Chapter 12
Dynamics of Complex Robotic Mechanical Systems

12.1 Introduction

The subject of this chapter is the dynamics of the class of robotic mechanical systems introduced in Chap. 10 under the generic name of *complex*. Notice that this class comprises serial manipulators not allowing a decoupling of the orientation from the positioning tasks. For purposes of dynamics, this decoupling is irrelevant and hence, was not a condition in the study of the dynamics of serial manipulators in Chap. 7. Thus, serial manipulators need not be further studied here, the focus being on parallel manipulators and rolling robots. The dynamics of walking machines and multifingered hands involves special features that render these systems more elaborate from the dynamics viewpoint, for they exhibit a time-varying topology. What this means is that these systems include kinematic loops that open when a leg takes off or when a finger releases an object and open chains that close when a leg touches ground or when a finger makes contact with an object. The implication here is that the degree of freedom of these systems is time-varying. The derivation of such a mathematical model is discussed in Pfeiffer et al. (1995), but is left out in this book.

The degree of freedom (dof) of the mechanical systems studied here is thus constant. Now, the two kinds of systems studied here pertain to very different types, for parallel manipulators fall into the realm of *holonomic*, while rolling robots into that of *nonholonomic*, mechanical systems. In order to better understand this essential difference between these two types of systems, we give below a summary of the classification of mechanical systems at large.

12.2 Classification of Robotic Mechanical Systems with Regard to Dynamics

Because robotic mechanical systems are a class of general mechanical systems, a classification of the latter will help us focus on the systems motivating this study. Mechanical systems can be classified according to various criteria, the most

J. Angeles, *Fundamentals of Robotic Mechanical Systems: Theory, Methods, and Algorithms*, Mechanical Engineering Series 124, DOI 10.1007/978-3-319-01851-5_12,

common one being based on the type of constraints to which these systems are subjected. In this context we find *holonomic* vs. *nonholonomic* and *scleronomic* vs. *rheonomic* constraints. Holonomic constraints are those that are expressed either as a system of algebraic equations in displacement variables, whether angular or translational, not involving any velocity variables, or as a system of equations in velocity variables that nevertheless can be integrated *as a whole* to produce a system of equations of the first type. Note that it is not necessary that every single scalar equation of velocity constraints be integrable; rather, the whole system must be integrable for the system of velocity constraints to lead to a system of displacement constraints. If the system of velocity constraints is not integrable, the constraints are said to be nonholonomic. Moreover, if a mechanical system is subject only to holonomic constraints, it is said to be holonomic; otherwise, it is nonholonomic. Manipulators composed of revolute and prismatic pairs are examples of holonomic systems, while wheeled robots are usually nonholonomic systems. On the other hand, if a mechanical system is subject to constraints that are not explicit functions of time, these constraints are termed scleronomic, while if the constraints are explicit functions of time, they are termed rheonomic. For our purposes, however, this distinction is irrelevant.

In order to understand better one more classification of mechanical systems, we recall the concepts of generalized coordinate and generalized speed that were introduced in Sect. 7.3.2. The generalized coordinates of a mechanical system are all those displacement variables, whether rotational or translational, that determine uniquely a configuration of the system. Note that the set of generalized coordinates of a system is not unique. Moreover, various sets of generalized coordinates of a mechanical system need not have the same number of elements, but there is a minimum number below which the set of generalized coordinates cannot define the configuration of the system. This minimum number corresponds, in the case of holonomic systems, to the *degree of freedom* of the system. Serial and parallel manipulators coupled only by revolute or prismatic pairs are holonomic, their joint variables, grouped in vector $\boldsymbol{\theta}$, playing the role of generalized coordinates, while their joint rates, grouped in vector $\dot{\boldsymbol{\theta}}$, in turn, play the role of generalized speeds. Note that in the case of parallel manipulators, not all joint variables are independent generalized coordinates. In the case of nonholonomic systems, on the other hand, the number of generalized coordinates needed to fully specify their configuration exceeds their degree of freedom by virtue of the lack of integrability of their kinematic constraints. This concept is best illustrated with the aid of examples, which are included in Sect. 12.5. Time-derivatives of the generalized coordinates, or linear combinations thereof, are termed the *generalized speeds* of the system. If the kinetic energy of a mechanical system is zero when all its generalized speeds are set equal to zero, the system is said to be *catastatic*. If, on the contrary, the kinetic energy of the system is nonzero even if all the generalized speeds are set equal to zero, the system is said to be *acatastatic*. All the systems that we will study in this chapter are catastatic. A light robot mounted on a heavy noninertial base that undergoes a *controlled* motion is an example of an acatastatic system, for the motion of the base can be assumed to be insensitive to the dynamics of the robot; however, the motion of the base does affect the dynamics of the robot.

Another criterion used in classifying mechanical systems, which pertains specifically to robotic mechanical systems, is based on the type of actuation. In general, a system needs at least as many independent actuators as degrees of freedom. However, instances arise in which the number of actuators is greater than the degree of freedom of the system. In these instances, we speak of *redundantly actuated systems*. In view of the fundamental character of this book, we will not study redundant actuation here; we will thus assume that the number of independent actuators equals the degree of freedom of the system.

The main results of this chapter are applicable to robotic mechanical systems at large. For brevity, we will frequently refer to the objects of our study simply as *systems*.

12.3 The Structure of the Dynamics Models of Holonomic Systems

We saw in Sect. 7.6 that the mathematical model of a manipulator of the serial type contains basically three terms, namely, one linear in the joint accelerations, one quadratic in the joint rates, and one arising from the environment, i.e., from actuators, dissipation, and potential fields such as gravity. We show in this section that in fact, the essential structure of this model still holds in the case of more general mechanical systems subject to holonomic constraints, if we regard the rates of the actuated joints as the *independent* generalized speeds of the system. Nonholonomic robotic systems are studied in Sect. 12.5.

First, we will assume that the mechanical system at hand is composed of r rigid bodies and its degree of freedom is n. Henceforth, we assume that these bodies are coupled in such a way that they may form kinematic loops; for this reason, such systems contain some *unactuated* joints. Definitions similar to those of Sect. 7.3.1 are henceforth adopted. In this vein, the manipulator mass matrix of that section becomes now, more generically, the $6r \times 6r$ *system mass matrix* $\mathbf{M}$, the $6r \times 6r$ *system angular velocity matrix* $\mathbf{W}$, and the $6r$-dimensional *system twist vector* $\mathbf{t}$ being defined likewise.

We assume further that the total number of joints, active and passive, is $m > n$. The m-dimensional array $\boldsymbol{\theta}$ of joint variables, associated with both actuated and unactuated joints, is thus naturally partitioned into two subarrays, the n-dimensional vector of actuated joint variables $\boldsymbol{\theta}_a$ and its m'-dimensional unactuated counterpart $\boldsymbol{\theta}_u$, with $m' \equiv m - n$, namely,

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\theta}_a \\ \boldsymbol{\theta}_u \end{bmatrix} \tag{12.1}$$

We can now set up the mathematical model of the system at hand using the natural orthogonal complement, as introduced in Sect. 7.5. Since the system under study has n degrees of freedom, the model sought must be a system of n second-order ordinary differential equations. We can proceed to derive this model as we did in Sect. 7.5, by regarding all joints first as if they were independent, but taking into account that only n of the total m joints are actuated. We do this by introducing a vector of constraint forces, as is done in the realm of Lagrangian dynamics (Török 2000). In this vein, we first represent the twists of all the moving links as linear transformations of the joint-rate vector $\dot{\boldsymbol{\theta}}$, then assemble all the individual six-dimensional twist arrays into the $6r$-dimensional array $\mathbf{t}$ defined above as the system twist. We thus end up with a relation of the form

$$\mathbf{t} = \mathbf{U}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{12.2}$$

where $\mathbf{U}(\boldsymbol{\theta})$ is the $6r \times m$ twist-shaping matrix, playing a role similar to that of matrix $\mathbf{T}$ of Sect. 7.5. Moreover, the constraints relating all joint rates can be cast in the form

$$\mathbf{A}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \mathbf{0}_p \tag{12.3}$$

where $\mathbf{A}(\boldsymbol{\theta})$ is a $p \times m$ matrix, whereby $p < m$, with nullity—the nullity of a matrix is the dimension of its null space—$\nu = n$, and $\mathbf{0}_p$ is the p-dimensional zero vector. Given the nullity of $\mathbf{A}(\boldsymbol{\theta})$, up to n of the m components of $\boldsymbol{\theta}$ can be assigned freely without violating the constraints (12.3), which is compatible with the assumption on the dof of the system. Note that, in setting up the foregoing p constraints on the joint rates, the number p depends on the *topology* of the system, i.e., on its number of links; on its number of joints; and on how the links are coupled, so as to form kinematic loops.

In applying the procedure of the natural orthogonal complement to the constrained system, we end up with a system of m second-order ordinary differential equations, namely, the Euler–Lagrange equations of a system constrained by the relations (12.3), which thus takes the form

$$\tilde{\mathbf{I}}\ddot{\boldsymbol{\theta}} + \tilde{\mathbf{C}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} = \tilde{\boldsymbol{\tau}} + \tilde{\boldsymbol{\delta}} + \tilde{\boldsymbol{\gamma}} + \tilde{\mathbf{J}}^T\mathbf{w}^W + \mathbf{A}^T\boldsymbol{\lambda} \tag{12.4a}$$

The above equation contains terms that are familiar from Sect. 7.5, except for the last term of the right-hand side. This term accounts for the generically termed *constraint forces* and amounting to constraint joint torques and forces that must be exerted at all joints in order to maintain the topology of the system. Vector $\boldsymbol{\lambda}$ is termed the *vector of Lagrange multipliers* in the realm of Lagrangian dynamics. In the above equation, the definitions below, similar to those of Eqs. (7.58) and (7.59), have been introduced:

$$\tilde{\mathbf{I}}(\boldsymbol{\theta}) \equiv \mathbf{U}^T\mathbf{M}\mathbf{U} \tag{12.4b}$$

$$\tilde{\mathbf{C}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \equiv \mathbf{U}^T\mathbf{M}\dot{\mathbf{U}} + \mathbf{U}^T\mathbf{W}\mathbf{M}\mathbf{U} \tag{12.4c}$$

$$\tilde{\boldsymbol{\tau}} \equiv \mathbf{U}^T\mathbf{w}^A, \quad \tilde{\boldsymbol{\delta}} \equiv \mathbf{U}^T\mathbf{w}^D, \quad \tilde{\boldsymbol{\gamma}} \equiv \mathbf{U}^T\mathbf{w}^G \tag{12.4d}$$

Moreover, $\mathbf{w}^A$, $\mathbf{w}^D$, $\mathbf{w}^G$, and $\mathbf{w}^W$ are the various types of wrenches acting on the system: exerted by the actuators; stemming from dissipation effects; due to the gravity field; and exerted by the environment, respectively. In turn, $\tilde{\mathbf{J}}$ is the $6 \times m$ Jacobian matrix mapping the system joint rates into the end-effector twist, while $\mathbf{w}^W$ is assumed applied onto the end-effector.

Upon resorting to the kinematics of the system, it is possible to express the vector of joint rates $\dot{\boldsymbol{\theta}}$ as a linear transformation of the vector of actuated joint rates $\dot{\boldsymbol{\theta}}_a$, namely,[1]

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Theta}(\boldsymbol{\theta}_a)\dot{\boldsymbol{\theta}}_a \tag{12.5}$$

where we have assumed that, from the geometry of the system, $\boldsymbol{\theta}_u$ has been solved for in terms of $\boldsymbol{\theta}_a$. Further, upon substitution of Eq. (12.5) into Eq. (12.3), we obtain

$$\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Theta}(\boldsymbol{\theta}_a)\dot{\boldsymbol{\theta}}_a = \mathbf{0}_p$$

which must hold for *any* $\dot{\boldsymbol{\theta}}_a$, given the dof of the system. As a consequence, then,

$$\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Theta}(\boldsymbol{\theta}_a) = \mathbf{O}_{pn} \tag{12.6}$$

and hence, $\boldsymbol{\Theta}(\boldsymbol{\theta}_a)$ is an orthogonal complement of $\mathbf{A}(\boldsymbol{\theta})$, which we can also call a natural orthogonal complement. Notice, however, that contrary to the natural orthogonal complement $\mathbf{U}$, which maps the joint-rate vector onto the system twist, $\boldsymbol{\Theta}$ maps the space of actuated joint rates into that of the system joint rates. Apparently,

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{\Theta}(\boldsymbol{\theta}_a)\ddot{\boldsymbol{\theta}}_a + \dot{\boldsymbol{\Theta}}(\boldsymbol{\theta}_a, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a \tag{12.7}$$

Upon substitution of Eq. (12.7) into Eq. (12.4a), we obtain

$$\tilde{\mathbf{I}}\boldsymbol{\Theta}\ddot{\boldsymbol{\theta}}_a + \tilde{\mathbf{I}}\dot{\boldsymbol{\Theta}}\dot{\boldsymbol{\theta}}_a + \tilde{\mathbf{C}}(\boldsymbol{\theta}_a, \dot{\boldsymbol{\theta}}_a)\boldsymbol{\Theta}\dot{\boldsymbol{\theta}}_a = \tilde{\boldsymbol{\tau}} + \tilde{\boldsymbol{\delta}} + \tilde{\boldsymbol{\gamma}} + \tilde{\mathbf{J}}^T\mathbf{w}^W + \mathbf{A}^T\boldsymbol{\lambda}$$

Further, the term of constraint forces is eliminated from the above equations upon multiplying both sides of the above equation by $\boldsymbol{\Theta}^T$ from the left, thus obtaining the mathematical model sought, i.e.,

$$\mathbf{I}\ddot{\boldsymbol{\theta}}_a + \mathbf{C}\dot{\boldsymbol{\theta}}_a = \boldsymbol{\tau} + \boldsymbol{\delta} + \boldsymbol{\gamma} + \mathbf{J}^T\mathbf{w}^W \tag{12.8a}$$

[1] $\boldsymbol{\Theta}$ is not to be confused with the matrix defined in Eqs. (10.54a and b).

with the definitions below:

$$\mathbf{I} = \mathbf{T}^T\mathbf{M}\mathbf{T}, \quad \mathbf{C} = \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T\mathbf{W}\mathbf{M}\mathbf{T}, \quad \mathbf{J} = \tilde{\mathbf{J}}\boldsymbol{\Theta}, \tag{12.8b}$$

$$\boldsymbol{\tau} = \boldsymbol{\Theta}^T\tilde{\boldsymbol{\tau}}, \quad \boldsymbol{\delta} = \boldsymbol{\Theta}^T\tilde{\boldsymbol{\delta}}, \quad \boldsymbol{\gamma} = \boldsymbol{\Theta}^T\tilde{\boldsymbol{\gamma}}, \tag{12.8c}$$

and

$$\mathbf{T} = \mathbf{U}\boldsymbol{\Theta} \tag{12.8d}$$

That is, the mathematical model governing the dynamics of any holonomic robotic mechanical system is formally identical to that of Eq. (7.61) obtained for serial manipulators.

