Chapter 27
Derivation of the Threshold Form of the Polytomous Rasch Model

We viewed various formats for ordered response categories in Chap. 2, described the threshold form of the Polytomous Rasch Model (PRM) and showed applications of the model in Chaps. 20–22. In this chapter, we derive the model from first principles. This derivation follows the original derivation of the threshold form of the PRM in Andrich (1978) which was built on Andersen (1977), which in turn was built on Rasch (1961). In doing so, we apply the concept of a *response space* that was described in Statistics Review 5. The derivation begins with an analogy between instruments of measurement and ordered response categories.

Measurement and Ordered Response Categories

In a prototype of measurement, an instrument is constructed in such a way that a linear continuum is partitioned by equidistant thresholds into categories called units. The thresholds are considered equally fine (same discrimination) and, relative to the size of the property being measured, fine enough that their own width can be ignored. Then the measurement is the *count* of the number of intervals, the units, from the chosen origin that the property maps onto the continuum. A prototype of measurement, the very familiar ruler partitioned into centimetres and millimetres, is shown in Fig. 27.1. To develop the analogy with ordered response categories, superimposed on the ruler are five ordered categories. We will see how the only differences are that the latter in general do not have equidistant thresholds and that floor and ceiling effects play a role, whereas in measurement they generally do not.

To make the development relatively concrete, Table 27.1 shows the scoring criteria for the assessment of essay writing in four ordered categories with respect to the criterion of *setting or context* (Harris, 1991). To simplify the notation, while retaining the order, the successive categories have been labelled as grades of Fail (F), Pass (P), Credit (C) and Distinction (D). The intended ordering of the proficiency of the four categories with respect to the criterion is clear: *Inadequate (F)* < *Discrete (P)* < *Integrated (C)* < *Manipulated (D)*.

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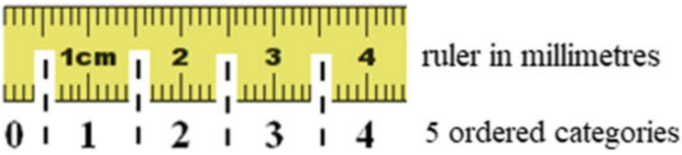

Fig. 27.1 A continuum partitioned in the prototype of measurement with five ordered categories

Table 27.1 Scoring criteria for the assessment of essay writing with respect to the criterion of setting (Harris, 1991)

0 (F)	**Inadequate setting**: Insufficient or irrelevant information given for the story
1 (P)	**Discrete setting**: Discrete setting as an introduction, with some details that show some linkage and organization
2 (C)	**Integrated setting**: There is a setting which, rather than being simply at the beginning, is introduced throughout the story
3 (D)	**Manipulated setting**: In addition to the setting being introduced throughout the story, pertinent information is woven or integrated so that this integration contributes to the story

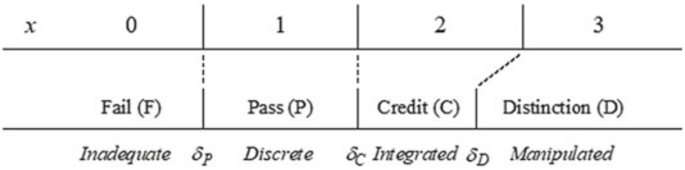

Fig. 27.2 A continuum partitioned into four, non-equidistant, ordered categories for assessing essays with respect to the criterion of setting

Figure 27.2 shows the continuum partitioned by three thresholds into four, non-equidistant, ordered categories for the grade classifications in Table 27.1. The extreme grades, *F* and *D*, are not bounded on the continuum and, as mentioned earlier, need not be, and in this case are not equidistant.

Minimum Proficiencies and Threshold Difficulty Order in the Full Space Ω

To derive the PRM in terms of the thresholds, their implied order is formalized. To simplify the notation, we do not subscript the person and item parameters in this derivation, emphasizing here that the response is with respect to a *single person* responding to a *single item*.

