 \quad\Longleftrightarrow\quad e^{2\pi i(b-a)} \;=\; 1.$$

From $e^{2\pi ix} = \cos 2\pi x + i \sin 2\pi x$ we see that the last equation is, in turn, equivalent to

$$\cos 2\pi(b-a) = 1 \;\text{ and }\; \sin 2\pi(b-a) = 0.$$

Since $\cos x = 1$ holds if and only if x is an integer multiple of 2π, we must have $b - a \in \mathbb{Z}$, and this also implies $\sin 2\pi(b-a) = 0$. □



The amount of black in the corner rectangle is $\min(x, \frac{1}{2}) \cdot \min(y, \frac{1}{2}) + \max(x - \frac{1}{2}, 0) \cdot \max(y - \frac{1}{2}, 0)$, and this is always greater than $\frac{1}{2}xy$.

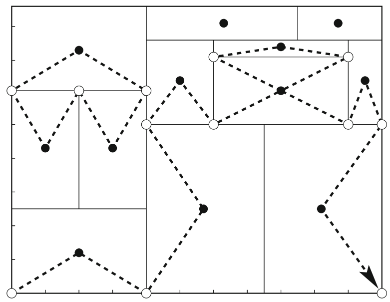
■ **Second Proof.** Color the plane in a checkerboard fashion with black/white squares of size $\frac{1}{2} \times \frac{1}{2}$, starting with a black square at $(0,0)$.

By the assumption on the tiling every small rectangle T_i must receive an equal amount of black and white, and therefore the big rectangle R too contains the same amount of black and white.

But this implies that R must have an integer side, since otherwise it can be split into four pieces, three of which have equal amounts of black and white, while the piece in the upper right-hand corner does not. Indeed, if $x = a - \lfloor a \rfloor$, $y = b - \lfloor b \rfloor$, so that $0 < x, y < 1$, then the amount of black is always greater than that of white.

This is illustrated in the figure in the margin. □

■ **Third proof.** Let C be the set of corners in the tiling for which both coordinates are integral (so, for example, $(0,0) \in C$), and let T be the set of tiles. Form a bipartite graph G on the vertex set $C \cup T$ by joining each corner $c \in C$ to all the tiles of which it is a corner. The hypothesis implies that each tile is joined to 0, 2, or 4 corners in C, since if one corner is in C, then so is also the other end of any integer side. Now look at C. Any $c \in C$ which is not a corner of R is joined to an *even* number of tiles, but the vertex $(0,0)$ is joined to only *one* tile. As the number of odd-degree vertices in any finite graph is even (as we have just observed on page 204), there must be another $c \in C$ of odd degree, and c can only be one of the other vertices of R — end of proof. □



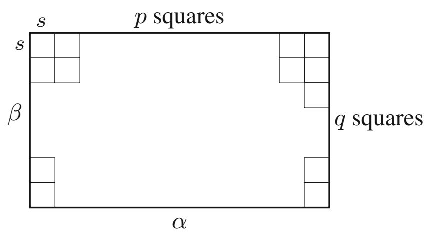
Here the bipartite graph G is drawn with vertices in C white, vertices in T black, and dashed edges.

All three proofs can quite easily be adapted to also yield an n-dimensional version of de Bruijn's result: Whenever an n-dimensional box R is tiled by boxes all of which have at least one integer side, then R has an integer side.

However, we want to keep our discussion in the plane (for this chapter), and look at a "companion result" to de Bruijn's, due to Max Dehn (many years earlier), which sounds quite similar, but asks for different ideas.

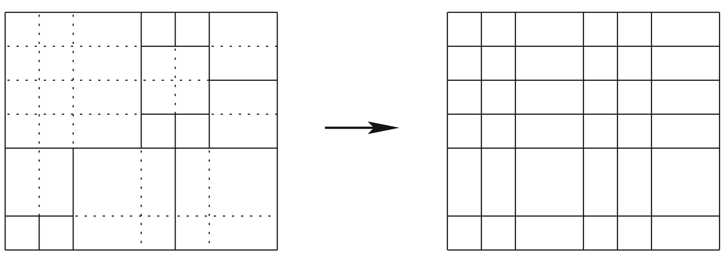
> **Theorem.** *A rectangle can be tiled with squares if and only if the ratio of its side lengths is a rational number.*

One half of the theorem is immediate. Suppose the rectangle R has side lengths α and β with $\frac{\alpha}{\beta} \in \mathbb{Q}$, that is, $\frac{\alpha}{\beta} = \frac{p}{q}$ with $p, q \in \mathbb{N}$. Setting $s := \frac{\alpha}{p} = \frac{\beta}{q}$, we can easily tile R with copies of the $s \times s$ square as shown in the margin.



