Van der Waerden's permanent conjecture

Chapter 24

Suppose $M = (m_{ij})$ is a real $n \times n$ matrix. If in the usual representation of the determinant we omit the signs of the permutations, we get the *permanent* per M,

$$\text{per}\, M \;:=\; \sum_{\sigma} m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)},$$

where σ runs through all permutations of $\{1, 2, \ldots, n\}$.

In contrast to the determinant, which can be quickly calculated (e.g. by Gaussian elimination), computation of the permanent is provably difficult. Therefore a lot of research about permanents concerned bounds and approximation; the book by Minc [7] gives an excellent overview of the subject.

We consider in this chapter the most famous theorem about permanents and its fabulous recent proof. A real matrix $M = (m_{ij})$ is called *doubly stochastic* if its entries are nonnegative, with each row sum and column sum equal to 1. In 1926 Bartel L. van der Waerden asked whether

$$\text{per}\, M \;\geq\; \frac{n!}{n^n}$$

holds for every doubly stochastic $n \times n$ matrix, the minimum being attained only by the matrix $M = (m_{ij})$, where $m_{ij} = \frac{1}{n}$ for all i and j.

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However, in 1969 Van der Waerden told his compatriot Van Lint that he had never heard of such a conjecture nor of his name being attached to it . . .

This "Van der Waerden conjecture" remained unsolved for over fifty years, until it was confirmed (more or less independently and more or less simultaneously) by G. P. Egorychev and D. I. Falikman in 1981. The paper [5] by Jacobus van Lint gives a very readable account of the history of the conjecture and the proofs.

The arguments of Egorychev and Falikman were rather involved, so it was a great surprise when in 2007 Leonid Gurvits presented a short, elegant, and completely different proof. In fact, he proved a stronger statement that included other previous results in this area as well.

Theorem. *Let $M = (m_{ij})$ be a doubly stochastic $n \times n$ matrix. Then*

$$\text{per}\, M \;\geq\; \frac{n!}{n^n},$$

and equality holds if and only if $m_{ij} = \frac{1}{n}$ for all i and j.

$$\text{per}\begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} = \frac{n!}{n^n}.$$

M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*, https://doi.org/10.1007/978-3-662-57265-8_24

For our presentation of the proof we follow closely the beautiful exposition of Gurvits' work by Monique Laurent and Alexander Schrijver in [4].

As first step let us translate matrices into polynomials. To every $n \times n$ matrix $M = (m_{ij})$ we associate the polynomial $p_M(x) \in \mathbb{R}[x_1, \dots, x_n]$,

$$p_M(x) = p_M(x_1, \dots, x_n) \;:=\; \prod_{i=1}^{n} \Big(\sum_{j=1}^{n} m_{ij} x_j \Big).$$

For example, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get

$$\begin{aligned} & p_M(x_1, x_2) \\ =\ & (ax_1 + bx_2)(cx_1 + dx_2) \\ =\ & acx_1^2 + (ad + bc)x_1x_2 + bdx_2^2. \end{aligned}$$

Since every term picks a variable from each row, $p_M(x)$ is *homogeneous* of degree n, meaning that every monomial $x_1^{k_1} \cdots x_n^{k_n}$ has total degree $k_1 + \cdots + k_n = n$. Note that $p_M(x)$ may be the zero-polynomial with all coeffcients equal to 0, which happens for example if M has a zero row. It is convenient to include this case and still regard $p_M(x) \equiv 0$ as homogeneous of degree n, when the set of variables is clear.

Next we define for $p(x) \in \mathbb{R}[x_1, \dots, x_n]$ its *derivative in* x_n:

$$p'(x_1, \dots, x_{n-1}) \;:=\; \frac{\partial\, p(x)}{\partial\, x_n}\Big|_{x_n = 0}.$$

Observe that if p is homogeneous of degree n in n variables, then p' is homogeneous of degree $n-1$ in $n-1$ variables. Indeed, since exactly the monomials of $p(x)$ that are linear in x_n survive in p', the degree decreases by 1.

In general, we define for $i = 0, 1, \dots, n$

$$q_i(x_1, \dots, x_i) \;:=\; \frac{\partial^{(n-i)}\, p(x)}{\partial\, x_n \cdots \partial\, x_{i+1}}\Big|_{x_n = x_{n-1} = \cdots = x_{i+1} = 0}.$$

In this way we get a chain $(q_n, q_{n-1}, \dots, q_0)$, where $q_n = p$ and $q_{i-1} = q_i'$ for $1 \le i \le n$, and finally q_0 is the coefficient of $x_1 x_2 \cdots x_n$ in p. Furthermore, if p is homogeneous of degree n, then q_i is homogeneous of degree i.