First, consider the *minimum* proficiency required to obtain the successive grades, *F*, *P*, *C*, *D*. We take that the minimum proficiency β_D to obtain a *D* is at the point on the continuum where the probability of success is 0.5. Let this point on the continuum be the threshold δ_D, giving in complete notation $\Pr\{D; \beta_D, \delta_D\} = 0.5$. Further, we take that this probability is characterized by the dichotomous Rasch model, $\Pr\{D; \beta, \delta_D\} = (e^{\beta-\delta_D})/\gamma_D$ where $\gamma_D = 1 + e^{\beta-\delta_D}$ is the usual normalizing factor which ensures that $\Pr\{D; \beta, \delta_D\} + \Pr\{not\ D; \beta, \delta_D\} = 1$. At the minimum proficiency, $\beta_D = \delta_D$, $\Pr\{D;\ \beta_D,\ \delta_D\} = 0.5$. Likewise, we take that the minimum proficiencies required to obtain *C, P,* respectively, are $\beta_C,\ \beta_P$ and that the thresholds $\delta_C,\ \delta_P$ on the continuum are such that $\Pr\{C;\ \beta_C,\ \delta_C\} = 0.5$, $\Pr\{P;\ \beta_P,\ \delta_P\} = 0.5$. Again, if the response probabilities are characterized by the Rasch model, $\beta_C = \delta_C$ and $\beta_P = \delta_P$. It is stressed that the thresholds $\delta_P,\ \delta_C,\ \delta_D$ are defined by their relationship to minimum proficiencies $\beta_P,\ \beta_C,\ \beta_D$. Therefore, the minimum proficiency required to achieve *P, C, D,* respectively, can be referred to as the proficiency at respective thresholds $\delta_P,\ \delta_C,\ \delta_D$ on the continuum. In achievement testing, they may be referred to as difficulties.

Second, we take it that the minimum proficiency required to achieve a *D* is greater than that to achieve a *C*, which in turn is greater than the minimum proficiency to achieve a *P*. These requirements reflect the intended order of degrees of proficiency, with the implication that $\beta_D > \beta_C > \beta_P$. The relationship here is a transitive one reflecting the very powerful constraint of order implied by the levels of proficiency. Now, given the relationships $\beta_D = \delta_D, \beta_C = \delta_C, \beta_P = \delta_P$, the implication is that $\delta_D > \delta_C > \delta_P$ with the parallel transitive relationship on these thresholds. These threshold locations are shown to conform to this order in Fig. 27.2. It is stressed that this order is a *requirement* which will be reflected in some way in the model. However, as we have seen in Chaps. 21 and 22, there is no guarantee that the data will reflect this requirement. If data do not satisfy this requirement, then it is a property of the data which will manifest itself by an incorrect ordering of the threshold estimates.

Before proceeding to derive the PRM, we note two related differences between Figs. 27.1 and 27.2. First, we have already indicated that unlike the thresholds in Fig. 27.1, those in Fig. 27.2 are not equidistant. Second, the thresholds in Fig. 27.1 are open ended in principle; those in Fig. 27.2 are finite in number. In the natural sciences, when the size of the property appears too close to the extreme measurements provided by the instrument, then an instrument that has a wider or better aligned range is sought and used. Specifically, if it is assumed that there are random errors of measurement, in which case the errors follow the normal distribution, then it is assumed, or required, that the instrument and property are so well aligned that the probability of an extreme measurement is zero (Stigler, 1986, p. 110). Such a luxury is not afforded in the case of a finite number of categories, and floor and ceiling effects are evident in items such as those shown above with the assessment of essays.

Specifying the Dichotomous Rasch Model for Responses at the Thresholds

To specify the dichotomous Rasch model for success at the proficiencies $\delta_P,\ \delta_C,\ \delta_D$, let y be the dichotomous variable which takes the values (0, 1), respectively, for failure and success at each minimum proficiency. This gives the responses $y_P,\ y_C,\ y_D$. Then, for example,

$$\Pr\{y_P = 1; \beta, \delta_P\} = e^{\beta - \delta_P}/\gamma_P; \Pr\{y_P = 0; \beta, \delta_P\} = 1/\gamma_P, \tag{27.1}$$

for any person with proficiency β.

