Chapter 3
# Fundamentals of Rigid-Body Mechanics

3.1 Introduction

The purpose of this chapter is to lay down the foundations of the kinetostatics and dynamics of rigid bodies, as needed in the study of multibody mechanical systems. With this background, we study the kinetostatics and dynamics of robotic manipulators of the serial type in Chaps. 5 and 7, respectively, while devoting Chap. 6 to the study of trajectory planning. The latter requires, additionally, the background of Chap. 4. A special feature of the current chapter is the study of the relations between the angular velocity of a rigid body and the time-rates of change of the various sets of rotation invariants introduced in Chap. 2. Similar relations between the angular acceleration and the second time-derivatives of the rotation invariants are also recalled, the corresponding derivations being outlined in Appendix A.

Furthermore, an introduction to the extremely useful analysis tool known as *screw theory* (Roth 1984) is included. In this context, the concepts of twist and wrench are introduced, which prove in subsequent chapters to be extremely useful in deriving the kinematic and static, i.e., the *kinetostatic*, relations among the various bodies of multibody mechanical systems.

3.2 General Rigid-Body Motion and Its Associated Screw

In this section we analyze the general motion of a rigid body. Thus, let A and P be two points of the same rigid body $\mathcal{B}$, the former being a particular reference point, whereas the latter is an arbitrary point of $\mathcal{B}$, as shown in Fig. 3.1. Moreover, the position vector of point A in the original configuration is $\mathbf{a}$, and the position vector of the same point in the displaced configuration, denoted by A', is $\mathbf{a}'$. Similarly, the position vector of point P in the original configuration is $\mathbf{p}$, while in the displaced configuration $\mathcal{B}'$, this point is P', its position vector being $\mathbf{p}'$. Furthermore, $\mathbf{p}'$ is to

Fig. 3.1 General rigid-body displacement
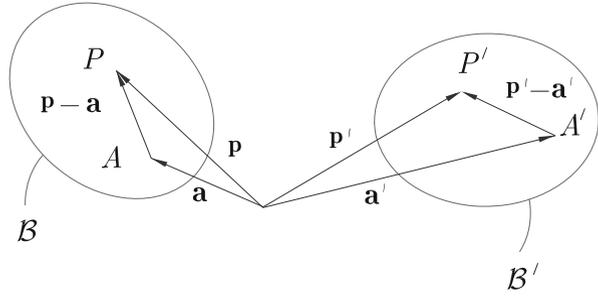
be determined, while $\mathbf{a}$, $\mathbf{a}'$, and $\mathbf{p}$ are given, along with the rotation matrix $\mathbf{Q}$. Vector $\mathbf{p}-\mathbf{a}$ can be considered to undergo a rotation $\mathbf{Q}$ about point A throughout the motion taking the body from the original to the final configuration. Since vector $\mathbf{p} - \mathbf{a}$ is mapped into $\mathbf{p}' - \mathbf{a}'$ under the above rotation, one can write

$$\mathbf{p}' - \mathbf{a}' = \mathbf{Q}(\mathbf{p} - \mathbf{a}) \tag{3.1}$$

and hence

$$\mathbf{p}' = \mathbf{a}' + \mathbf{Q}(\mathbf{p} - \mathbf{a}) \tag{3.2}$$

which is the relationship sought. Moreover, let $\mathbf{d}_A$ and $\mathbf{d}_P$ denote the displacements of A and P, respectively, i.e.,

$$\mathbf{d}_A \equiv \mathbf{a}' - \mathbf{a}, \quad \mathbf{d}_P \equiv \mathbf{p}' - \mathbf{p} \tag{3.3}$$

From Eqs. (3.2) and (3.3) one can readily obtain an expression for $\mathbf{d}_P$, namely,

$$\begin{aligned} \mathbf{d}_P &= \mathbf{a}' - \mathbf{p} + \mathbf{Q}(\mathbf{p} - \mathbf{a}) \\ &= \mathbf{a}' - \mathbf{a} - \mathbf{p} + \mathbf{Q}(\mathbf{p} - \mathbf{a}) + \mathbf{a} \\ &= \mathbf{d}_A + \underbrace{(\mathbf{Q} - \mathbf{1})(\mathbf{p} - \mathbf{a})}_{\mathbf{d}_{P/A}} \end{aligned} \tag{3.4}$$

where $\mathbf{d}_{P/A}$ denotes the *relative displacement* of P with respect to A. Notice that this displacement is caused only by the rotation. What Eq. (3.4) states is that the displacement of an arbitrary point P of a rigid body, of position vector $\mathbf{p}$ in an original configuration, is determined by the displacement of one certain point A and the concomitant rotation $\mathbf{Q}$, as depicted in Fig. 3.2. In this figure, the final configuration $\mathcal{B}'$ is attained via an intermediate configuration $\mathcal{B}''$, which is in turn attained from $\mathcal{B}$ by a pure translation.[1] Then, $\mathcal{B}'$ is attained from $\mathcal{B}''$ by a pure rotation $\mathbf{Q}$, of axis parallel to vector $\mathbf{e}$ and of angle ϕ. Apparently, once the

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[1] A body undergoes a pure translation when all its points move under the same displacement.

Fig. 3.2 General rigid-body displacement attained by a combination of translation and rotation
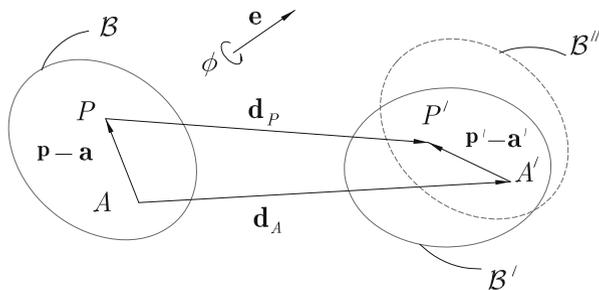
displacement of P is known, its position vector $\mathbf{p}'$ can be readily determined. An interesting result in connection with the foregoing discussion is summarized below:

Theorem 3.2.1. *Let a rigid body $\mathcal{B}$ move under a general displacement, of rotation $\mathbf{Q}$ with axis parallel to the unit vector $\mathbf{e}$. Then, the displacements of all the points of $\mathcal{B}$ have the same component along $\mathbf{e}$.*

Proof. Multiply both sides of Eq. (3.4) by $\mathbf{e}^T$, thereby obtaining

$$\mathbf{e}^T \mathbf{d}_P = \mathbf{e}^T \mathbf{d}_A + \mathbf{e}^T (\mathbf{Q} - \mathbf{1})(\mathbf{p} - \mathbf{a})$$

Now, the second term of the right-hand side of the above equation vanishes because $\mathbf{Q}\mathbf{e} = \mathbf{e}$, and hence, $\mathbf{Q}^T \mathbf{e} = \mathbf{e}$; the said equation thus leads to

$$\mathbf{e}^T \mathbf{d}_P = \mathbf{e}^T \mathbf{d}_A \equiv d_0 \tag{3.5}$$

thereby showing that the displacements of all points of the body have the same projection d_0 onto the axis of rotation, q.e.d.

As a consequence of the foregoing result, we have the classical *Mozzi–Chasles Theorem* (Mozzi 1763; Chasles 1830; Ceccarelli 1995), namely,

Theorem 3.2.2 (Mozzi 1763; Chasles 1830). *Given a rigid body moving with a general displacement, a set of its points, located on a line $\mathcal{L}$, undergo identical displacements of minimum magnitude. Moreover, line $\mathcal{L}$ and the minimum-magnitude displacement are parallel to the axis of the rotation involved, as illustrated in Fig. 3.3.*

Proof. The proof is straightforward in light of Theorem 3.2.1, which allows us to express the displacement of an arbitrary point P as the sum of two orthogonal components, namely, one parallel to the axis of rotation, independent of P and denoted by $\mathbf{d}_\parallel$—read "d-par"—and one perpendicular to this axis, denoted by $\mathbf{d}_\perp$—read "d-perp"—i.e.,

$$\mathbf{d}_P = \mathbf{d}_\parallel + \mathbf{d}_\perp \tag{3.6a}$$

Fig. 3.3 The Mozzi–Chasles
theorem
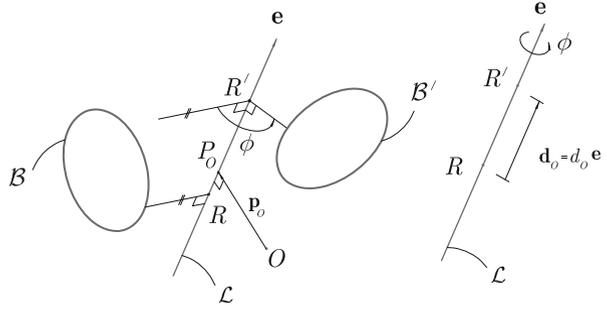
where

$$\mathbf{d}_\parallel = \mathbf{e}\mathbf{e}^T\mathbf{d}_P = d_0\mathbf{e} \equiv \mathbf{d}_0, \quad \mathbf{d}_\perp = (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_P \tag{3.6b}$$

and clearly, d_0 is a constant, as defined in Eq. (3.5), while $\mathbf{d}_\parallel$ and $\mathbf{d}_\perp$ are mutually
orthogonal. Indeed, orthogonality is apparent by definition, but is also verified
below:

$$\mathbf{d}_\parallel \cdot \mathbf{d}_\perp = d_0\mathbf{e}^T(\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_P = d_0(\mathbf{e}^T - \mathbf{e}^T)\mathbf{d}_P = 0$$

Now, by virtue of the orthogonality of the two components of $\mathbf{d}_P$, it is apparent that

$$\|\mathbf{d}_P\|^2 = \|\mathbf{d}_\parallel\|^2 + \|\mathbf{d}_\perp\|^2 = d_0^2 + \|\mathbf{d}_\perp\|^2$$

for the displacement $\mathbf{d}_P$ of any point of the body. Hence, in order to minimize $\|\mathbf{d}_P\|$
we have to make $\|\mathbf{d}_\perp\|$, and hence, $\mathbf{d}_\perp$ itself, equal to zero, i.e., we must have $\mathbf{d}_P$
parallel to $\mathbf{e}$:

$$\mathbf{d}_P = \alpha\mathbf{e}$$

for a certain scalar α. That is, the displacements of minimum magnitude of the
body under study are parallel to the axis of $\mathbf{Q}$, thereby proving the first part of the
Mozzi–Chasles Theorem. The second part is also readily proven by noticing that
if P^* is a point of minimum displacement magnitude, of position vector $\mathbf{p}^*$, its
component perpendicular to the axis of rotation must vanish, and hence,

$$\mathbf{d}_\perp^* \equiv (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_{P^*}$$
$$= (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_A + (\mathbf{1} - \mathbf{e}\mathbf{e}^T)(\mathbf{Q} - \mathbf{1})(\mathbf{p}^* - \mathbf{a}) = \mathbf{0}$$

Upon expansion of the above expression for $\mathbf{d}_\perp^*$, we obtain

$$(\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}^* - \mathbf{a}) = \mathbf{0}$$

Now it is apparent that if we define a line $\mathcal{L}$ passing through P^* and parallel to $\mathbf{e}$, then the position vector $\mathbf{p}^* + \lambda\mathbf{e}$ of any of its points P satisfies the foregoing equation. As a consequence, all points of minimum displacement magnitude lie in a line parallel to the axis of rotation of $\mathbf{Q}$, q.e.d.

An important implication of the foregoing theorem is that a rigid body can attain an arbitrary configuration from a given original one, following a screw-like motion of axis $\mathcal{L}$ and pitch p, the latter being defined presently. Thus, it seems appropriate to call $\mathcal{L}$ the *screw axis* of the rigid-body motion.

Note that d_0, as defined in Eq. (3.5), is an invariant of the motion at hand. Thus, associated with a rigid-body motion, one can then define a *screw* of axis $\mathcal{L}$ and pitch p. Of course, the pitch is defined as

$$ p \equiv \frac{d_0}{\phi} = \frac{\mathbf{d}_P^T \mathbf{e}}{\phi} \qquad \text{or} \qquad p \equiv \frac{2\pi d_0}{\phi} \tag{3.7} $$

which has units of m/rad or, correspondingly, of m/turn. Moreover, the angle ϕ of the rotation involved can be regarded as one more feature of this motion. This angle is, in fact, the *amplitude* associated with the said motion. We will come across screws in discussing velocities and forces acting on rigid bodies, along with their pitches and amplitudes. Thus, it is convenient to introduce this concept at this stage.