For the proof of the converse Max Dehn used an elegant argument that he had already successfully employed in his solution of Hilbert's third problem (see Chapter 10). In fact, the two papers appeared in successive years in the *Mathematische Annalen*.

■ **Proof.** Suppose R is tiled by squares of possibly different sizes. By scaling we may assume that R is an $a \times 1$ rectangle. Let us assume $a \notin \mathbb{Q}$ and derive a contradiction from this. The first step is to extend the sides of the squares to the full width resp. height of R as in the figure.



R is now decomposed into a number of small rectangles; let $a_1, a_2, \ldots, a_M$ be their side lengths (in any order), and consider the set

$$A := \{1, a, a_1, \ldots, a_M\} \subseteq \mathbb{R}.$$

Next comes a linear algebra part. We define $V(A)$ as the vector space of all linear combinations of the numbers in A with rational coefficients. Note that $V(A)$ contains all side lengths of the squares in the original tiling, since any such side length is the sum of some a_i's. As the number a is not rational, we may extend $\{1, a\}$ to a basis B of $V(A)$,

$$B = \{b_1 = 1, b_2 = a, b_3, \dots, b_m\}.$$

Define the function $f : B \to \mathbb{R}$ by

$$f(1) := 1, \quad f(a) := -1, \quad \text{and} \quad f(b_i) := 0 \;\text{ for }\; i \geq 3,$$

and extend it linearly to $V(A)$.

Linear extension:
$f(q_1 b_1 + \cdots + q_m b_m) :=$
$\quad q_1 f(b_1) + \cdots + q_m f(b_m)$
for $q_1, \dots, q_m \in \mathbb{Q}$.

The following definition of "area" of rectangles finishes the proof in three quick steps: For $c, d \in V(A)$ the area of the $c \times d$ rectangle is defined as

$$\text{area}(\underset{c}{\boxed{\quad\quad}}\,d) = f(c)f(d).$$

(1) area $(\underset{c_1}{\boxed{\quad\quad}}\!\underset{c_2}{\boxed{\quad}}\,d) = \text{area}(\underset{c_1}{\boxed{\quad\quad}}\,d) + \text{area}(\underset{c_2}{\boxed{\quad}}\,d).$

This follows immediately from the linearity of f. The analogous result holds, of course, for vertical strips.

(2) $\text{area}(R) = \sum\limits_{\text{squares}} \text{area}(\boxed{\quad})$, where the sum runs through the squares in the tiling.

Just note that by (1) area(R) equals the sum of the areas of all small rectangles in the extended tiling. Since any such rectangle is in exactly one square of the original tiling, we see (again by (1)) that this sum is also equal to the right-hand side of (2).

(3) We have

$$\text{area}(R) = f(a)f(1) = -1,$$

whereas for a square of side length t, area$(\underset{t}{\boxed{\quad}}) = f(t)^2 \geq 0$, and so

$$\sum_{\text{squares}} \text{area}(\boxed{\quad}) \geq 0,$$

and this is our desired contradiction. $\square$

For those who want to go for further excursions into the world of tilings the beautiful survey paper [1] by Federico Ardila and Richard Stanley is highly recommended.

References

[1] F. ARDILA & R. P. STANLEY: *Tilings,* Math. Intelligencer (4)**32** (2010), 32-43.

[2] N. G. DE BRUIJN: *Filling boxes with bricks,* Amer. Math. Monthly **76** (1969), 37-40.

[3] M. DEHN: *Über die Zerlegung von Rechtecken in Rechtecke,* Mathematische Annalen **57** (1903), 314-332.

[4] S. WAGON: *Fourteen proofs of a result about tiling a rectangle,* Amer. Math. Monthly **94** (1987), 601-617.

"The new hopscotch:
Don't hit the integers!"