In the above specifications, any response at the threshold proficiency is assumed to be independent of any other response at any other threshold. That is, it is as if a decision at different thresholds is made by a different judge. We do not need actual independent responses to proceed with the logic of the model's development, Eq. (27.1) being a definition of a probability. To show the effect of this implication on the continuum, the structure of the continuum in Fig. 27.2 has been resolved in Fig. 27.3 into three distinct continuums.

With independent responses assumed at each threshold, we set out the full set of possible probabilities. For efficiency of exposition, denote $P_y = \Pr\{y = 1; \beta, \delta_y\}$ and its complement $Q_y = \Pr\{y = 0; \beta, \delta_y\}$. Further, because the derivation of the model involves response spaces and subspaces, it is efficient to make clear the response space—it is denoted Ω. The responses and the response space, which has $2^3 = 8$ elements, are set out in Table 27.2. To be specific, the space is $\Omega \equiv \{(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}$. The last row of Table 27.2 provides the sum of the probabilities of the above set of response elements, which as required is 1.

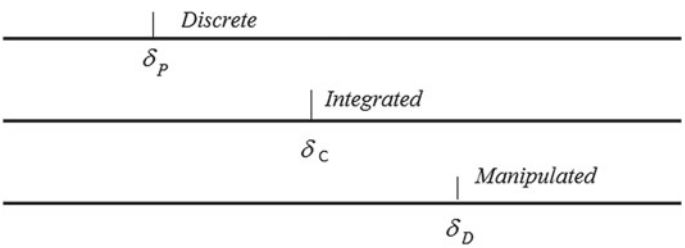

Fig. 27.3 A resolved structure for a decision at each threshold proficiency

Table 27.2 The independent response space Ω and the Guttman subspace Ω^G for the responses from Fig. 27.3

y_P	y_C	y_D	x	Grade		
0 Q_P	0 Q_C	0 Q_D	0	F	Ω^G	Ω
1 P_P	0 Q_C	0 Q_D	1	P		
1 P_P	1 P_C	0 Q_D	2	C		
1 P_P	1 P_C	1 P_D	3	D		
0 Q_P	1 P_C	0 Q_D				
0 Q_P	0 Q_C	1 P_D				
1 P_P	0 Q_C	1 P_D				
0 Q_P	1 P_C	1 P_D				

$$\sum_{\Omega} \Pr\{(y_P, y_C, y_D)\} = \sum_{\Omega} \Pr\{y_P \mid \Omega\} \Pr\{y_C \mid \Omega\} \Pr\{y_D \mid \Omega\} = 1$$

The Response Subspace Ω^G

In addition to the space Ω, Table 27.2 also shows another space, the subspace Ω^G. This subspace arises from the following reasoning. Consider a response in the original ordered format structure of Table 27.1 and Fig. 27.2. In this example, there can be only one response in one of four categories. Suppose first that the response is deemed a D. This implies a success at threshold δ_D. However, because of the required ordering of the thresholds, $\delta_D > \delta_C > \delta_P$, this response necessarily implies a success also at both thresholds δ_C, δ_P. That is, if a performance is deemed a success at a Distinction, then it is also deemed, simultaneously, a success at both Credit and Pass. This is analogous to implications of a measurement. For example, if an object is deemed to be 5 cm in length, then it implies that it is deemed also to be greater than 4, 3, 2 and 1 cm in length.