For the polynomial in the margin above, we obtain $q_1(x_1) = (ad + bc)x_1$ and $q_0 = ad + bc$.

Let us look at the chain generated by $p_M(x)$,

$$p_M(x) = q_n, \dots, q_i, \dots, q_0.$$

The following two facts will be important:

A. per M *is the coefficient of* $x_1 x_2 \cdots x_n$ *in* q_n, *thus* $q_0 =$ per M.

This holds by the definition of the permanent.

B. *For* $i = 1, \dots, n$ *we have*

$$\deg_i q_i \le \min\{i, \lambda_M(i)\}, \tag{1}$$

where $\deg_j q_i$ *denotes the degree of* x_j *in* $q_i(x_1, \dots, x_i)$ *and* $\lambda_M(i)$ *records the number of nonzero entries in the* i*th column of* M.

Indeed, we have $\deg_i q_i \le i$, because q_i is homogeneous of degree i, while $\deg_i q_i \le \deg_i q_n \le \lambda_M(i)$ is clear from the definition of $p_M(x)$.

Here comes the main idea of the proof: We associate a parameter to every polynomial p and bound it from below when passing from p to p'.

Before going on, let us fix some notation. We will let $\mathbb{R}_+$ denote the nonnegative reals, and $p(x) \in \mathbb{R}_+[x_1, \dots, x_n]$ means that all coefficents of $p(x)$ are nonnegative. For a complex number $z \in \mathbb{C}$, let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ be the real and the imaginary part, respectively. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ and $\mathbb{C}_{++} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ denote closed and open right complex half-planes. The notation extends to $\mathbb{R}_+^n$ and $\mathbb{C}_+^n$, etc. Thus, for example, $z = (z_1, \dots, z_n) \in \mathbb{C}_{++}^n$ holds if $\mathrm{Re}(z_i) > 0$ for all i.

For every polynomial $p(x) \in \mathbb{R}_+[x_1, \dots, x_n]$ define the *capacity* $\mathrm{cap}(p)$ by

$$\mathrm{cap}(p) \quad := \quad \inf \Big\{ p(x) : x \in \mathbb{R}_+^n,\ \prod_{i=1}^{n} x_i = 1 \Big\}.$$

In particular, $\mathrm{cap}(p) \geq 0$ as p has only nonnegative coefficients, and if p is the constant polynomial $p(x) \equiv c$, then $\mathrm{cap}(p) = c$.

We also need the function $g : \mathbb{N}_0 \to \mathbb{R}$, defined by $g(0) := 1$ and

$$g(k) := \Big(\frac{k-1}{k} \Big)^{k-1} \quad \text{for} \quad k \geq 1.$$

Using $1 + x \leq e^x$ twice, which holds strictly except for $x = 0$, we get

$$\frac{g(k+1)}{g(k)} = \frac{k}{k+1} \Big(\frac{k^2}{k^2-1} \Big)^{k-1} < e^{-\frac{1}{k+1}} e^{\frac{1}{k^2-1}(k-1)} = 1$$

for $k \geq 1$. Thus g is non-increasing, $g(0) = g(1) > g(2) > \cdots$.

Writing $g(k) = (1 - \frac{1}{k})^{k-1}$, we see that $\lim_{k \to \infty} g(k) = \frac{1}{e}$.

Call a polynomial $p(x) \in \mathbb{R}[x_1, \dots, x_n]$ *H-stable* if it has no roots in $\mathbb{C}_{++}^n$.

The following result is the key step. We postpone its proof for the moment and show first how it immediately implies the Theorem.

Gurvits' Proposition.
If $p(x) \in \mathbb{R}_+[x_1, \dots, x_n]$ *is H-stable and homogeneous of degree* n*, then either* $p' \equiv 0$*, or* p' *is H-stable and homogeneous of degree* $n-1$*. In either case*

$$\mathrm{cap}(p') \quad \geq \quad \mathrm{cap}(p) \cdot g(\deg_n p).$$

■ **Proof of the Theorem.** Let $M = (m_{ij})$ be a doubly stochastic $n \times n$ matrix. We already know that $p_M(x)$ is homogeneous of degree n.

