Chapter 22
The Polytomous Rasch Model III

This chapter elaborates further on aspects of the Polytomous Rasch model (PRM). First, it considers a reparameterization of the thresholds which has some advantages in the estimation of the parameters of the model. It is a reparameterization used in RUMM2030. Second, we consolidate the interpretation of responses in terms of an independent response space that is relevant in the case that the model is used with responses in ordered categories. Third, we elaborate on the rescoring of ordered categories in the case of problems with the operation of the categories. Fourth, we compare the Rasch model with the other model used for ordered categories, commonly known as the Graded Response Model (GRM).

Reparameterisation of the Thresholds

Andrich (1985) and Andrich and Luo (2003) reparameterized the thresholds with some advantages for parameter estimation. These advantages include that the estimation can be carried out and all thresholds estimated, even if some categories have zero frequency. Suppose that we assume that the thresholds for an item are equally spaced.

We start with the PRM in the form from the last chapter,

$$\Pr\{k, \beta, \delta, \underline{\tau}\} = \frac{1}{\gamma} \exp\left[\kappa_x + x(\beta - \delta)\right] \tag{22.1}$$

where $\gamma = \sum_{k=0}^{m} \exp[\kappa_k + k(\beta - \delta)]$ is a normalizing factor which is the sum of all the numerators ensuring that the sum of the probabilities is 1, and

$$\kappa_x = -\tau_0 - \tau_1 - \cdots - \tau_x. \tag{22.2}$$

We continue to drop the subscripts n and i, recognizing that we are referring to just one person responding to one item.

D. Andrich and I. Marais, *A Course in Rasch Measurement Theory*,
Springer Texts in Education, https://doi.org/10.1007/978-981-13-7496-8_22

Equidistant Thresholds

To start with, suppose there are just three thresholds and four categories ($m = 3$).

Then let

$$\tau_2 - \tau_1 = \tau_3 - \tau_2 = 2\lambda$$

where λ is the half distance between successive thresholds, that is, $(\tau_2 - \tau_1)/2 = (\tau_3 - \tau_2)/2 = \lambda$.

Then it can be shown that $\kappa_x = -\sum_{k=0}^{x} \tau_k = -\tau_0 \ldots - \tau_x = x(m - x)\lambda$.

We show below that the result is true. Remember $\tau_0 \equiv 0$ and $\kappa_m = -\sum_{k=0}^{m=3} \tau_k = -\tau_0 - \tau_1 - \tau_2 - \tau_3 = 0$ in the form of Eq. (22.2) above. The thresholds in this form are deviations from the item's overall location δ and therefore sum to zero: $\kappa_m = -\sum_{k=0}^{m=3} \tau_k = -(\tau_0 + \tau_1 + \tau_2 + \tau_3) = -0 = 0$.

To make the illustration concrete, suppose $\tau_1 = -1.5$, $\tau_2 = 0$ and $\tau_3 = 1.5$.

Then $\lambda = (\tau_2 - \tau_1)/2 = (\tau_3 - \tau_2)/2 = 0.75$.

Table 22.1 shows the structure of $\kappa_x = -\sum_{k=0}^{x} \tau_k = -\tau_0 \cdots - \tau_x = x(m - x)\lambda$.

Then the model can be written as

$$\Pr\{x; \beta, \delta, \theta\} = \frac{1}{\gamma} \exp\left[x(m - x)\lambda + x(\beta - \delta)\right] \tag{22.3}$$

Table 22.1 The structure of the category coefficients for equal threshold distances

x	κ_x	$x(m - x)\lambda;\ m = 3$
0	$\kappa_0 = -\sum_{k=0}^{0} \tau_k = -\tau_0 \equiv 0$	$0(3 - 0)\lambda = (0)(3)\lambda$ $= 0$
1	$\kappa_1 = -\sum_{k=0}^{1} \tau_k = -\tau_0 - \tau_1 = -\tau_1$ $= -(-1.5)$ $= 1.5$	$1(3 - 1)\lambda = (1)(2)\lambda$ $= 2(0.75)$ $= 1.5$
2	$\kappa_2 = -\sum_{k=0}^{2} \tau_k = -\tau_0 - \tau_1 - \tau_2$ $= -(-1.5) - 0$ $= 1.5$	$2(3 - 2)\lambda = (2)(1)\lambda$ $= 2(0.75)$ $= 1.5$
3	$\kappa_3 = -\sum_{k=0}^{3} \tau_k = -\tau_0 - \tau_1 - \tau_2 - \tau_3$ $= -(\tau_1 + \tau_2 + \tau_3) \equiv 0$	$3(3 - 3)\lambda = (3)(0)\lambda$ $= 0$

The parameter λ can be considered to characterize the *spread* of the responses. The greater the value of λ, the *narrower* the spread of responses (that is, the greater the proportion of responses in the middle category); the smaller the value of λ, the greater the spread of responses (that is, the greater the proportion of responses in the extreme categories). In RUMM2030, this parameter is called the *spread*.