3.2.1   The Screw of a Rigid-Body Motion

The screw axis $\mathcal{L}$ is totally specified by a given point P_0 of $\mathcal{L}$ that can be defined, for example, as that lying closest to the origin, and a unit vector $\mathbf{e}$ defining its direction. Expressions for the position vector $\mathbf{p}_0$ of P_0 in terms of $\mathbf{a}$, $\mathbf{a}'$ and $\mathbf{Q}$, are derived below:

Since P_0 was defined as the point of $\mathcal{L}$ lying closest to the origin, $\mathbf{p}_0$ obviously is perpendicular to $\mathbf{e}$, i.e.,

$$ \mathbf{e}^T \mathbf{p}_0 = 0 \tag{3.8} $$

Moreover, the displacement $\mathbf{d}_0$ of P_0 is parallel to the vector of $\mathbf{Q}$, and hence, is identical to $\mathbf{d}_\parallel$ defined in Eq. (3.6b), i.e., it satisfies

$$ (\mathbf{Q} - \mathbf{1})\mathbf{d}_0 = \mathbf{0} $$

where $\mathbf{d}_0$ can be expressed using the general expression for the displacement, Eq. (3.4), namely,

$$ \mathbf{d}_0 = \mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}_0 - \mathbf{a}) \tag{3.9a} $$

Now, since $\mathbf{d}_0$ is identical to $\mathbf{d}_\parallel$, we have, from Eq. (3.6b),

$$\mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}_0 - \mathbf{a}) = \mathbf{d}_\parallel \equiv \mathbf{e}\mathbf{e}^T\mathbf{d}_0$$

But from Theorem 3.2.1,

$$\mathbf{e}^T\mathbf{d}_0 = \mathbf{e}^T\mathbf{d}_A$$

and so

$$\mathbf{d}_A + (\mathbf{Q} - \mathbf{1})(\mathbf{p}_0 - \mathbf{a}) = \mathbf{e}\mathbf{e}^T\mathbf{d}_A$$

or, after rearranging terms,

$$(\mathbf{Q} - \mathbf{1})\mathbf{p}_0 = (\mathbf{Q} - \mathbf{1})\mathbf{a} - (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_A \tag{3.9b}$$

which cannot be solved for $\mathbf{p}_0$ because, as the reader is invited to show, its matrix coefficient, $\mathbf{Q} - \mathbf{1}$, is singular, with null space spanned by vector $\mathbf{e}$. Nevertheless, an expression for $\mathbf{p}_0$ can be found if Eq. (3.8) is adjoined to Eq. (3.9b), thereby obtaining

$$\mathbf{A}\mathbf{p}_0 = \mathbf{b} \tag{3.10}$$

where $\mathbf{A}$ is a 4×3 matrix and $\mathbf{b}$ is a four-dimensional vector, both given by

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{Q} - \mathbf{1} \\ \mathbf{e}^T \end{bmatrix}, \quad \mathbf{b} \equiv \begin{bmatrix} (\mathbf{Q} - \mathbf{1})\mathbf{a} - (\mathbf{1} - \mathbf{e}\mathbf{e}^T)\mathbf{d}_A \\ 0 \end{bmatrix} \tag{3.11}$$

Equation (3.10) cannot be solved for $\mathbf{p}_0$ directly, because $\mathbf{A}$ is not a square matrix. In fact, that equation represents an *overdetermined* system of four equations and three unknowns. Thus, in general, that system does not admit a solution. However, the four equations are compatible, and hence, in this particular case, a solution of that equation, which turns out to be *unique*, can be determined. In fact, if both sides of Eq. (3.10) are multiplied from the left by $\mathbf{A}^T$, we have

$$\mathbf{A}^T\mathbf{A}\mathbf{p}_0 = \mathbf{A}^T\mathbf{b} \tag{3.12}$$

Moreover, if the product $\mathbf{A}^T\mathbf{A}$, which is a 3×3 matrix, is invertible, then $\mathbf{p}_0$ can be computed from Eq. (3.12). In fact, the said product is not only invertible, but also admits an inverse that is rather simple to derive, as shown below. Now the rotation matrix $\mathbf{Q}$ is recalled in terms of its *natural invariants*, namely, the unit vector $\mathbf{e}$ parallel to its axis of rotation and the angle of rotation ϕ about this axis, as given in Eq. (2.49), reproduced below for quick reference:

$$\mathbf{Q} = \mathbf{e}\mathbf{e}^T + \cos\phi(\mathbf{1} - \mathbf{e}\mathbf{e}^T) + \sin\phi\mathbf{E}$$

where $\mathbf{1}$ represents the 3×3 identity matrix and $\mathbf{E}$ the *cross-product matrix* of $\mathbf{e}$, as introduced in Eq. (2.38). Further, Eq. (2.49) is substituted into Eq. (3.11), which yields

$$\mathbf{A}^T\mathbf{A} = 2(1 - \cos\phi)\mathbf{1} - (1 - 2\cos\phi)\mathbf{e}\mathbf{e}^T \tag{3.13}$$

It is now apparent that the foregoing product is a linear combination of $\mathbf{1}$ and $\mathbf{e}\mathbf{e}^T$. This suggests that its inverse is very likely a linear combination of these two matrices as well. If this is in fact true, then one can write

$$(\mathbf{A}^T\mathbf{A})^{-1} = \alpha\mathbf{1} + \beta\mathbf{e}\mathbf{e}^T \tag{3.14}$$

coefficients α and β being determined from the condition that the product of $\mathbf{A}^T\mathbf{A}$ by its inverse should be $\mathbf{1}$, which leads to

$$\alpha = \frac{1}{2(1 - \cos\phi)}, \quad \beta = \frac{1 - 2\cos\phi}{2(1 - \cos\phi)} \tag{3.15}$$

and hence,

$$(\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{2(1 - \cos\phi)}\mathbf{1} + \frac{1 - 2\cos\phi}{2(1 - \cos\phi)}\mathbf{e}\mathbf{e}^T \tag{3.16}$$

On the other hand, from Eq. (3.11),

$$\mathbf{A}^T\mathbf{b} = (\mathbf{Q} - \mathbf{1})^T[(\mathbf{Q} - \mathbf{1})\mathbf{a} - \mathbf{d}_A] \tag{3.17}$$

Upon solving Eq. (3.12) for $\mathbf{p}_0$ and substituting relations (3.16) and (3.17) into the expression thus resulting, one finally obtains

$$\mathbf{p}_0 = \frac{(\mathbf{Q} - \mathbf{1})^T(\mathbf{Q}\mathbf{a} - \mathbf{a}')}{2(1 - \cos\phi)}, \quad \text{for } \phi \neq 0 \tag{3.18}$$

Apparently, $\mathbf{p}_0$ can be computed as long as $\phi \neq 0$. If $\phi = 0$, then we are in the presence of a pure translation, all the points of the body undergoing the same displacement. Therefore, any point of the body is of minimum-magnitude displacement.

We have thus defined a line $\mathcal{L}$ of the rigid body under study that is completely defined by its point P_0 of position vector $\mathbf{p}_0$ and a unit vector $\mathbf{e}$ determining its direction. Moreover, we have already defined the pitch of the associated motion, Eq. (3.7). The line thus defined, along with the pitch, determines the screw of the motion under study.

Fig. 3.4 A line $\mathcal{L}$ passing
through two points
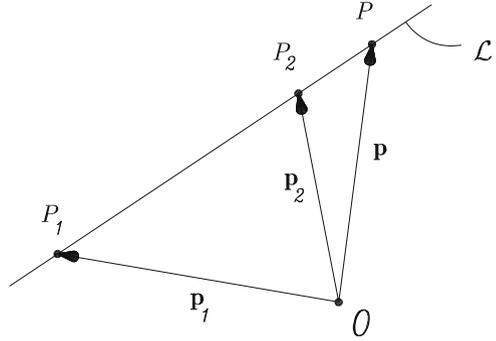
3.2.2 The Plücker Coordinates of a Line

Alternatively, the screw axis, and any line for that matter, can be defined more conveniently by its *Plücker coordinates*. In motivating this concept, we recall the equation of a line $\mathcal{L}$ passing through two points P_1 and P_2 of position vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, as shown in Fig. 3.4.

If point P lies in $\mathcal{L}$, then, it must be collinear with P_1 and P_2, a property that is expressed as

$$(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p} - \mathbf{p}_1) = \mathbf{0}$$

or upon expansion,

$$(\mathbf{p}_2 - \mathbf{p}_1) \times \mathbf{p} + \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{0} \tag{3.19}$$

If we now introduce the cross-product matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ of vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ in the above equation, we have an alternative expression for the equation of the line, namely,

$$(\mathbf{P}_2 - \mathbf{P}_1)\mathbf{p} + \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{0}$$

The above equation can be regarded as a linear equation in the homogeneous coordinates of point P, namely,

$$\begin{bmatrix} \mathbf{P}_2 - \mathbf{P}_1 & \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \mathbf{0} \tag{3.20}$$

It is now apparent that the line is defined completely by two vectors, the difference $\mathbf{p}_2 - \mathbf{p}_1$, or its cross-product matrix for that matter, and the cross product $\mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1)$. We will thus define a six-dimensional array $\boldsymbol{\gamma}_{\mathcal{L}}$ containing these two vectors, namely,

$$\boldsymbol{\gamma}_{\mathcal{L}} \equiv \begin{bmatrix} \mathbf{p}_2 - \mathbf{p}_1 \\ \mathbf{p}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) \end{bmatrix} \tag{3.21}$$

whose six scalar entries are the Plücker coordinates of $\mathcal{L}$. Moreover, if we let

$$\mathbf{e} \equiv \frac{\mathbf{p}_2 - \mathbf{p}_1}{\|\mathbf{p}_2 - \mathbf{p}_1\|}, \quad \mathbf{n} \equiv \mathbf{p}_1 \times \mathbf{e} \tag{3.22}$$

then we can write

$$\boldsymbol{\gamma}_{\mathcal{L}} = \|\mathbf{p}_2 - \mathbf{p}_1\| \begin{bmatrix} \mathbf{e} \\ \mathbf{n} \end{bmatrix}$$

The six scalar entries of the above array are the *normalized Plücker coordinates* of $\mathcal{L}$. Vector $\mathbf{e}$ determines the direction of $\mathcal{L}$, while $\mathbf{n}$ determines its location; $\mathbf{n}$ can be interpreted as the moment of a unit force parallel to $\mathbf{e}$ and of line of action $\mathcal{L}$. Hence, $\mathbf{n}$ is called the *moment* of $\mathcal{L}$. Henceforth, only the normalized Plücker coordinates of lines will be used. For brevity, we will refer to these simply as the Plücker coordinates of the line under study. The Plücker coordinates thus defined will be stored in a *Plücker array $\boldsymbol{\kappa}$* in the form

$$\boldsymbol{\kappa} = \begin{bmatrix} \mathbf{e} \\ \mathbf{n} \end{bmatrix} \tag{3.23}$$

where for conciseness, we have dropped the subscript $\mathcal{L}$, while assuming that the line under discussion is self-evident.

Note, however, that the six components of the Plücker array, i.e., the *Plücker coordinates* of line $\mathcal{L}$, are not independent, for they obey

$$\mathbf{e} \cdot \mathbf{e} = 1, \quad \mathbf{n} \cdot \mathbf{e} = 0 \tag{3.24}$$

and hence, any line $\mathcal{L}$ has only four independent Plücker coordinates. In the foregoing paragraphs, we have talked about the Plücker array of a line, and not about the Plücker vector; the reason for this distinction is given below. The set of Plücker arrays is a clear example of an array of real numbers not constituting a vector space. What disables Plücker arrays from being vectors are the two constraints that their components must satisfy, namely, (a) the sum of the squares of the first three components of a Plücker array is unity, and (b) the unit vector of a line is normal to the moment of the line. Nevertheless, we can perform with Plücker arrays certain operations that pertain to vectors, as long as we keep in mind the essential differences. For example, we can multiply Plücker arrays by matrices of the suitable dimension, with entries having appropriate units, as we will show presently.

It must be pointed out that a Plücker array is dependent upon the location of the point with respect to which the moment of the line is measured. Indeed, let $\boldsymbol{\kappa}_A$ and $\boldsymbol{\kappa}_B$ denote the Plücker arrays of the same line $\mathcal{L}$ when its moment is measured at points A and B, respectively. Moreover, this line passes through a point P of position vector $\mathbf{p}$ for a particular origin O. Now, let the moment of $\mathcal{L}$ with respect to A and B be denoted by $\mathbf{n}_A$ and $\mathbf{n}_B$, respectively, i.e.,

$$\mathbf{n}_A \equiv (\mathbf{p} - \mathbf{a}) \times \mathbf{e}, \quad \mathbf{n}_B \equiv (\mathbf{p} - \mathbf{b}) \times \mathbf{e} \tag{3.25}$$

and hence,

$$\boldsymbol{\kappa}_A \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{n}_A \end{bmatrix}, \quad \boldsymbol{\kappa}_B \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{n}_B \end{bmatrix} \tag{3.26}$$

Obviously,

$$\mathbf{n}_B - \mathbf{n}_A = (\mathbf{a} - \mathbf{b}) \times \mathbf{e} \tag{3.27}$$

i.e.,

$$\boldsymbol{\kappa}_B = \begin{bmatrix} \mathbf{e} \\ \mathbf{n}_A + (\mathbf{a} - \mathbf{b}) \times \mathbf{e} \end{bmatrix} \tag{3.28}$$

which can be rewritten as

$$\boldsymbol{\kappa}_B = \mathbf{U}\boldsymbol{\kappa}_A \tag{3.29a}$$

with the 6×6 matrix $\mathbf{U}$ defined as

$$\mathbf{U} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{A} - \mathbf{B} & \mathbf{1} \end{bmatrix} \tag{3.29b}$$

while $\mathbf{A}$ and $\mathbf{B}$ are, respectively, the cross-product matrices of vectors $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{O}$ denotes the 3×3 zero matrix. Given the lower-triangular structure of matrix $\mathbf{U}$, its determinant is simply the product of its diagonal entries, which are all unity. Hence,

$$\det(\mathbf{U}) = 1 \tag{3.30}$$

$\mathbf{U}$ thus belonging to the *unimodular group* of 6×6 matrices. These matrices are rather simple to invert. In fact, as one can readily prove,

$$\mathbf{U}^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{B} - \mathbf{A} & \mathbf{1} \end{bmatrix} \tag{3.31}$$

Relations (3.29a and b) can then be called the *Plücker-coordinate transfer formulas*.

Note that upon multiplication of both sides of Eq. (3.27) by $(\mathbf{a} - \mathbf{b})$,

$$(\mathbf{a} - \mathbf{b})^T \mathbf{n}_B = (\mathbf{a} - \mathbf{b})^T \mathbf{n}_A \tag{3.32}$$

and hence, the moments of the same line $\mathcal{L}$ with respect to two points are not independent, for they have the same component along the line joining the two points.

A special case of a line, of interest in kinematics, is a *line at infinity*. This is a line with undefined orientation, but with a defined direction of its moment;

this moment is, moreover, *independent* of the point with respect to which it is measured. Very informally, the Plücker coordinates of a line at infinity can be derived from the general expression, Eq. (3.23), if we rewrite it in the form

$$\boldsymbol{\kappa} = \|\mathbf{n}\| \begin{bmatrix} \mathbf{e}/\|\mathbf{n}\| \\ \mathbf{n}/\|\mathbf{n}\| \end{bmatrix}$$

where clearly $\mathbf{n}/\|\mathbf{n}\|$ is a unit vector; henceforth, this vector will be denoted by $\mathbf{f}$. Now let us take the limit of the above expression as P goes to infinity, i.e., when $\|\mathbf{p}\| \to \infty$, and consequently, as $\|\mathbf{n}\| \to \infty$. Thus,

$$\lim_{\|\mathbf{n}\| \to \infty} \boldsymbol{\kappa} = \left(\lim_{\|\mathbf{n}\| \to \infty} \|\mathbf{n}\| \right) \left(\lim_{\|\mathbf{n}\| \to \infty} \begin{bmatrix} \mathbf{e}/\|\mathbf{n}\| \\ \mathbf{f} \end{bmatrix} \right)$$

whence

$$\lim_{\|\mathbf{n}\| \to \infty} \boldsymbol{\kappa} = \left(\lim_{\|\mathbf{n}\| \to \infty} \|\mathbf{n}\| \right) \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$$

The six-dimensional array appearing in the above equation is defined as the Plücker array of a line at infinity, $\boldsymbol{\kappa}_\infty$, namely,

$$\boldsymbol{\kappa}_\infty = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \tag{3.33}$$

Note that a line at infinity of *unit moment* $\mathbf{f}$ can be thought of as being a line lying in a plane perpendicular to the unit vector $\mathbf{f}$, but otherwise with an indefinite location in the plane, except that it is an infinitely large distance from the origin. Thus, lines at infinity vary only in the orientation of the plane in which they lie.