12.4 Dynamics of Parallel Manipulators

We illustrate the modeling techniques of mechanical systems with kinematic loops via a class of systems known as *parallel manipulators*. While parallel manipulators can take on a large variety of forms, we focus here on those termed *platform manipulators*, with an architecture similar to that of flight simulators. In platform manipulators we can distinguish two special links, namely, the base $\mathcal{B}$ and the moving platform $\mathcal{M}$. Moreover, these two links are coupled via six *legs*, with each leg constituting a six-axis kinematic chain of the serial type, as shown in Fig. 12.1, whereby a wrench $\mathbf{w}^W$, represented by a double-headed arrow, acts on $\mathcal{M}$ and is applied at $C_{\mathcal{M}}$, the mass center of $\mathcal{M}$. This figure shows the axes of the revolutes coupling the legs to the two platforms as forming regular polygons. However, the modeling discussed below is not restricted to this particular geometry. As a matter of fact, these axes need not even be coplanar. On the other hand, the architecture of Fig. 12.1 is very general, for it includes more specific types of platform manipulators, such as flight simulators. In these, the first three revolute axes stemming from the base platform have intersecting axes, thereby giving rise to a spherical kinematic pair, while the upper two axes intersect at right angles, thus constituting a universal joint. Moreover, the intermediate joint in flight simulators is not a revolute, but rather a prismatic pair, which is the actuated joint of the leg. A leg kinematically equivalent to that of flight simulators can be obtained from that of the manipulator of Fig. 12.1, if the intermediate revolute has an axis perpendicular to the line connecting the centers of the spherical and the universal joints of the corresponding leg, as shown in Fig. 12.2. In flight simulators, the pose of the moving platform is controlled by hydraulic actuators that vary the distance between these two centers. In the revolute-coupled equivalent leg, the length of the same line is controlled by the rotation of the intermediate revolute.

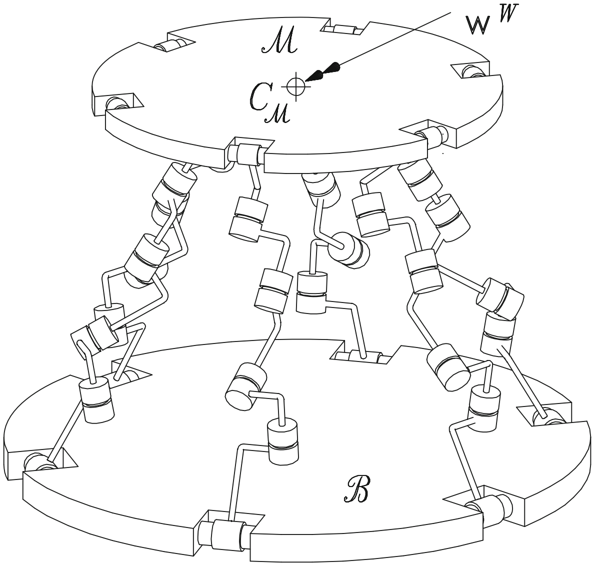

Fig. 12.1 A platform-type parallel manipulator

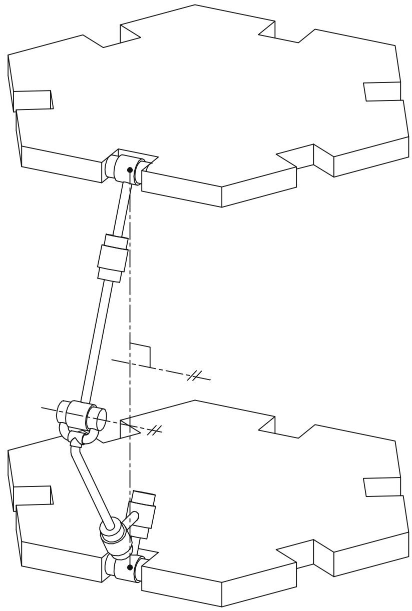

Fig. 12.2 A leg of a simple platform-type parallel manipulator

Shown in Fig. 12.3 is the graph of the system depicted in Fig. 12.1. In that graph, the nodes denote rigid links, while the edges denote joints. By application of *Euler's formula* for graphs (Harary 1972), the number ι of independent loops of a system with many kinematic loops is given by

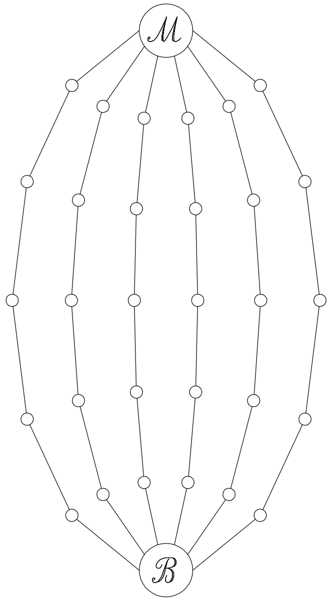

Fig. 12.3 The graph of the flight simulator

$$\iota = j - l + 1 \tag{12.9}$$

where j is the number of revolute and prismatic joints and l is the number of links.

Thus, if we apply Euler's formula to the system of Fig. 12.1, we conclude that its kinematic chain contains five independent loops. Hence, while the chain apparently contains six distinct loops, only five of these are independent. Moreover, the degree of freedom of the manipulator is six. Indeed, the total number of links of the manipulator is $l = 6 \times 5 + 2 = 32$. Of these, one is fixed, and hence, we have 31 moving links, each with six degrees of freedom prior to coupling. Thus, we have a total of $31 \times 6 = 186$ degrees of freedom at our disposal. Upon coupling, each revolute removes five degrees of freedom, and hence, the 36 kinematic pairs remove 180 degrees of freedom, the manipulator thus being left with 6 degrees of freedom. We derive below the mathematical model governing the motion of the overall system in terms of the independent generalized coordinates associated with the actuated joints of the legs.

We assume, henceforth, that each leg is a six-axis open kinematic chain with either revolute or prismatic pairs, only one of which is actuated, and we thus have as many actuated joints as degrees of freedom. Furthermore, we label the legs with Roman numerals $I, II, \ldots, VI$ and denote the mass center of the mobile platform $\mathcal{M}$ by $C_{\mathcal{M}}$, with the twist of $\mathcal{M}$ denoted by $\mathbf{t}_{\mathcal{M}}$ and defined at the mass center. That is, if $\mathbf{c}_{\mathcal{M}}$ denotes the position vector of $C_{\mathcal{M}}$ in an inertial frame and $\dot{\mathbf{c}}_{\mathcal{M}}$ its velocity, while $\boldsymbol{\omega}_{\mathcal{M}}$ is the angular velocity of $\mathcal{M}$, then

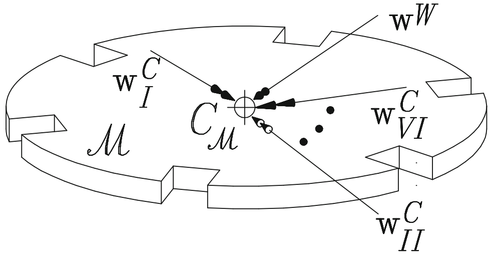

Fig. 12.4 The free-body diagram of $\mathcal{M}$

$$\mathbf{t}_{\mathcal{M}} \equiv \begin{bmatrix} \boldsymbol{\omega}_{\mathcal{M}} \\ \dot{\mathbf{c}}_{\mathcal{M}} \end{bmatrix} \tag{12.10}$$

Next, the Newton–Euler equations of $\mathcal{M}$ are derived from the free-body diagram shown in Fig. 12.4. In this figure, the legs have been replaced by the *constraint wrenches* $\{\mathbf{w}_J^C\}_I^{VI}$ acting at point $C_{\mathcal{M}}$, the governing equation thus taking the form of Eq. (7.5c), namely,

$$\mathbf{M}_{\mathcal{M}}\dot{\mathbf{t}}_{\mathcal{M}} = -\mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{t}_{\mathcal{M}} + \mathbf{w}^W + \sum_{J=I}^{VI} \mathbf{w}_J^C \tag{12.11}$$

with $\mathbf{w}^W$ denoting the external wrench acting on $\mathcal{M}$. Furthermore, let us denote by q_J the variable of the actuated joint of the Jth leg, all variables of the six actuated joints being grouped in the six-dimensional array $\mathbf{q}$, i.e.,

$$\mathbf{q} \equiv \begin{bmatrix} q_I \; q_{II} \; \cdots \; q_{VI} \end{bmatrix}^T \tag{12.12}$$

Now, we derive a relation between the twist $\mathbf{t}_{\mathcal{M}}$ and the active joint rates, $\dot{q}_J$, for $J = I, II, \ldots, VI$. To this end, we resort to Fig. 12.5, depicting the Jth leg as a serial-type, six-axis manipulator, whose twist–shape relations are readily expressed as in Eq. (5.9), namely,

$$\mathbf{J}_J\dot{\boldsymbol{\theta}}_J = \mathbf{t}_{\mathcal{M}}, \quad J = I, II, \ldots, VI \tag{12.13}$$

where $\mathbf{J}_J$ is the 6×6 Jacobian matrix of the Jth leg.

In Fig. 12.5, the moving platform $\mathcal{M}$ has been replaced by the constraint wrench transmitted by the moving platform onto the end link of the Jth leg, $-\mathbf{w}_J^C$, whose sign is the opposite of that transmitted by this leg onto $\mathcal{M}$ by virtue of Newton's third law. The dynamics model of the manipulator of Fig. 12.5 then takes the form

$$\mathbf{I}_J\ddot{\boldsymbol{\theta}}_J + \mathbf{C}_J(\boldsymbol{\theta}_J, \dot{\boldsymbol{\theta}}_J)\dot{\boldsymbol{\theta}}_J = \boldsymbol{\tau}_J - \mathbf{J}_J^T\mathbf{w}_J^C, \quad J = I, II, \ldots, VI \tag{12.14}$$

where $\mathbf{I}_J$ is the 6×6 inertia matrix of the manipulator, while $\mathbf{C}_J$ is the matrix coefficient of the inertia terms that are quadratic in the joint rates. Moreover, $\boldsymbol{\theta}_J$ and $\boldsymbol{\tau}_J$ denote the six-dimensional vectors of joint variables and joint torques, namely,

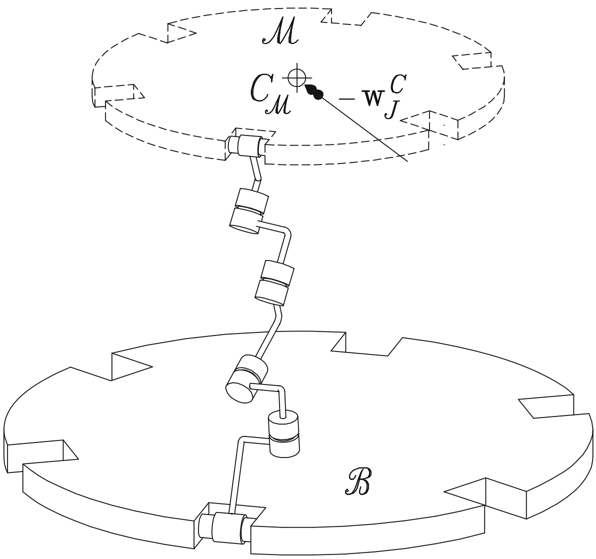

Fig. 12.5 The serial manipulator of the Jth leg

$$\boldsymbol{\theta}_J \equiv \begin{bmatrix} \theta_{J1} \\ \theta_{J2} \\ \vdots \\ \theta_{J6} \end{bmatrix}, \quad \boldsymbol{\tau}_J \equiv \begin{bmatrix} 0 \\ \vdots \\ \tau_{Jk} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{12.15}$$

with subscript Jk denoting in turn the only actuated joint of the Jth leg, namely, the kth joint of the leg. If we now introduce $\mathbf{e}_{Jk}$, defined as a unit vector all of whose entries are zero except for the kth entry, which is unity, then we can write

$$\boldsymbol{\tau}_J = f_J \mathbf{e}_{Jk} \tag{12.16}$$

If the actuated joint is prismatic, as is the case in flight simulators, then f_J is a force; if this joint is a revolute, then f_J is a torque.

Now, since the dimension of $\mathbf{q}$ coincides with the degree of freedom of the manipulator, it is possible to find, within the framework of the natural orthogonal complement, a 6×6 matrix $\mathbf{L}_J$ mapping the vector of actuated joint rates $\dot{\mathbf{q}}$ into the vector of Jth-leg joint-rates, namely,

$$\dot{\boldsymbol{\theta}}_J = \mathbf{L}_J \dot{\mathbf{q}}, \quad J = I, II, \ldots, VI \tag{12.17}$$

The calculation of $\mathbf{L}_J$ will be illustrated with an example.

Moreover, if the manipulator of Fig. 12.5 is not at a singular configuration, then we can solve for $\mathbf{w}_J^C$ from Eq. (12.14), i.e.,

$$\mathbf{w}_J^C = \mathbf{J}_J^{-T}(\boldsymbol{\tau}_J - \mathbf{I}_J\ddot{\boldsymbol{\theta}}_J - \mathbf{C}_J\dot{\boldsymbol{\theta}}_J) \tag{12.18}$$

in which the superscript $-T$ stands for the transpose of the inverse, or equivalently, the inverse of the transpose, while $\mathbf{I}_J = \mathbf{I}_J(\boldsymbol{\theta}_J)$ and $\mathbf{C}_J = \mathbf{C}_J(\boldsymbol{\theta}_J, \dot{\boldsymbol{\theta}}_J)$. Further, we substitute $\mathbf{w}_J^C$ as given by Eq. (12.18) into Eq. (12.11), thereby obtaining the Newton–Euler equations of the moving platform free of constraint wrenches. Additionally, the equations thus resulting now contain inertia terms and joint torques pertaining to the Jth leg, namely,

$$\mathbf{M}_{\mathcal{M}}\dot{\mathbf{t}}_{\mathcal{M}} = -\mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{t}_{\mathcal{M}} + \mathbf{w}^W + \sum_{J=I}^{VI}\mathbf{J}_J^{-T}(\boldsymbol{\tau}_J - \mathbf{I}_J\ddot{\boldsymbol{\theta}}_J - \mathbf{C}_J\dot{\boldsymbol{\theta}}_J) \tag{12.19}$$

Still within the framework of the natural orthogonal complement, we set up the relation between the twist $\mathbf{t}_{\mathcal{M}}$ and the vector of actuated joint rates $\dot{\mathbf{q}}$ as

$$\mathbf{t}_{\mathcal{M}} = \mathbf{T}\dot{\mathbf{q}} \tag{12.20}$$

which upon differentiation with respect to time, yields

$$\dot{\mathbf{t}}_{\mathcal{M}} = \mathbf{T}\ddot{\mathbf{q}} + \dot{\mathbf{T}}\dot{\mathbf{q}} \tag{12.21}$$

In the next step, we substitute $\mathbf{t}_{\mathcal{M}}$ and its time-derivative as given by Eqs. (12.20 and 12.21) into Eq. (12.19), thereby obtaining

$$\mathbf{M}_{\mathcal{M}}(\mathbf{T}\ddot{\mathbf{q}} + \dot{\mathbf{T}}\dot{\mathbf{q}}) + \mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{T}\dot{\mathbf{q}} + \sum_{J=I}^{VI}\mathbf{J}_J^{-T}(\mathbf{I}_J\ddot{\boldsymbol{\theta}}_J + \mathbf{C}_J\dot{\boldsymbol{\theta}}_J) = \mathbf{w}^W + \sum_{J=I}^{VI}\mathbf{J}_J^{-T}\boldsymbol{\tau}_J \tag{12.22}$$

Further, we recall relation (12.17), which upon differentiation with respect to time, yields

$$\ddot{\boldsymbol{\theta}}_J = \mathbf{L}_J\ddot{\mathbf{q}} + \dot{\mathbf{L}}_J\dot{\mathbf{q}} \tag{12.23}$$

Next, relations (12.17 and 12.23) are substituted into Eq. (12.22), thereby obtaining the model sought in terms only of actuated joint variables. After simplification, this model takes the form

$$\mathbf{M}_{\mathcal{M}}\mathbf{T}\ddot{\mathbf{q}} + \mathbf{M}_{\mathcal{M}}\dot{\mathbf{T}}\dot{\mathbf{q}} + \mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{T}\dot{\mathbf{q}} + \sum_{J=I}^{J=VI}\mathbf{J}_J^{-T}(\mathbf{I}_J\mathbf{L}_J\ddot{\mathbf{q}} + \mathbf{I}_J\dot{\mathbf{L}}_J\dot{\mathbf{q}} + \mathbf{C}_J\mathbf{L}_J\dot{\mathbf{q}}) = \mathbf{w}^W + \sum_{J=I}^{VI}\mathbf{J}_J^{-T}\boldsymbol{\tau}_J \tag{12.24}$$

where now $\mathbf{I}_J = \mathbf{I}_J(\mathbf{q})$ and $\mathbf{C}_J = \mathbf{C}_J(\mathbf{q}, \dot{\mathbf{q}})$.