Even if there are more than three thresholds, $m > 3$, and they are assumed to be equidistant, that is $\lambda = (\tau_2 - \tau_1)/2 = (\tau_3 - \tau_2)/2 = \cdots = (\tau_m - \tau_{m-1})/2$, the same formula Eq. (22.3) holds.

In this case, the number of parameters estimated is less than the number of thresholds. Only one parameter for thresholds is estimated, the average distance between them, even though there might be more than three thresholds.

Recovering the Thresholds

To obtain a value for each threshold, we note that with an estimate of λ we have an estimate of all of the category coefficients, $\hat{\kappa}_x = x(m - x)\hat{\lambda}$. Then, we apply Eq. (22.2) in the following way:

From

$$\kappa_x = -\tau_0 - \tau_1 - \cdots - \tau_x,$$
$$\kappa_x - \kappa_{x+1} = (-\tau_0 - \tau_1 - \cdots - \tau_x) - (-\tau_0 - \tau_1 - \cdots - \tau_x - \tau_{x+1})$$
$$= \tau_{x+1}.$$

Thus suppose that we had obtained the estimate $\hat{\lambda} = 0.75$. Then

$$\hat{\kappa}_0 = x(m - x)\hat{\lambda} = 0(3 - 0)(0.75) = 0,$$
$$\hat{\kappa}_1 = x(m - x)\hat{\lambda} = 1(3 - 1)(0.75) = 1.5,$$
$$\hat{\kappa}_2 = x(m - x)\hat{\lambda} = 2(3 - 2)(0.75) = 1.5,$$
$$\hat{\kappa}_m = x(m - x)\hat{\lambda} = 3(3 - 3)(0.75) = 0,$$

and

$$\hat{\kappa}_0 - \hat{\kappa}_1 = \hat{\tau}_1 = 0 - 1.5 = -1.5,$$
$$\hat{\kappa}_1 - \hat{\kappa}_2 = \hat{\tau}_2 = 1.5 - 1.5 = 0.0,$$
$$\hat{\kappa}_2 - \hat{\kappa}_3 = \hat{\tau}_3 = 1.5 - 0 = 1.5.$$

Non-equidistant Thresholds

Now suppose that the thresholds are not equidistant, that is, they are skewed.

For example, suppose $\tau_1 = -1.7$, $\tau_2 = -0.6$, $\tau_3 = 2.3$. Then $\kappa_0 = \kappa_m = 0$ as before, but now $\tau_2 - \tau_1 = 1.1$ and $\tau_3 - \tau_2 = 2.9$. We can still find the average distance between the successive thresholds $[(\tau_2 - \tau_1) + (\tau_3 - \tau_2)]/2 = (1.1 + 2.9)/2 = 4/2 = 2$. Therefore, the half distance λ is $2/2 = 1$.

Suppose we consider the thresholds from the perspective of a deviation of each from equidistance.

We let $1.1 = \tau_2 - \tau_1 = 2\lambda - 6\eta$ and $2.9 = \tau_3 - \tau_2 = 2\lambda - 6\eta$.

Then

$$\begin{aligned} 1.1 = \tau_2 - \tau_1 = 2\lambda - 6\eta &= 2(1) - 6\eta \\ 6\eta &= 2 - 1.1 = 0.9 \\ \eta &= 0.15 \end{aligned}$$

This is consistent with the second of the above expressions. Inserting $\eta = 0.15$ gives

$$\begin{aligned} 2.9 = \tau_3 - \tau_2 = 2\lambda - 6\eta &= 2(1) - 6(0.15) \\ &= 2 + 0.9 \\ &= 2.9 \end{aligned}$$

The coefficient 6 of η eliminates the need for fractions in the following expression:

$$\kappa_x = x(m - x)\lambda + x(m - x)(2x - m)\eta.$$

The parameter η characterizes the deviation of the thresholds from equidistance. It is therefore an indicator of the *skewness* of the thresholds. The greater its value, the greater the deviation of the successive thresholds distances from equidistance. Table 22.2 demonstrates that the expression gives the required values with the above example.