3.2.3 The Pose of a Rigid Body

A possible form of describing a general rigid-body motion, then, is through a set of eight real numbers, namely, the six Plücker coordinates of its screw axis, its pitch, and its amplitude, i.e., its angle. Hence, *a rigid-body motion is fully described by six independent parameters*. Moreover, the pitch can attain values from $-\infty$ to $+\infty$. Alternatively, a rigid-body displacement can be described by seven dependent parameters as follows: four invariants of the concomitant rotation—the linear invariants, the natural invariants, or the Euler–Rodrigues parameters, introduced in Sect. 2.3—and the three components of the displacement of an arbitrary point. Since those invariants are not independent, but subject to one constraint, this description consistently involves six independent parameters. Thus, let a rigid body undergo a

general motion, of rotation $\mathbf{Q}$ from a reference configuration $\mathcal{C}_0$. If a landmark point A of the body undergoes a displacement $\mathbf{d}_A$, then the *pose array*, or simply the *pose* $\mathbf{s}$, of the body in configuration $\mathcal{C}$ is defined as a seven-dimensional array, namely,

$$\mathbf{s} \equiv \begin{bmatrix} \mathbf{q} \\ q_0 \\ \mathbf{d}_A \end{bmatrix} \tag{3.34}$$

where the three-dimensional vector $\mathbf{q}$ and the scalar q_0 are *any* four invariants of $\mathbf{Q}$. For example, if these are the Euler–Rodrigues parameters, then

$$\mathbf{q} \equiv \sin\left(\frac{\phi}{2}\right)\mathbf{e}, \quad q_0 \equiv \cos\left(\frac{\phi}{2}\right)$$

If alternatively, we work with the linear invariants, then

$$\mathbf{q} \equiv (\sin\phi)\mathbf{e}, \quad q_0 \equiv \cos\phi$$

and, of course, if we work instead with the natural invariants, then

$$\mathbf{q} \equiv \mathbf{e}, \quad q_0 \equiv \phi$$

In the first two cases, the constraint mentioned above is

$$\|\mathbf{q}\|^2 + q_0^2 = 1 \tag{3.35}$$

In the last case, the constraint is simply

$$\|\mathbf{e}\|^2 = 1 \tag{3.36}$$

Notice that the pose of a rigid body is an array of numbers quantifying the displacement of the body from a reference configuration $\mathcal{C}_0$ to a current configuration $\mathcal{C}$. As such, then, the pose is a *relative* concept.

An important problem in kinematics is the computation of the screw parameters, i.e., the components of $\mathbf{s}$, as given in Eq. (3.34), from coordinate measurements over a certain finite set of points. From the foregoing discussion, it should be clear that the computation of the attitude of a rigid body, given by matrix $\mathbf{Q}$ or its invariants, is crucial in solving this problem. Moreover, besides its theoretical importance, this problem, known as *pose estimation*, has also practical relevance. Actually, the problem of interest can be regarded as one of *relative-pose estimation*, as the pose is, in reality, a relative concept. In space applications, for example, a common operation is the *rendezvousing* of two objects, e.g., the now-defunct *Space Shuttle* with the *International Space Station*. Another example is the grabbing of objects floating in space, e.g., a drifting satellite, by a robotic gripper. In order to properly orient the gripper, the relative pose of the satellite with respect to the gripper palm

Fig. 3.5 MDA Space Vision Marker System (SVMS) at MDA's Space Vision Laboratory in Brampton, Ontario, Canada. Developed under contract with the Canadian Space Agency
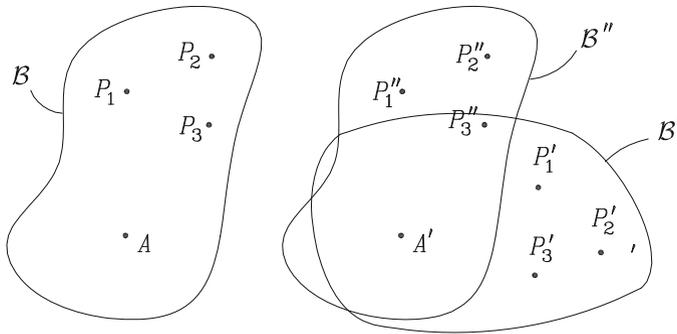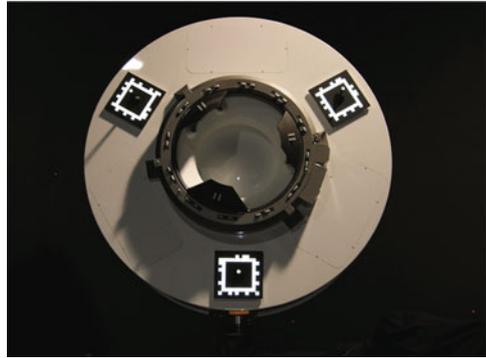
Fig. 3.6 Decomposition of the displacement of a rigid body

is needed. This can be estimated by means of markers previously attached on the satellite surface, that identify corresponding points thereof. These markers are not simple points, but arrays of visual features similar to the simple one-dimensional and two-dimensional barcodes, although more evolved, to package a larger amount of information in the same area. The purpose here is to add robustness to the estimation algorithm in use. A new concept, the Space Vision Marker System (SVMS), was introduced by a MDA Team (Bondy et al. 2007), that uses markers of the kind shown in Fig. 3.5, along with estimation algorithms. These markers are aimed at marker-detection throughout "a wide range of viewing distances and angles, and illumination (including direct sun light and shadows); encoded redundant features allow identification even with a partial data loss."

A straightforward method of computing the screw parameters consists in regarding the motion as follows: Choose a certain point A of the body, of position vector $\mathbf{a}$, and track it as the body moves to a displaced configuration, at which point A moves to A', of position vector $\mathbf{a}'$. Assume that the body reaches the displaced configuration $\mathcal{B}'$, passing through an intermediate one $\mathcal{B}''$, which is attained by a pure translation, as depicted in Fig. 3.2. Next, configuration $\mathcal{B}'$ is reached by rotating the body about point A', as indicated in Fig. 3.6.

Matrix $\mathbf{Q}$ can now be readily determined. To do this, define three points of the body, P_1, P_2, and P_3, in such a way that the three vectors defined below are orthonormal and form a right-hand system:

$$\mathbf{e}_1 \equiv \overrightarrow{AP_1}, \quad \mathbf{e}_2 \equiv \overrightarrow{AP_2}, \quad \mathbf{e}_3 \equiv \overrightarrow{AP_3} \tag{3.37}$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \tag{3.38}$$

where δ_{ij} is the *Kronecker delta*, defined as 1 if $i = j$, and as 0 otherwise. Now, let the set $\{\mathbf{e}_i\}_1^3$ be labeled $\{\mathbf{e}_i'\}_1^3$ and $\{\mathbf{e}_i''\}_1^3$ in configurations $\mathcal{B}'$ and $\mathcal{B}''$, respectively. Moreover, let q_{ij} denote the entries of the matrix representation of the rotation $\mathbf{Q}$ in a frame X, Y, Z with origin at A and such that the foregoing axes are parallel to vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively. It is clear, from Definition 2.2.1, that

$$q_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j' \tag{3.39}$$

i.e.,

$$[\mathbf{Q}] = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1' & \mathbf{e}_1 \cdot \mathbf{e}_2' & \mathbf{e}_1 \cdot \mathbf{e}_3' \\ \mathbf{e}_2 \cdot \mathbf{e}_1' & \mathbf{e}_2 \cdot \mathbf{e}_2' & \mathbf{e}_2 \cdot \mathbf{e}_3' \\ \mathbf{e}_3 \cdot \mathbf{e}_1' & \mathbf{e}_3 \cdot \mathbf{e}_2' & \mathbf{e}_3 \cdot \mathbf{e}_3' \end{bmatrix} \tag{3.40}$$

Note that all $\mathbf{e}_i$ and $\mathbf{e}_i'$ appearing in Eq. (3.40) must be represented in the same coordinate frame. Once $\mathbf{Q}$ is determined, computing the remaining screw parameters is straightforward. One can use, for example, Eq. (3.18) to determine the point of the screw axis that lies closest to the origin, which would thus allow one to compute the Plücker coordinates of the screw axis.

3.3   Rotation of a Rigid Body About a Fixed Point

In this section, the motion of a rigid body having a point fixed is analyzed. This motion is fully described by a rotation matrix $\mathbf{Q}$ that is proper orthogonal. Now, $\mathbf{Q}$ will be assumed to be a smooth function of time, and hence, the position vector of a point P in an original configuration, denoted here by $\mathbf{p}_0$, is mapped smoothly into a new vector $\mathbf{p}(t)$, namely,

$$\mathbf{p}(t) = \mathbf{Q}(t)\mathbf{p}_0 \tag{3.41}$$

The velocity of P is computed by differentiating both sides of Eq. (3.41) with respect to time, thus obtaining

$$\dot{\mathbf{p}}(t) = \dot{\mathbf{Q}}(t)\mathbf{p}_0 \tag{3.42}$$

which is not a very useful expression, because it requires knowledge of the original position of P. An alternative expression can be derived if Eq. (3.41) is solved for $\mathbf{p}_0$ and the expression thus resulting is substituted into Eq. (3.42), which yields

$$\dot{\mathbf{p}} = \dot{\mathbf{Q}}\mathbf{Q}^T \mathbf{p} \tag{3.43}$$

where the argument t has been dropped for the sake of simplicity, but one must keep in mind that all quantities are now time-varying. The product $\dot{\mathbf{Q}}\mathbf{Q}^T$ is known as the *angular-velocity matrix* of the rigid-body motion and is denoted by $\boldsymbol{\Omega}$, i.e.,

$$\boldsymbol{\Omega} \equiv \dot{\mathbf{Q}}\mathbf{Q}^T \tag{3.44}$$

As a consequence of the orthogonality of $\mathbf{Q}$, one has a basic result, namely,

Theorem 3.3.1. *The angular-velocity matrix is skew-symmetric.*

Proof. This follows directly from definition (3.44).

In order to derive the *angular-velocity vector* of a rigid-body motion, we recall the concept of *axial vector*, or simply *vector*, of a 3×3 matrix, introduced in Sect. 2.3.3. Thus, the angular-velocity vector $\boldsymbol{\omega}$ of the rigid-body motion under study is defined as the vector of $\boldsymbol{\Omega}$, i.e.,

$$\boldsymbol{\omega} \equiv \text{vect}(\boldsymbol{\Omega}) \tag{3.45}$$

and hence, Eq. (3.43) can be written as

$$\dot{\mathbf{p}} = \boldsymbol{\Omega}\mathbf{p} = \boldsymbol{\omega} \times \mathbf{p} \tag{3.46}$$

from which it is apparent that *the velocity of any point P of a body moving with a point O fixed is perpendicular to line OP.*

3.4 General Instantaneous Motion of a Rigid Body

If a rigid body now undergoes the most general motion, none of its points remains fixed, and the position vector of any of these, P, in a displaced configuration is given by Eq. (3.2). Let $\mathbf{a}_0$ and $\mathbf{p}_0$ denote the position vectors of points A and P of Sect. 3.2, respectively, in the reference configuration $\mathcal{C}_0$, $\mathbf{a}(t)$ and $\mathbf{p}(t)$ being the position vectors of the same points in the displaced configuration $\mathcal{C}$. Moreover, if $\mathbf{Q}(t)$ denotes the rotation matrix, then

$$\mathbf{p}(t) = \mathbf{a}(t) + \mathbf{Q}(t)(\mathbf{p}_0 - \mathbf{a}_0) \tag{3.47}$$

Now, the velocity of P is computed by differentiating both sides of Eq. (3.47) with respect to time, thus obtaining

$$\dot{\mathbf{p}}(t) = \dot{\mathbf{a}}(t) + \dot{\mathbf{Q}}(t)(\mathbf{p}_0 - \mathbf{a}_0) \tag{3.48}$$

which again, as expression (3.48), is not very useful, for it requires the values of the position vectors of A and P in the original configuration. However, if Eq. (3.47) is solved for $\mathbf{p}_0 - \mathbf{a}_0$ and the expression thus resulting is substituted into Eq. (3.48), we obtain

$$\dot{\mathbf{p}} = \dot{\mathbf{a}} + \boldsymbol{\Omega}(\mathbf{p} - \mathbf{a}) \tag{3.49}$$

or in terms of the angular-velocity vector,

$$\dot{\mathbf{p}} = \dot{\mathbf{a}} + \boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a}) \tag{3.50}$$

where the argument t has been dropped for brevity but is implicit, since all variables of the foregoing equation are now functions of time. Furthermore, from Eq. (3.50), it is apparent that

$$(\dot{\mathbf{p}} - \dot{\mathbf{a}}) \cdot (\mathbf{p} - \mathbf{a}) = 0 \tag{3.51}$$

which can be summarized as

Theorem 3.4.1. *The relative velocity of two points of the same rigid body is perpendicular to the line joining them.*

Moreover, similar to the outcome of Theorem 3.2.1, one now has an additional result that is derived upon dot-multiplying both sides of Eq. (3.50) by $\boldsymbol{\omega}$, namely,

$$\boldsymbol{\omega} \cdot \dot{\mathbf{p}} = \boldsymbol{\omega} \cdot \dot{\mathbf{a}}$$

and hence,

Corollary 3.4.1. *The projections of the velocities of all the points of a rigid body onto the angular-velocity vector are identical.*

Similar to the Mozzi–Chasles theorem, we have

Theorem 3.4.2. *Given a rigid body under general motion, a set of its points located on a line $\mathcal{L}'$ undergoes the identical minimum-magnitude velocity $\mathbf{v}_0$ parallel to the angular velocity.*

Definition 3.4.1. The line containing the points of a rigid body undergoing minimum-magnitude velocities is called the *instant screw axis* (ISA) of the body under the given motion.