Claim 1. $p_M(x)$ *is H-stable.*

Suppose x is a root of $p_M(x)$. From $p_M(x) = \prod_{i=1}^n \big(\sum_{j=1}^n m_{ij} x_j \big) = 0$ it follows that $\sum_{j=1}^n m_{ij} x_j = 0$ for some i, thus $\sum_{j=1}^n m_{ij} \mathrm{Re}(x_j) = 0$. But this precludes $x \in \mathbb{C}_{++}^n$, since $m_{i\ell} > 0$ for some ℓ.

Recall for this that $m_{ij} \geq 0$ and $\sum_{i=1}^n m_{ij} = 1$.

Claim 2. $\mathrm{cap}(p_M) = 1$.

Take any $x \in \mathbb{R}_+^n$ with $\prod_{j=1}^n x_j = 1$. By the inequality of the arithmetic and the geometric mean (see Chapter 20) we have

The AM-GM inequality: For $a_1, \dots, a_n, p_1, \dots, p_n \in \mathbb{R}_+$ with $\sum_{i=1}^n p_i = 1$ we get $\sum_{i=1}^n p_i a_i \geq a_1^{p_1} \dots a_n^{p_n}$.

$$\begin{aligned} p_M(x) &= \prod_{i=1}^n \Big(\sum_{j=1}^n m_{ij} x_j \Big) \geq \prod_{i=1}^n \prod_{j=1}^n x_j^{m_{ij}} = \prod_{j=1}^n \prod_{i=1}^n x_j^{m_{ij}} \\ &= \prod_{j=1}^n x_j^{\sum_{i=1}^n m_{ij}} = \prod_{j=1}^n x_j = 1, \end{aligned}$$

and thus cap$(p_M) \geq 1$.
On the other hand,

$$p_M(1,1,\dots,1) = \prod_{i=1}^n \Big(\sum_{j=1}^n m_{ij} \Big) = \prod_{i=1}^n 1 = 1,$$

which proves Claim 2.

Since $p_M(x)$ is H-stable, we may apply Gurvits' Proposition repeatedly to conclude that all the polynomials q_i are H-stable, and obtain for each i that

$$\text{cap}(q_{i-1}) \geq \text{cap}(q_i)\, g(\deg_i q_i) \geq \text{cap}(q_i)\, g\big(\min\{i, \lambda_M(i)\}\big), \qquad (2)$$

where the second inequality follows from (1) and the fact that g is non-increasing.
Iterating (2) we get, starting with cap$(p_M) = 1$, that

$$\begin{aligned} \text{per}\, M = q_0 &\geq \prod_{i=1}^n g\big(\min\{i, \lambda_M(i)\}\big) \geq \prod_{i=1}^n g(i) \qquad (3) \\ &= \prod_{i=1}^n \Big(\frac{i-1}{i} \Big)^{i-1} = \prod_{i=1}^n i \, \frac{(i-1)^{i-1}}{i^i} = \frac{n!}{n^n}, \end{aligned}$$

which is our desired inequality.

It remains to prove the uniqueness part. Suppose that per $M = \frac{n!}{n^n}$, where we may assume that $n \geq 2$. From the fact that we have equality in (3), we conclude that $i \leq \lambda_M(i)$ for all i, and hence $n = \lambda_M(n)$. By symmetry, we conclude that all entries of M are nonzero. Thus it suffices to prove, again by symmetry, that all entries in the *last* column are equal to $\frac{1}{n}$.
Since we have equality in Gurvits's Proposition applied to p_M and p'_M, and since cap$(p_M) = 1$, we find

$$\inf_y p'_M(y) = \text{cap}(p'_M) = g(n) = \Big(\frac{n-1}{n} \Big)^{n-1},$$

where y ranges over all $y \in \mathbb{R}_+^{n-1}$ with $\prod_{j=1}^{n-1} y_j = 1$. Take any such y.
In the following chain of inequalities the indices i and k range from 1 to n, while j goes from 1 to $n-1$. From

$$p_M(x) = \prod_{i=1}^n \Big(\sum_{j=1}^n m_{ij} x_j \Big)$$

we infer that

$$p'_M(y) \;=\; \frac{\partial p_M(x)}{\partial x_n}\bigg|_{x=(y,0)} \;=\; \sum_k m_{kn} \prod_{i\neq k} \Big(\sum_j m_{ij} y_j\Big)$$