The equation for the model may then be expressed as

$$\Pr\{x; \beta, \delta, \lambda, \eta\} = \frac{1}{\gamma}[x(m - x)\lambda + x(m - x)(2x - m)\eta + x(\beta - \delta)]. \quad (22.4)$$

This equation can hold for any number of thresholds, three or greater. However, if there are more than three thresholds, then this equation estimates a smaller number of parameters than the possible number of thresholds that can be estimated. The maximum number of parameters is the number of thresholds.

With estimates of λ and η, we have an estimate of the category coefficients κ_x and from these we can again use Eq. (22.2) to recover the actual thresholds.

Table 22.2 The structure of the category coefficients for non-equal threshold distances

x	κ_x	$x(m-x)\lambda + x(m-x)(2x-m)\eta$
0	$\kappa_0 = -\sum_{k=0}^{0} \tau_k = -\tau_0 \equiv 0$	$0(3-0)1 + 0(3-0)(2(0)-3)(0.15)$ $= 0$
1	$\kappa_1 = -\sum_{k=0}^{1} \tau_k = -\tau_0 - \tau_1 = -\tau_1$ $= -(-1.7)$ $= 1.7$	$1(3-1)1 + 1(3-1)(2(1)-3)(0.15)$ $= 2 + 2(-1)(0.15)$ $= 2 - 0.3 = 1.7$
2	$\kappa_2 = -\sum_{k=0}^{2} \tau_k = -\tau_0 - \tau_1 - \tau_2$ $= -(-1.7) - (-0.6)$ $= 2.3$	$2(3-2)1 + 2(3-2)(2(2)-3)(0.15)$ $= 2 + 2(4-3)(0.15)$ $= 2 + 2(0.15) = 2.3$
3	$\kappa_3 = -\sum_{k=0}^{3} \tau_k = -\tau_0 - \tau_1 - \tau_2 - \tau_3$ $= -(\tau_1 + \tau_2 + \tau_3) \equiv 0$	$3(3-3)1 + 3(3-3)(2(3)-3)(0.15)$ $= 0$

Thus suppose we have as estimates $\hat{\lambda} = 1$ and $\hat{\eta} = 0.15$ for a four-category item (three thresholds) as above. Then

$$
\begin{aligned}
\hat{\kappa}_x &= x(m-x)\hat{\lambda} + x(m-x)(2x-m)\hat{\eta} \\
\hat{\kappa}_0 &= 0(3-0)\hat{\lambda} + 0(3-0)(2(0)-3)\hat{\eta} = 0\hat{\lambda} + 0\hat{\eta} = 0, \\
\hat{\kappa}_1 &= 1(3-1)\hat{\lambda} + 1(3-1)(2(1)-3)\hat{\eta} = 2\hat{\lambda} + 1(2)(-1)\hat{\eta} \\
&= 2(1) - 2(0.15) = 2 - 0.30 = 1.7, \\
\hat{\kappa}_2 &= 2(3-2)\hat{\lambda} + 2(3-2)(2(2)-3)\hat{\eta} = 2\hat{\lambda} + 2(1)(1)\hat{\eta} \\
&= 2(1) + 2(0.15) = 2 + 0.30 = 2.3, \\
\hat{\kappa}_3 &= 3(3-3)\hat{\lambda} + 0(3-3)(2(3)-3)\hat{\eta} = 0\hat{\lambda} + 0\hat{\eta} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\kappa}_0 - \hat{\kappa}_1 &= \hat{\tau}_1 = 0 - 1.7 = -1.7, \\
\hat{\kappa}_1 - \hat{\kappa}_2 &= \hat{\tau}_2 = 1.7 - 2.3 = -0.6, \\
\hat{\kappa}_2 - \hat{\kappa}_3 &= \hat{\tau}_3 = 2.3 - 0 = 2.3.
\end{aligned}
$$

In summary, with two thresholds we can have, in addition to the location parameter δ, the spread parameter λ; with three thresholds, as we showed above, we can have, in addition to the location parameter δ, up to a spread and skewness parameters λ, η.

With four thresholds we can have, in addition to the location parameter, up to a spread, skewness and kurtosis parameter. This is four parameters for the thresholds. With five thresholds we can have another parameter, and so on. We call these the *principal components* of the thresholds following the use of the term by Guttman (1950). This is discussed in some detail in Andrich (1985).

In each case, we can have less parameters estimated than the number of thresholds, but not more parameters than the number of thresholds. However, each threshold has an estimated value as shown above. In RUMM2030, the number of parameters estimated is up to the kurtosis parameter even if the number of thresholds is greater than four. The thresholds are then recovered from the category coefficients as shown above.

Inference of an Independent Response Space

Andrich (2010) shows that, in the case of a single response in more than two ordered categories, for example, some kind of an ordered category response, the analysis can be interpreted as if there was a separate response at each of the thresholds and the analysis carried out with the dichotomous Rasch model.