3.4.1 The Instant Screw of a Rigid-Body Motion

From Theorem 3.4.2, the instantaneous motion of a body is equivalent to that of the nut of a screw of axis $\mathcal{L}'$, the ISA. As the body moves, the ISA changes, for which reason the motion of the body is called an *instantaneous screw*. Moreover, since $\mathbf{v}_0$ is parallel to $\boldsymbol{\omega}$, it can be expressed in the form

$$\mathbf{v}_0 = v_0 \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \tag{3.52}$$

where v_0 is a scalar quantity denoting the signed magnitude of $\mathbf{v}_0$ and bears the sign of $\mathbf{v}_0 \cdot \boldsymbol{\omega}$. Furthermore, the pitch p' of the instantaneous screw is defined as

$$p' \equiv \frac{v_0}{\|\boldsymbol{\omega}\|} \equiv \frac{\dot{\mathbf{p}} \cdot \boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} \quad \text{or} \quad p' \equiv \frac{2\pi v_0}{\|\boldsymbol{\omega}\|} \tag{3.53}$$

which thus bears units of m/rad or, correspondingly, of m/turn.

Again, the ISA $\mathcal{L}'$ can be specified uniquely through its Plücker coordinates, stored in the $\mathbf{p}_{\mathcal{L}'}$ array defined as

$$\mathbf{p}_{\mathcal{L}'} \equiv \begin{bmatrix} \mathbf{e}' \\ \mathbf{n}' \end{bmatrix} \tag{3.54}$$

where $\mathbf{e}'$ and $\mathbf{n}'$ are, respectively, the unit vector defining the direction of $\mathcal{L}'$ and its moment about the origin, i.e.,

$$\mathbf{e}' \equiv \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}, \quad \mathbf{n}' \equiv \mathbf{p} \times \mathbf{e}' \tag{3.55}$$

$\mathbf{p}$ being the position vector of any point of the ISA. Clearly, $\mathbf{e}'$ is defined uniquely but becomes trivial when the rigid body instantaneously undergoes a pure translation, i.e., a motion during which, instantaneously, $\boldsymbol{\omega} = \mathbf{0}$ and all body-points move with the same velocity $\mathbf{v}_0$. In this case, $\mathbf{e}'$ is defined as the unit vector parallel to $\mathbf{v}_0$. Thus, an instantaneous rigid-body motion is defined by a line $\mathcal{L}'$, a pitch p', and an amplitude $\|\boldsymbol{\omega}\|$. Such a motion is, then, fully determined by six independent parameters, namely, the four independent Plücker coordinates of $\mathcal{L}'$, its pitch, and its amplitude. A line supplied with a pitch is, in general, called a *screw*; a screw supplied with an amplitude representing the magnitude of an angular velocity provides the representation of an instantaneous rigid-body motion that is sometimes called the *twist*, an item that will be discussed in more detail below.

Hence, the instantaneous screw is fully defined by six independent real numbers. Moreover, such as in the case of the screw motion, the pitch of the instantaneous screw can attain values from $-\infty$ to $+\infty$.

The ISA can be alternatively described in terms of the position vector $\mathbf{p}_0'$ of its point lying closest to the origin. Expressions for $\mathbf{p}_0'$ in terms of the position and the velocity of an arbitrary body-point and the angular velocity are derived below. To this end, we decompose $\dot{\mathbf{p}}$ into two orthogonal components, $\dot{\mathbf{p}}_\parallel$ and $\dot{\mathbf{p}}_\perp$, along and transverse to the angular-velocity vector, respectively—see Eqs. (2.6a) and (2.6b). Hence, $\dot{\mathbf{a}}$ is first decomposed into two such orthogonal components, $\dot{\mathbf{a}}_\parallel$ and $\dot{\mathbf{a}}_\perp$, the former being parallel, the latter normal to the ISA, i.e.,

$$\dot{\mathbf{a}} \equiv \dot{\mathbf{a}}_\parallel + \dot{\mathbf{a}}_\perp \tag{3.56}$$

These orthogonal components are given as

$$\dot{\mathbf{a}}_{\parallel} \equiv \dot{\mathbf{a}} \cdot \boldsymbol{\omega} \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} \equiv \frac{\boldsymbol{\omega}\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|^2} \dot{\mathbf{a}}, \quad \dot{\mathbf{a}}_{\perp} \equiv \left(\mathbf{1} - \frac{\boldsymbol{\omega}\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|^2}\right) \dot{\mathbf{a}} \equiv -\frac{1}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\Omega}^2 \dot{\mathbf{a}} \qquad (3.57)$$

In the derivation of Eq. (3.57) we have used the identity introduced in Eq. (2.40), which means, in our case:

$$\boldsymbol{\Omega}^2 \equiv \boldsymbol{\omega}\boldsymbol{\omega}^T - \|\boldsymbol{\omega}\|^2 \mathbf{1} \qquad (3.58)$$

Upon substitution of Eqs. (3.57) into Eq. (3.50), we obtain

$$\dot{\mathbf{p}} = \underbrace{\frac{\boldsymbol{\omega}\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|^2} \dot{\mathbf{a}}}_{\dot{\mathbf{p}}_{\parallel}} \underbrace{- \frac{1}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\Omega}^2 \dot{\mathbf{a}} + \boldsymbol{\Omega}(\mathbf{p} - \mathbf{a})}_{\dot{\mathbf{p}}_{\perp}} \qquad (3.59)$$

Of the three components of $\dot{\mathbf{p}}$, the first, henceforth referred to as its *axial component*, is parallel, the last two being normal to $\boldsymbol{\omega}$. The sum of the last two components is referred to as the *normal component* of $\dot{\mathbf{p}}$. From Eq. (3.59) it is apparent that the axial component is independent of $\mathbf{p}$, while the normal component is a linear function of $\mathbf{p}$. An obvious question now arises: *For an arbitrary motion, is it possible to find a certain point of position vector* $\mathbf{p}$ *whose velocity normal component vanishes?* The vanishing of the normal component obviously implies the minimization of the magnitude of $\dot{\mathbf{p}}$. The condition under which this happens can now be stated as

$$\dot{\mathbf{p}}_{\perp} = \mathbf{0}$$

or

$$\boldsymbol{\Omega}(\mathbf{p} - \mathbf{a}) - \frac{1}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\Omega}^2 \dot{\mathbf{a}} = \mathbf{0} \qquad (3.60)$$

which can be further expressed as a vector equation linear in $\mathbf{p}$, namely,

$$\boldsymbol{\Omega}\mathbf{p} = \boldsymbol{\Omega}\left(\mathbf{a} + \frac{1}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\Omega}\dot{\mathbf{a}}\right) \qquad (3.61)$$

or

$$\boldsymbol{\Omega}(\mathbf{p} - \mathbf{r}) = \mathbf{0} \qquad (3.62a)$$

with $\mathbf{r}$ defined as

$$\mathbf{r} \equiv \mathbf{a} + \frac{1}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\Omega}\dot{\mathbf{a}} \qquad (3.62b)$$

and hence, a possible solution of the foregoing problem is

$$\mathbf{p} = \mathbf{r} = \mathbf{a} + \frac{1}{\|\boldsymbol{\omega}\|^2} \boldsymbol{\Omega} \dot{\mathbf{a}} \tag{3.63}$$

However, this solution is not unique, for Eq. (3.62a) does not require that $\mathbf{p} - \mathbf{r}$ be zero, only that this difference lie in the null space of $\boldsymbol{\Omega}$, i.e., that $\mathbf{p} - \mathbf{r}$ be linearly dependent with $\boldsymbol{\omega}$. In other words, if a vector $\alpha\boldsymbol{\omega}$ is added to $\mathbf{p}$ as given in Eq. (3.63), then the sum also satisfies Eq. (3.61). It is then apparent that Eq. (3.61) does not determine a single point whose normal velocity component vanishes, but a set of points lying on the ISA, and thus, other solutions are possible. For example, we can find the point of the ISA lying closest to the origin. To this end, let $\mathbf{p}'_0$ be the position vector of that point. This vector is obviously perpendicular to $\boldsymbol{\omega}$, i.e.,

$$\boldsymbol{\omega}^T \mathbf{p}'_0 = 0 \tag{3.64}$$

Next, Eq. (3.61) is rewritten for $\mathbf{p}'_0$, and Eq. (3.64) is adjoined to it, thereby deriving an expanded linear system of equations, namely,

$$\mathbf{A}\mathbf{p}'_0 = \mathbf{b} \tag{3.65}$$

where $\mathbf{A}$ is a 4×3 matrix and $\mathbf{b}$ is a four-dimensional vector, both being given below:

$$\mathbf{A} \equiv \begin{bmatrix} \boldsymbol{\Omega} \\ \boldsymbol{\omega}^T \end{bmatrix}, \quad \mathbf{b} \equiv \begin{bmatrix} \boldsymbol{\Omega}\mathbf{a} + (1/\|\boldsymbol{\omega}\|^2)\boldsymbol{\Omega}^2\dot{\mathbf{a}} \\ 0 \end{bmatrix} \tag{3.66}$$

This system is of the same nature as that appearing in Eq. (3.10), and hence, it can be solved for $\mathbf{p}'_0$ following the same procedure. Thus, both sides of Eq. (3.65) are multiplied from the left by $\mathbf{A}^T$, thereby obtaining

$$\mathbf{A}^T\mathbf{A}\mathbf{p}'_0 = \mathbf{A}^T\mathbf{b} \tag{3.67}$$

where

$$\mathbf{A}^T\mathbf{A} = \boldsymbol{\Omega}^T\boldsymbol{\Omega} + \boldsymbol{\omega}\boldsymbol{\omega}^T = -\boldsymbol{\Omega}^2 + \boldsymbol{\omega}\boldsymbol{\omega}^T \tag{3.68}$$

Moreover, from Eq. (3.58), the rightmost side of the foregoing relation becomes $\|\boldsymbol{\omega}\|^2\mathbf{1}$, and hence, the matrix coefficient of the left-hand side of Eq. (3.67) and the right-hand side of the same equation reduce, respectively, to

$$\mathbf{A}^T\mathbf{A} = \|\boldsymbol{\omega}\|^2\mathbf{1}, \quad \mathbf{A}^T\mathbf{b} = \boldsymbol{\Omega}(\dot{\mathbf{a}} - \boldsymbol{\Omega}\mathbf{a}) \tag{3.69}$$

Upon substitution of Eq. (3.69) into Eq. (3.67) and further solving for $\mathbf{p}'_0$, the desired expression is derived:

$$\mathbf{p}'_0 = \frac{\boldsymbol{\Omega}(\dot{\mathbf{a}} - \boldsymbol{\Omega}\mathbf{a})}{\|\boldsymbol{\omega}\|^2} \equiv \frac{\boldsymbol{\omega} \times (\dot{\mathbf{a}} - \boldsymbol{\omega} \times \mathbf{a})}{\|\boldsymbol{\omega}\|^2} \tag{3.70}$$

Thus, the instantaneous screw is fully defined by an alternative set of six independent scalars, namely, the three components of its angular velocity $\boldsymbol{\omega}$ and the three components of the velocity of an arbitrary body point A, denoted by $\dot{\mathbf{a}}$. As in the case of the screw motion, we can also represent the instantaneous screw by a line and two additional parameters, as we explain below.

3.4.2 The Twist of a Rigid Body

A line, as we saw earlier, is fully defined by its six-dimensional Plücker array, which contains only four independent components. Now, if a pitch p is added as a fifth feature to the line or, correspondingly, to its Plücker array, we obtain a screw $\mathbf{s}$, namely,

$$\mathbf{s} \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{p} \times \mathbf{e} + p\mathbf{e} \end{bmatrix} \tag{3.71}$$

An *amplitude* is any scalar A multiplying the foregoing screw. The amplitude produces a twist or a *wrench*, to be discussed presently, depending on its units. The twist or the wrench thus defined can be regarded as an eight-parameter array. These eight parameters, of which only six are independent, are the amplitude, the pitch, and the six Plücker coordinates of the associated line. Clearly, a twist or a wrench is defined completely by six independent real numbers. More generally, a twist can be regarded as a six-dimensional array defining completely the velocity field of a rigid body; it comprises the three components of the angular velocity and the three components of the velocity of any of the points of the body.

Below we elaborate on the foregoing concepts. Upon multiplication of the screw appearing in Eq. (3.71) by the amplitude A representing the magnitude of an angular velocity, we obtain a twist $\mathbf{t}$, namely,

$$\mathbf{t} \equiv \begin{bmatrix} A\mathbf{e} \\ \mathbf{p} \times (A\mathbf{e}) + p(A\mathbf{e}) \end{bmatrix}$$

where the product $A\mathbf{e}$ can be readily identified as the angular velocity $\boldsymbol{\omega}$ parallel to vector $\mathbf{e}$, of magnitude $|A|$. Moreover, the lower part of $\mathbf{t}$ can be readily identified with the velocity of a point of a rigid body. Indeed, if we regard the line $\mathcal{L}$ and point O as sets of points of a rigid body $\mathcal{B}$ moving with an angular velocity $\boldsymbol{\omega}$ and such that point P moves with a velocity $p\boldsymbol{\omega}$ parallel to the angular velocity, then the lower vector of $\mathbf{t}$, denoted by $\mathbf{v}$, represents the velocity of point O, i.e.,

$$\mathbf{v} = -\boldsymbol{\omega} \times \mathbf{p} + p\boldsymbol{\omega}$$

We can thus express the twist $\mathbf{t}$ as

$$\mathbf{t} \equiv \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{bmatrix} \tag{3.72}$$

If the pitch is zero, the twist is a pure rotation; if infinite, the twist is a pure translation, in which case the twist is

$$\mathbf{t} \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix} = \|\mathbf{v}\| \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \end{bmatrix} \tag{3.73}$$

Then, the *screw of infinite pitch* $\mathbf{s}_\infty$ is defined as the six-dimensional array appearing in the above equation, namely,

$$\mathbf{s}_\infty \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \end{bmatrix} \tag{3.74}$$

Note that this screw array is identical to the Plücker array of a line at infinity lying in a plane of unit normal $\mathbf{e}$, as displayed in Eq. (3.33).

The twist array, as defined in Eq. (3.72), with $\boldsymbol{\omega}$ on top, represents the *ray coordinates* of the twist. An exchange of the order of the two Cartesian vectors of this array, in turn, gives rise to the *axis coordinates* of the twist.

The foregoing twist was also termed *motor* by Everett (1875). As Phillips (1990) points out, the word motor is an abbreviation of *mo*ment and vec*tor*. An extensive introduction into motor algebra was published by von Mises (1924), a work that is now available in English (von Mises 1996). Roth (1984), in turn, provided a summary of these concepts, as applicable to robotics. The foregoing array goes also by other names, such as the German *Kinemate* or the French *torseur cinématique*.