Here we make use of the Leibniz rule for differentiating products: $(f_1 f_2 \cdots f_n)' = \sum_k f'_k \prod_{i\neq k} f_i$.

and thus obtain the following chain:

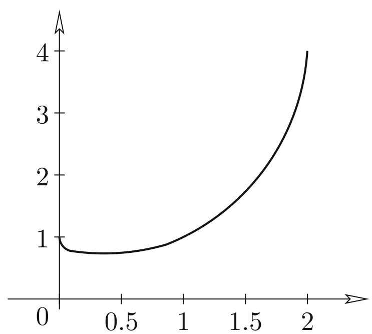
$$\begin{aligned} p'_M(y) &= \sum_k m_{kn} \prod_{i\neq k} \Big(\sum_j m_{ij} y_j\Big) \\ &\overset{\text{AM-GM}}{\geq} \prod_k \prod_{i\neq k} \Big(\sum_j m_{ij} y_j\Big)^{m_{kn}} \\ &= \prod_i \prod_{k\neq i} \Big(\sum_j m_{ij} y_j\Big)^{m_{kn}} \\ &= \prod_i \Big(\sum_j m_{ij} y_j\Big)^{1-m_{in}} \\ &= \prod_i \Big[(1-m_{in}) \sum_j \frac{m_{ij}}{1-m_{in}} y_j\Big]^{1-m_{in}} \\ &\overset{\text{AM-GM}}{\geq} \prod_i \Big[(1-m_{in})^{1-m_{in}} \prod_j y_j^{m_{ij}}\Big] \\ &= \prod_i (1-m_{in})^{1-m_{in}} \prod_j \prod_i y_j^{m_{ij}} \\ &= \prod_i (1-m_{in})^{1-m_{in}} \underbrace{\prod_j y_j}_{=1} \\ &= \prod_i (1-m_{in})^{1-m_{in}} \\ &\geq \Big(\frac{n-1}{n}\Big)^{n-1}. \end{aligned}$$

Note that $m_{in} \neq 1$, since $n \geq 2$ and the i-th row has no zeros.

For the last inequality in this chain we exploit the log-convexity of the function x^x for $x > 0$. For this recall that a real function f is *convex* if $\frac{1}{n}\big(f(x_1) + \cdots + f(x_n)\big) \geq f\big(\frac{x_1+\cdots+x_n}{n}\big)$; a function $f(x)$ is *log-convex* if $\log f(x)$ is convex. Then $\frac{1}{n}\sum_i \log f(x_i) \geq \log f\big(\frac{x_1+\cdots+x_n}{n}\big)$, or $f(x_1)\cdots f(x_n) \geq f\big(\frac{x_1+\ldots+x_n}{n}\big)^n$. For the function x^x we have

$$x_1^{x_1} x_2^{x_2} \cdots x_n^{x_n} \geq \Big(\frac{x_1+\cdots+x_n}{n}\Big)^{\sum_i x_i}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. In our case $x_i = 1 - m_{in}$ with $x_1 + \ldots + x_n = n-1$, thus $\big(\frac{x_1+\cdots+x_n}{n}\big)^{\sum_i x_i} \geq \big(\frac{n-1}{n}\big)^{n-1}$.



The function $f(x) = x^x$ is log-convex

And here is the punch line: Since this chain of inequalities holds for every such y, and since $\inf p'_M(y) = \big(\frac{n-1}{n}\big)^{n-1}$, the last inequality (which is independent of y) must be an equality, and from this we conclude that $1 - m_{1n} = \cdots = 1 - m_{nn}$, that is, $m_{1n} = \cdots = m_{nn} = \frac{1}{n}$. □

In our work towards a proof of Gurvits's proposition, assume now that the polynomial $p(x) \in \mathbb{R}_+[x_1, \dots, x_n]$ is H-stable and homogeneous of degree n. We already know that $p'(x_1, \dots, x_{n-1})$ is homogeneous of degree $n-1$.