Our final step in this formulation consists in deriving a reduced 6×6 model in terms only of actuated joint variables. Prior to this step, we note that from Eqs. (12.13), (12.17), and (12.20),

$$\mathbf{L}_J = \mathbf{J}_J^{-1}\mathbf{T} \tag{12.25}$$

Upon substitution of the above relation into Eq. (12.24) and multiplication of both sides of Eq. (12.24) by $\mathbf{T}^T$ from the left, we obtain the desired model in the form of Eqs. (12.8a), namely,

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \boldsymbol{\tau}^W + \sum_{J=I}^{VI} \mathbf{L}_J \boldsymbol{\tau}_J \tag{12.26}$$

with the 6×6 matrices $\mathbf{M}(\mathbf{q})$, $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$, and vector $\boldsymbol{\tau}^W$ defined as

$$\mathbf{M}(\mathbf{q}) \equiv \mathbf{T}^T \mathbf{M}_{\mathcal{M}} \mathbf{T} + \sum_{J=I}^{VI} \mathbf{L}_J^T \mathbf{I}_J \mathbf{L}_J \tag{12.27a}$$

$$\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \equiv \mathbf{T}^T(\mathbf{M}_{\mathcal{M}}\dot{\mathbf{T}} + \mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{T}) + \sum_{J=I}^{VI} \mathbf{L}_J^T(\mathbf{I}_J\dot{\mathbf{L}}_J + \mathbf{C}_J\mathbf{L}_J) \tag{12.27b}$$

$$\boldsymbol{\tau}^W \equiv \mathbf{T}^T \mathbf{w}^W \tag{12.27c}$$

Alternatively, the foregoing variables can be expressed in a more compact form that will shed more light on the above model. To do this, we define the 36×36 matrices $\mathbf{I}$ and $\mathbf{C}$ as well as the 6×36 matrix $\mathbf{L}$, the 6×6 matrix $\boldsymbol{\Lambda}$, and the six-dimensional vector $\boldsymbol{\phi}$ as

$$\mathbf{I} \equiv \mathrm{diag}(\mathbf{I}_I, \mathbf{I}_{II}, \ldots, \mathbf{I}_{VI}) \tag{12.28a}$$

$$\mathbf{C} \equiv \mathrm{diag}(\mathbf{C}_I, \mathbf{C}_{II}, \ldots, \mathbf{C}_{VI}) \tag{12.28b}$$

$$\mathbf{L} \equiv \begin{bmatrix}\mathbf{L}_I & \mathbf{L}_{II} & \ldots & \mathbf{L}_{VI}\end{bmatrix} \tag{12.28c}$$

$$\boldsymbol{\Lambda} \equiv \begin{bmatrix}\mathbf{L}_I\mathbf{e}_{Ik} & \mathbf{L}_{II}\mathbf{e}_{IIk} & \ldots & \mathbf{L}_{VI}\mathbf{e}_{VIk}\end{bmatrix} \tag{12.28d}$$

$$\boldsymbol{\phi} \equiv \begin{bmatrix}f_I & f_{II} & \ldots & f_{VI}\end{bmatrix}^T \tag{12.28e}$$

and hence,

$$\mathbf{M}(\mathbf{q}) \equiv \mathbf{T}^T \mathbf{M}_{\mathcal{M}} \mathbf{T} + \mathbf{L}^T \mathbf{I} \mathbf{L} \tag{12.29a}$$

$$\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \equiv \mathbf{T}^T(\mathbf{M}_{\mathcal{M}}\dot{\mathbf{T}} + \mathbf{W}_{\mathcal{M}}\mathbf{M}_{\mathcal{M}}\mathbf{T}) + \mathbf{L}^T\mathbf{I}\dot{\mathbf{L}} + \mathbf{L}^T\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{L} \tag{12.29b}$$

$$\sum_{J=I}^{VI} \mathbf{L}_J^T \boldsymbol{\tau}_J \equiv \boldsymbol{\Lambda}\boldsymbol{\phi} \tag{12.29c}$$

whence the mathematical model of Eq. (12.26) takes on a more familiar form, namely,

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \boldsymbol{\tau}^W + \boldsymbol{\Lambda}\boldsymbol{\phi} \tag{12.30}$$

Thus, for inverse dynamics, we want to determine $\boldsymbol{\phi}$ for a motion given by $\mathbf{q}$ and $\dot{\mathbf{q}}$, which can be done from the above equation, namely,

$$\boldsymbol{\phi} = \boldsymbol{\Lambda}^{-1}[\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} - \boldsymbol{\tau}^W] \tag{12.31}$$

Notice, however, that the foregoing solution is not recursive, and since it requires linear-equation solving, it is of order n^3, which thus yields a rather high numerical complexity. It should be possible to produce a recursive algorithm for the computation of $\boldsymbol{\phi}$, but this issue will not be pursued here. Moreover, given the parallel structure of the manipulator, the associated recursive algorithm should be parallelizable with multiple processors.

For purposes of direct dynamics, on the other hand, we want to solve for $\ddot{\mathbf{q}}$ from Eq. (12.30). Moreover, for simulation purposes, we need to derive the state-variable equations of the system at hand. This can be readily done if we define $\mathbf{r} \equiv \dot{\mathbf{q}}$, the state-variable model thus taking on the form

$$\dot{\mathbf{q}} = \mathbf{r} \tag{12.32a}$$

$$\dot{\mathbf{r}} = \mathbf{M}^{-1}[-\mathbf{N}(\mathbf{q}, \mathbf{r})\mathbf{r} + \boldsymbol{\tau}^W + \boldsymbol{\Lambda}\boldsymbol{\phi}] \tag{12.32b}$$

In light of the matrix inversion of the foregoing model, then, the complexity of the forward dynamics computations is also of order n^3.

Example 12.4.1. Derive matrix $\mathbf{L}_J$ of Eq. (12.17) for a manipulator having six identical legs like that of Fig. 12.2, the actuators being placed at the fourth joint.

Solution: We attach coordinate frames to the links of the serial chain of the Jth leg following the Denavit–Hartenberg notation, while noting that the first three joints intersect at a common point, and hence, $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}_3$. According to this notation, we recall, vector $\mathbf{r}_i$ is directed from the origin O_i of the ith frame to the operation point of the manipulator, which in this case, is $C_{\mathcal{M}}$. The Jacobian matrix of the Jth leg then takes the form

$$\mathbf{J}_J = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{e}_6 \\ \mathbf{e}_1 \times \mathbf{r}_1 & \mathbf{e}_2 \times \mathbf{r}_1 & \mathbf{e}_3 \times \mathbf{r}_1 & \mathbf{e}_4 \times \mathbf{r}_4 & \mathbf{e}_5 \times \mathbf{r}_5 & \mathbf{e}_6 \times \mathbf{r}_5 \end{bmatrix}_J$$

the subscript J of the array in the right-hand side reminding us that the vectors inside it pertain to the Jth leg. Thus, matrix $\mathbf{J}_J$ maps the joint-rate vector of the Jth leg, $\dot{\boldsymbol{\theta}}_J$, into the twist $\mathbf{t}_{\mathcal{M}}$ of the platform, i.e.,

$$\mathbf{J}_J\dot{\boldsymbol{\theta}}_J = \mathbf{t}_{\mathcal{M}}$$

Clearly, the joint-rate vector of the Jth leg is defined as

$$\dot{\boldsymbol{\theta}}_J \equiv \left[\dot{\theta}_{J1}\ \dot{\theta}_{J2}\ \dot{\theta}_{J3}\ \dot{\theta}_{J4}\ \dot{\theta}_{J5}\ \dot{\theta}_{J6}\right]^T$$

Now, note that except for $\dot{\theta}_{J4}$, all joint-rates of this leg are passive and thus need not appear in the mathematical model of the whole manipulator. Hence, we should aim at eliminating all joint-rates from the above twist-rate relation, except for the one associated with the active joint. We can achieve this if we realize that

$$\mathbf{r}_{J1} \times \mathbf{e}_{Ji} + \mathbf{e}_{Ji} \times \mathbf{r}_{J1} = \mathbf{0}, \quad i = 1, 2, 3$$

Further, we define a 3×6 matrix $\mathbf{A}_J$ as

$$\mathbf{A}_J \equiv \left[\mathbf{R}_{J1}\ \mathbf{1}\right]$$

with $\mathbf{R}_{J1}$ defined, in turn, as the cross-product matrix of $\mathbf{r}_{J1}$. Now, upon multiplication of $\mathbf{J}_J$ by $\mathbf{A}_J$ from the left, we obtain a 3×6 matrix whose first three columns vanish, namely,

$$\mathbf{A}_J\mathbf{J}_J = \left[\mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{e}_4 \times (\mathbf{r}_4 - \mathbf{r}_1)\ \mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)\ \mathbf{e}_6 \times (\mathbf{r}_5 - \mathbf{r}_1)\right]_J$$

and hence, if we multiply both sides of the above twist–shape equation by $\mathbf{A}_J$ from the left, we will obtain a new twist–shape equation that is free of the first three joint rates. Moreover, this equation is three-dimensional, i.e.,

$$[\mathbf{e}_4 \times (\mathbf{r}_4 - \mathbf{r}_1)\dot{\theta}_4 + \mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)\dot{\theta}_5 + \mathbf{e}_6 \times (\mathbf{r}_5 - \mathbf{r}_1)\dot{\theta}_6]_J = -\boldsymbol{\omega}_{\mathcal{M}} \times \mathbf{r}_{J1} + \dot{\mathbf{c}}_{\mathcal{M}}$$

where the subscript J attached to the brackets enclosing the whole left-hand side again reminds us that all quantities therein are to be understood as pertaining to the Jth leg. For example, $\mathbf{e}_4$ is to be read $\mathbf{e}_{J4}$. Furthermore, only $\dot{\theta}_{J4}$ is associated with an active joint and denoted, henceforth, by $\dot{q}_J$, i.e.,

$$q_J \equiv \theta_{J4} \tag{12.33}$$

It is noteworthy that the foregoing method of elimination of passive joint rates is not ad hoc at all. While we applied it here to the elimination of the three joint rates of a spherical joint, it has been formalized and generalized to all six lower kinematic pairs (Angeles 1994).

We have now to eliminate both $\dot{\theta}_{J5}$ and $\dot{\theta}_{J6}$ from the foregoing equation. This can be readily accomplished if we dot-multiply both sides of the same equation by vector $\mathbf{u}_J$ defined as the cross product of the vector coefficients of the two passive joint rates, i.e.,

$$\mathbf{u}_J \equiv [\mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1)]_J \times [\mathbf{e}_6 \times (\mathbf{r}_5 - \mathbf{r}_1)]_J$$

We thus obtain a third twist–shape relation that is scalar and free of passive joint rates, namely,

$$\mathbf{u}_J \cdot [\mathbf{e}_4 \times (\mathbf{r}_4 - \mathbf{r}_1)\dot{\theta}_4]_J = \mathbf{u}_J \cdot (-\boldsymbol{\omega}_{\mathcal{M}} \times \mathbf{r}_{J1} + \dot{\mathbf{c}}_{\mathcal{M}})$$

The above equation is clearly of the form

$$\zeta_J \dot{q}_J = \mathbf{y}_J^T \mathbf{t}_{\mathcal{M}}, \quad \dot{q}_J \equiv (\dot{\theta}_4)_J, \quad J = I, II, \ldots, VI$$

with ζ_J and $\mathbf{y}_J$ defined, in turn, as

$$\zeta_J \equiv \mathbf{u}_J \cdot \mathbf{e}_{J4} \times (\mathbf{r}_{J4} - \mathbf{r}_{J1}) \tag{12.34a}$$

$$\mathbf{y}_J \equiv \begin{bmatrix} -\mathbf{r}_{J1} \times \mathbf{u}_J \\ \mathbf{u}_J \end{bmatrix} \tag{12.34b}$$

Upon assembling the foregoing six scalar twist–shape relations, we obtain a six-dimensional twist–shape relation between the active joint rates of the manipulator and the twist of the moving platform, namely,

$$\mathbf{Z}\dot{\mathbf{q}} = \mathbf{Y}\mathbf{t}_{\mathcal{M}}$$

with the obvious definitions for the two 6×6 matrices $\mathbf{Y}$ and $\mathbf{Z}$ given below:

$$\mathbf{Y} \equiv \begin{bmatrix} \mathbf{y}_I^T \\ \mathbf{y}_{II}^T \\ \vdots \\ \mathbf{y}_{VI}^T \end{bmatrix}, \quad \mathbf{Z} \equiv \text{diag}(\zeta_I, \zeta_{II}, \ldots, \zeta_{VI})$$

We now can determine matrix $\mathbf{T}$ of the procedure described above, as long as $\mathbf{Y}$ is invertible, in the form

$$\mathbf{T} = \mathbf{Y}^{-1}\mathbf{Z}$$

whence the leg-matrix $\mathbf{L}_J$ of the same procedure is readily determined, namely,

$$\mathbf{L}_J = \mathbf{J}_J^{-1}\mathbf{T}$$

Therefore, all we need now is an expression for the inverse of the leg Jacobian $\mathbf{J}_J$. This Jacobian is clearly full, which might discourage the reader from attempting its closed-form inversion. However, a closer look reveals that this Jacobian is similar to that of decoupled manipulators, studied in Sect. 5.2, and hence, its closed-form inversion should be reducible to that of a 3×3 matrix. Indeed, if we recall the twist-transfer formula of Eqs. (5.12a and b), we can then write $\mathbf{J}_J$ as

$$\mathbf{J}_J \equiv \mathbf{U}_J \mathbf{K}_J$$

where $\mathbf{U}_J$ is a unimodular 6×6 matrix and $\mathbf{K}_J$ is the Jacobian of the same Jth leg, but now defined with its operation point located at the center of the spherical joint. Thus,

$$\mathbf{U}_J \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O}_{J1} - \mathbf{C}_{\mathcal{M}} & \mathbf{1} \end{bmatrix}, \quad \mathbf{K}_J \equiv \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{O} & \mathbf{K}_{22} \end{bmatrix}_J$$

the superscript J indicating the Jth leg and with the definitions below:

O: the 3×3 zero matrix;
1: the 3×3 identity matrix;
$\mathbf{O}_{J1}$: the cross-product matrix of $\mathbf{o}_{J1}$, the position vector of the center of the spherical joint;
$\mathbf{C}_{\mathcal{M}}$: the cross product matrix of $\mathbf{c}_{\mathcal{M}}$, the position vector of $C_{\mathcal{M}}$.