The relationships between the angular-velocity vector and the time derivatives of the invariants of the associated rotation are linear. Indeed, let the three sets of four invariants of rotation, namely, the natural invariants, the linear invariants, and the Euler–Rodrigues parameters be grouped in the four-dimensional arrays $\boldsymbol{\nu}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\eta}$, respectively, i.e.,

$$\boldsymbol{\nu} \equiv \begin{bmatrix} \mathbf{e} \\ \phi \end{bmatrix}, \quad \boldsymbol{\lambda} \equiv \begin{bmatrix} (\sin \phi)\mathbf{e} \\ \cos \phi \end{bmatrix}, \quad \boldsymbol{\eta} \equiv \begin{bmatrix} [\sin(\phi/2)]\mathbf{e} \\ \cos(\phi/2) \end{bmatrix} \tag{3.75}$$

We then have the linear relations derived in full detail elsewhere (Angeles 1988), and outlined in Appendix A for quick reference, namely,

$$\dot{\boldsymbol{\nu}} = \mathbf{N}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\lambda}} = \mathbf{L}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\eta}} = \mathbf{H}\boldsymbol{\omega} \tag{3.76a}$$

with $\mathbf{N}$, $\mathbf{L}$, and $\mathbf{H}$ defined as

$$\mathbf{N} \equiv \begin{bmatrix} [\sin\phi/(2(1 - \cos\phi))](\mathbf{1} - \mathbf{e}\mathbf{e}^T) - (1/2)\mathbf{E} \\ \mathbf{e}^T \end{bmatrix}, \tag{3.76b}$$

$$\mathbf{L} \equiv \begin{bmatrix} (1/2)[\mathrm{tr}(\mathbf{Q})\mathbf{1} - \mathbf{Q}] \\ -(\sin\phi)\mathbf{e}^T \end{bmatrix}, \tag{3.76c}$$

$$\mathbf{H} \equiv \frac{1}{2} \begin{bmatrix} \cos(\phi/2)\mathbf{1} - \sin(\phi/2)\mathbf{E} \\ -\sin(\phi/2)\mathbf{e}^T \end{bmatrix} \tag{3.76d}$$

where, it is recalled, $\mathrm{tr}(\cdot)$ denotes the trace of its square matrix argument $(\cdot)$, i.e., the sum of the diagonal entries of that matrix.

The inverse relations of those shown in Eqs. (3.76a) are to be derived by resorting to the approach introduced when solving Eq. (3.65) for $\mathbf{p}_0'$, thereby obtaining

$$\boldsymbol{\omega} = \tilde{\mathbf{N}}\dot{\boldsymbol{\nu}} = \tilde{\mathbf{L}}\dot{\boldsymbol{\lambda}} = \tilde{\mathbf{H}}\dot{\boldsymbol{\eta}} \tag{3.77a}$$

the 3×4 matrices $\tilde{\mathbf{N}}$, $\tilde{\mathbf{L}}$, and $\tilde{\mathbf{H}}$ being defined below:

$$\tilde{\mathbf{N}} \equiv \left[(\sin\phi)\mathbf{1} + (1 - \cos\phi)\mathbf{E} \ \mathbf{e} \right], \tag{3.77b}$$

$$\tilde{\mathbf{L}} \equiv \left[\mathbf{1} + [(\sin\phi)/(1 + \cos\phi)]\mathbf{E} \ -[(\sin\phi)/(1 + \cos\phi)]\mathbf{e} \right], \tag{3.77c}$$

$$\tilde{\mathbf{H}} \equiv 2\left[[\cos(\phi/2)]\mathbf{1} + [\sin(\phi/2)]\mathbf{E} \ -[\sin(\phi/2)]\mathbf{e} \right] \tag{3.77d}$$

Caveat *The angular velocity vector is **not** a time-derivative, i.e., no Cartesian vector exists whose time-derivative is the angular-velocity vector.*

However, matrices $\mathbf{N}$, $\mathbf{L}$, and $\mathbf{H}$ of Eqs. (3.76b–d) can be regarded as *integration factors* that yield time-derivatives.

Now we can write the relationship between the twist and the time-rate of change of the seven-dimensional pose array $\mathbf{s}$, namely,

$$\dot{\mathbf{s}} = \mathbf{T}\mathbf{t} \tag{3.78}$$

where

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{F} & \mathbf{O}_{43} \\ \mathbf{O} & \mathbf{1} \end{bmatrix} \tag{3.79}$$

in which $\mathbf{O}$ and $\mathbf{O}_{43}$ are the 3×3 and the 4×3 zero matrices, while $\mathbf{1}$ is the 3×3 identity matrix and $\mathbf{F}$ is, correspondingly, $\mathbf{N}$, $\mathbf{L}$, or $\mathbf{H}$, depending upon the invariant representation chosen for the rotation. The inverse relationship of Eq. (3.78) takes the form

$$\mathbf{t} = \mathbf{S}\dot{\mathbf{s}} \tag{3.80a}$$

where

$$\mathbf{S} \equiv \begin{bmatrix} \tilde{\mathbf{F}} & \mathbf{O} \\ \mathbf{O}_{34} & \mathbf{1} \end{bmatrix} \tag{3.80b}$$

in which $\mathbf{O}_{34}$ is the 3×4 zero matrix. Moreover, $\tilde{\mathbf{F}}$ is one of $\tilde{\mathbf{N}}$, $\tilde{\mathbf{L}}$, or $\tilde{\mathbf{H}}$, depending on the rotation representation adopted, namely, the natural invariants, the linear invariants, or the Euler–Rodrigues parameters, respectively.

A formula that relates the twist of the same rigid body at two different points is now derived. Let A and P be two arbitrary points of a rigid body. The twist at each of these points is defined as

$$\mathbf{t}_A = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_A \end{bmatrix}, \quad \mathbf{t}_P = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_P \end{bmatrix} \tag{3.81}$$

Moreover, Eq. (3.50) can be rewritten as

$$\mathbf{v}_P = \mathbf{v}_A + (\mathbf{a} - \mathbf{p}) \times \boldsymbol{\omega} \tag{3.82}$$

Combining Eq. (3.81) with Eq. (3.82) yields

$$\mathbf{t}_P = \mathbf{U}\mathbf{t}_A \tag{3.83a}$$

where

$$\mathbf{U} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{A} - \mathbf{P} & \mathbf{1} \end{bmatrix} \tag{3.83b}$$

with the 6×6 matrix $\mathbf{U}$ defined as in Eq. (3.29b), while $\mathbf{A}$ and $\mathbf{P}$ denote the cross-product matrices of vectors $\mathbf{a}$ and $\mathbf{p}$, respectively. Thus, Eqs. (3.83a and b) can be fairly called the *twist-transfer formulas.*

3.5 Acceleration Analysis of Rigid-Body Motions

Upon differentiation of both sides of Eq. (3.49) with respect to time, one obtains

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + \dot{\boldsymbol{\Omega}}(\mathbf{p} - \mathbf{a}) + \boldsymbol{\Omega}(\dot{\mathbf{p}} - \dot{\mathbf{a}}) \tag{3.84}$$

Now, Eq. (3.49) is solved for $\dot{\mathbf{p}} - \dot{\mathbf{a}}$, and the expression thus resulting is substituted into Eq. (3.84), thereby obtaining

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + (\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}^2)(\mathbf{p} - \mathbf{a}) \tag{3.85}$$

where the matrix sum in parentheses is termed the *angular-acceleration matrix* of the rigid-body motion and is represented by $\mathbf{W}$, i.e.,

$$\mathbf{W} \equiv \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}^2 \tag{3.86}$$

Apparently, the first term of the right-hand side of Eq. (3.86) is skew-symmetric, whereas the second one is symmetric. Thus,

$$\text{vect}(\mathbf{W}) = \text{vect}(\dot{\boldsymbol{\Omega}}) = \dot{\boldsymbol{\omega}} \tag{3.87}$$

$\dot{\boldsymbol{\omega}}$ being termed the *angular-acceleration vector* of the rigid-body motion. We have now an interesting result, namely,

$$\begin{aligned} \text{tr}(\mathbf{W}) = \text{tr}(\boldsymbol{\Omega}^2) &= \text{tr}(-\|\boldsymbol{\omega}\|^2 \mathbf{1} + \boldsymbol{\omega}\boldsymbol{\omega}^T) \\ &= -\|\boldsymbol{\omega}\|^2 \text{tr}(\mathbf{1}) + \boldsymbol{\omega} \cdot \boldsymbol{\omega} = -2\|\boldsymbol{\omega}\|^2 \end{aligned} \tag{3.88}$$

Moreover, Eq. (3.85) can be written as

$$\ddot{\mathbf{p}} = \ddot{\mathbf{a}} + \dot{\boldsymbol{\omega}} \times (\mathbf{p} - \mathbf{a}) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{p} - \mathbf{a})] \tag{3.89}$$

On the other hand, the time derivative of $\mathbf{t}$, henceforth referred to as the *twist rate*, is displayed below:

$$\dot{\mathbf{t}} \equiv \begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \dot{\mathbf{v}} \end{bmatrix} \tag{3.90}$$

in which $\dot{\mathbf{v}}$ is the acceleration of a point of the body. The relationship between the twist rate and the second time derivative of the screw is derived by differentiation of both sides of Eq. (3.78), which yields

$$\ddot{\mathbf{s}} = \mathbf{T}\dot{\mathbf{t}} + \dot{\mathbf{T}}\mathbf{t} \tag{3.91}$$

where

$$\dot{\mathbf{T}} \equiv \begin{bmatrix} \dot{\mathbf{F}} & \mathbf{O}_{43} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.92}$$

and $\mathbf{F}$ is one of $\mathbf{N}$, $\mathbf{L}$, or $\mathbf{H}$, accordingly. The inverse relationship of Eq. (3.91) is derived by differentiating both sides of Eq. (3.80a) with respect to time, which yields

$$\dot{\mathbf{t}} = \mathbf{S}\ddot{\mathbf{s}} + \dot{\mathbf{S}}\dot{\mathbf{s}} \tag{3.93}$$

where

$$\dot{\mathbf{S}} = \begin{bmatrix} \dot{\tilde{\mathbf{F}}} & \mathbf{O} \\ \mathbf{O}_{34} & \mathbf{O} \end{bmatrix} \tag{3.94}$$

with $\mathbf{O}$ and $\mathbf{O}_{34}$ already defined in Eq. (3.80b) as the 3×3 and the 3×4 zero matrices, respectively, while $\dot{\tilde{\mathbf{F}}}$ is one of $\dot{\tilde{\mathbf{N}}}$, $\dot{\tilde{\mathbf{L}}}$, or $\dot{\tilde{\mathbf{H}}}$, according with the type of rotation representation at hand.

Before we embark in differentiating the foregoing matrices, we introduce a few definitions: Let

$$\boldsymbol{\lambda} \equiv \begin{bmatrix} \mathbf{u} \\ u_0 \end{bmatrix}, \quad \boldsymbol{\eta} \equiv \begin{bmatrix} \mathbf{r} \\ r_0 \end{bmatrix} \tag{3.95a}$$

i.e.,

$$\mathbf{u} \equiv \sin\phi\,\mathbf{e}, \quad u_0 \equiv \cos\phi, \quad \mathbf{r} \equiv \sin\left(\frac{\phi}{2}\right)\mathbf{e}, \quad r_0 \equiv \cos\left(\frac{\phi}{2}\right) \tag{3.95b}$$

Thus, the time derivatives sought take on the forms

$$\dot{\mathbf{N}} = \frac{1}{4(1 - \cos\phi)} \begin{bmatrix} \mathbf{B} \\ \dot{\mathbf{e}} \end{bmatrix} \tag{3.96a}$$

$$\dot{\mathbf{L}} = \begin{bmatrix} (1/2)[\mathbf{1}\mathrm{tr}(\dot{\mathbf{Q}}) - \dot{\mathbf{Q}}] \\ -(1/2)\boldsymbol{\omega}^T[\mathbf{1}\mathrm{tr}(\mathbf{Q}) - \mathbf{Q}^T]] \end{bmatrix}$$

$$= \begin{bmatrix} -(\boldsymbol{\omega} \cdot \mathbf{u})\mathbf{1} - (1/2)\boldsymbol{\Omega}\mathbf{Q} \\ -(1/2)\boldsymbol{\omega}^T[\mathbf{1}\mathrm{tr}(\mathbf{Q}) - \mathbf{Q}^T]] \end{bmatrix} \tag{3.96b}$$

$$\dot{\mathbf{H}} = \frac{1}{2} \begin{bmatrix} \dot{r}_0 \mathbf{1} - \dot{\mathbf{R}} \\ -\dot{\mathbf{r}}^T \end{bmatrix} \tag{3.96c}$$

where we have used the identities below, which are derived in Appendix A.

$$\mathrm{tr}(\dot{\mathbf{Q}}) \equiv \mathrm{tr}(\boldsymbol{\Omega}\mathbf{Q}) \equiv -2\boldsymbol{\omega}^T\mathbf{u} \tag{3.96d}$$

Furthermore, $\mathbf{R}$ denotes the cross-product matrix of $\mathbf{r}$, and $\mathbf{B}$ is defined as

$$\mathbf{B} \equiv -2(\mathbf{e} \cdot \boldsymbol{\omega})\mathbf{1} + 2(3 - \cos\phi)(\mathbf{e} \cdot \boldsymbol{\omega})\mathbf{e}\mathbf{e}^T - 2(1 + \sin\phi)\boldsymbol{\omega}\mathbf{e}^T$$
$$-(2\cos\phi + \sin\phi)\mathbf{e}\boldsymbol{\omega}^T - (\sin\phi)[\boldsymbol{\Omega} - (\mathbf{e} \cdot \boldsymbol{\omega})\mathbf{E}] \tag{3.96e}$$

Moreover,

$$\dot{\mathbf{N}} = [\dot{\phi}(\cos\phi)\mathbf{1} + \dot{\phi}(\sin\theta)\mathbf{E} \ \dot{\mathbf{e}}] \tag{3.97a}$$

$$\dot{\mathbf{L}} = [\mathbf{V}/D \ \dot{\mathbf{u}}] \tag{3.97b}$$

$$\dot{\mathbf{H}} = [\dot{r}_0\mathbf{1} + \dot{\mathbf{R}} \ -\dot{\mathbf{r}}] \tag{3.97c}$$

where $\mathbf{V}$ and D are defined below:

$$\mathbf{V} \equiv \dot{\mathbf{U}} - (\dot{\mathbf{u}}\mathbf{u}^T + \mathbf{u}\dot{\mathbf{u}}^T) - \frac{\dot{u}_0}{D}(\mathbf{U} - \mathbf{u}\mathbf{u}^T) \tag{3.97d}$$

$$D \equiv 1 + u_0 \tag{3.97e}$$

with $\mathbf{U}$ denoting, now, the cross-product matrix of $\mathbf{u}$.