Lemma 1. *For each $x \in \mathbb{C}^n_+$,*

$$|p(x)| \geq |p(\mathrm{Re}(x))|.$$

We may assume that $x \in \mathbb{C}^n_{++}$ by continuity. Since $p(x)$ is H-stable, we have $p(\mathrm{Re}(x)) \neq 0$. Fix x and consider the set $\{p\big(x + s\mathrm{Re}(x)\big) : s \in \mathbb{C}\}$ as a function of s. As $p(x)$ is homogeneous of degree n, we may write

$$p\big(x + s\mathrm{Re}(x)\big) = p\big(\mathrm{Re}(x)\big) \prod_{i=1}^{n} (s - b_i),$$

for some complex numbers $b_1, \dots, b_n$. Since $p\big(x + b_i\mathrm{Re}(x)\big) = 0$ for each i, we infer that $x + b_i\mathrm{Re}(x) \notin \mathbb{C}_{++}$, which implies that

$$\mathrm{Re}\big(x + b_i\mathrm{Re}(x)\big) \;=\; \mathrm{Re}(x)\big(1 + \mathrm{Re}(b_i)\big) \;\leq\; 0,$$

hence $\mathrm{Re}(b_i) \leq -1$, and thus $|b_i| \geq 1$.

Since $\mathrm{Re}(x) > 0$, we have $1 + \mathrm{Re}(b_i) \leq 0$.

It follows that

$$|p(x)| \;=\; |p\big(x + 0 \cdot \mathrm{Re}(x)\big)| \;=\; |p\big(\mathrm{Re}(x)\big)| \prod_{i=1}^{n} |b_i| \;\geq\; |p\big(\mathrm{Re}(x)\big)|,$$

as claimed. □

Lemma 2. *Let $y \in \mathbb{C}^{n-1}_{++}$ and $\prod_{j=1}^{n-1} \mathrm{Re}(y_j) = 1$. Then*

$$\mathrm{cap}(p) \;\leq\; \frac{p\big(\mathrm{Re}(y), t\big)}{t} \quad \textit{for every } t > 0.$$

For the proof set $\lambda := t^{-\frac{1}{n}}$ and $\bar{x} := \lambda\big(\mathrm{Re}(y), t\big) \in \mathbb{R}^n_{++}$. Then

$$\prod_{i=1}^{n} \bar{x}_i \;=\; \lambda^n \Big(\underbrace{\prod_{j=1}^{n-1} \mathrm{Re}(y_j)}_{=1} \Big) t \;=\; 1,$$

and thus, using that $p(x)$ is homogeneous of degree n,

$$\mathrm{cap}(p) \;\leq\; p(\bar{x}) \;=\; \lambda^n p\big(\mathrm{Re}(y), t\big) \;=\; \frac{p\big(\mathrm{Re}(y), t\big)}{t}.$$ □

■ **Proof Gurvits' Proposition.** It now suffices to show (by scaling, see the margin) that for $y \in \mathbb{C}^{n-1}_{++}$ with $\prod_{j=1}^{n-1} \mathrm{Re}(y_j) = 1$ the following two conditions hold:

Suppose $y \in \mathbb{C}^{n-1}_{++}$ and set $\bar{y} = \frac{1}{\lambda} y$, where $\lambda = \big[\prod_{j=1}^{n-1} \mathrm{Re}(y_j)\big]^{\frac{1}{n-1}}$. Then $\prod_{j=1}^{n-1} \mathrm{Re}(\bar{y}_j) = 1$, and $p'(y) = \lambda^{n-1} p'(\bar{y})$. Hence y is a root of p' if and only if $\bar{y}$ is.

(I) *If $p'(y) = 0$, then $p' \equiv 0$.*

(II) *If $y \in \mathbb{R}^{n-1}_{++}$, then $p'(y) \geq \mathrm{cap}(p) \cdot g(\deg_n p)$.*

Case 1. $p(y, 0) = 0$.

We have $p(\text{Re}(y), 0) = 0$ by Lemma 1. Furthermore,

$$p'(y) = \lim_{t \searrow 0} \frac{p(y,t) - p(y,0)}{t} = \lim_{t \searrow 0} \frac{p(y,t)}{t},$$

Remember that $p'(x) = \frac{\partial p(x)}{\partial x_n}\Big|_{x_n=0}$.

and similarly

$$p'(\text{Re}(y)) = \lim_{t \searrow 0} \frac{p(\text{Re}(y),t)}{t}.$$

Since $p(\text{Re}(y), t) \le |p(y,t)|$, again by Lemma 1, we infer from Lemma 2 that

$$\text{cap}(p) \le \lim_{t \searrow 0} \frac{p(\text{Re}(y),t)}{t} = p'(\text{Re}(y)) \le \lim_{t \searrow 0} \frac{|p(y,t)|}{t} = |p'(y)|.$$

If $p'(y) = 0$, then $p'(\text{Re}(y)) = 0$, and thus $p' \equiv 0$ since all coefficients of p' are nonnegative. This proves (I) in this case, and (II) is trivially true since $g(k) \le 1$ for all k.