In the example of classifying an essay as a D, the three implied independent, dichotomous responses from Table 27.2 and Fig. 27.3 at the three successive thresholds are $\{y_P, y_C, y_D\} = \{1, 1, 1\}$. We notice that the number of thresholds at which there is a success is 3, which is simply the sum $\{y_P + y_C + y_D\} = \{1 + 1 + 1\} = 3$. Now suppose that the response from the format of Table 27.1 is C. This response implies, not only a success at δ_C, but because of the order $\delta_D > \delta_C > \delta_P$, it implies a success at δ_P and a failure at δ_D. The three implied responses from Table 27.2 at the three successive thresholds are $\{y_P, y_C, y_D\} = \{1, 1, 0\}$. We note immediately that the number of thresholds at which there is a success is 2, again simply the sum $\{y_P + y_C + y_D\} = \{1 + 1 + 0\} = 2$.

Suppose next that the response from the format of Table 27.1 is P. This response implies, not only a success at δ_P, but because of the order $\delta_D > \delta_C > \delta_P$, it implies a failure at both δ_C and δ_D. The three implied responses from Table 27.2 at the three successive thresholds are $\{y_P, y_C, y_D\} = \{1, 0, 0\}$, where we again note immediately that the number of thresholds at which there is a success is 1, simply the sum $\{y_P + y_C + y_D\} = \{1 + 0 + 0\} = 1$. Finally, suppose that the response from the format of Table 27.1 is F. This response implies, not only a failure at δ_P, but because of the order $\delta_D > \delta_C > \delta_P$, it implies a failure at both δ_C and δ_D. The three implied responses from Table 27.2 at the three successive thresholds are $\{y_P, y_C, y_D\} = \{0, 0, 0\}$, where we again note immediately that the number of thresholds at which there is a success is 0, simply the sum $\{y_P + y_C + y_D\} = \{0 + 0 + 0\} = 0$.

Taking these possible responses of successes and failures, we see that they are from the subspace of responses in Table 27.2. We notate this subspace as $\Omega^G \equiv \{(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1)\}$. Relative to the required order of the threshold proficiencies, $\delta_P < \delta_C < \delta_D$, this subspace is the set of Guttman patterns which we studied in Chap. 5. As noted above, and studied in that chapter, the sum of the responses within each set gives the total number of successes at the thresholds taken in their order of proficiency. Let $x = \{y_P + y_C + y_D\}$ in Ω^G, and let $x = 0, 1, 2, 3$ be this total score. The value x indicates, not only the total number of successes, but the identity of the specific thresholds at which the implied response was a success. It therefore also indicates the category of the response in Fig. 27.2, for example, $x = 2$ implies a grade of C. Tables 27.2 and 27.3 show this count of successes and the equivalent grade classification.

This count of the successes is analogous to the count of the number of units an object exceeds from an origin in typical measurement considered above. However, because they are estimated, the thresholds do not have to be equidistant as they are in measurement.

The full space Ω has other response elements which we may notate $\Omega^{\tilde{G}} \equiv \{(0, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}$, where $\tilde{G}$ stands for the subspace *not* G. These responses are incompatible with the required order. Thus, for example, the response set $\{y_P, y_C, y_D\} = \{0, 1, 0\}$ implies simultaneously a success at C but failure at P which is of lesser proficiency than C. Therefore, the responses in $\tilde{G}$ are excluded from the possible set of implied dichotomous responses at the thresholds and we focus on Ω^G.

Table 27.3 Probabilities of the Guttman subspace Ω^G in the space Ω

y_P	y_C	y_D	x	Grade	
0 $1/\gamma_P$	0 $1/\gamma_C$	0 $1/\gamma_D$	0	F	
1 $e^{\beta-\delta_P}/\gamma_P$	0 $1/\gamma_C$	0 $1/\gamma_D$	1	P	
1 $e^{\beta-\delta_P}/\gamma_P$	1 $e^{\beta-\delta_C}/\gamma_C$	0 $1/\gamma_D$	2	C	Ω^G
1 $e^{\beta-\delta_P}/\gamma_P$	1 $e^{\beta-\delta_C}/\gamma_C$	1 $e^{\beta-\delta_D}/\gamma_D y$	3	D	
$\sum_{\Omega^G} \Pr\{(y_P, y_C, y_D)\} =$ $\sum_{\Omega^G} \Pr\{y_P\|\Omega\} \Pr\{y_C\|\Omega\} \Pr\{y_D\|\Omega\} = \Gamma < 1$					