3.6   Rigid-Body Motion Referred to Moving Coordinate Axes

Although in kinematics no "preferred" coordinate system exists, in dynamics the governing equations of rigid-body motions are valid only in *inertial frames*, the concept to be defined in Sect. 3.8. Thus, it is important to refer vectors and matrices to inertial frames, but sometimes it is not possible to do so directly. For instance, a space vessel can be supplied with instruments to measure the velocity and the acceleration of a satellite drifting in space, but the measurements taken from the space vessel will be referred to a coordinate frame fixed to it, which is not inertial. If the motion of the vessel with respect to an inertial coordinate frame is recorded, e.g., from an Earth-based station, then the acceleration of the satellite with respect to an inertial frame can be computed using the foregoing information. How to do this is the subject of this section.

In the realm of kinematics, it is not necessary to distinguish between inertial and noninertial coordinate frames, and hence, it will suffice to call the coordinate systems involved *fixed* and *moving*. Thus, consider the fixed coordinate frame X, Y, Z, which will be labeled $\mathcal{F}$, and the moving coordinate frame $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, which will be labeled $\mathcal{M}$, both being depicted in Fig. 3.7. Moreover, let $\mathbf{Q}$ be the rotation matrix taking frame $\mathcal{F}$ into the orientation of $\mathcal{M}$, and $\mathbf{o}$ the position vector of the origin of $\mathcal{M}$ from the origin of $\mathcal{F}$. Further, let $\mathbf{p}$ be the position vector of point P from the origin of $\mathcal{F}$ and $\boldsymbol{\rho}$ the position vector of the same point from the origin of $\mathcal{M}$. From Fig. 3.7 one has

$$[\mathbf{p}]_{\mathcal{F}} = [\mathbf{o}]_{\mathcal{F}} + [\boldsymbol{\rho}]_{\mathcal{F}} \tag{3.98}$$

where it will be assumed that $\boldsymbol{\rho}$ is not available in frame $\mathcal{F}$, but in $\mathcal{M}$. Hence,

$$[\boldsymbol{\rho}]_{\mathcal{F}} = [\mathbf{Q}]_{\mathcal{F}}[\boldsymbol{\rho}]_{\mathcal{M}} \tag{3.99}$$
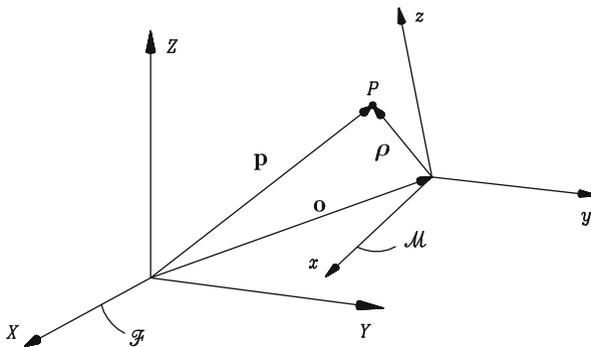

Fig. 3.7   Fixed and moving coordinate frames

Substitution of Eq. (3.99) into Eq. (3.98) yields

$$[\mathbf{p}]_{\mathcal{F}} = [\mathbf{o}]_{\mathcal{F}} + [\mathbf{Q}]_{\mathcal{F}}[\boldsymbol{\rho}]_{\mathcal{M}} \tag{3.100}$$

Now, in order to compute the velocity of P, both sides of Eq. (3.100) are differentiated with respect to time, which leads to

$$[\dot{\mathbf{p}}]_{\mathcal{F}} = [\dot{\mathbf{o}}]_{\mathcal{F}} + [\dot{\mathbf{Q}}]_{\mathcal{F}}[\boldsymbol{\rho}]_{\mathcal{M}} + [\mathbf{Q}]_{\mathcal{F}}[\dot{\boldsymbol{\rho}}]_{\mathcal{M}} \tag{3.101}$$

Furthermore, from the definition of $\boldsymbol{\Omega}$, Eq. (3.44), we have

$$[\dot{\mathbf{Q}}]_{\mathcal{F}} = [\boldsymbol{\Omega}]_{\mathcal{F}}[\mathbf{Q}]_{\mathcal{F}} \tag{3.102}$$

Upon substitution of the foregoing relation into Eq. (3.101), we obtain

$$[\dot{\mathbf{p}}]_{\mathcal{F}} = [\dot{\mathbf{o}}]_{\mathcal{F}} + [\boldsymbol{\Omega}]_{\mathcal{F}}[\mathbf{Q}]_{\mathcal{F}}[\boldsymbol{\rho}]_{\mathcal{M}} + [\mathbf{Q}]_{\mathcal{F}}[\dot{\boldsymbol{\rho}}]_{\mathcal{M}} \tag{3.103}$$

which is an expression for the velocity of P in $\mathcal{F}$ in terms of the velocity of P in $\mathcal{M}$ and the twist of $\mathcal{M}$ with respect to $\mathcal{F}$. Next, the acceleration of P in $\mathcal{F}$ is derived by differentiation of both sides of Eq. (3.103) with respect to time, which yields

$$[\ddot{\mathbf{p}}]_{\mathcal{F}} = [\ddot{\mathbf{o}}]_{\mathcal{F}} + [\dot{\boldsymbol{\Omega}}]_{\mathcal{F}}[\mathbf{Q}]_{\mathcal{F}}[\boldsymbol{\rho}]_{\mathcal{M}} + [\boldsymbol{\Omega}]_{\mathcal{F}}[\dot{\mathbf{Q}}]_{\mathcal{F}}[\boldsymbol{\rho}]_{\mathcal{M}}$$
$$+ [\boldsymbol{\Omega}]_{\mathcal{F}}[\mathbf{Q}]_{\mathcal{F}}[\dot{\boldsymbol{\rho}}]_{\mathcal{M}} + [\dot{\mathbf{Q}}]_{\mathcal{F}}[\dot{\boldsymbol{\rho}}]_{\mathcal{M}} + [\mathbf{Q}]_{\mathcal{F}}[\ddot{\boldsymbol{\rho}}]_{\mathcal{M}} \tag{3.104}$$

Further, upon substitution of identity (3.102) into Eq. (3.104), we obtain

$$[\ddot{\mathbf{p}}]_{\mathcal{F}} = [\ddot{\mathbf{o}}]_{\mathcal{F}} + ([\dot{\boldsymbol{\Omega}}]_{\mathcal{F}} + [\boldsymbol{\Omega}^2]_{\mathcal{F}})[\mathbf{Q}]_{\mathcal{F}}[\boldsymbol{\rho}]_{\mathcal{M}}$$
$$+ 2[\boldsymbol{\Omega}]_{\mathcal{F}}[\mathbf{Q}]_{\mathcal{F}}[\dot{\boldsymbol{\rho}}]_{\mathcal{M}} + [\mathbf{Q}]_{\mathcal{F}}[\ddot{\boldsymbol{\rho}}]_{\mathcal{M}} \tag{3.105}$$

Moreover, from the results of Sect. 3.5, it is apparent that the first two terms of the right-hand side of Eq. (3.105) represent the acceleration of P as a point of $\mathcal{M}$, whereas the fourth term is the acceleration of P measured from $\mathcal{M}$. The third term is known as the *Coriolis acceleration*, for it was first pointed out by the French mathematician Gustave Gaspard Coriolis (1835).

3.7 Static Analysis of Rigid Bodies

Germane to the velocity analysis of rigid bodies is their force-and-moment analysis. In fact, striking similarities exist between the velocity relations associated with rigid bodies and the forces and moments acting on them. From elementary statics it is known that the resultant of all external actions, i.e., forces and moments, exerted on a rigid body can be reduced to a force $\mathbf{f}$ acting at a point, say A, and a moment
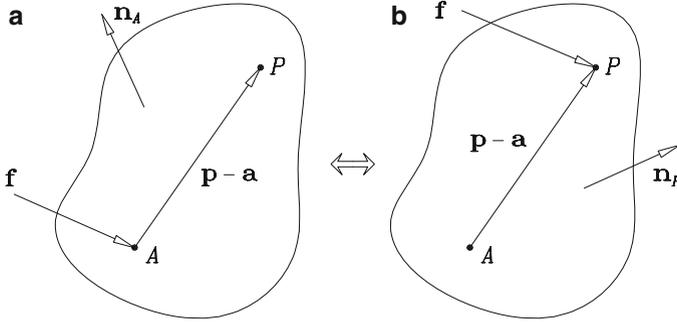
Fig. 3.8   Equivalent systems of force and moment acting on a rigid body

$\mathbf{n}_A$. Alternatively, the same force $\mathbf{f}$ can be defined as acting at an arbitrary point P of the body, as depicted in Fig. 3.8, but then the resultant moment $\mathbf{n}_P$ changes correspondingly.

In order to establish a relationship between $\mathbf{n}_A$ and $\mathbf{n}_P$, the moment of the first system of force and moment with respect to point P is equated to the moment about the same point of the second system, thus obtaining

$$\mathbf{n}_P = \mathbf{n}_A + (\mathbf{a} - \mathbf{p}) \times \mathbf{f} \tag{3.106}$$

which can be rewritten as

$$\mathbf{n}_P = \mathbf{n}_A + \mathbf{f} \times (\mathbf{p} - \mathbf{a}) \tag{3.107}$$

whence the analogy with Eq. (3.50) is apparent. Indeed, $\mathbf{n}_P$ and $\mathbf{n}_A$ of Eq. (3.107) play the role of the velocities of P and A, $\dot{\mathbf{p}}$ and $\dot{\mathbf{a}}$, respectively, whereas $\mathbf{f}$ of Eq. (3.107) plays the role of $\boldsymbol{\omega}$ of Eq. (3.50). Thus, similar to Theorem 3.4.2, one has

Theorem 3.7.1. *For a given system of forces and moments acting on a rigid body, if the resultant force is applied at any point of a particular line $\mathcal{L}''$, then the resultant moment is of minimum magnitude. Moreover, that minimum-magnitude moment is parallel to the resultant force.*

Hence, the resultant of the system of forces and moments is equivalent to a force $\mathbf{f}$ acting at a point of $\mathcal{L}''$ and a moment $\mathbf{n}$, with both $\mathbf{f}$ and $\mathbf{n}$ parallel to $\mathcal{L}''$. Paraphrasing the definition of the ISA, one defines line $\mathcal{L}''$ as the *axis of the wrench* acting on the body. Let $\mathbf{n}_0$ be the minimum-magnitude moment. Paraphrasing Eq. (3.52) in turn, $\mathbf{n}_0$ can be expressed as

$$\mathbf{n}_0 = n_0 \frac{\mathbf{f}}{\|\mathbf{f}\|}, \quad n_0 \equiv \frac{\mathbf{n}_P \cdot \mathbf{f}}{\|\mathbf{f}\|} \tag{3.108}$$

Moreover, the *pitch of the wrench*, p'', is defined as

$$p'' \equiv \frac{n_0}{\|\mathbf{f}\|} = \frac{\mathbf{n}_P \cdot \mathbf{f}}{\|\mathbf{f}\|^2} \quad \text{or} \quad p'' = \frac{2\pi \mathbf{n}_P \cdot \mathbf{f}}{\|\mathbf{f}\|^2} \tag{3.109}$$

which, again, has units of m/rad or, correspondingly, of m/turn. Of course, the wrench axis can be defined by its Plücker array, $\mathbf{p}_{\mathcal{L}''}$, i.e.,

$$\mathbf{p}_{\mathcal{L}''} \equiv \begin{bmatrix} \mathbf{e}'' \\ \mathbf{n}'' \end{bmatrix}, \quad \mathbf{e}'' = \frac{\mathbf{f}}{\|\mathbf{f}\|}, \quad \mathbf{n}'' = \mathbf{p} \times \mathbf{e}'' \tag{3.110}$$

where $\mathbf{e}''$ is the unit vector parallel to $\mathcal{L}''$, $\mathbf{n}''$ is the moment of $\mathcal{L}''$ about the origin, and $\mathbf{p}$ is the position vector of any point on $\mathcal{L}''$.

The wrench axis is fully specified, then, by the direction of $\mathbf{f}$ and point P_0'' of position vector $\mathbf{p}_0''$ lying closest to the origin, which can be derived by analogy with Eq. (3.70), namely, as

$$\mathbf{p}_0'' = \frac{1}{\|\mathbf{f}\|^2} \mathbf{f} \times (\mathbf{n}_A - \mathbf{f} \times \mathbf{a}) \tag{3.111}$$

Similar to Theorem 3.4.1, one has

Theorem 3.7.2. *Consider a system of moments and forces acting on a rigid body, with the resultant force applied at an arbitrary point of the body. The projection of the resultant moment onto the wrench axis is identical for all points.*

From the foregoing discussion, then, the wrench applied to a rigid body can be fully specified by the resultant force $\mathbf{f}$ acting at an arbitrary point P and the associated moment, $\mathbf{n}_P$. We shall derive presently the counterpart of the six-dimensional array of the twist, namely, the wrench array. Upon multiplication of the screw of Eq. (3.71) by an amplitude A with units of force, what we will obtain would be a wrench $\mathbf{w}$, i.e., a six-dimensional array with its first three components having units of force and its last components units of moment. We would like to be able to obtain the power developed by the wrench on the body moving with the twist $\mathbf{t}$ by a simple inner product of the two arrays. However, because of the form the wrench $\mathbf{w}$ has taken, the inner product of these two arrays would be meaningless, for it would involve the sum of two scalar quantities with different units; moreover, none of the two quantities has an immediate physical meaning. In fact, the first scalar would have units of force by frequency (angular velocity by force), while the second would have units of moment of moment multiplied by frequency (velocity by moment), thereby leading to a physically meaningless result. This inconsistency can be resolved if we redefine the wrench not simply as the product of a screw by an amplitude, but as a linear transformation of that screw involving the 6×6 array $\boldsymbol{\Gamma}$ defined as

$$\boldsymbol{\Gamma} \equiv \begin{bmatrix} \mathbf{O} & \mathbf{1} \\ \mathbf{1} & \mathbf{O} \end{bmatrix} \tag{3.112}$$

where $\mathbf{O}$ and $\mathbf{1}$ denote, respectively, the 3×3 zero and identity matrices. Now we define the wrench as a linear transformation of the screw $\mathbf{s}$ defined in Eq. (3.71). This transformation is obtained upon multiplying $\mathbf{s}$ by the product $A\mathbf{\Gamma}$, the amplitude A having units of force, i.e.,

$$\mathbf{w} \equiv A\mathbf{\Gamma}\mathbf{s} \equiv \begin{bmatrix} \mathbf{p} \times (A\mathbf{e}) + p(A\mathbf{e}) \\ A\mathbf{e} \end{bmatrix}$$

The foregoing wrench is given in axis coordinates, while the twist was given in ray coordinates.

