Chapter 7
Invariance of Comparisons—Separation of Person and Item Parameters

Statistics Review 8: Conditional probability
Statistics Review 9: Independence

You should start this chapter by reviewing conditional probability in *Statistics Review 8*. Make sure you understand the concrete example of the tossing of two coins illustrating the use of *conditional probabilities*. In this chapter, we set up a parallel argument with the Rasch model. We then show through an example that the probability that a person answers one of two items correctly depends only on the relative difficulties of the items and is independent of the proficiency of the person. To help appreciate this example, which involves the expression of the model and conditional probabilities, understanding the simpler example of the tossing of two coins is very important. Because the Rasch model implies *statistical independence of responses*, the probability of answering both items correctly equals the product of the probabilities of answering the separate items correctly. This condition of independence of responses is touched on briefly towards the end of this chapter but in more detail in subsequent chapters of the book. It is an extremely important condition, both in the construction of instruments and in the Rasch model for analysing data from the instruments. *Statistics Review 9* reviews independence in set theory. Finally, we elaborate on the important principle of measurement, namely *invariance of comparisons*, which we introduced in the previous chapter.

A feature of CTT is that its various properties depend on the distribution of the proficiencies of the persons. Indeed, many of the statistics depend on the assumption that the true scores of people are normally distributed. A feature of the Rasch measurement model is that no assumptions need to be made about this distribution, and indeed, the distribution of proficiencies may be studied empirically. In order to appreciate how this works algebraically, and then conceptually, we begin by considering that a person responds to two items which are simply scored correct or incorrect.

D. Andrich and I. Marais, *A Course in Rasch Measurement Theory*,
Springer Texts in Education, https://doi.org/10.1007/978-981-13-7496-8_7

Conditional Probabilities with Two Items in the Rasch Model

The calculations below are exactly as in the tables in *Statistics Review 8* except that, instead of having numerical probabilities in the expressions, we have theoretical ones according to the model. We then use some specific values in the equations derived from the dichotomous Rasch model. We continue to use the concepts of proficiency and difficulty in the exposition, though *locations* of persons and items could be substituted readily for these expressions.

Let the proficiency of person n be β_n and the difficulty of item i be δ_i. Then the probabilities of this person answering two items, $i = 1$ and 2, correctly or incorrectly are shown in Table 7.1. To save space, we note that in the Rasch model, the probability of the two outcomes $x_{ni} = 1$ (correct) and $x_{ni} = 0$ (incorrect) are given respectively by

$$\Pr\{x_{ni} = 1\} = \frac{e^{\beta_n - \delta_i}}{1 + e^{\beta_n - \delta_i}} \text{ and } \Pr\{x_{ni} = 0\} = \frac{1}{1 + e^{\beta_n - \delta_i}}$$

in which the denominator in both expressions is $1 + e^{\beta_n - \delta_i}$.

For convenience, we let $\gamma_{ni} = 1 + e^{\beta_n - \delta_i}$ and use this as the denominator throughout.

Now we proceed with the same conditional probability argument as with the two coins, except that once again we use the probabilities according to the model rather than the numerical probabilities.

This means that we consider only those outcomes where one is correct (1) and the other is incorrect (0). This subset of outcomes is shown in Table 7.2.

Table 7.1 Probabilities of responses of a person to two dichotomously scored items

Item 1 (Probability)	Item 2 (Probability)	Joint outcomes (Probability)
1 $(e^{\beta_n - \delta_1})/\gamma_{n1}$	1 $(e^{\beta_n - \delta_2})/\gamma_{n2}$	$(e^{\beta_n - \delta_1}/\gamma_{n1})\,(e^{\beta_n - \delta_2}/\gamma_{n2})$
0 $1/\gamma_{n1}$	1 $(e^{\beta_n - \delta_2})/\gamma_{n2}$	$1/\gamma_{n1}\,(e^{\beta_n - \delta_2}/\gamma_{n2})$
1 $(e^{\beta_n - \delta_1})/\gamma_{n1}$	0 $1/\gamma_{n2}$	$(e^{\beta_n - \delta_1}/\gamma_{n1})\,1/\gamma_{n2}$
0 $1/\gamma_{n1}$	0 $1/\gamma_{n2}$	$1/\gamma_{n1}\,1/\gamma_{n2}$
		Total = 1.00

Table 7.2 Probabilities of one item correct and the other incorrect

Item 1 (Probability)	Item 2 (Probability)	Joint outcomes (Probability)
0 $1/\gamma_{n1}$	1 $(e^{\beta_n - \delta_2})/\gamma_{n2}$	$(e^{\beta_n - \delta_2})/\gamma_{n1}\gamma_{n2}$
1 $(e^{\beta_n - \delta_1})/\gamma_{n1}$	0 $1/\gamma_{n2}$	$(e^{\beta_n - \delta_1})/\gamma_{n1}\gamma_{n2}$
	Total probability	$(e^{\beta_n - \delta_2})/\gamma_{n1}\gamma_{n2} + (e^{\beta_n - \delta_1})/\gamma_{n1}\gamma_{n2}$ $= (e^{\beta_n - \delta_1} + e^{\beta_n - \delta_2})/\gamma_{n1}\gamma_{n2}$