Formalizing the Response Space Ω^G

To construct the PRM, all that is required now is that the probabilities of the responses in Ω^G sum to 1. It is necessary to impose this constraint for two reasons. First, given a response in one category, the sum of the probabilities of responses in all categories must sum to 1. Second, because the sum of the probabilities of the responses in Ω sums to 1, the probabilities of the responses of the subspace Ω^G are less than 1 in the full space Ω. Table 27.3 shows these probabilities in terms of the dichotomous Rasch model with Γ the sum of the probabilities of the responses in Ω^G.

We have that the total number of successes, x, at thresholds defines, not only each Guttman pattern, but as shown in Table 27.3, the corresponding grade. To develop the form of the PRM, Table 27.4 shows the probabilities $\Pr\{x\}$ from the full space Ω in detail.

The terms in the last column in Table 27.4 can be written more generally as

$$\Pr\{x = 0; \beta, (\delta)|\Omega\} = 1/\gamma_P\gamma_C\gamma_D;$$
$$\Pr\{x; \beta, (\delta)|\Omega\} = e^{x\beta-\sum_{k=1}^{x}\delta_k}/\gamma_P\gamma_C\gamma_D, x = 1, 2, 3. \tag{27.2}$$

Table 27.4 Explicit expressions for probabilities of the subspace Ω^G in the space Ω

$\Pr\{x\|\Omega\} = \Pr\{y_P, y_C, y_D\|\Omega\}$	$= \Pr\{y_P\|\Omega\} \Pr\{y_C\|\Omega\} \Pr\{y_D\|\Omega\}$	
$\Pr\{x = 0\} = \Pr\{(0, 0, 0)\}$	$= 1.1.1/\gamma_P\gamma_C\gamma_D$	$= e^{0\beta}/\gamma_P\gamma_C\gamma_D$
$\Pr\{x = 1\} = \Pr\{(1, 0, 0)\}$	$= e^{\beta-\delta_P}.1.1/\gamma_P\gamma_C\gamma_D$	$= e^{1\beta-\delta_P}/\gamma_P\gamma_C\gamma_D$
$\Pr\{x = 2\} = \Pr\{(1, 1, 0)\}$	$= e^{\beta-\delta_P}.e^{\beta-\delta_C}.1/\gamma_P\gamma_C\gamma_D$	$= e^{2\beta-\delta_P-\delta_C}/\gamma_P\gamma_C\gamma_D$
$\Pr\{x = 3\} = \Pr\{(1, 1, 1)\}$	$= e^{\beta-\delta_P}.e^{\beta-\delta_C}.e^{\beta-\delta_D}/\gamma_P\gamma_C\gamma_D$	$= e^{3\beta-\delta_P-\delta_C-\delta_D}/\gamma_P\gamma_C\gamma_D$
$\sum_{\Omega^G} \Pr\{(y_P, y_C, y_D)\|\Omega\} = \sum_{\Omega^G} \Pr\{y_P\|\Omega\} \Pr\{y_C\|\Omega\} \Pr\{y_D\|\Omega\} = \Gamma < 1$		