Now, the first three components of the foregoing array can be readily identified as the moment of a force of magnitude A acting along a line of action given by the Plücker array of Eq. (3.110), with respect to a point P, to which a moment parallel to that line and of magnitude pA is added. Moreover, the last three components of that array pertain apparently to a force of magnitude A and parallel to the same line. We denote here the above-mentioned moment by $\mathbf{n}$ and the force by $\mathbf{f}$, i.e.,

$$\mathbf{f} \equiv A\mathbf{e}, \quad \mathbf{n} \equiv \mathbf{p} \times \mathbf{f} + p\mathbf{f}$$

The wrench $\mathbf{w}$ is then defined as

$$\mathbf{w} \equiv \begin{bmatrix} \mathbf{n} \\ \mathbf{f} \end{bmatrix} \tag{3.113}$$

which can thus be interpreted as a representation of a system of forces and moments acting on a rigid body, with the force acting at point P of the body $\mathcal{B}$ defined above and a moment $\mathbf{n}$. Under these circumstances, we say that $\mathbf{w}$ *acts* at point P of $\mathcal{B}$.

With the foregoing definitions it is now apparent that the wrench has been defined so that the *inner product* $\mathbf{t}^T\mathbf{w}$ will produce the power Π developed by $\mathbf{w}$ acting at P when $\mathcal{B}$ moves with a twist $\mathbf{t}$ defined at the same point, i.e.,

$$\Pi = \mathbf{t}^T\mathbf{w} \tag{3.114}$$

When a wrench $\mathbf{w}$ that acts on a rigid body moving with the twist $\mathbf{t}$ develops zero power onto the body, we say that the wrench and the twist are *reciprocal* to each other. By the same token, the screws associated with that wrench–twist pair are said to be *reciprocal*. More specifically, let the wrench and the twist be given in terms of their respective screws, $\mathbf{s}_w$ and $\mathbf{s}_t$, as

$$\mathbf{w} = W\mathbf{\Gamma}\mathbf{s}_w, \quad \mathbf{t} = T\mathbf{s}_t, \tag{3.115}$$

where W and T are the amplitudes of the wrench and the twist, respectively, while $\mathbf{\Gamma}$ is as defined in Eq. (3.112). Thus, the two screws $\mathbf{s}_w$ and $\mathbf{s}_t$ are reciprocal if

$$(\mathbf{\Gamma}\mathbf{s}_w)^T\mathbf{s}_t \equiv \mathbf{s}_w^T\mathbf{\Gamma}^T\mathbf{s}_t = 0 \tag{3.116}$$

and by virtue of the symmetry of $\mathbf{\Gamma}$, the foregoing relation can be further expressed as

$$\mathbf{s}_w^T \mathbf{\Gamma} \mathbf{s}_t = 0 \quad \text{or} \quad \mathbf{s}_t^T \mathbf{\Gamma} \mathbf{s}_w = 0 \tag{3.117}$$

The product $\mathbf{s}_w^T \mathbf{\Gamma} \mathbf{s}_t$ or its equivalent $\mathbf{s}_t^T \mathbf{\Gamma} \mathbf{s}_w$ is termed the *reciprocal product* of the two screws $\mathbf{s}_t$ and $\mathbf{s}_w$.

Now, if A and P are arbitrary points of a rigid body, we define the wrench at these points as

$$\mathbf{w}_A \equiv \begin{bmatrix} \mathbf{n}_A \\ \mathbf{f} \end{bmatrix}, \quad \mathbf{w}_P \equiv \begin{bmatrix} \mathbf{n}_P \\ \mathbf{f} \end{bmatrix} \tag{3.118}$$

Therefore, Eq. (3.106) leads to

$$\mathbf{w}_P = \mathbf{V} \mathbf{w}_A \tag{3.119a}$$

where

$$\mathbf{V} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{A} - \mathbf{P} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{3.119b}$$

with $\mathbf{A}$ and $\mathbf{P}$ already defined in Eq. (3.83b) as the cross-product matrices of vectors $\mathbf{a}$ and $\mathbf{p}$, respectively. Thus, $\mathbf{w}_P$ is a linear transformation of $\mathbf{w}_A$. By analogy with the twist-transfer formulas of Eqs. (3.83a and b), Eqs. (3.119a and b) are termed here the *wrench-transfer formulas*.

Multiplying the transpose of Eq. (3.83a) by Eq. (3.119a) yields

$$\mathbf{t}_P^T \mathbf{w}_P = \mathbf{t}_A^T \mathbf{U}^T \mathbf{V} \mathbf{w}_A \tag{3.120}$$

where

$$\mathbf{U}^T \mathbf{V} = \begin{bmatrix} \mathbf{1} & -\mathbf{A} + \mathbf{P} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{A} - \mathbf{P} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \mathbf{1}_{6\times6} \tag{3.121}$$

with $\mathbf{1}_{6\times6}$ denoting the 6×6 identity matrix. Thus, $\mathbf{t}_P^T \mathbf{w}_P = \mathbf{t}_A^T \mathbf{w}_A$, as expected, since the wrench develops the same amount of power, regardless of where the force is assumed to be applied. Also note that an interesting relation between $\mathbf{U}$ and $\mathbf{V}$ follows from Eq. (3.121), namely,

$$\mathbf{V}^{-1} = \mathbf{U}^T \tag{3.122}$$

3.8 Dynamics of Rigid Bodies

The equations governing the motion of rigid bodies are recalled in this section and cast into a form suitable to multibody dynamics. A rigid body $\mathcal{B}$ of mass density ρ, which need not be uniform, is assumed at the outset. It is further assumed that the

body is under motion, with angular velocity $\boldsymbol{\omega}$. The kinetic energy of $\mathcal{B}$ is defined as the integral of the *elementary* kinetic energy of a differential element $d\mathcal{B}$ of the body over the whole body. The mass of the body is thus given by

$$m = \int_{\mathcal{B}} \rho \, d\mathcal{B} \tag{3.123}$$

$\mathcal{B}$ denoting also the region of the three-dimensional space occupied by the body. Given that the laws of *classical mechanics*—as exemplified by the Newton–Euler equations, to be recalled presently—hold only in inertial frames, the concept is now introduced:

> An inertial frame is a coordinate system that translates with uniform velocity and constant orientation with respect to the stars.

Now, if $\mathbf{p}$ denotes the position vector of an arbitrary point P of the body, from the origin O of a given inertial frame, the *mass first moment* of the body with respect to O, $\mathbf{q}_O$, is defined as

$$\mathbf{q}_O \equiv \int_{\mathcal{B}} \rho \mathbf{p} \, d\mathcal{B} \tag{3.124}$$

Upon recalling the *Mean-value Theorem* of integral calculus (Brand 1965), there exists a point C, of position vector $\mathbf{c}$, such that

$$\int_{\mathcal{B}} \rho \mathbf{p} \, d\mathcal{B} = m\mathbf{c}$$

point C thus being defined as the *center of mass* of $\mathcal{B}$. That is,

$$\mathbf{c} \equiv \frac{\mathbf{q}_O}{m} \tag{3.125}$$

Notice that point C need not be located within the physical boundary of $\mathcal{B}$—think, for example, of a homogeneous torus.

Further, the *mass second moment* of the body with respect to O is introduced. To this end, the *kinetic energy* of the rigid body, denoted by T, is defined as the integral of the kinetic energy of the mass element $dm \equiv \rho d\mathcal{B}$ of the body over the whole body, i.e.,

$$T \equiv \int_{\mathcal{B}} \frac{1}{2} \rho \|\dot{\mathbf{p}}\|^2 d\mathcal{B} \tag{3.126}$$

where $\dot{\mathbf{p}}$ is the velocity of P and $\mathbf{p}$ is the position vector of P in the same inertial frame, whose origin, for the sake of simplicity, is assumed to be stationary. Next, the velocity of P is expressed as in Eq. (3.50), using a reference—or base—point; in that equation, an arbitrary point A was used as a reference. It will prove convenient to use the center of mass C as reference in the sequel. Thus,

$$\dot{\mathbf{p}} = \dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{r}, \quad \mathbf{r} \equiv \mathbf{p} - \mathbf{c} \tag{3.127}$$

or, if matrix $\mathbf{R} \equiv \mathrm{CPM}(\mathbf{r})$ is introduced, then

$$\dot{\mathbf{p}} = \dot{\mathbf{c}} - \mathbf{R}\boldsymbol{\omega} \qquad (3.128)$$

Upon substitution of the above expression into Eq. (3.126), an alternative expression for the kinetic energy is obtained:

$$T = \frac{1}{2} \int_{\mathcal{B}} \rho[\|\dot{\mathbf{c}}\|^2 - 2\dot{\mathbf{c}}^T \mathbf{R}\boldsymbol{\omega} + \boldsymbol{\omega}^T \mathbf{R}^T \mathbf{R}\boldsymbol{\omega}]d\mathcal{B}$$

The foregoing expression is now rewritten as the sum of three integrals:

$$T = \frac{1}{2}\|\dot{\mathbf{c}}\|^2 \underbrace{\int_{\mathcal{B}} \rho d\mathcal{B}}_{m} - \dot{\mathbf{c}}^T \left(\int_{\mathcal{B}} \rho \mathbf{R} d\mathcal{B} \right) \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}^T \left(\int_{\mathcal{B}} \rho \underbrace{\mathbf{R}^T \mathbf{R}}_{-\mathbf{R}^2} d\mathcal{B} \right) \boldsymbol{\omega} \quad (3.129)$$

Now, since $\mathbf{R}$ is skew-symmetric, its square expands according to Eq. (2.40), namely,

$$\mathbf{R}^2 = -\|\mathbf{r}\|^2 \mathbf{1} + \mathbf{r}\mathbf{r}^T$$

Furthermore, the integral appearing in the second term of Eq. (3.129) vanishes. Indeed,

$$\int_{\mathcal{B}} \rho \mathbf{R} d\mathcal{B} = \int_{\mathcal{B}} \rho \mathrm{CPM}(\mathbf{r}) d\mathcal{B} \equiv \mathrm{CPM}(\int_{\mathcal{B}} \rho \mathbf{r} d\mathcal{B})$$

where the integral and the $\mathrm{CPM}(\cdot)$ operators have been interchanged because they are both linear and smooth.[2] Further, if the definition of $\mathbf{r}$ is recalled from Eq. (3.127), the above integral expands to

$$\int_{\mathcal{B}} \rho \mathbf{R} d\mathcal{B} = \mathrm{CPM}(\underbrace{\int_{\mathcal{B}} \rho \mathbf{p} d\mathcal{B}}_{m\mathbf{c}} - \mathbf{c} \underbrace{\int_{\mathcal{B}} \rho d\mathcal{B}}_{m})$$

which indeed vanishes. Therefore, the kinetic energy of $\mathcal{B}$ reduces to

$$T \equiv \frac{1}{2}m\|\dot{\mathbf{c}}\|^2 + \frac{1}{2}\boldsymbol{\omega}^T \underbrace{[\int_{\mathcal{B}} \rho(\|\mathbf{r}\|^2 \mathbf{1} - \mathbf{r}\mathbf{r}^T)d\mathcal{B}]}_{\mathbf{I}_C} \boldsymbol{\omega} \qquad (3.130)$$

---

[2]Smoothness means that they do not introduce jump discontinuities upon differentiation.

which is known to be a positive-definite quantity, in that it (a) is positive and (b) only vanishes when the body is stationary in an inertial frame, i.e., when both $\dot{\mathbf{c}}$ and $\boldsymbol{\omega}$ vanish. In fact, the first term of the above expression is apparently positive-definite, for its first factor, the mass, is positive, its second factor being the square of a real quantity, namely, a Euclidean norm—which is itself positive. The second term is a quadratic form in the angular-velocity vector, associated with matrix $\mathbf{I}_C$, defined as the *moment of inertia of $\mathcal{B}$ at the center of mass*. As this is a matrix, $\mathbf{I}_C$ is also referred to as the *moment-of-inertia matrix* . This matrix is the integral of the quantity in parentheses that appears multiplied by the density ρ, an inherently positive quantity. The term in parentheses, however, is apparently *positive-semidefinite*, as it equals the product $\mathbf{R}\mathbf{R}^T$; this product would be positive-definite, were it not for the cross-product matrix $\mathbf{R}$, which is known to be singular, of rank 2—see Excercise 2.11. Therefore, $\mathbf{I}_C$ is at least positive-semidefinite. As a matter of fact, the only case in which $\mathbf{I}_C$ is semidefinite occurs when $\mathcal{B}$ is one-dimensional, i.e., a "long" body with two of its dimensions negligible with respect to its length. This would be a body filling the space of a line segment, its moment-of-inertia matrix having a nullity—dimension of its null space—of 1, with null space spanned by a vector parallel to the line. Other than this rather unlikely case, $\mathbf{I}_C$ is positive-definite.

Therefore, $\mathbf{I}_C$ has three real eigenvalues, in general all three positive, known as the *principal moments of inertia*, and three mutually orthogonal eigenvectors, defining three *principal axes of inertia* of $\mathcal{B}$, all three concurring at the center of mass.