Case 2. $p(y,t)$ *has degree at most* 1 *as a polynomial in* t.

Since $p(\text{Re}(y), t) \le |p(y,t)|$ for all $t > 0$ by Lemma 1, we conclude that $p(\text{Re}(y), t)$ also has degree at most 1 in t. Thus

$$p'(y) = \lim_{t \to \infty} \frac{p(y,t)}{t}, \qquad p'(\text{Re}(y)) = \lim_{t \to \infty} \frac{p(\text{Re}(y),t)}{t},$$

and Lemma 2 tells us that

$$\text{cap}(p) \le \lim_{t \to \infty} \frac{p(\text{Re}(y),t)}{t} = p'(\text{Re}(y)) \le \lim_{t \to \infty} \frac{|p(y,t)|}{t} = |p'(y)|,$$

and we infer (I) and (II) as before.

Case 3. $p(y,0) \ne 0$*, and* $p(y,t)$ *has degree at least* 2 *in* t.

This implies that $k := \deg_n p \ge 2$, and we can write

$$p(y,t) = p(y,0) \prod_{i=1}^{k} (1 + a_i t) \tag{4}$$

for some complex numbers $a_1, \dots, a_k$. Hence

$$p'(y) = p(y,0) \sum_{i=1}^{k} a_i$$

where not all a_i are equal to 0, since $p(y,t)$ has degree at least 2 in t.

The following result is the heart of the proof.

Claim. *If* $a_i \ne 0$*, then the inverse* a_i^{-1} *is a nonnegative linear combination of the complex numbers* $y_1, \dots, y_{n-1}$.

To see this we need the famous Lemma of Farkas from linear optimization. (See for example Schrijver [8, Sect. 7.3].)

Farkas Lemma. Let $A \in \mathbb{R}^{r\times s}$ be a real matrix and $b \in \mathbb{R}^r$ a vector. Then exactly one of the following alternatives holds:
(i) $Ax = b$, $x \in \mathbb{R}^s$, $x \geq 0$ is solvable,
(ii) $A^T z > 0$, $z \in \mathbb{R}^r$, $b^T z < 0$ is solvable.

For our problem take $r = 2$, $s = n-1$, and such an $a_i \neq 0$, and set

$$A = \begin{pmatrix} \mathrm{Re}(y_1) & \cdots & \mathrm{Re}(y_{n-1}) \\ \mathrm{Im}(y_1) & \cdots & \mathrm{Im}(y_{n-1}) \end{pmatrix}, \qquad b = \begin{pmatrix} \mathrm{Re}(a_i^{-1}) \\ \mathrm{Im}(a_i^{-1}) \end{pmatrix}.$$

Alternative (i) is exactly what we want:

$$a_1^{-1} = x_1 y_1 + \cdots + x_{n-1} y_{n-1}, \quad x \in \mathbb{R}_+^{n-1}. \tag{5}$$

Assume the opposite: Let $z = \binom{c}{d}$ be a solution and set $\lambda = c - id \in \mathbb{C}$. Then

$$\begin{aligned} (A^T z)_j \;&=\; \mathrm{Re}(y_j)\,\mathrm{Re}(\lambda) - \mathrm{Im}(y_j)\,\mathrm{Im}(\lambda) \;=\; \mathrm{Re}(y_j\lambda) \;>\; 0, \\ & \qquad\qquad b^T z = \mathrm{Re}(a_i^{-1}\lambda) \;<\; 0. \end{aligned}$$

This means that $(\lambda y, -\lambda a_i^{-1})$ lies in $\mathbb{C}_{++}^n$. However, by (4)

$$p(\lambda y, -\lambda a_i^{-1}) = \lambda^n p(y, -a_i^{-1}) = 0,$$

contradicting the H-stability of $p(x)$. This proves the claim.

Since $y \in \mathbb{C}_{++}^{n-1}$, we have $\mathrm{Re}(a_i^{-1}) > 0$ in (5) and thus $\mathrm{Re}(a_i) > 0$ for all nonzero a_i. Looking at (4), we conclude that $p'(y) \neq 0$, which proves (I).