Furthermore, the 3×3 blocks of $\mathbf{K}_J$ are defined, in turn, as

$$(\mathbf{K}_{11})_J \equiv \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix}_J$$

$$(\mathbf{K}_{12})_J \equiv \begin{bmatrix} \mathbf{e}_4 & \mathbf{e}_5 & \mathbf{e}_6 \end{bmatrix}_J$$

$$(\mathbf{K}_{22})_J \equiv \begin{bmatrix} \mathbf{e}_4 \times (\mathbf{r}_4 - \mathbf{r}_1) & \mathbf{e}_5 \times (\mathbf{r}_5 - \mathbf{r}_1) & \mathbf{e}_6 \times (\mathbf{r}_5 - \mathbf{r}_1) \end{bmatrix}_J$$

Now, if the inverse of a block matrix is recalled, we have

$$\mathbf{K}_J^{-1} = \begin{bmatrix} \mathbf{K}_{11}^{-1} & -\mathbf{K}_{11}^{-1}\mathbf{K}_{12}\mathbf{K}_{22}^{-1} \\ \mathbf{O} & \mathbf{K}_{22}^{-1} \end{bmatrix}_J$$

where the superscript of the blocks has been transferred to the whole matrix, in order to ease the notation. The problem of inverting $\mathbf{K}_J$ has now been reduced to that of inverting two of its 3×3 blocks. These can be inverted explicitly if we recall the concept of *reciprocal bases* (Brand 1965). Thus,

$$(\mathbf{K}_{11}^{-1})_J = \frac{1}{\Delta_{11}^J} \begin{bmatrix} (\mathbf{e}_2 \times \mathbf{e}_3)^T \\ (\mathbf{e}_3 \times \mathbf{e}_1)^T \\ (\mathbf{e}_1 \times \mathbf{e}_2)^T \end{bmatrix}_J$$

$$(\mathbf{K}_{22}^{-1})_J = \frac{1}{\Delta_{22}^J} \begin{bmatrix} [(\mathbf{e}_5 \times \mathbf{s}_5) \times (\mathbf{e}_6 \times \mathbf{s}_5)]^T \\ [(\mathbf{e}_6 \times \mathbf{s}_5) \times (\mathbf{e}_4 \times \mathbf{s}_4)]^T \\ [(\mathbf{e}_4 \times \mathbf{s}_4) \times (\mathbf{e}_5 \times \mathbf{s}_5)]^T \end{bmatrix}_J$$

with $\mathbf{s}_{J4}$, $\mathbf{s}_{J5}$, Δ_{11}^J, and Δ_{22}^J defined as

$$\mathbf{s}_{J4} \equiv \mathbf{r}_{J4} - \mathbf{r}_{J1}$$

$$\mathbf{s}_{J5} \equiv \mathbf{r}_{J5} - \mathbf{r}_{J1}$$

$$\Delta_{11}^J \equiv \det(\mathbf{K}_{11}^J) = (\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3)_J$$

$$\Delta_{22}^J \equiv \det(\mathbf{K}_{22}^J) = [(\mathbf{e}_4 \times \mathbf{s}_4) \times (\mathbf{e}_{\times}\mathbf{s}_5) \cdot (\mathbf{e}_6 \times \mathbf{s}_5)]_J$$

the subscripted brackets and parentheses still reminding us that all vectors involved pertain to the Jth leg. Moreover, since $\mathbf{U}_J$ is unimodular, its inverse is simply

$$\mathbf{U}_J^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{C}_{\mathcal{M}} - \mathbf{O}_{J1} & \mathbf{1} \end{bmatrix}$$

and hence,

$$\mathbf{J}_J^{-1} = \begin{bmatrix} \mathbf{K}_{11}^{-1} - \mathbf{K}_{11}^{-1}\mathbf{K}_{12}\mathbf{K}_{22}^{-1}(\mathbf{C}_{\mathcal{M}} - \mathbf{O}_{J1}) & -\mathbf{K}_{11}^{-1}\mathbf{K}_{12}\mathbf{K}_{22}^{-1} \\ \mathbf{K}_{22}^{-1}(\mathbf{C}_{\mathcal{M}} - \mathbf{O}_{J1}) & \mathbf{K}_{22}^{-1} \end{bmatrix}_J$$

the matrix sought, $\mathbf{L}_J$, then being calculated as

$$\mathbf{L}_J = \mathbf{J}_J^{-1}\mathbf{Y}^{-1}\mathbf{Z}$$

While we have a closed-form inverse of $\mathbf{J}_J$, we do not have one for $\mathbf{Y}$, which is full and does not bear any particular structure that would allow us its inversion explicitly. Therefore, matrix $\mathbf{L}_J$ should be calculated numerically.

12.5 Dynamics of Rolling Robots

The dynamics of rolling robots, similar to that of other robotic mechanical systems, comprises two main problems, inverse and direct dynamics. We will study both using the same mathematical model. Hence, the main task here is to derive this model. It turns out that while rolling robots usually are nonholonomic mechanical systems, their mathematical models are formally identical to those of holonomic systems. The difference between holonomic and nonholonomic systems lies in that, in the former, the number of independent actuators equals the necessary and sufficient number of variables—independent generalized coordinates in Lagrangian mechanics—defining a posture (configuration) of the system. In nonholonomic systems, however, the necessary and sufficient number of variables defining a posture of the system exceeds the number of independent actuators. As a consequence, in holonomic systems the dof equals the number of independent actuators. In nonholonomic systems, the dof is usually defined as the necessary and sufficient number of variables defining the system posture, while the number of independent actuators is termed the system *mobility*, which thus turns out to be smaller than the system dof. Therefore, relations between these dependent and independent variables will be needed and will be derived in the course of our discussion. Moreover, we will study robots with both conventional and omnidirectional wheels. Of the latter, we will focus on robots with Mekanum wheels.

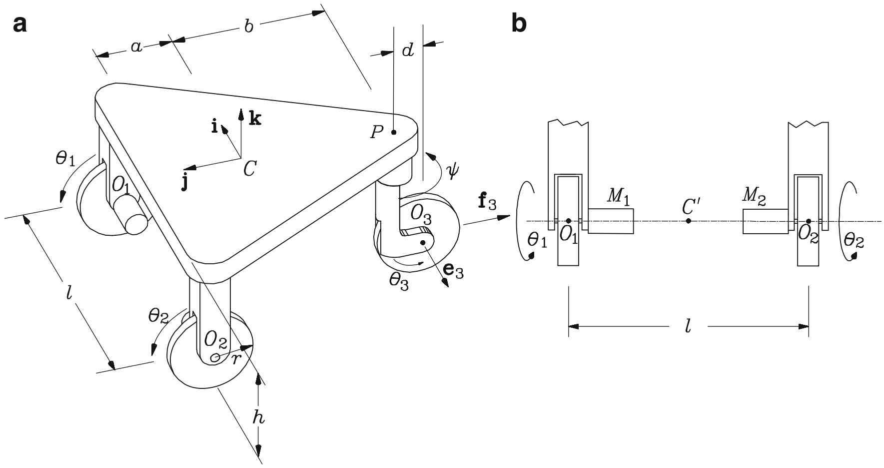

Fig. 12.6 A two-dof rolling robot: (**a**) its general layout; and (**b**) a detail of its actuated wheels

12.5.1 Robots with Conventional Wheels

We study here the robot of Fig. 10.17, under the assumption that it is driven by motors collocated at the axes of its two coaxial wheels, indicated as M_1 and M_2 in Fig. 10.17b. For quick reference, we repeat this figure here as Fig. 12.6.

Our approach will be one of multibody dynamics; for this reason, we distinguish five rigid bodies composing the robotic mechanical system at hand. These are the three wheels (two actuated and one caster wheels), the bracket carrying the caster wheel, and the platform. We label these bodies with numbers from 1 to 5, in the foregoing order, while noticing that bodies 4 and 5, the bracket and the platform, undergo planar motion, and hence, deserve special treatment. The 6×6 mass matrices of the first three bodies are labeled $\mathbf{M}_1$ to $\mathbf{M}_3$, with a similar labeling for their corresponding six-dimensional twists, the counterpart items for bodies 4 and 5 being denoted by $\mathbf{M}'_4$, $\mathbf{M}'_5$, $\mathbf{t}'_4$, and $\mathbf{t}'_5$, the primes indicating 3×3—as opposed to 6×6 in the general case—mass matrices and three-dimensional—as opposed to six-dimensional in the general case—twist arrays.

We undertake the formulation of the mathematical model of the mechanical system under study, which is of the general form of Eq. (12.8a) derived for holonomic systems. The nonholonomy of the system brings about special features that will be highlighted in the derivations below.

As a first step in our formulation, we distinguish between *actuated* and *unactuated joint variables*, grouped into vectors $\boldsymbol{\theta}_a$ and $\boldsymbol{\theta}_u$, respectively, their time-derivatives being the *actuated* and *unactuated joint rates*, $\dot{\boldsymbol{\theta}}_a$ and $\dot{\boldsymbol{\theta}}_u$, respectively. From the kinematic analysis of this system in Sect. 10.5.1, it is apparent that the foregoing vectors are all two-dimensional, namely,

$$\boldsymbol{\theta}_a \equiv \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad \boldsymbol{\theta}_u \equiv \begin{bmatrix} \theta_3 \\ \psi \end{bmatrix} \tag{12.35}$$

Further, we set to deriving expressions for the twists of the five moving bodies in terms of the actuated joint rates, i.e., we write those twists as linear transformations of $\dot{\boldsymbol{\theta}}_a$, i.e.,

$$\mathbf{t}_i = \mathbf{T}_i \dot{\boldsymbol{\theta}}_a, \quad i = 1, 2, 3 \tag{12.36a}$$

and

$$\mathbf{t}'_i = \mathbf{T}'_i \dot{\boldsymbol{\theta}}_a, \quad i = 4, 5 \tag{12.36b}$$

where, from Eqs. (10.41a and b), (10.44a and b), and (10.48),

$$\mathbf{T}_1 = \begin{bmatrix} -\mathbf{i} + \rho\delta\mathbf{k} & -\rho\delta\mathbf{k} \\ r\mathbf{j} & \mathbf{0} \end{bmatrix} \tag{12.37}$$

$$\mathbf{T}_2 = \begin{bmatrix} \rho\delta\mathbf{k} & -(\mathbf{i} + \rho\delta\mathbf{k}) \\ \mathbf{0} & r\mathbf{j} \end{bmatrix} \tag{12.38}$$

$$\mathbf{T}_3 = \begin{bmatrix} \boldsymbol{\Theta}_3 \\ \mathbf{G}_3 \end{bmatrix} \tag{12.39}$$

$$\mathbf{T}'_4 = \begin{bmatrix} \boldsymbol{\theta}_4^T \\ \mathbf{G}_4 \end{bmatrix} \tag{12.40}$$

$$\mathbf{T}'_5 = \begin{bmatrix} \rho\delta & -\rho\delta \\ r(\lambda\mathbf{i} + (1/2)\mathbf{j}) & r(-\lambda\mathbf{i} + (1/2)\mathbf{j}) \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\theta}_5^T \\ \mathbf{C}_5 \end{bmatrix} \tag{12.41}$$

with $\boldsymbol{\Theta}_3$, $\mathbf{G}_3$, $\boldsymbol{\theta}_4$ and $\mathbf{G}_4$ yet to be derived. In the sequel, we will find convenient to work with a few nondimensional parameters, α, δ, ρ—already defined in Eq. (10.53)—and λ, which is introduced now, and displayed below with the first three parameters for quick reference:

$$\alpha \equiv \frac{a+b}{l}, \quad \delta \equiv \frac{d}{l}, \quad \rho \equiv \frac{r}{d}, \quad \lambda \equiv \frac{a}{l} \tag{12.42}$$

In the derivations below, we resort to the notation introduced in Sect. 10.5.1. First, we note that, from Eqs. (10.45), (10.52a and b), we can write, with θ_{ij} denoting the (i, j) entry of $\boldsymbol{\Theta}$, as derived in Sect. 10.5.1,

$$\boldsymbol{\omega}_3 = (\theta_{11}\dot{\theta}_1 + \theta_{12}\dot{\theta}_2)\mathbf{e}_3 + [\rho\delta(\dot{\theta}_1 - \dot{\theta}_2) + \theta_{21}\dot{\theta}_1 + \theta_{22}\dot{\theta}_2]\mathbf{k} \tag{12.43}$$

or

$$\boldsymbol{\omega}_3 = \boldsymbol{\Theta}_3 \dot{\boldsymbol{\theta}}_a \tag{12.44}$$

with $\boldsymbol{\Theta}_3$ defined as

$$\boldsymbol{\Theta}_3 = \left[\theta_{11}\mathbf{e}_3 + (\theta_{21} + \rho\delta)\mathbf{k} \;\; \theta_{12}\mathbf{e}_3 + (\theta_{22} - \rho\delta)\mathbf{k}\right]$$

In more compact form,

$$\boldsymbol{\Theta}_3 = \left[\theta_{11}\mathbf{e}_3 + \overline{\theta}_{21}\mathbf{k} \;\; \theta_{12}\mathbf{e}_3 + \overline{\theta}_{22}\mathbf{k}\right] \tag{12.45a}$$

with $\overline{\theta}_{21}$ and $\overline{\theta}_{22}$ defined, in turn, as

$$\overline{\theta}_{21} \equiv \theta_{21} + \rho\delta, \quad \overline{\theta}_{22} \equiv \theta_{22} - \rho\delta \tag{12.45b}$$

Moreover,

$$\dot{\mathbf{c}}_3 = -r\dot{\theta}_3\mathbf{f}_3 = -r(\theta_{11}\dot{\theta}_1 + \theta_{12}\dot{\theta}_2)\mathbf{f}_3$$

and hence,

$$\mathbf{G}_3 = r\left[-\theta_{11}\mathbf{f}_3 \;\; -\theta_{12}\mathbf{f}_3\right] \tag{12.46}$$

Further, it is apparent from Fig. 12.6 that the scalar angular velocity of the bracket, ω_4, is given by

$$\omega_4 = \omega + \dot{\psi}$$

and hence,

$$\omega_4 = \rho\delta(\dot{\theta}_1 - \dot{\theta}_2) + \theta_{21}\dot{\theta}_1 + \theta_{22}\dot{\theta}_2 = \overline{\theta}_{21}\dot{\theta}_1 + \overline{\theta}_{22}\dot{\theta}_2$$

Therefore, we can write

$$\omega_4 = \boldsymbol{\theta}_4^T\dot{\boldsymbol{\theta}}_a \tag{12.47a}$$

where $\boldsymbol{\theta}_4$ is defined as

$$\boldsymbol{\theta}_4 \equiv \left[\overline{\theta}_{21} \;\; \overline{\theta}_{22}\right]^T \tag{12.47b}$$

Now, since we are given the inertial properties of the bracket in bracket coordinates, it makes sense to express $\dot{\mathbf{c}}_4$ in those coordinates, taking into account that point C_4 lies in the middle of the line $\overline{PO_3}$. Such an expression is obtained below:

$$\dot{\mathbf{c}}_4 = \dot{\mathbf{o}}_3 + \boldsymbol{\omega}_4 \times \frac{1}{2}[-d\mathbf{f}_3 + (h - r)\mathbf{k}] = -r\dot{\theta}_3\mathbf{f}_3 + \frac{d}{2}(\omega + \dot{\psi})\mathbf{e}_3$$

Upon expressing $\dot{\theta}_3$ and $\dot{\psi}$ in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$, we obtain

$$\dot{\mathbf{c}}_4 = d\left(\frac{1}{2}\overline{\theta}_{21}\mathbf{e}_3 - \rho\theta_{11}\mathbf{f}_3\right)\dot{\theta}_1 + d\left(\frac{1}{2}\overline{\theta}_{22}\mathbf{e}_3 - \rho\theta_{12}\mathbf{f}_3\right)\dot{\theta}_2 \tag{12.48}$$

whence it is apparent that

$$\mathbf{G}_4 = d\left[(1/2)\overline{\theta}_{21}\mathbf{e}_3 - \rho\theta_{11}\mathbf{f}_3 \;\; (1/2)\overline{\theta}_{22}\mathbf{e}_3 - \rho\theta_{12}\mathbf{f}_3\right] \tag{12.49}$$

Therefore,

$$\mathbf{T}'_4 = \begin{bmatrix} \overline{\theta}_{21} & \overline{\theta}_{22} \\ d[(1/2)\overline{\theta}_{21}\mathbf{e}_3 - \rho\theta_{11}\mathbf{f}_3] & d[(1/2)\overline{\theta}_{22}\mathbf{e}_3 - \rho\theta_{12}\mathbf{f}_3] \end{bmatrix} \tag{12.50}$$

thereby completing all needed twist-shaping matrices.