Of course, we do not have such a data set, but it is remarkable that when we analyse ordered category data with the PRM, it is as if we had independent, dichotomous responses at the thresholds, and that we have analysed these dichotomous responses using the dichotomous Rasch model. It is this inference that permits us to realize that when we have reversed thresholds and that we have a problem with the empirical ordering of the categories. The derivation of the model which explains this inference is shown in Chap. 27.

Rescoring Items

One of the apparently surprising features of the PRM is that if the data fit the model perfectly for some number of categories, then combining a pair of adjacent categories by summing their frequencies destroys the fit of the responses to the Rasch model with the fewer number of categories.

Only in the case that a threshold does not discriminate between a pair of adjacent categories, it is theoretically justified to pool the frequencies of adjacent categories. Often when the threshold estimates are reversed from their natural order, it is reasonable to consider combining categories. As part of this consideration, the discrimination at the thresholds can be considered. Sometimes, the thresholds might not be reversed, but the discrimination at a pair of thresholds might be close to zero and the categories could be combined.

RUMM2030 permits you to study the discrimination at the thresholds graphically. Before considering an example where a threshold did not discriminate, we consider

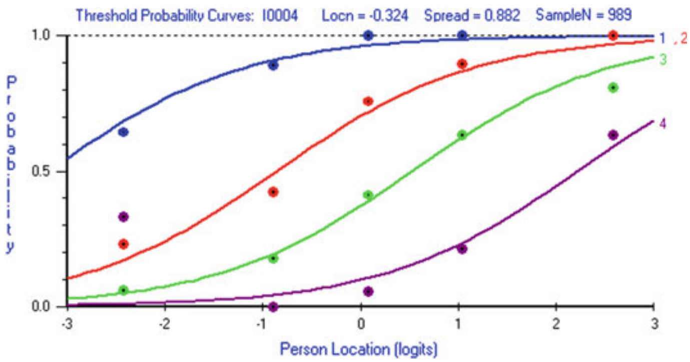

Fig. 22.1 An item where all four thresholds operated successfully

how we infer whether a threshold is discriminating properly or not. The inference about discrimination at the threshold is identical to the inference from dichotomous items. In the example below, where an item had five categories, there are four thresholds and at each threshold we can infer a dichotomous response. For each threshold x, we consider the proportion of responses in two adjacent categories for each class interval

$$\frac{\text{Proportion }(x)}{\text{Proportion }(x-1)+\text{Proportion }(x)}$$

These should follow the dichotomous Rasch model at the thresholds. Figure 22.1 shows such an example for an item with five categories where all the thresholds operated as required. The persons were classified into five class intervals and the above proportions calculated for each pair of adjacent categories. The dots show the proportions, which should be close to the respective theoretical threshold probability curves.

Below is an output in which the discrimination at a threshold was close to zero. Only the discrimination at threshold 3 is shown. All other thresholds discriminated well. Therefore, categories 2 and 3 should be combined. RUMM2030 permits you to do another analysis by rescoring the item (Fig. 22.2).

It is important to appreciate that this kind of combining of categories from an analysis should be treated as hypothesis testing of the possible reconstruction of the categories for future administration of the instrument. The combining of categories after the data are collected does not imply formal equivalence with combining categories and defining a new category from the original two categories.

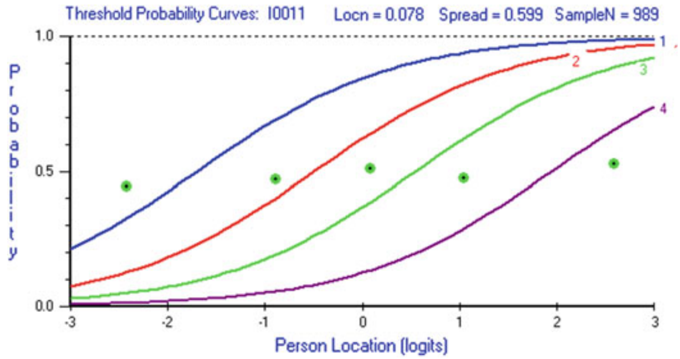

Fig. 22.2 An item where threshold 3 did not discriminate

Exercises

Exercise 4: Advanced analysis of polytomous responses in Appendix C.

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Further Reading

Andrich, D. (2010). Understanding the response structure and process in the polytomous Rasch model. In M. Nering & R. Ostini (Eds.), *Handbook of polytomous item response theory models: Developments and applications* (pp. 123–152). Mahwah, New Jersey: Lawrence Erlbaum Associates Inc.