The total probability of one of the items being correct and the other incorrect is shown at the bottom of Table 7.2. Relative to this total probability (of either one of the items being answered correctly and the other incorrectly), the probability that the first item is correct and the second is incorrect is given by the ratio

$$\begin{aligned}
&\Pr\{(x_{n1}=1,\ x_{n2}=0)|(x_{n1}=1,\ x_{n2}=0)\ \text{ or }\ (x_{n1}=0,\ x_{n2}=1)\}\\
&=\frac{(e^{\beta_n-\delta_1})/\gamma_{n1}\gamma_{n2}}{(e^{\beta_n-\delta_1})/\gamma_{n1}\gamma_{n2}+(e^{\beta_n-\delta_2})/\gamma_{n1}\gamma_{n2}}\\
&=\frac{(e^{\beta_n-\delta_1})/\gamma_{n1}\gamma_{n2}}{(e^{\beta_n-\delta_1}+e^{\beta_n-\delta_2})/\gamma_{n1}\gamma_{n2}}\\
&=\frac{e^{\beta_n-\delta_1}}{e^{\beta_n-\delta_1}+e^{\beta_n-\delta_2}}=\frac{e^{\beta_n}e^{-\delta_1}}{e^{\beta_n}(e^{-\delta_1}+e^{-\delta_2})}\\
&=\frac{e^{-\delta_1}}{(e^{-\delta_1}+e^{-\delta_2})}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\Pr\{(x_{n1}=1,\ x_{n2}=0)|(x_{n1}=1,\ x_{n2}=0)\ \text{ or }\ (x_{n1}=0,\ x_{n2}=1)\}\\
&=\frac{e^{-\delta_1}}{(e^{-\delta_1}+e^{-\delta_2})}.
\end{aligned} \tag{7.1}$$

Thus the probability that the first item is correct, when only one is correct and the other is incorrect, depends only on the relative difficulties of the items, and does not depend on the proficiency of the person. This is a profound equation and indicates that the relative difficulties of the items can be found without assuming anything about the value of the person's proficiency.

The probability of the second item being correct when the first is incorrect is the complement of the above result, which you might like to show.

$$\begin{aligned}
&\Pr\{(x_{n1}=0,\ x_{n2}=1)|(x_{n1}=1,\ x_{n2}=0)\ \text{ or }\ (x_{n1}=0,\ x_{n2}=1)\}\\
&=\frac{e^{-\delta_2}}{(e^{-\delta_1}+e^{-\delta_2})}.
\end{aligned} \tag{7.2}$$

A similar equation can be developed if two persons $n=1$ and $n=2$ respond to one item i.

$$\begin{aligned}
&\Pr\{(x_{1i}=1,\ x_{2i}=0)|(x_{1i}=1,\ x_{2i}=0)\ \text{ or }\ (x_{1i}=0,\ x_{2i}=1)\}\\
&=\frac{e^{\beta_1}}{(e^{\beta_1}+e^{\beta_2})}.
\end{aligned} \tag{7.3}$$

This means that the comparison of the difficulties between two items can be made independently of the proficiency of any person, and the comparison between people can be made independently of the difficulties of the items.

We will see how these equations might be applied in the next chapter.

Example

To consolidate the above result, below is a calculation from first principles and from Eqs. (7.1) and (7.2) for the following case: person n with proficiency $\beta_n = 0.5$ responds to item 1 with difficulty $\delta_1 = 0.5$ and item 2 with difficulty $\delta_2 = 1.5$.

From the probabilities in Table 7.3, we have the following:

The conditional probability that item 1 is correct and item 2 is incorrect is given by

$$\Pr\{(1,\ 0)|(1,\ 0) \text{ or } (0,\ 1)\} = \frac{0.365}{0.500} = 0.73$$

From Eq. (7.1) directly,

$$\Pr\{(1,\ 0)|(1,\ 0) \text{ or } (0,\ 1)\} = \frac{e^{-\delta_1}}{e^{-\delta_1} + e^{-\delta_2}} = \frac{e^{-0.5}}{e^{-0.5} + e^{-1.5}} = \frac{0.61}{0.61 + 0.22} = 0.73$$

which clearly is the same as from Table 7.3.

From Table 7.3, the conditional probability that item 1 is incorrect and item 2 is correct is given by

$$\Pr\{(0,\ 1)|(0,\ 1) \text{ or } (1, 0)\} = \frac{0.135}{0.500} = 0.27$$

From Eq. (7.2) directly,

$$\Pr\{(0,\ 1)|(0,\ 1) \text{ or } (1,\ 0)\} = \frac{e^{-\delta_2}}{e^{-\delta_1} + e^{-\delta_2}} = \frac{e^{-1.5}}{e^{-0.5} + e^{-1.5}} = \frac{0.22}{0.61 + 0.22} = 0.27$$

which is also clearly the same as from Table 7.3.

The detailed calculations for the probabilities in Table 7.3 are shown below.