The 2×2 matrix of generalized inertia, $\mathbf{I}(\boldsymbol{\theta})$, is now obtained. Here we have written this matrix as a function of all variables, independent and dependent, arrayed in the four-dimensional vector $\boldsymbol{\theta}$, because we cannot obtain an expression for $\boldsymbol{\theta}_u$ in terms of $\boldsymbol{\theta}_a$, given the nonholonomy of the system at hand. Therefore, $\mathbf{I}$ is, in general, a function of θ_1, θ_2, θ_3, and ψ. To be sure, from the above expressions for the twist-shaping matrices $\mathbf{T}_i$ and $\mathbf{T}'_i$, it is apparent that the said inertia matrix is an explicit function of ψ only, its dependence on θ_1 and θ_2 being implicitly given via vectors $\mathbf{e}_3$ and $\mathbf{f}_3$. We derive the expression sought for $\mathbf{I}$ starting from the kinetic energy, namely,

$$T = \sum_1^3 \frac{1}{2}\mathbf{t}_i^T\mathbf{M}_i\mathbf{t}_i + \frac{1}{2}\sum_4^5 (\mathbf{t}'_i)^T\mathbf{M}'_i\mathbf{t}'_i$$

or

$$T = \frac{1}{2}\dot{\boldsymbol{\theta}}_a^T\left(\sum_1^3 \mathbf{T}_i^T\mathbf{M}_i\mathbf{T}_i\right)\dot{\boldsymbol{\theta}}_a + \frac{1}{2}\dot{\boldsymbol{\theta}}_a^T\left(\sum_4^5 (\mathbf{T}'_i)^T\mathbf{M}'_i\mathbf{T}'_i\right)\dot{\boldsymbol{\theta}}_a \tag{12.51}$$

and hence,

$$\mathbf{I} = \sum_1^3 \mathbf{T}_i^T\mathbf{M}_i\mathbf{T}_i + \sum_4^5 (\mathbf{T}'_i)^T\mathbf{M}'_i\mathbf{T}'_i \tag{12.52}$$

In order to expand the foregoing expression, we let $\mathbf{J}_w$ and $\mathbf{J}_c$ be the 3×3 inertia matrices of the two actuated wheels and the caster wheel, respectively, the scalar moments of inertia of the bracket and the platform, which undergo planar motion, being denoted by I_b and I_p. Likewise, we let m_w, m_b, m_c, and m_p denote the masses of the corresponding bodies. Therefore,

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{J}_w & \mathbf{O} \\ \mathbf{O} & m_w \mathbf{1}_3 \end{bmatrix} = \mathbf{M}_2$$

$$\mathbf{M}_3 = \begin{bmatrix} \mathbf{J}_c & \mathbf{O} \\ \mathbf{O} & m_c \mathbf{1}_3 \end{bmatrix}$$

$$\mathbf{M}_4' = \begin{bmatrix} I_b & \mathbf{0}^T \\ \mathbf{0} & m_b \mathbf{1}_2 \end{bmatrix}$$

$$\mathbf{M}_5' = \begin{bmatrix} I_p & \mathbf{0}^T \\ \mathbf{0} & m_p \mathbf{1}_2 \end{bmatrix}$$

with $\mathbf{O}$ and $\mathbf{1}_3$ denoting the 3×3 zero and identity matrices, while $\mathbf{0}$ and $\mathbf{1}_2$ the two-dimensional zero vector and the 2×2 identity matrix. Furthermore, under the assumption that the actuated wheels are dynamically balanced, we have

$$\mathbf{J}_w = \begin{bmatrix} I & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{bmatrix}$$

Moreover, we assume that the caster wheel can be modeled as a rigid disk of uniform material of the given mass m_c and radius r, and hence, in bracket-fixed coordinates $\{\mathbf{e}_3, \mathbf{f}_3, \mathbf{k}\}$,

$$\mathbf{J}_c = \frac{1}{4} m_c r^2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is now a simple matter to calculate

$$\mathbf{T}_1^T \mathbf{M}_1 \mathbf{T}_1 = \begin{bmatrix} I + (\rho\delta)^2 H + m_w r^2 & -(\rho\delta)^2 H \\ -(\rho\delta)^2 H & (\rho\delta)^2 H \end{bmatrix}$$

$$\mathbf{T}_2^T \mathbf{M}_2 \mathbf{T}_2 = \begin{bmatrix} (\rho\delta)^2 H & -(\rho\delta)^2 H \\ -(\rho\delta)^2 H & I + (\rho\delta)^2 H + m_w r^2 \end{bmatrix}$$

where the *symmetry* between the two foregoing expressions is to be highlighted: that is, the second expression is derived if the diagonal entries of the first expression are exchanged, which is physically plausible, because such an exchange is equivalent to a relabeling of the two wheels. The calculation of the remaining products is less straightforward but can be readily obtained. From the expressions for $\mathbf{T}_3$ and $\mathbf{M}_3$, we have

$$\mathbf{T}_3^T \mathbf{M}_3 \mathbf{T}_3 = \begin{bmatrix} \boldsymbol{\Theta}_3^T & \mathbf{G}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{J}_c & \mathbf{O} \\ \mathbf{O} & m_c \mathbf{1}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Theta}_3 \\ \mathbf{G}_3 \end{bmatrix} = \boldsymbol{\Theta}_3^T \mathbf{J}_c \boldsymbol{\Theta}_3 + m_c \mathbf{G}_3^T \mathbf{G}_3$$

In order to calculate the foregoing products, we write $\mathbf{J}_c$ and $\boldsymbol{\Theta}_3$ in component form, i.e.,

$$\mathbf{J}_c\boldsymbol{\Theta}_3 = \frac{1}{4}m_c r^2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_{11} & \theta_{12} \\ 0 & 0 \\ \overline{\theta}_{21} & \overline{\theta}_{22} \end{bmatrix} = \frac{1}{4}m_c r^2 \begin{bmatrix} 2\theta_{11} & 2\theta_{12} \\ 0 & 0 \\ \overline{\theta}_{21} & \overline{\theta}_{22} \end{bmatrix}$$

and hence,

$$\boldsymbol{\Theta}_3^T\mathbf{J}_c\boldsymbol{\Theta}_3 = \frac{1}{4}m_c r^2 \begin{bmatrix} 2\theta_{11}^2 + \overline{\theta}_{21}^2 & 2\theta_{11}\theta_{12} + \overline{\theta}_{21}\overline{\theta}_{22} \\ 2\theta_{11}\theta_{12} + \overline{\theta}_{21}\overline{\theta}_{22} & 2\theta_{12}^2 + \overline{\theta}_{22}^2 \end{bmatrix}$$

Likewise,

$$m_3\mathbf{G}_3^T\mathbf{G}_3 = m_c r^2 \begin{bmatrix} \theta_{11}^2 & \theta_{11}\theta_{12} \\ \theta_{11}\theta_{12} & \theta_{12}^2 \end{bmatrix}$$

Further,

$$(\mathbf{T}_4')^T\mathbf{M}_4'\mathbf{T}_4' = \begin{bmatrix} \boldsymbol{\theta}_4 & \mathbf{G}_4^T \end{bmatrix} \begin{bmatrix} I_b & \mathbf{0}^T \\ \mathbf{0} & m_b\mathbf{1}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_4^T \\ \mathbf{G}_4 \end{bmatrix} = I_b\boldsymbol{\theta}_4\boldsymbol{\theta}_4^T + m_b\mathbf{G}_4^T\mathbf{G}_4$$

Upon expansion, we have

$$(\mathbf{T}_4')^T\mathbf{M}_4'\mathbf{T}_4' = I_b \begin{bmatrix} \overline{\theta}_{21}^2 & \overline{\theta}_{21}\overline{\theta}_{22} \\ \overline{\theta}_{21}\overline{\theta}_{22} & \overline{\theta}_{22}^2 \end{bmatrix} + \frac{1}{4}m_b d^2 \begin{bmatrix} \overline{\theta}_{21}^2 + 4\rho^2\theta_{11}^2 & \overline{\theta}_{21}\overline{\theta}_{22} + 4\rho^2\theta_{11}\theta_{12} \\ \overline{\theta}_{21}\overline{\theta}_{22} + 4\rho^2\theta_{11}\theta_{12} & \overline{\theta}_{22}^2 + 4\rho^2\theta_{12}^2 \end{bmatrix}$$

Finally,

$$(\mathbf{T}_5')^T\mathbf{M}_5'\mathbf{T}_5' = \begin{bmatrix} \boldsymbol{\theta}_5 & \mathbf{G}_5^T \end{bmatrix} \begin{bmatrix} I_p & \mathbf{0}^T \\ \mathbf{0} & m_p\mathbf{1}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_5^T \\ \mathbf{G}_5 \end{bmatrix} = I_p\boldsymbol{\theta}_5\boldsymbol{\theta}_5^T + m_p\mathbf{G}_5^T\mathbf{G}_5$$

which can be readily expanded as

$$(\mathbf{T}_5')^T\mathbf{M}_5'\mathbf{T}_5' = I_p(\rho\delta)^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + m_p r^2 \begin{bmatrix} (1/4) + \lambda^2 & (1/4) - \lambda^2 \\ (1/4) - \lambda^2 & (1/4) + \lambda^2 \end{bmatrix}$$

We can thus express the generalized inertia matrix as

$$\mathbf{I} = \mathbf{I}_w + \mathbf{I}_c + \mathbf{I}_b + \mathbf{I}_p$$

where $\mathbf{I}_w$, $\mathbf{I}_c$, $\mathbf{I}_b$, and $\mathbf{I}_p$ denote the contributions of the actuated wheels, the caster wheel, the bracket, and the platform, respectively, i.e.,

$$\mathbf{I}_w = \sum_1^2 \mathbf{T}_i^T \mathbf{M}_i \mathbf{T}_i = \begin{bmatrix} I + 2(\rho\delta)^2 H + m_w r^2 & -2(\rho\delta)^2 H \\ -2(\rho\delta)^2 H & I + 2(\rho\delta)^2 H + m_w r^2 \end{bmatrix}$$

$$\mathbf{I}_c = \frac{m_c r^2}{4} \begin{bmatrix} 6\theta_{11}^2 + \overline{\theta}_{21}^2 & 6\theta_{11}\theta_{12} + \overline{\theta}_{21}\overline{\theta}_{22} \\ 6\theta_{11}\theta_{12} + \overline{\theta}_{21}\overline{\theta}_{22} & 6\theta_{12}^2 + \overline{\theta}_{22}^2 \end{bmatrix}$$

$$\mathbf{I}_b = I_b \begin{bmatrix} \overline{\theta}_{21}^2 & \overline{\theta}_{21}\overline{\theta}_{22} \\ \overline{\theta}_{21}\overline{\theta}_{22} & \overline{\theta}_{22}^2 \end{bmatrix} + \frac{1}{4} m_b d^2 \begin{bmatrix} \overline{\theta}_{21}^2 + 4\rho^2\theta_{11}^2 & \overline{\theta}_{21}\overline{\theta}_{22} + 4\rho^2\theta_{11}\theta_{12} \\ \overline{\theta}_{21}\overline{\theta}_{22} + 4\rho^2\theta_{11}\theta_{12} & \overline{\theta}_{22}^2 + 4\rho^2\theta_{12}^2 \end{bmatrix}$$

$$\mathbf{I}_p = I_p(\rho\delta)^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + m_p r^2 \begin{bmatrix} (1/4) + \lambda^2 & (1/4) - \lambda^2 \\ (1/4) - \lambda^2 & (1/4) + \lambda^2 \end{bmatrix}$$

It is now apparent that the contributions of the actuated wheels and the platform are constant, while those of the caster wheel and the bracket are configuration-dependent. Therefore, only the latter contribute to the Coriolis and centrifugal generalized forces. We thus have

$$\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}} = \mathbf{T}_3^T \mathbf{M}_3 \dot{\mathbf{T}}_3 + (\mathbf{T}_4')^T \mathbf{M}_4' \dot{\mathbf{T}}_4'$$

From the expression for $\mathbf{T}_3^T \mathbf{M}_3 \mathbf{T}_3$, we obtain

$$\mathbf{T}_3^T \mathbf{M}_3 \dot{\mathbf{T}}_3 = \mathbf{\Theta}_3^T \mathbf{J}_c \dot{\mathbf{\Theta}}_3 + m_3 \mathbf{C}_3^T \dot{\mathbf{C}}_3$$

the time-derivatives being displayed below:

$$\dot{\mathbf{\Theta}}_3 = \left[\dot{\theta}_{11}\mathbf{e}_3 + \theta_{11}\omega_4\mathbf{f}_3 + \dot{\theta}_{21}\mathbf{k} \;\; \dot{\theta}_{12}\mathbf{e}_3 + \theta_{12}\omega_4\mathbf{f}_3 + \dot{\theta}_{22}\mathbf{k}\right]$$

$$\dot{\mathbf{C}}_3 = r\left[-\dot{\theta}_{11}\mathbf{f}_3 + \theta_{11}\omega_4\mathbf{e}_3 \;\; -\dot{\theta}_{12}\mathbf{f}_3 + \theta_{12}\omega_4\mathbf{e}_3\right]$$

with the time-derivatives of the entries of $\mathbf{\Theta}$ given as

$$\dot{\mathbf{\Theta}} = \dot{\psi} \begin{bmatrix} -\alpha \sin\psi + (\cos\psi)/2 & \alpha \sin\psi + (\cos\psi)/2 \\ \rho[-\alpha\cos\psi - (\sin\psi)/2] & \rho[\alpha\cos\psi - (\sin\psi)/2] \end{bmatrix} \tag{12.53}$$

its parameters being defined in Eq. (12.42). Upon expansion, the products appearing in the expression for $\mathbf{T}_3^T \mathbf{M}_3 \dot{\boldsymbol{T}}_3$ become

$$\boldsymbol{\Theta}_3^T \mathbf{J}_c \dot{\boldsymbol{\Theta}}_3 = \frac{m_c r^2}{4} \begin{bmatrix} 2\theta_{11}\dot{\theta}_{11} + \overline{\theta}_{21}\dot{\theta}_{21} & 2\theta_{11}\dot{\theta}_{12} + \overline{\theta}_{21}\dot{\theta}_{22} \\ 2\theta_{12}\dot{\theta}_{11} + \overline{\theta}_{22}\dot{\theta}_{21} & 2\theta_{12}\dot{\theta}_{12} + \overline{\theta}_{22}\dot{\theta}_{22} \end{bmatrix}$$

$$m_3 \mathbf{C}_3^T \dot{\mathbf{C}}_3 = m_c r^2 \begin{bmatrix} \theta_{11}\dot{\theta}_{11} & \theta_{11}\dot{\theta}_{12} \\ \theta_{12}\dot{\theta}_{11} & \theta_{12}\dot{\theta}_{12} \end{bmatrix}$$

Therefore,

$$\mathbf{T}_3^T \mathbf{M}_3 \dot{\mathbf{T}}_3 = \frac{m_c r^2}{4} \begin{bmatrix} 6\theta_{11}\dot{\theta}_{11} + \overline{\theta}_{21}\dot{\theta}_{21} & 6\theta_{11}\dot{\theta}_{12} + \overline{\theta}_{21}\dot{\theta}_{22} \\ 6\theta_{12}\dot{\theta}_{11} + \overline{\theta}_{22}\dot{\theta}_{21} & 6\theta_{12}\dot{\theta}_{12} + \overline{\theta}_{22}\dot{\theta}_{22} \end{bmatrix}$$