The sum of the terms in Eq. (27.2) can now be written as $\Gamma = (1 + \sum_{x=1}^{3} e^{x\beta - \sum_{k=1}^{x}\delta_k})/\gamma_P\gamma_C\gamma_D$, $x = 0, 1, 2, 3$. To ensure that the probabilities in the subspace Ω^G sum to 1, each $\Pr\{x;\beta,(\delta)|\Omega\}$, $x = 0, 1, 2, 3$ needs to be divided by their sum Γ. Note that the denominator $\gamma_P\gamma_C\gamma_D$ of each term $\Pr\{x;\beta,(\delta)|\Omega\}$ is the same, and is also the same as the denominator of Γ. Therefore, in this division $\gamma_P\gamma_C\gamma_D$ cancels, leaving the numerators of the terms as $\Pr\{x = 0;\beta,(\delta)|\Omega^G\} = 1$, $\Pr\{x;\beta,(\delta)|\Omega^G\} = e^{x\beta - \sum_{k=1}^{x}\delta_k}$, $x = 1, 2, 3$, and their denominator as simply their sum $\gamma = 1 + \sum_{x=1}^{3} e^{x\beta - \sum_{k=1}^{x}\delta_k}$; the sum γ has no threshold subscripts. The division of each term in Ω^G by their sum Γ ensures they sum to 1 and therefore form a probability space.

Generalizing the Notation of Grade Classification

We now generalize the specific notation of $\delta_P, \delta_C, \delta_D$ to one indexed with successive integers identifying the successive thresholds, $\delta_1, \delta_2, \delta_3$. For ease of notation, we also define a threshold $\delta_0 \equiv 0$, giving the general expression of the PRM as

$$\Pr\{x;\beta,(\delta)|\Omega^G\} = e^{x\beta - \sum_{k=0}^{x}\delta_k}/\gamma,\ x = 0, 1, 2, 3. \tag{27.3}$$

For the remainder of the chapter, we also generalize the number of categories to $m + 1$, with a maximum score of m, and reintroduce the person and item subscripts to give

$$\Pr\{x;\beta_n,(\delta_i)|\Omega^G\} = e^{x\beta_n - \sum_{k=0}^{x}\delta_{ik}}/\gamma_{ni},\ x = 0, 1, 2, 3, \ldots, m_i. \tag{27.4}$$

Equation (27.4) is a general form of the PRM. We encountered it as Eq. (21.6) in Chap. 21.

A Fundamental Identity of the PRM

We are now in a position to revisit and expand on the understanding of the structure of the PRM introduced in Chap. 21. This involves a fundamental identity in the full space Ω and the Guttman subspace Ω^G.

The Full Space Ω

First, recall that the thresholds $\delta_P, \delta_C, \delta_D$ in the assessment of essays were defined in terms of the minimum proficiencies $\beta_P, \beta_C, \beta_D$ required to succeed at each of them, giving $\delta_P = \beta_P, \delta_C = \beta_C, \delta_D = \beta_D$. We then replaced the specific notation of the three thresholds $\delta_P, \delta_C, \delta_D$ for the four categories of Fail, Pass, Credit and Distinction to integer subscripts giving $\delta_1,\ \delta_2,\ \delta_3$. Then consistent with this notation, the minimum proficiencies may be notated $\beta_1, \beta_2, \beta_3$, respectively. This notation was generalized to $\delta_{i1}, \delta_{i2}, \delta_{i3}, \ldots, \delta_{ix}, \ldots, \delta_{im}$ for an item i with $m_i + 1$ categories. The minimum proficiency at δ_{ix} threshold is then β_x.

Second, we defined the probability of success according to the dichotomous Rasch model. In the threshold notation with integer subscripts above, and retaining the person and item subscripts, this implies that for any β_n,

$$\Pr\{y_{nix} = 1; \beta_n, \delta_{ix}|\Omega\} = e^{\beta_n - \delta_{ix}}/\gamma_{ni}, y = 1, 2, \ldots m_i \tag{27.5}$$

where to consolidate the meaning of this relationship, Eq. (27.5) includes the full response space Ω.

At minimum proficiency, $\beta_x = \delta_{ix}$, $\Pr\{y_{nix} = 1|\Omega\} = 0.5$. Equation (27.5) emphasizes that the thresholds are defined in terms of probabilities that have no constraints placed on them.