To see (II), pick $y \in \mathbb{R}_+^{n-1}$ with $\prod_{j=1}^{n-1} y_j = 1$. In this case all nonzero a_i are positive reals by (5), thus $\sum_{i=1}^k a_i > 0$, and $\frac{p'(y)}{p(y,0)} = \sum_{i=1}^k a_i > 0$. Set $t = \frac{k}{k-1}\frac{p(y,0)}{p'(y)} > 0$. Using once more the AM-GM inequality we infer

$$\begin{aligned} \frac{p(y,t)}{p(y,0)} \;&=\; \prod_{i=1}^{k}(1 + a_i t) \;\leq\; \Big[\frac{1}{k}\sum_{i=1}^{k}(1 + a_i t)\Big]^k \\ &=\; \Big[\frac{1}{k}\Big(k + \frac{p'(y)}{p(y,0)}t\Big)\Big]^k \;=\; \Big[1 + \frac{1}{k-1}\Big]^k \;=\; \Big(\frac{k}{k-1}\Big)^k. \end{aligned}$$

Lemma 2 applied to $t = \frac{k}{k-1}\frac{p(y,0)}{p'(y)}$ therefore gives

$$\begin{aligned} \mathrm{cap}(p) \;&\leq\; \frac{p(y,t)}{t} \;=\; p'(y)\frac{k-1}{k}\frac{p(y,t)}{p(y,0)} \\ &\leq\; p'(y)\frac{k-1}{k}\Big(\frac{k}{k-1}\Big)^k \;=\; \frac{p'(y)}{g(k)}, \end{aligned}$$

or $p'(y) \geq \mathrm{cap}(p)\cdot g(k)$, and the proof is complete. □

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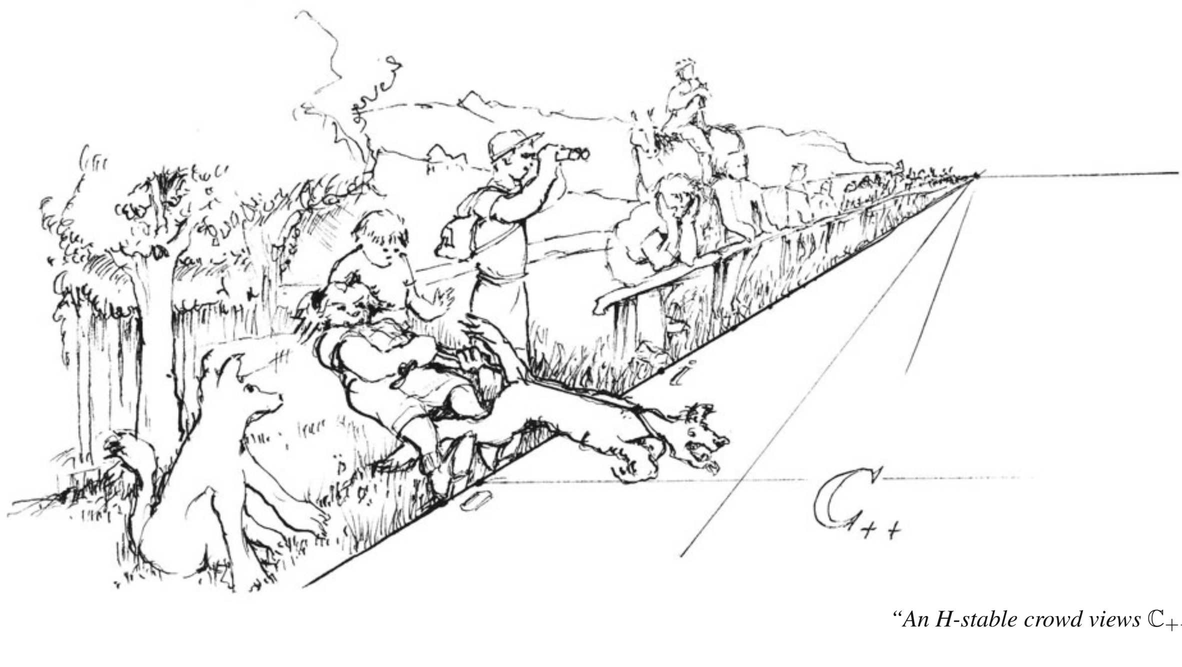

"An H-stable crowd views $\mathbb{C}_{++}$"