Likewise,

$$(\mathbf{T}'_4)^T \mathbf{M}'_4 \dot{\mathbf{T}}'_4 = I_b \boldsymbol{\theta}_4 \dot{\boldsymbol{\theta}}_4^T + m_b \mathbf{C}_4^T \dot{\mathbf{C}}_4$$

the above time-derivatives being

$$\dot{\boldsymbol{\theta}}_4^T = \begin{bmatrix} \dot{\theta}_{21} & \dot{\theta}_{22} \end{bmatrix}$$

$$\dot{\mathbf{C}}_4 = d \begin{bmatrix} c_{11}\mathbf{e}_3 + c_{12}\mathbf{f}_3 & c_{21}\mathbf{e}_3 + c_{22}\mathbf{f}_3 \end{bmatrix}$$

with coefficients $c_{i,j}$ given below:

$$c_{11} = \frac{1}{2}\dot{\theta}_{21} + \rho\theta_{11}\omega_4 , \quad c_{12} = \frac{1}{2}\overline{\theta}_{21}\omega_4 - \rho\dot{\theta}_{11}$$

$$c_{21} = \frac{1}{2}\dot{\theta}_{22} + \rho\theta_{12}\omega_4 , \quad c_{22} = \frac{1}{2}\overline{\theta}_{22}\omega_4 - \rho\dot{\theta}_{12}$$

Hence,

$$I_b \boldsymbol{\theta}_4 \dot{\boldsymbol{\theta}}_4^T = I_b \begin{bmatrix} \theta_{21}\dot{\theta}_{21} & \theta_{21}\dot{\theta}_{22} \\ \theta_{22}\dot{\theta}_{21} & \theta_{22}\dot{\theta}_{22} \end{bmatrix}$$

$$m_b \mathbf{C}_4^T \dot{\mathbf{C}}_4 = \frac{1}{2} m_b d^2 \begin{bmatrix} \overline{\theta}_{21}c_{11} - 2\rho\theta_{11}c_{12} & \overline{\theta}_{21}c_{21} - 2\rho\theta_{11}c_{22} \\ \overline{\theta}_{22}c_{11} - 2\rho\theta_{12}c_{12} & \overline{\theta}_{22}c_{21} - 2\rho\theta_{12}c_{22} \end{bmatrix}$$

Therefore,

$$(\mathbf{T}'_4)^T \mathbf{M}'_4 \dot{\mathbf{T}}'_4 = I_b \begin{bmatrix} \theta_{21}\dot{\theta}_{21} & \theta_{21}\dot{\theta}_{22} \\ \theta_{22}\dot{\theta}_{21} & \theta_{22}\dot{\theta}_{22} \end{bmatrix} + \frac{1}{2} m_b d^2 \begin{bmatrix} \overline{\theta}_{21}c_{11} - 2\rho\theta_{11}c_{12} & \overline{\theta}_{21}c_{21} - 2\rho\theta_{11}c_{22} \\ \overline{\theta}_{22}c_{11} - 2\rho\theta_{12}c_{12} & \overline{\theta}_{22}c_{21} - 2\rho\theta_{12}c_{22} \end{bmatrix}$$

In the final steps, we calculate $\mathbf{T}^T\mathbf{WMT}$. As we saw earlier, only the caster wheel and the bracket can contribute to this term, for the contributions of the other bodies to the matrix of generalized inertia are constant. However, the bracket undergoes planar motion, and according to Exercise 7.10, its contribution to this term vanishes. Therefore,

$$\mathbf{T}^T\mathbf{W}\mathbf{M}\mathbf{T} = \mathbf{T}_3^T\mathbf{W}_3\mathbf{M}_3\mathbf{T}_3$$

Upon expansion of the foregoing product, we have

$$\begin{aligned}\mathbf{T}_3^T\mathbf{W}_3\mathbf{M}_3\mathbf{T}_3 &= \begin{bmatrix}\boldsymbol{\Theta}_3^T & \mathbf{G}_3^T\end{bmatrix}\begin{bmatrix}\boldsymbol{\Omega}_3 & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{I}_c & \mathbf{O}\\ \mathbf{O} & m_c\mathbf{1}_3\end{bmatrix}\begin{bmatrix}\boldsymbol{\Theta}_3\\ \mathbf{G}_3\end{bmatrix}\\ &= \boldsymbol{\Theta}_3^T\boldsymbol{\Omega}_3\mathbf{I}_c\boldsymbol{\Theta}_3 \end{aligned} \tag{12.54}$$

The foregoing term vanishes, as we prove below. First, notice that

$$\boldsymbol{\Omega}_3\boldsymbol{\omega}_3 = \mathbf{0}$$

However, from Eq. (12.44),

$$\boldsymbol{\omega}_3 = \boldsymbol{\Theta}_3\dot{\boldsymbol{\theta}}_a$$

and hence,

$$\boldsymbol{\Omega}_3\boldsymbol{\Theta}_3\dot{\boldsymbol{\theta}}_a = \mathbf{0}$$

for *every* $\dot{\boldsymbol{\theta}}_a$, whence

$$\boldsymbol{\Omega}_3\boldsymbol{\Theta}_3 = \mathbf{O}_{32}$$

with $\mathbf{O}_{32}$ denoting the 3×2 zero matrix. Upon transposing the foregoing expression, we obtain

$$\boldsymbol{\Theta}_3^T\boldsymbol{\Omega}_3 = \mathbf{O}_{23}$$

where we have recalled that $\boldsymbol{\Omega}_3$ is skew-symmetric.

Substitution of the above expression into Eq. (12.54) readily shows that the term in question indeed vanishes, i.e.,

In summary, the Coriolis and centrifugal force terms of the system at hand take the form

$$\begin{aligned}\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a = &\frac{m_c r^2}{4}\begin{bmatrix}6\theta_{11}(\dot{\theta}_{11}\dot{\theta}_1 + \dot{\theta}_{12}\dot{\theta}_2) + \overline{\theta}_{21}(\dot{\theta}_{12}\dot{\theta}_1 + \dot{\theta}_{22}\dot{\theta}_2)\\ 6\theta_{12}(\dot{\theta}_{11}\dot{\theta}_1 + \dot{\theta}_{12}\dot{\theta}_2) + \overline{\theta}_{22}(\dot{\theta}_{12}\dot{\theta}_1 + \dot{\theta}_{22}\dot{\theta}_2)\end{bmatrix}\\ &+ I_b(\dot{\theta}_{21}\dot{\theta}_1 + \dot{\theta}_{22}\dot{\theta}_2)\begin{bmatrix}\theta_{21}\\ \theta_{22}\end{bmatrix} + \frac{1}{2}m_b d^2(c_{11}\dot{\theta}_1 + c_{21}\dot{\theta}_2)\begin{bmatrix}\overline{\theta}_{21}\\ \overline{\theta}_{22}\end{bmatrix}\\ &- m_b d^2\rho(c_{12}\dot{\theta}_1 + c_{22}\dot{\theta}_2)\begin{bmatrix}\theta_{11}\\ \theta_{12}\end{bmatrix}\end{aligned}$$

If we recall that the c_{ij} coefficients are linear in the joint rates, then the foregoing expression clearly shows the quadratic nature of the Coriolis and centrifugal terms with respect to the joint rates.

The derivation of the forces supplied by the actuators is straightforward:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

The dissipative generalized force is less straightforward, but its calculation is not too lengthy. In fact, if we assume linear dashpots at all joints, then the dissipation function is

$$\Delta = \frac{1}{2}c_1\dot{\theta}_1^2 + \frac{1}{2}c_2\dot{\theta}_2^2 + \frac{1}{2}c_3\dot{\theta}_3^2 + \frac{1}{2}c_4\dot{\psi}^2 = \frac{1}{2}\dot{\boldsymbol{\theta}}_a^T\mathbf{C}_{12}\dot{\boldsymbol{\theta}}_a + \frac{1}{2}\dot{\boldsymbol{\theta}}_u^T\mathbf{C}_{34}\dot{\boldsymbol{\theta}}_u$$

with $\mathbf{C}_{12}$ and $\mathbf{C}_{34}$ defined as

$$\mathbf{C}_{12} \equiv \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \quad \mathbf{C}_{34} \equiv \begin{bmatrix} c_3 & 0 \\ 0 & c_4 \end{bmatrix}$$

Now, if we recall the expression for $\dot{\boldsymbol{\theta}}_u$ in terms of $\dot{\boldsymbol{\theta}}_a$, we end up with

$$\Delta = \frac{1}{2}\dot{\boldsymbol{\theta}}_a^T\mathbf{D}\dot{\boldsymbol{\theta}}_a$$

$\mathbf{D}$ being defined, in turn, as the equivalent damping matrix, given by

$$\mathbf{D} = \mathbf{C}_{12} + \boldsymbol{\Theta}^T\mathbf{C}_{34}\boldsymbol{\Theta}$$

Since $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\psi)$, $\mathbf{D} = \mathbf{D}(\psi)$, the dynamics model under study thus taking the form

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a = \boldsymbol{\tau} - \mathbf{D}(\psi)\dot{\boldsymbol{\theta}}_a$$

with $\mathbf{I}$ and $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$ given, such as in the case of holonomic systems, as

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{T}^T\mathbf{M}\mathbf{T}$$
$$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) = \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T\mathbf{W}\mathbf{M}\mathbf{T}$$

thereby completing the mathematical model governing the motion of the system at hand. Note here that $\boldsymbol{\theta}$ denotes the four-dimensional vector of joint variables containing all four angles appearing as components of $\boldsymbol{\theta}_a$ and $\boldsymbol{\theta}_u$. Because of the nonholonomy of the system, an expression for the latter in terms of the former cannot be derived, and thus the whole four-dimensional vector $\boldsymbol{\theta}$ is left as an argument of both $\mathbf{I}$ and $\mathbf{C}$.

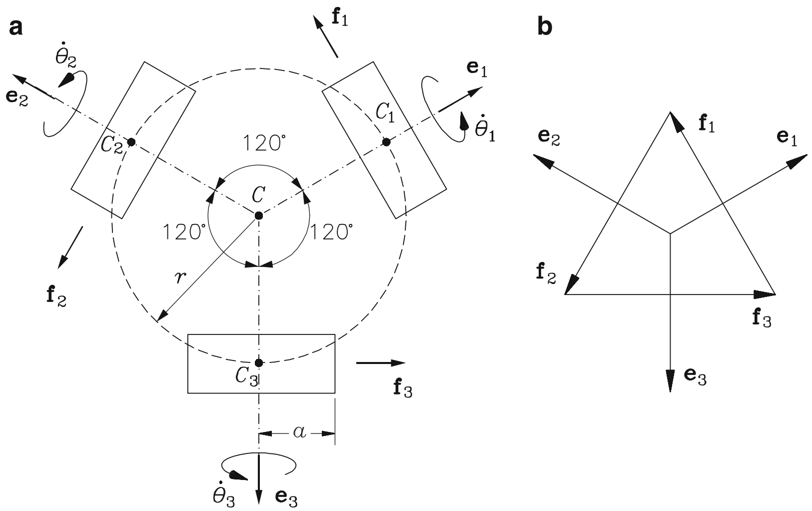

Fig. 12.7 Rolling robot with ODWs in a Δ-array

Note that calculating the torque $\boldsymbol{\tau}$ required for a given motion—inverse dynamics—of the rolling robot under study is straightforward from the above model. However, given the strong coupling among all variables involved, a recursive algorithm in this case is not apparent. On the other hand, the determination of the motion produced by a given history of joint torques requires (a) the calculation of $\mathbf{I}$, which can be achieved symbolically; (b) the inversion of $\mathbf{I}$, which can be done symbolically because this is a 2×2 matrix; (c) the calculation of the Coriolis and centrifugal terms, as well as the dissipative forces; and (d) the integration of the initial-value problem resulting once initial values to $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}_a$ have been assigned.

12.5.2 Robots with Omnidirectional Wheels

We now consider a three-dof robot with three actuated wheels of the Mekanum type, as shown in Fig. 10.19, with the configuration of Fig. 12.7, which will be termed, henceforth, the Δ-*array*. This system is illustrated in Fig. 12.8.

Below we will adopt the notation of Sect. 10.5.2, with $\alpha = \pi/2$ and $n = 3$. We now recall that the twist of the platform was represented in planar form as

$$\mathbf{t}' \equiv \begin{bmatrix} \omega \\ \dot{\mathbf{c}} \end{bmatrix} \tag{12.55}$$

where ω is the scalar angular velocity of the platform and $\dot{\mathbf{c}}$ is the two-dimensional position vector of its mass center, which will be assumed to coincide with the

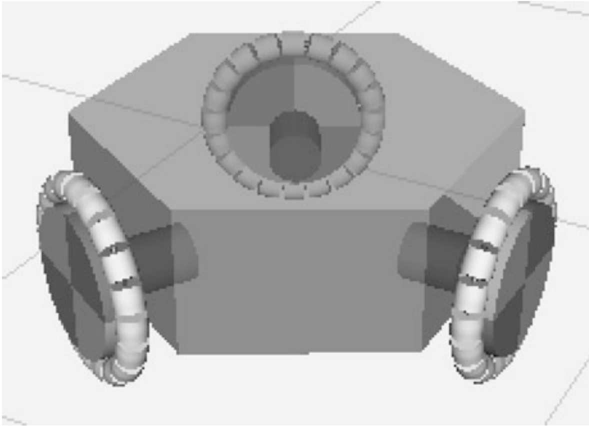

Fig. 12.8 A view of the three-wheeled robot with Mekanum wheels in a Δ-array

centroid of the set of points $\{ C_i \}_1^3$. Moreover, the three wheels are actuated, and hence, the three-dimensional vector of actuated joint rates is defined as

$$\dot{\boldsymbol{\theta}}_a \equiv \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{12.56}$$

The relation between $\dot{\boldsymbol{\theta}}_a$ and $\mathbf{t}'$ was derived in general in Sect. 10.5.2. As pertaining to the robot of Fig. 12.7, we have

$$\mathbf{J}\dot{\boldsymbol{\theta}}_a = \mathbf{K}\mathbf{t}' \tag{12.57a}$$

with the two 3×3 Jacobians $\mathbf{J}$ and $\mathbf{K}$ defined as

$$\mathbf{J} \equiv -a\mathbf{1}, \quad \mathbf{K} \equiv \begin{bmatrix} r\ \mathbf{f}_1^T \\ r\ \mathbf{f}_2^T \\ r\ \mathbf{f}_3^T \end{bmatrix} \tag{12.57b}$$

where, it is recalled, a is the height of the axis of the wheel hub and r is the horizontal distance of the points of contact with the ground to the mass center C of the platform, as indicated in Fig. 12.7a. Moreover, vectors $\{ \mathbf{e}_i \}_1^3$ and $\{ \mathbf{f}_i \}_1^3$, defined in Sect. 10.5.2, are displayed in Fig. 12.7. Below we derive expressions for ω and $\dot{\mathbf{c}}$, from Eq. (12.57a), in terms of the joint rates. To this end, we expand these three equations, thus obtaining

$$r\omega + \mathbf{f}_1^T\dot{\mathbf{c}} = -a\dot{\theta}_1 \tag{12.58a}$$

$$r\omega + \mathbf{f}_2^T\dot{\mathbf{c}} = -a\dot{\theta}_2 \tag{12.58b}$$

$$r\omega + \mathbf{f}_3^T\dot{\mathbf{c}} = -a\dot{\theta}_3 \tag{12.58c}$$

Upon adding corresponding sides of the three foregoing equations, we obtain

$$3r\omega + \dot{\mathbf{c}}^T \sum_1^3 \mathbf{f}_i = -a \sum_1^3 \dot{\theta}_i \tag{12.59}$$