The Guttman Subspace Ω^G

From Eq. (27.4), we now simplify the ratio of the probability of response in category x relative to the response in adjacent categories $x - 1$ and x in the subspace Ω^G:

$$\begin{aligned}
&\frac{\Pr\{x; \beta_n, (\delta_i)|\Omega^G\}}{\Pr\{x-1; \beta_n, (\delta_i)|\Omega^G\} + \Pr\{x; \beta_n, (\delta_i)|\Omega^G\}} \\
&= \frac{e^{x\beta_n - \sum_{k=0}^{x}\delta_{ik}}/\gamma_{ni}}{e^{(x-1)\beta_n - \sum_{k=0}^{x-1}\delta_{ik}}/\gamma_{ni} + e^{x\beta_n - \sum_{k=0}^{x}\delta_{ik}}/\gamma_{ni}} \\
&= \frac{e^{x\beta_n - \sum_{k=0}^{x}\delta_{ik}}}{e^{(x-1)\beta_n - \sum_{k=0}^{x-1}\delta_{ik}} + e^{x\beta_n - \sum_{k=0}^{x}\delta_{ik}}} \\
&= \frac{e^{x\beta_n - \sum_{k=0}^{x-1}\delta_{ik} - \delta_{ix}}}{e^{x\beta_n - \beta_n - \sum_{k=0}^{x-1}\delta_{ik}} + e^{x\beta_n - \sum_{k=0}^{x-1}\delta_{ik} - \delta_{ix}}} \\
&= \frac{e^{x\beta_n - \sum_{k=0}^{x-1}\delta_{ik}}e^{-\delta_{ix}}}{e^{x\beta_n - \sum_{k=0}^{x-1}\delta_{ik}}e^{-\beta_n} + e^{x\beta_n - \sum_{k=0}^{x-1}\delta_{ik} - \delta_{ix}}} \\
&= \frac{e^{x\beta_n - \sum_{k=0}^{x-1}\delta_{ik}}e^{-\delta_{ix}}}{e^{x\beta_n - \sum_{k=0}^{x-1}\delta_{ik}}(e^{-\beta_n} + e^{-\delta_{ix}})}
\end{aligned}$$

$$= \frac{e^{-\delta_{ix}}}{e^{-\beta_n} + e^{-\delta_{ix}}}$$
$$= \frac{e^{\beta_n - \delta_{ix}}}{1 + e^{\beta_n - \delta_{ix}}}, x = 1, 2, 3, \ldots m_i.$$

The above ratio is derived from the probabilities *within* categories x – 1 and x *within* the subspace Ω^G. Therefore, we may define these adjacent pairs of categories as a subspace within Ω^G and notate it as $\Omega^G_{x-1,x}$. In summary, using this notation, the derivation above gives

$$\Pr\{x; \beta_n, (\delta_i)|\Omega^G_{x-1,x}\} = \frac{e^{\beta_n - \delta_{ix}}}{1 + e^{\beta_n - \delta_{ix}}} = e^{\beta_n - \delta_{ix}}/\gamma_{nix}, x = 1, 2, 3, \ldots, m_i. \quad (27.6)$$

Equation (27.6) is the *conditional* probability of success at threshold δ_{ix} as the probability of a response in category x relative to a response in the adjacent categories $x - 1$ and x in the PRM. These are parameters δ_{ix} estimated in the application of the PRM.

The Dichotomous Rasch Model Identity in Ω and $\Omega^G_{x-1,x}$

Equation (27.6) derived from the PRM is identical to Eq. (27.5), that is,

$$\Pr\{y_{nix} = 1; \beta_n, (\delta_i)|\Omega\} \equiv \Pr\{x|\Omega^G_{x-1,x}\} = e^{\beta_n - \delta_{ix}}/\gamma_{nix}. \quad (27.7)$$

The identity of the probability of a successful response in the two spaces Ω and $\Omega^G_{x-1,x}$ is fundamental to the interpretation of the PRM. The identity defines the thresholds of the PRM in terms of the proficiency required to succeed at the thresholds *unconstrained* by any subspace. Thus, the thresholds estimated by the PRM are the minimum proficiencies β_x required to succeed at the thresholds δ_{ix} giving $\beta_x = \delta_{ix}$. And these require that $\beta_{x+1} > \beta_x, x = 1, 2, \ldots, m_i$. Therefore, it is required that $\delta_{ix+1} > \delta_{ix}, x = 1, 2, \ldots, m_i$.