Table 7.3 Example of probabilities for $\beta_n = 0.5$ with $\delta_1 = 0.5$ and $\delta_2 = 1.5$

Item 1 (Probability)	Item 2 (Probability)	Joint outcomes (Probability)
0 (0.50)	1 (0.27)	(0.50) (0.27) = 0.135
1 (0.50)	0 (0.73)	(0.50) (0.73) = 0.365
		Pr{(0 1) or (1, 0)} = 0.135 + 0.365 = 0.500

For item 1:

$$\begin{aligned}\Pr\{x_{n1}=1\} &= \tfrac{e^{\beta_n-\delta_1}}{1+e^{\beta_n-\delta_1}} & \text{and } \Pr\{x_{n1}=0\} &= \tfrac{1}{1+e^{\beta_n-\delta_1}}\\ &= \tfrac{e^{0.5-0.5}}{1+e^{0.5-0.5}} & &= \tfrac{1}{1+e^{0.5-0.5}}\\ &= \tfrac{e^{0}}{1+e^{0}} & &= \tfrac{1}{1+e^{0}}\\ &= \tfrac{1}{1+1} & &= \tfrac{1}{1+1}\\ &= \tfrac{1}{2}=0.5 & &= \tfrac{1}{2}=0.5\end{aligned}$$

Clearly also,

$$\Pr\{x_{n1}=0\}=\frac{1}{1+e^{\beta_n-\delta_1}}=1-\Pr\{x_{n1}=1\}=1-\frac{e^{\beta_n-\delta_1}}{1+e^{\beta_n-\delta_1}}=1.0-0.5=0.5.$$

For item 2:

$$\begin{aligned}\Pr\{x_{n2}=1\} &= \frac{e^{\beta_n-\delta_2}}{1+e^{\beta_n-\delta_2}}=\frac{e^{0.5-1.5}}{1+e^{0.5-1.5}}=\frac{e^{-1.0}}{1+e^{-1.0}}\\ &= \frac{0.37}{1+0.37}=\frac{0.37}{1.37}=0.27 \quad \text{and} \quad \Pr\{x_{n2}=0\}=\frac{1}{1+e^{\beta_n-\delta_2}}=1-0.27=0.73\end{aligned}$$

You should check that you follow these calculations.

The Condition of Local Independence

Presenting the results as we have in Tables 7.1, 7.2 and 7.3 is possible because the Rasch model implies statistical *independence of responses* in the sense that

$$\Pr\{((x_{ni}))\}=\prod_n\prod_i \Pr\{x_{ni}\}$$

where $((x_{ni}))$ denotes the matrix of responses $X_{ni}=x,\ n=1\ldots N,\ I=1\ldots I.$

That is, the probability of the set of responses to the items of an instrument equals the product of the probabilities of the responses to each of the items. For example, that is why we could write in Table 7.1 that the probability of a joint outcome of answering both items correctly is the product of the probabilities of answering each item correctly, that is $(e^{\beta_n-\delta_1}/\gamma_{n1})\ (e^{\beta_n-\delta_2}/\gamma_{n2})$.

The Principle of Invariant Comparisons

Rasch (1961) used the term *specific objectivity* to describe this important principle of *invariant comparison* which we summarized immediately following Eq. (7.3).

> The comparison between two stimuli should be independent of which particular individuals were instrumental for the comparison; and it should also be independent of which other stimuli within the considered class were or might also have been compared.
>
> Symmetrically, a comparison between two individuals should be independent of which particular stimuli within the class considered were instrumental for the comparison; and it should also be independent of which other individuals were also compared, on the same or some other occasion. (Rasch, 1961, p. 332).

Rasch referred to such comparisons as *objective*. Further, to highlight that this invariance is always constrained relative to a *specific* frame of reference, he referred to the objectivity of the comparisons as *specifically objective*. From Eqs. (7.1), (7.2) and (7.3), we saw how the comparison of the difficulties between two items can be made independently of the proficiency of any person, and the comparison between people can be made independently of the difficulties of the items.

Exercises

Suppose responses of person n to dichotomously scored items i, where $x_{ni} = 1$ represents a correct response and $x_{ni} = 0$ represents an incorrect response, conform to the Rasch model. That is, suppose

$$\Pr\{x_{ni} = 1\} = \frac{e^{\beta_n - \delta_i}}{1 + e^{\beta_n - \delta_i}} \text{ and } \Pr\{x_{ni} = 0\} = \frac{1}{1 + e^{\beta_n - \delta_i}}$$

Let $\beta_n = 1.0$, $\delta_1 = 0.5$ and $\delta_2 = 1.5$.

1. Which item is more difficult, item 1 or item 2?
2. What is the probability that the person answers each of the items correctly? i.e. find $\Pr\{x_{n1} = 1\}$ and find $\Pr\{x_{n2} = 1\}$.
3. What is the probability that the person will answer the first item correctly, given that the person has answered only one of the two items correctly?
4. Suppose another person with $\beta_n = 0.5$ responds to the two items. What is this person's probability of answering the first item correctly, given that the person has answered only one item correctly?
5. What do you notice in comparing your answers to (3) and (4) above?

Reference

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Further Reading

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