But from Fig. 12.7b, it is apparent that

$$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \mathbf{0} \tag{12.60a}$$

$$\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 = \mathbf{0} \tag{12.60b}$$

Likewise,

$$\mathbf{e}_1 = \frac{\sqrt{3}}{3}(\mathbf{f}_3 - \mathbf{f}_2), \ \mathbf{e}_2 = \frac{\sqrt{3}}{3}(\mathbf{f}_1 - \mathbf{f}_3), \ \mathbf{e}_3 = \frac{\sqrt{3}}{3}(\mathbf{f}_2 - \mathbf{f}_1) \tag{12.60c}$$

$$\mathbf{f}_1 = \frac{\sqrt{3}}{3}(\mathbf{e}_2 - \mathbf{e}_3), \ \mathbf{f}_2 = \frac{\sqrt{3}}{3}(\mathbf{e}_3 - \mathbf{e}_1), \ \mathbf{f}_3 = \frac{\sqrt{3}}{3}(\mathbf{e}_1 - \mathbf{e}_2) \tag{12.60d}$$

and hence, the above equation for ω and $\dot{\mathbf{c}}$ leads to

$$\omega = -\frac{a}{3r} \sum_1^3 \dot{\theta}_i \tag{12.61}$$

Now we derive an expression for $\dot{\mathbf{c}}$ in terms of the actuated joint rates. We do this by subtracting, sidewise, Eq. (12.58b) from Eq. (12.58a) and Eq. (12.58c) from Eq. (12.58b), thus obtaining a system of two linear equations in two unknowns, the two components of the two-dimensional vector $\dot{\mathbf{c}}$, namely,

$$\mathbf{A}\dot{\mathbf{c}} = \mathbf{b}$$

with matrix $\mathbf{A}$ and vector $\mathbf{b}$ defined as

$$\mathbf{A} \equiv \begin{bmatrix} (\mathbf{f}_1 - \mathbf{f}_2)^T \\ (\mathbf{f}_2 - \mathbf{f}_3)^T \end{bmatrix} \equiv -\sqrt{3} \begin{bmatrix} \mathbf{e}_3^T \\ \mathbf{e}_1^T \end{bmatrix}, \quad \mathbf{b} \equiv -a \begin{bmatrix} \dot{\theta}_1 - \dot{\theta}_2 \\ \dot{\theta}_2 - \dot{\theta}_3 \end{bmatrix}$$

where we have used relations (12.60c). Since $\mathbf{A}$ is a 2×2 matrix, its inverse can be readily found with the aid of Facts 5.7.3 and 5.7.4, which yield

$$\dot{\mathbf{c}} = \frac{2}{3}a \begin{bmatrix} -\mathbf{E}\mathbf{e}_1 & \mathbf{E}\mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 - \dot{\theta}_2 \\ \dot{\theta}_2 - \dot{\theta}_3 \end{bmatrix}$$

Now, from Fig. 12.7b,

$$\mathbf{E}\mathbf{e}_1 = \mathbf{f}_1, \quad \mathbf{E}\mathbf{e}_3 = \mathbf{f}_3$$

and hence, $\dot{\mathbf{c}}$ reduces to

$$\dot{\mathbf{c}} = \frac{2}{3}a[(\dot{\theta}_2 - \dot{\theta}_1)\mathbf{f}_1 + (\dot{\theta}_2 - \dot{\theta}_3)\mathbf{f}_3] \equiv \frac{2}{3}a[\dot{\theta}_2(\mathbf{f}_1 + \mathbf{f}_3) - \dot{\theta}_1\mathbf{f}_1 - \dot{\theta}_3\mathbf{f}_3]$$

But by virtue of Eq. (12.60b),

$$\mathbf{f}_1 + \mathbf{f}_3 = -\mathbf{f}_2$$

the above expression for $\dot{\mathbf{c}}$ thus becoming

$$\dot{\mathbf{c}} = -\frac{2a}{3}\sum_1^3 \dot{\theta}_i\mathbf{f}_i \tag{12.62}$$

Thus, ω is proportional to the mean value of $\{\dot{\theta}_i\}_1^3$, while $\dot{\mathbf{c}}$ is proportional to the mean value of $\{\dot{\theta}_i\mathbf{f}_i\}_1^3$. In deriving the mathematical model of the robot at hand, we will resort to the natural orthogonal complement, and therefore, we will require expressions for the twists of all bodies involved in terms of the actuated wheel rates. We start by labeling the wheels as bodies 1, 2, and 3, with the platform being body 4. Moreover, we will neglect the inertia of the rollers, and so no labels need be attached to these. Furthermore, the wheel hubs undergo rotations with angular velocities in two orthogonal directions, and hence, a full six-dimensional twist representation of these will be required. Henceforth, we will regard the angular velocity of the platform and the velocity of its mass center as three-dimensional vectors. Therefore,

$$\mathbf{t}_4 \equiv \mathbf{T}_4\dot{\boldsymbol{\theta}}_a, \quad \mathbf{T}_4 \equiv -\lambda\begin{bmatrix}\mathbf{k} & \mathbf{k} & \mathbf{k}\\ 2r\mathbf{f}_1 & 2r\mathbf{f}_2 & 2r\mathbf{f}_3\end{bmatrix} \tag{12.63}$$

with λ defined, in turn, as the ratio

$$\lambda \equiv \frac{a}{3r} \tag{12.64}$$

Now, the wheel angular velocities are given simply as

$$\boldsymbol{\omega}_i = \dot{\theta}_i\mathbf{e}_i + \omega\mathbf{k} = \dot{\theta}_i\mathbf{e}_i - \lambda\left(\sum_1^3 \dot{\theta}_i\right)\mathbf{k} \tag{12.65}$$

or

$$\boldsymbol{\omega}_1 = (\mathbf{e}_1 - \lambda\mathbf{k})\dot{\theta}_1 - \lambda\dot{\theta}_2\mathbf{k} - \lambda\dot{\theta}_3\mathbf{k} \tag{12.66a}$$

$$\boldsymbol{\omega}_2 = -\lambda\dot{\theta}_1\mathbf{k} + (\mathbf{e}_2 - \lambda\mathbf{k})\dot{\theta}_2 - \lambda\dot{\theta}_3\mathbf{k} \tag{12.66b}$$

$$\boldsymbol{\omega}_3 = -\lambda\dot{\theta}_1\mathbf{k} - \lambda\dot{\theta}_2\mathbf{k} + (\mathbf{e}_3 - \lambda\mathbf{k})\dot{\theta}_3 \tag{12.66c}$$

Similar expressions are derived for vectors $\dot{\mathbf{c}}_i$. To this end, we resort to the geometry of Fig. 12.7, from which we derive the relations

$$\dot{\mathbf{c}}_i = \dot{\mathbf{c}} + \omega r \mathbf{f}_i = -2\lambda r \left(\sum_1^3 \dot{\theta}_j \mathbf{f}_j\right) - \lambda r \left(\sum_1^3 \dot{\theta}_j\right) \mathbf{f}_i$$

and hence,

$$\dot{\mathbf{c}}_1 = -\lambda r[(3\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)\mathbf{f}_1 + 2(\dot{\theta}_2\mathbf{f}_2 + \dot{\theta}_3\mathbf{f}_3)] \tag{12.67a}$$

$$\dot{\mathbf{c}}_2 = -\lambda r[2\dot{\theta}_1\mathbf{f}_1 + (\dot{\theta}_1 + 3\dot{\theta}_2 + \dot{\theta}_3)\mathbf{f}_2 + 2\dot{\theta}_3\mathbf{f}_3] \tag{12.67b}$$

$$\dot{\mathbf{c}}_3 = -\lambda r[2(\dot{\theta}_1\mathbf{f}_1 + \dot{\theta}_2\mathbf{f}_2) + (\dot{\theta}_1 + \dot{\theta}_2 + 3\dot{\theta}_3)\mathbf{f}_3] \tag{12.67c}$$

From the foregoing relations, and those for the angular velocities of the wheels, Eqs. (12.66a–c), we can now write the twists of the wheels in the form

$$\mathbf{t}_i = \mathbf{T}_i \dot{\boldsymbol{\theta}}_a, \quad i = 1, 2, 3 \tag{12.68}$$

where

$$\mathbf{T}_1 \equiv \begin{bmatrix} \mathbf{e}_1 - \lambda\mathbf{k} & -\lambda\mathbf{k} & -\lambda\mathbf{k} \\ -3\lambda r\mathbf{f}_1 & -\lambda r(\mathbf{f}_1 + 2\mathbf{f}_2) & -\lambda r(\mathbf{f}_1 + 2\mathbf{f}_3) \end{bmatrix}$$

$$\mathbf{T}_2 \equiv \begin{bmatrix} -\lambda\mathbf{k} & \mathbf{e}_2 - \lambda\mathbf{k} & -\lambda\mathbf{k} \\ -\lambda r(\mathbf{f}_2 + 2\mathbf{f}_1) & -3\lambda r\mathbf{f}_2 & -\lambda r(\mathbf{f}_2 + 2\mathbf{f}_3) \end{bmatrix}$$

$$\mathbf{T}_3 \equiv \begin{bmatrix} -\lambda\mathbf{k} & -\lambda\mathbf{k} & \mathbf{e}_3 - \lambda\mathbf{k} \\ -\lambda r(\mathbf{f}_3 + 2\mathbf{f}_1) & -\lambda r(\mathbf{f}_3 + 2\mathbf{f}_2) & -3\lambda r\mathbf{f}_3 \end{bmatrix}$$

On the other hand, similar to what we have in Eq. (12.62), an interesting relationship among angular velocities of the wheels arises here. Indeed, upon adding the corresponding sides of the three equations (12.66a–c), we obtain

$$\sum_1^3 \boldsymbol{\omega}_i = \sum_1^3 \dot{\theta}_i \mathbf{e}_i - 3\lambda\mathbf{k}\sum_1^3 \dot{\theta}_i$$

Further, we dot-multiply the two sides of the foregoing equation by $\mathbf{k}$, which yields, upon rearrangement of terms,

$$3\lambda \sum_1^3 \dot{\theta}_i = -\mathbf{k} \cdot \sum_1^3 \boldsymbol{\omega}_i$$

and by virtue of Eq. (12.61),

$$\omega = \mathbf{k} \cdot \overline{\boldsymbol{\omega}}, \quad \overline{\boldsymbol{\omega}} \equiv \frac{1}{3}\sum_1^3 \boldsymbol{\omega}_i \tag{12.69}$$

that is, *the vertical component of the mean wheel angular velocity equals the scalar angular velocity of the platform.*

Now we proceed to establish the mathematical model governing the dynamics of the system under study. The generalized inertia matrix is then calculated as

$$\mathbf{I} = \sum_{1}^{4} \mathbf{T}_i^T \mathbf{M}_i \mathbf{T}_i \tag{12.70}$$

where, if $\mathbf{I}_w$ and m_w denote the moment-of-inertia matrix, in body-fixed coordinates, and the mass of each of the three wheels, with similar definitions for $\mathbf{I}_p$ and m_p as pertaining to the platform,

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{I}_w & \mathbf{O} \\ \mathbf{O} & m_w \mathbf{1} \end{bmatrix}, \quad i = 1, 2, 3, \quad \mathbf{M}_4 = \begin{bmatrix} \mathbf{I}_p & \mathbf{O} \\ \mathbf{O} & m_p \mathbf{1} \end{bmatrix} \tag{12.71}$$

We will also need the angular-velocity dyads, $\mathbf{W}_i$, which are calculated as

$$\mathbf{W}_i = \begin{bmatrix} \mathbf{\Omega}_i & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \quad i = 1, 2, 3 \tag{12.72}$$

where $\mathbf{W}_4$ will not be needed, since the platform undergoes planar motion. We have

$$\mathbf{M}_1 \mathbf{T}_1 = \begin{bmatrix} \mathbf{I}_w(\mathbf{e}_1 - \lambda \mathbf{k}) & -\lambda \mathbf{I}_w \mathbf{k} & -\lambda \mathbf{I}_w \mathbf{k} \\ -3 m_w \lambda r \mathbf{f}_1 & -m_w \lambda r (\mathbf{f}_1 + 2\mathbf{f}_2) & -m_w \lambda r (\mathbf{f}_1 + 2\mathbf{f}_3) \end{bmatrix}$$

Moreover, we assume that in a local coordinate frame $\{\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}\}$,

$$\mathbf{I}_w = \begin{bmatrix} I & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}$$

in which I and J are constants. Hence,

$$\mathbf{T}_1^T \mathbf{M}_1 \mathbf{T}_1 = \begin{bmatrix} I + \lambda^2 K & \lambda^2 J & \lambda^2 J \\ \lambda^2 J & \lambda^2 L & \lambda^2 M \\ \lambda^2 J & \lambda^2 M & \lambda^2 L \end{bmatrix}$$

where

$$K \equiv J + 9 m_w r^2$$
$$L \equiv J + 3 m_w r^2$$
$$M \equiv J - 3 m_w r^2$$

Likewise,

$$\mathbf{T}_2^T \mathbf{M}_2 \mathbf{T}_2 = \begin{bmatrix} \lambda^2 L & \lambda^2 J & \lambda^2 M \\ \lambda^2 J & I + \lambda^2 K & \lambda^2 J \\ \lambda^2 M & \lambda^2 J & \lambda^2 L \end{bmatrix}$$

$$\mathbf{T}_3^T \mathbf{M}_3 \mathbf{T}_3 = \begin{bmatrix} \lambda^2 L & \lambda^2 J & \lambda^2 J \\ \lambda^2 J & \lambda^2 L & \lambda^2 J \\ \lambda^2 J & \lambda^2 J & I + \lambda^2 K \end{bmatrix}$$

Furthermore,

$$\mathbf{M}_4 \mathbf{T}_4 = -\lambda \begin{bmatrix} \mathbf{I}_p \mathbf{k} & \mathbf{I}_p \mathbf{k} & \mathbf{I}_p \mathbf{k} \\ 2m_p r \mathbf{f}_1 & 2m_p r \mathbf{f}_2 & 2m_p r \mathbf{f}_3 \end{bmatrix}$$

It is apparent that, by virtue of the planar motion undergone by the platform, only its moment of inertia H about the vertical passing through its mass center is needed. Then,

$$\mathbf{T}_4^T \mathbf{M}_4 \mathbf{T}_4 = \lambda^2 \begin{bmatrix} H + 4m_p r^2 & H - 2m_p r^2 & H - 2m_p r^2 \\ H - 2m_p r^2 & H + 4m_p r^2 & H - 2m_p r^2 \\ H - 2m_p r^2 & H - 2m_p r^2 & H + 4m_p r^2 \end{bmatrix}$$

Upon summing all four products computed above, we obtain

$$\mathbf{I} = \begin{bmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{bmatrix}$$

with the definitions below:

$$\alpha \equiv I + \lambda^2 (H + 3J + 15 m_w r^2 + 4 m_p r^2)$$

$$\beta \equiv \lambda^2 \left(H + 3J - 3 m_w r^2 - 2 m_p r^2 \right)$$

which is a constant matrix. Moreover, note that the geometric and inertial symmetry assumed at the outset is apparent in the form of the foregoing inertia matrix, its inverse being readily obtained in closed form, namely,

$$\mathbf{I}^{-1} = \frac{1}{\Delta} \begin{bmatrix} \alpha + \beta & -\beta & -\beta \\ -\beta & \alpha + \beta & -\beta \\ -\beta & -\beta & \alpha + \beta \end{bmatrix}, \quad \Delta \equiv (\alpha + \beta)\alpha - 2\beta^2$$