This relationship is made concrete in Andrich (2016). Responses were simulated for two sets of two dichotomous items according to the Rasch model. Then, a subset of responses which satisfied the Guttman structure according to the hypothesized ordering of the thresholds was taken. The original full set of dichotomous responses is the set Ω above, while the chosen subset of responses is Ω^G above. The former set was analysed with the dichotomous Rasch model and the latter with the PRM. The dichotomous item parameter estimates, and the corresponding PRM threshold estimates were within standard errors of estimates, reflecting that they are estimates of identical parameters $\delta_{ix}, x = 1, 2, \ldots, m_i$ from different sets of data.

It is possible to derive the PRM beginning from Eq. (27.6), that is, the conditional probability of a response in category x given the response is in either category $x - 1$

or x. Then, it is required to ensure that the sum of the probabilities is 1. If the implied sample spaces are made explicit, then the Guttman subspace Ω^G is shown to be implied, and the independent response space Ω can be inferred as the space of which Ω^G is a subspace. This derivation is shown in detail in Andrich (2013).

As we saw in Chaps. 20–22, it is possible that thresholds estimates from responses are not in this required order. In that case, there is some malfunctioning of the operation of the ordering of the categories. However, because the reversals can result from many different sources of malfunctioning, the reversed threshold estimates as such do not tell the specific source. The source or sources must be identified with further study of the format, the content, and so on, of the item.

Finally, the derivation of the PRM can begin with the conditional probabilities in the subspace $\Omega^G_{x-1,x}$ of Eq. (27.6) resulting in Eq. (27.4) of the PRM. Providing the implied sample spaces are taken into account, the Guttman space of implied successes at the thresholds is recovered (Andrich, 2013).

A general and a specific point regarding fit in relation to reversed threshold estimates is stressed. First, fit to the Rasch model is understood to be a necessary condition for invariance of comparisons and for measurement, not both a necessary and sufficient condition. Thus, other statistical and empirical properties of measurement are not revoked or bypassed by the Rasch model and fit to the Rasch model (Duncan, 1984). Second, and in any case, because reversed threshold estimates are used to recover the data in the usual test of fit, the responses may fit the model even when threshold estimates are reversed. Therefore, although fit is a necessary condition for all the properties of the Rasch model to hold, fit in itself does not bear on evidence of the malfunctioning of ordered categories of items. It is, however, possible for fit and reversed threshold estimates to interact.

Exercises

Suppose the minimum proficiencies of $\beta_P, \beta_C, \beta_D$ for achieving Pass (P), Credit (C) and Distinction (D), respectively, on an item i with four ordered categories are $-0.50, -0.10, 0.60$.

(a) What are the threshold values $\delta_{iP}, \delta_{iC}, \delta_{iD}$?
(b) Assume a person has a proficiency of $\beta_n = 0.0$. Complete the probabilities $\Pr\{y_{niP}|\Omega\}$, $\Pr\{y_{niC}|\Omega\}$ and $\Pr\{y_{niD}|\Omega\}$ of Table 27.2.
(c) Calculate the probabilities $\Pr\{(y_{niP}, y_{niC}, y_{niD})|\Omega\}$ for the subset of Guttman patterns of Table 27.4.
(d) Normalize the subset of probabilities calculated in (c) to give the probabilities $\Pr\{x; \beta_n, (\delta_i)|\Omega^G\}$, $x = 0, 1, 2, 3$ (that is, ensure their sum is 1).
(e) Calculate $\Pr\{x; \beta_n, (\delta_i)|\Omega^G_{x-1,x}\}$ for $x = 1, 2, 3$.
(f) Which of the probabilities you calculated in (e) above are, respectively, identical to the probabilities you calculated in (b) above.

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