Next, we turn to the calculation of the $\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}}$ term. This is readily found to be

$$\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}} = \sum_1^4 \mathbf{T}_i^T \mathbf{M}_i \dot{\mathbf{T}}_i$$

each of the foregoing products being expanded below. We have, first,

$$\dot{\mathbf{T}}_1 = \begin{bmatrix} \omega\mathbf{f}_1 & \mathbf{0} & \mathbf{0} \\ 3\lambda r\omega\mathbf{e}_1 & -\lambda r\omega(\mathbf{e}_3 - \mathbf{e}_2) & \lambda r\omega(\mathbf{e}_3 - \mathbf{e}_2) \end{bmatrix}$$

$$\dot{\mathbf{T}}_2 = \begin{bmatrix} \mathbf{0} & \omega\mathbf{f}_2 & \mathbf{0} \\ \lambda r\omega(\mathbf{e}_1 - \mathbf{e}_3) & 3\lambda r\omega\mathbf{e}_2 & -\lambda r\omega(\mathbf{e}_1 - \mathbf{e}_3) \end{bmatrix}$$

$$\dot{\mathbf{T}}_3 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \omega\mathbf{f}_3 \\ -\lambda r\omega(\mathbf{e}_2 - \mathbf{e}_1) & \lambda r\omega(\mathbf{e}_2 - \mathbf{e}_1) & 3\lambda r\omega\mathbf{e}_3 \end{bmatrix}$$

$$\dot{\mathbf{T}}_4 = \lambda \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2r\omega\mathbf{e}_1 & 2r\omega\mathbf{e}_2 & 2r\omega\mathbf{e}_3 \end{bmatrix}$$

Hence, for the first wheel,

$$\mathbf{M}_1\dot{\mathbf{T}}_1 = \begin{bmatrix} \mathbf{I}_w\omega\mathbf{f}_1 & \mathbf{0} & \mathbf{0} \\ 3\lambda m_w r\omega\mathbf{e}_1 & -\lambda m_w r\omega(\mathbf{e}_3 - \mathbf{e}_2) & \lambda m_w r\omega(\mathbf{e}_3 - \mathbf{e}_2) \end{bmatrix}$$

Therefore,

$$\mathbf{T}_1^T\mathbf{M}_1\dot{\mathbf{T}}_1 = 3\sqrt{3}\lambda^2 m_w r^2\omega \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

where the skew-symmetric matrix is the cross product matrix of vector $[\,0,\ 1,\ 1\,]^T$. By symmetry, the other two products, $\mathbf{T}_i^T\mathbf{M}_i\dot{\mathbf{T}}_i$, for $i = 1, 2$, take on similar forms, with the skew-symmetric matrix, becoming, correspondingly, the cross-product matrix of vectors $[\,1,\ 0,\ 1\,]^T$ and $[\,1,\ 1,\ 0\,]^T$. This means that the first of these three products is affected by the rotation of the second and the third wheels, but not by that of the first one; the second of those products is affected by the rotation of the first and the third wheels, but not by the second; the third product is affected, in turn, by the rotation of the first two wheels, but not by that of the third wheel. We thus have

$$\mathbf{T}_2^T\mathbf{M}_2\dot{\mathbf{T}}_2 = 3\sqrt{3}\lambda^2 m_w r^2\omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{T}_3^T\mathbf{M}_3\dot{\mathbf{T}}_3 = 3\sqrt{3}\lambda^2 m_w r^2\omega \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Furthermore,

$$\mathbf{M}_4\dot{\mathbf{T}}_4 = \lambda \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -2m_p r\omega\mathbf{e}_1 & -2m_p r\omega\mathbf{e}_2 & -2m_p r\omega\mathbf{e}_3 \end{bmatrix}$$

and hence,

$$\mathbf{T}_4^T \mathbf{M}_4 \dot{\mathbf{T}}_4 = 2\sqrt{3}\lambda^2 m_p r^2 \omega \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \tag{12.74a}$$

whose skew-symmetric matrix is readily identified as the cross-product matrix of vector $[\,1,\ 1,\ 1\,]^T$, thereby indicating an equal participation of all three wheels in this term, a rather plausible result. Upon adding all four products calculated above, we obtain

$$\mathbf{T}^T \mathbf{M} \dot{\mathbf{T}} = 2\sqrt{3}\lambda^2 (3m_w + m_p) r^2 \omega \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \tag{12.75}$$

The equal participation of all three wheels in the foregoing product is apparent. Moreover, notice that the term in parentheses can be regarded as an equivalent mass, which is merely the sum of all four masses involved, the moments of inertia of the wheels playing no role in this term.

We now turn to the calculation of the $\mathbf{T}^T \mathbf{W} \mathbf{M} \mathbf{T}$ term, which can be expressed as a sum, namely,

$$\mathbf{T}^T \mathbf{W} \mathbf{M} \mathbf{T} = \sum_1^3 \mathbf{T}_i^T \mathbf{W}_i \mathbf{M}_i \mathbf{T}_i$$

where we have not considered the contribution of the platform, because this undergoes planar motion. Moreover, matrices $\mathbf{W}_i$, for $i = 1$, 2, and 3, take the obvious forms

$$\mathbf{W}_i \equiv \begin{bmatrix} \mathbf{\Omega}_i & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

We then have, for the first wheel,

$$\mathbf{W}_1 \mathbf{M}_1 \mathbf{T}_1 = \begin{bmatrix} \boldsymbol{\omega}_1 \times [\mathbf{I}_w(\mathbf{e}_1 - \lambda \mathbf{k})] & -\boldsymbol{\omega}_1 \times (\lambda \mathbf{I}_w \mathbf{k}) & -\boldsymbol{\omega}_1 \times (\lambda \mathbf{I}_w \mathbf{k}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Now, it does not require too much effort to calculate the complete first product, which merely vanishes, i.e.,

$$\mathbf{T}_1^T \mathbf{W}_1 \mathbf{M}_1 \mathbf{T}_1 = \mathbf{O}_{33}$$

with $\mathbf{O}_{33}$ defined as the 3×3 zero matrix. By symmetry, the remaining two products also vanish, and hence, the sum also does, i.e.,

$$\mathbf{T}^T \mathbf{W} \mathbf{M} \mathbf{T} = \mathbf{O}_{33} \tag{12.76}$$

Now, calculating the dissipative and active generalized forces is straightforward. We will neglect here the dissipation of energy occurring at the bearings of the rollers, and hence, if we assume that the lubricant of the wheel hubs produces linear dissipative torques, then we have

$$\boldsymbol{\delta} = c \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \tag{12.77}$$

where c is the common damping coefficient for all three wheel hubs. We now have all the elements needed to set up the mathematical model governing the dynamics of the robot, namely,

$$\mathbf{I}\ddot{\boldsymbol{\theta}}_a + \mathbf{C}(\omega)\dot{\boldsymbol{\theta}}_a = \boldsymbol{\tau} - \boldsymbol{\delta} \tag{12.78}$$

where $\mathbf{C}(\omega) \equiv \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} + \mathbf{T}^T\mathbf{W}\mathbf{M}\mathbf{T}$; from Eqs. (12.75) and (12.76), this term becomes

$$\mathbf{C}(\omega) = 2\sqrt{3}\lambda^2(3m_w + m_p)r^2\omega \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \tag{12.79}$$

Since $\omega = -a/(3r)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$, the quadratic nature of the second term of Eq. (12.78) in the joint rates becomes apparent. It is also apparent that the mathematical model derived above does not depend on $\boldsymbol{\theta}_a$. What this means is that the mathematical model allows the integration of the actuated joint accelerations to yield joint-rate histories $\dot{\boldsymbol{\theta}}_a(t)$, but this model cannot provide joint-variable histories $\boldsymbol{\theta}_a(t)$. To obtain these, for given initial conditions, the joint-rate histories have to be integrated, which can be done by numerical quadrature.

Finally, in order to obtain the Cartesian histories of the platform pose, given by the angle σ that a specific line of the platform makes with a line fixed in an inertial frame, and the position vector of the mass center, $\mathbf{c}$, Eqs. (12.61) and (12.62) have to be integrated. While the integration of the former can be readily done by quadrature, that of the latter requires knowledge of vectors $\mathbf{f}_i$, for $i = 1, 2, 3$, and these vectors depend on σ. Thus, the integration of Eq. (12.61) can be done once the joint-rate histories are known; that of Eq. (12.62) requires knowledge of angle σ. These features are inherent to nonholonomic systems.

12.6 Exercises

12.1 Show that the mathematical model of an arbitrary robotic mechanical system, whether holonomic or nonholonomic, with r rigid bodies and n degrees of freedom, can be cast in the general form

$$\mathbf{I}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}_a + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)\dot{\boldsymbol{\theta}}_a = \boldsymbol{\tau}^A + \boldsymbol{\gamma} + \boldsymbol{\delta}$$

where

$\boldsymbol{\theta}$: the m-dimensional vector of variables associated with all joints, actuated and unactuated;
$\dot{\boldsymbol{\theta}}_a$: the n-dimensional vector of actuated joint variables, $n \leq m$;
$\boldsymbol{\tau}^A$: the n-dimensional vector of actuator torques;
$\boldsymbol{\gamma}$: the n-dimensional vector of gravity torques;
$\boldsymbol{\delta}$: the n-dimensional vector of dissipative torques;
$\mathbf{I}(\boldsymbol{\theta})$: the $n \times n$ matrix of generalized inertia;
$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$: the $n \times n$ matrix of Coriolis and centrifugal forces;

with $\mathbf{I}(\boldsymbol{\theta})$ and $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$ given by

$$\mathbf{I}(\boldsymbol{\theta}) \equiv \mathbf{T}^T\mathbf{M}\mathbf{T}$$

$$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) \equiv \frac{1}{2}[\dot{\mathbf{I}} + \mathbf{T}^T\mathbf{M}\dot{\mathbf{T}} - \dot{\mathbf{T}}^T\mathbf{M}\mathbf{T} + \mathbf{T}^T(\mathbf{W}\mathbf{M} + \mathbf{M}\mathbf{W})\mathbf{T}]$$

in which

M: the $6r \times 6r$ matrix of system mass;
T: the $n \times 6r$ twist-shaping matrix that maps the n-dimensional vector of actuated joint rates into the $6r$-dimensional vector of system twist $\mathbf{t}$;
W: the $6r \times 6r$ matrix of system angular velocity.

12.2 For the system of Exercise 12.1, show that the matrix difference $\dot{\mathbf{I}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a) - 2\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}_a)$ is skew-symmetric. This is a well-known result for holonomic systems (Spong et al. 2006).

12.3 For the rolling robot with conventional wheels of Sect. 12.5.1, find the generalized inertia matrix of the robot under the maneuvers described below:

(a) pure translation;
(b) midpoint of segment O_1O_2 stationary.

In each case, give a physical interpretation of the matrix thus obtained.

12.4 With reference to the same robot of Exercise 12.3, state the conditions on its geometric parameters that yield $\mathbf{I}_w$ and $\mathbf{I}_p$ isotropic, these two 2×2 matrices having been defined in Sect. 12.5.1.

12.5 Derive the mathematical model governing the motion of a two-dof rolling robot with conventional wheels, similar to that of Fig. 10.17, but with two caster wheels instead. The vertical axes of the caster wheels are a distance l apart and a distance $a + b$ from the common axis of the driving wheels. What is the characteristic length of this robot?

12.6 Find the conditions under which the three-wheeled robot with omnidirectional wheels analyzed in Sect. 12.5.2 has an isotropic inertia matrix. Discuss the advantages of such an inertially isotropic robot.

12.7 With reference to the omnidirectional robot of Sect. 12.5.2, show that the mathematical model can be manipulated to yield a single first-order ordinary differential equation in ω, of the form

$$\dot{\omega} + k\omega = f(t)$$

in which k is a constant with units of frequency, its inverse being the time-constant of the system. Find expressions for k and $f(t)$. Then, integrate the above equation in closed form, to obtain the time-history of ω for a given time-history $f(t)$ and given initial condition $\omega(0)$.

12.8 Establish the conditions on the actuated joint rates under which the three-wheeled robot with omnidirectional wheels of Sect. 12.5.2 undergoes pure translation. Under these conditions, the robot has only two degrees of freedom and, hence, a 2×2 inertia matrix. Derive an expression for its inertia matrix. *Hint: The constraint for pure translation can be written as*

$$\mathbf{a}^T \dot{\boldsymbol{\theta}}_a = 0$$

and hence, if the 3×2 *matrix* $\mathbf{L}$ *is an orthogonal complement of* $\mathbf{a}$*, i.e., if* $\mathbf{a}^T \mathbf{L} = \mathbf{0}_2^T$*, where* $\mathbf{0}_2$ *is the two-dimensional zero vector, then the underlying Euler–Lagrange equations of the constrained system can be derived by multiplying the two sides of the mathematical model found in Sect. 12.5.2 by* $\mathbf{L}^T$*:*

$$\mathbf{L}^T \mathbf{I} \ddot{\boldsymbol{\theta}}_a + \mathbf{L}^T \mathbf{C} \dot{\boldsymbol{\theta}}_a = \mathbf{L}^T \boldsymbol{\tau} - \mathbf{L}^T \boldsymbol{\delta}$$

Further, upon writing $\dot{\boldsymbol{\theta}}_a$ *as a linear transformation of a two-dimensional vector* $\boldsymbol{u}$*, namely, as*

$$\dot{\boldsymbol{\theta}}_a = \mathbf{L}\mathbf{u}$$

we obtain

$$\mathbf{L}^T \mathbf{I} \mathbf{L} \dot{\mathbf{u}} + \mathbf{L}^T \mathbf{C} \mathbf{L} \mathbf{u} = \mathbf{L}^T \boldsymbol{\tau} - \mathbf{L}^T \boldsymbol{\delta}$$

and hence, the generalized inertia matrix under pure translation is $\mathbf{L}^T \mathbf{I} \mathbf{L}$.

12.9 Find the maneuver(s) under which the Coriolis and centrifugal forces of the robot analyzed in Sect. 12.5.2 vanish. Note that in general, these forces do not vanish, even though the generalized inertia matrix of the robot is constant.

12.10 Find the eigenvalues and eigenvectors of the matrix of generalized inertia of the three-dof rolling robot with omnidirectional wheels analyzed in Sect. 12.5.2.

12.11 The Euler–Lagrange equations derived for holonomic mechanical systems in Sect. 12.3, termed the *Euler–Lagrange equations of the second kind*, require that the generalized coordinates describing the system be *independent*. In nonholonomic mechanical systems, a set of kinematic constraints is not integrable, which prevents us from solving for dependent from independent generalized coordinates, the application of the Euler–Lagrange equations as described in that section thus not being possible. However, dependent generalized coordinates can be used if the *Euler–Lagrange equations of the first kind* are recalled. To this end, we let $\mathbf{q}$ be the m-dimensional vector of generalized coordinates that are subject to p differential constraints of the form

$$\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{b}(\mathbf{q}, t)$$

where $\mathbf{A}$ is a $p \times m$ *matrix of constraints* and $\mathbf{b}$ is a p-dimensional vector depending on the configuration $\mathbf{q}$ and, possibly, on time explicitly. When $\mathbf{b}$ does not contain t explicitly, the constraints are termed *scleronomic*; otherwise, *rheonomic*. Furthermore, let $n \equiv m - p$ be the mobility of the system. The Euler–Lagrange equations of the first kind of the system at hand take on the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\phi} + \mathbf{A}^T \boldsymbol{\lambda}$$

where $\boldsymbol{\lambda}$ is a p-dimensional vector of *Lagrange multipliers* that are chosen so as to satisfy the kinematic constraints. Thus, we regard the m dependent generalized coordinates grouped in vector $\mathbf{q}$ as independent, their constraints giving rise to the constraint forces $\mathbf{A}^T \boldsymbol{\lambda}$.

Use the Euler–Lagrange equations of the first kind to set up the mathematical model of the rolling robot with omnidirectional wheels studied in Sect. 12.5.2.