Note, however, that the principal moments and the principal axes of inertia of a rigid body depend on the point with respect to which the moment of inertia is defined. Indeed, the moment of inertia can be defined about an arbitrary point of the body, whether within its physical boundary or without. If, in Eq. (3.126), the moment of inertia is taken with respect to an arbitrary point A of $\mathcal{B}$, then, the moment of inertia $\mathbf{I}_A$ with respect to A is defined as

$$\mathbf{I}_A \equiv \int_{\mathcal{B}} \rho[\|\mathbf{p}-\mathbf{a}\|^2\mathbf{1} - (\mathbf{p}-\mathbf{a})(\mathbf{p}-\mathbf{a})^T]d\mathcal{B} \tag{3.131}$$

Next, notice that, with the definition of $\mathbf{r}$ in Eq. (3.127), $\mathbf{p}-\mathbf{a}$ can be expressed as

$$\mathbf{p}-\mathbf{a} = \mathbf{r} - (\mathbf{a}-\mathbf{c}) \tag{3.132}$$

whence $\mathbf{I}_A$ becomes

$$\mathbf{I}_A = \int_{\mathcal{B}} \rho\{\|\mathbf{r}-(\mathbf{a}-\mathbf{c})\|^2\mathbf{1} - [\mathbf{r}-(\mathbf{a}-\mathbf{c})][\mathbf{r}-(\mathbf{a}-\mathbf{c})]^T\}d\mathcal{B}$$

which expands to

$$\mathbf{I}_A = \underbrace{\int_{\mathcal{B}} \rho(\|\mathbf{r}\|^2 \mathbf{1} - \mathbf{r}\mathbf{r}^T) d\mathcal{B}}_{\mathbf{I}_C} + 2\,\underbrace{(-\int_{\mathcal{B}} \mathbf{r}^T \rho d\mathcal{B})}_{\mathbf{0}^T}(\mathbf{a} - \mathbf{c})\mathbf{1} + (\mathbf{a} - \mathbf{c})\underbrace{\int_{\mathcal{B}} \mathbf{r}^T \rho d\mathcal{B}}_{\mathbf{0}^T}$$

$$+ \underbrace{(\int_{\mathcal{B}} \mathbf{r}\rho d\mathcal{B})}_{\mathbf{0}}(\mathbf{a} - \mathbf{c})^T + \underbrace{\int_{\mathcal{B}} \rho[\|\mathbf{a} - \mathbf{c}\|^2 \mathbf{1} - (\mathbf{a} - \mathbf{c})(\mathbf{a} - \mathbf{c})^T] d\mathcal{B}}_{m[\|\mathbf{a}-\mathbf{c}\|^2 \mathbf{1} - (\mathbf{a}-\mathbf{c})(\mathbf{a}-\mathbf{c})^T]}$$

the final expression for $\mathbf{I}_A$ thus being

$$\mathbf{I}_A = \mathbf{I}_C + m[\|\mathbf{a} - \mathbf{c}\|^2 \mathbf{1} - (\mathbf{a} - \mathbf{c})(\mathbf{a} - \mathbf{c})^T] \tag{3.133a}$$

which, in light of Eq. (2.39), can be cast in the form

$$\mathbf{I}_A = \mathbf{I}_C + m(\mathbf{A} - \mathbf{C})(\mathbf{A} - \mathbf{C})^T \tag{3.133b}$$

a result known as *Steiner's Theorem*, a.k.a. the *Theorem of Parallel Axes* .

Next, the *Newton–Euler equations* governing the motion of a rigid body are recalled, with the body $\mathcal{B}$ acted upon by a wrench of force $\mathbf{f}$ applied at its center of mass, and of moment $\mathbf{n}_C$ about the center of mass. The *Newton equation* then takes the form

$$\mathbf{f} = m\ddot{\mathbf{c}} \tag{3.134a}$$

whereas the *Euler equation* is

$$\mathbf{n}_C = \mathbf{I}_C \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_C \boldsymbol{\omega} \tag{3.134b}$$

The *momentum* $\mathbf{m}$ and the *angular momentum* $\mathbf{h}_C$ of a rigid body moving with a velocity of its center of mass $\dot{\mathbf{c}}$ and an angular velocity $\boldsymbol{\omega}$ are defined below, the angular momentum being defined, moreover, with respect to the center of mass. These momenta are

$$\mathbf{m} \equiv m\dot{\mathbf{c}}, \quad \mathbf{h}_C \equiv \mathbf{I}_C \boldsymbol{\omega} \tag{3.135}$$

Furthermore, the time-derivatives of the foregoing quantities are readily computed (see Exercise 3.17) as

$$\dot{\mathbf{m}} = m\ddot{\mathbf{c}}, \quad \dot{\mathbf{h}}_C = \mathbf{I}_C \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_C \boldsymbol{\omega} \tag{3.136}$$

and hence, Eqs. (3.134a and b) take the forms

$$\mathbf{f} = \dot{\mathbf{m}}, \quad \mathbf{n}_C = \dot{\mathbf{h}}_C \tag{3.137}$$

The set of equations (3.134a and b) are known as the Newton–Euler equations . These can be written in a more compact form as described below. First, a 6×6 matrix $\mathbf{M}$ is introduced that, following von Mises (1924), is termed here the *inertia dyad*, namely,

$$\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_C & \mathbf{O} \\ \mathbf{O} & m\mathbf{1} \end{bmatrix} \tag{3.138}$$

where $\mathbf{O}$ and $\mathbf{1}$ denote the 3×3 zero and identity matrices. A similar 6×6 matrix was defined by von Mises under the above name. However, von Mises's inertia dyad is full, while the matrix defined above is block-diagonal. Both matrices, nevertheless, denote the same physical property of a rigid body, i.e., its mass and moment of inertia. Now the Newton–Euler equations can be written as

$$\mathbf{M}\dot{\mathbf{t}} + \mathbf{WMt} = \mathbf{w} \tag{3.139}$$

in which matrix $\mathbf{W}$, henceforth termed the *angular-velocity dyad*, by similarity with the inertia dyad, is defined in turn as

$$\mathbf{W} \equiv \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{3.140}$$

with $\boldsymbol{\Omega}$ already defined as the angular-velocity matrix, namely, as $\mathrm{CPM}(\boldsymbol{\omega})$. Note that the twist of a rigid body lies in the null space of its angular-velocity dyad, i.e.,

$$\mathbf{Wt} = \mathbf{0} \tag{3.141}$$

Further definitions are introduced below: the *momentum screw* of the rigid body about the center of mass is the six-dimensional vector $\boldsymbol{\mu}$ defined as

$$\boldsymbol{\mu} \equiv \begin{bmatrix} \mathbf{I}_C \boldsymbol{\omega} \\ m\dot{\mathbf{c}} \end{bmatrix} = \mathbf{Mt} \tag{3.142}$$

Moreover, from Eqs. (3.136) and definition (3.142), the time-derivative of $\boldsymbol{\mu}$ can be readily derived as[3]

$$\dot{\boldsymbol{\mu}} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\boldsymbol{\mu} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{WMt} \tag{3.143}$$

The kinetic energy of a rigid body undergoing a motion in which its center of mass moves with velocity $\dot{\mathbf{c}}$ and rotates with an angular velocity $\boldsymbol{\omega}$ is given by Eq. (3.130), repeated below for quick reference:

$$T = \frac{1}{2} m \|\dot{\mathbf{c}}\|^2 + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I}_C \boldsymbol{\omega} \tag{3.144}$$

---

[3]See Exercise 3.17.

From the foregoing definitions, then, the kinetic energy can now be written as a quadratic form of the body twist, namely,

$$T = \frac{1}{2}\mathbf{t}^T\mathbf{M}\mathbf{t} \tag{3.145}$$

while the Newton–Euler equations can be written in a form more compact than that of Eq. (3.139):

$$\dot{\boldsymbol{\mu}} = \mathbf{w} \tag{3.146}$$

which is a six-dimensional vector equation.

Properly speaking, $\mathbf{M}$, as given by Eq. (3.138), should be subscripted with C, to emphasize that the moment of inertia in the upper left block of $\mathbf{M}$ is taken with respect to the center of mass. For brevity, we will dispense with this subscript whenever the moment of inertia is *centroidal*. If the moment of inertia is taken with respect to any other arbitrary point A, then the corresponding inertia dyad is represented with $\mathbf{M}_A$. The reader is asked to verify that $\mathbf{M}_A$ takes the form

$$\mathbf{M}_A = \begin{bmatrix} \mathbf{I}_A & m(\mathbf{C} - \mathbf{A}) \\ m(\mathbf{C} - \mathbf{A})^T & m\mathbf{1} \end{bmatrix} \tag{3.147}$$

Notice that, contrary to the inertia dyad defined with respect to the center of mass in Eq. (3.138), which is *block-diagonal*, $\mathbf{M}_A$ is full. It should not be too difficult to prove that, regardless of the point with respect to which the moment of inertia is defined, the inertia dyad is positive-definite—see Exercise 3.20. The proof, as suggested in that exercise, hinges on a decomposition of $\mathbf{M}_A$ in the form

$$\mathbf{M}_A = \mathbf{M}_C + m\mathbf{P}\mathbf{P}^T \tag{3.148}$$

The reader is invited to find an expression for $\mathbf{P}$. The foregoing expression for $\mathbf{M}_A$ can be fairly referred to as *Steiner's Theorem in dyad form*.

3.9 Exercises

3.1 The cube of Fig. 3.9 is displaced from configuration $AB \ldots H$ into configuration $A'B' \ldots H'$.

    (a) Determine the matrix representing the rotation $\mathbf{Q}$ undergone by the cube, in X, Y, Z coordinates.

    (b) Find the Plücker coordinates of line $\mathcal{L}$ of the cube undergoing displacements of minimum magnitude.

    (c) Find the intersections of $\mathcal{L}$ with the coordinate planes.
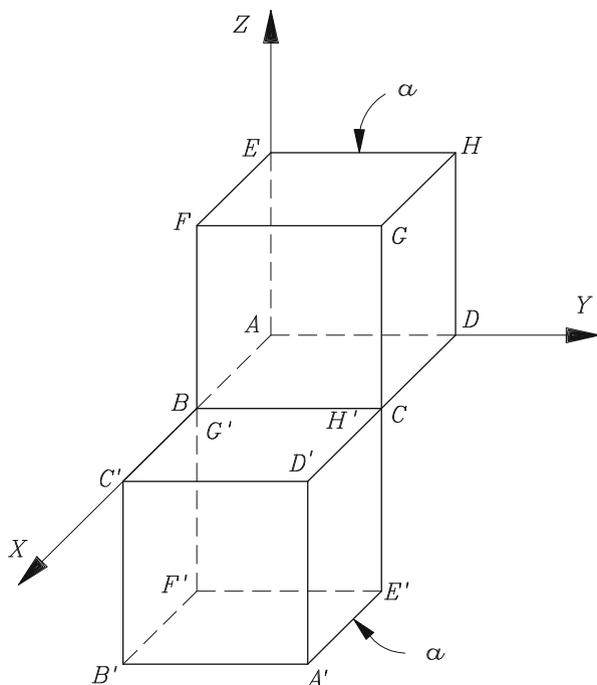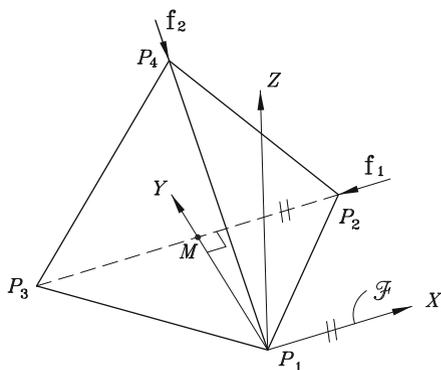
Fig. 3.9 Motion of a cube

Fig. 3.10 A regular
tetrahedron

3.2 Two unit forces, $\mathbf{f}_1$ and $\mathbf{f}_2$, are applied to the regular tetrahedron of unit-length
    edges displayed in Fig. 3.10 in such a way that $\mathbf{f}_1$ is directed from P_2 to P_3,
    whereas $\mathbf{f}_2$ is directed from P_4 to P_1. The effect of the foregoing system of
    forces on the rigid tetrahedron is obtained by application of the resultant of
    the two forces on a certain point P and a moment $\mathbf{n}$. Find the location of point
    P lying closest to P_4 that will make the magnitude of $\mathbf{n}$ a minimum.

3.3 The *moment* of a line $\mathcal{L}_1$ about a second line $\mathcal{L}_2$ is a scalar μ defined as

$$\mu = \mathbf{n}_1 \cdot \mathbf{e}_2$$

where $\mathbf{n}_1$ is the moment of $\mathcal{L}_1$ about an arbitrary point P of $\mathcal{L}_2$, while $\mathbf{e}_2$ is a unit vector parallel to line $\mathcal{L}_2$. Apparently, the necessary and sufficient condition for two lines to intersect is that the moment of one about the other vanish.

Using the above concept, show that the locus of all lines $\mathcal{L}$ intersecting three given lines $\{\mathcal{L}_k\}_1^3$ is a *quadric*, i.e., a surface defined by a function that is quadratic in the position vector $\mathbf{p}$ of a point of the surface: $f(\mathbf{p}) = \mathbf{p}^T \mathbf{M} \mathbf{p} + \mathbf{g}^T \mathbf{p} + h = 0$, $h > 0$. Notice that, the surface being generated by the motion of a line constrained to touch the three given lines, the surface is bound to be *ruled*. Now, there are only two ruled, second-order surfaces (Pottmann and Wallner 2001), the *one-sheet hyperboloid* and the *hyperbolic paraboloid*. Geometrically, the former is a closed surface, the latter open. Algebraically, the former is characterized by a matrix coefficient $\mathbf{M}$ with $\det(\mathbf{M}) < 0$, meaning that the matrix has two positive and one negative eigenvalues. The hyperbolic paraboloid is characterized by a singular $\mathbf{M}$, i.e., $\det(\mathbf{M}) = 0$. Show, furthermore, that $\det(\mathbf{M})$ is nonzero—showing that $\det(\mathbf{M}) < 0$ is far more challenging!—and hence, the quadric is a one-sheet hyperboloid. *Hint: Deriving the given expression for $f(\mathbf{p})$ should be done without resorting to components. Showing that $\mathbf{M}$ is singular requires resorting to components. To this end, choose the coordinate axes appropriately so as to avoid too cumbersome expressions.*

3.4 A robotic gripper is provided with two redundant sensors that are meant to measure a wrench acting on the gripper. The ith sensor, moreover, has its own coordinate frame, labeled $\mathcal{F}_i$, for $i = 1, 2$. Sensor i reported the ith measurement of the wrench $\mathbf{w}_P$, where subscript P indicates that the force is applied at point P, as $[\mathbf{w}_P]_i \equiv [\mathbf{n}^T, \mathbf{f}^T]_i^T$, for $i = 1, 2$. These measurements are given as

$$[\mathbf{n}]_1 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \quad [\mathbf{f}]_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$[\mathbf{n}]_2 = \begin{bmatrix} -5/3 \\ -10/3 \\ 10/3 \end{bmatrix}, \quad [\mathbf{f}]_2 = \begin{bmatrix} -4/3 \\ 4/3 \\ 2/3 \end{bmatrix}$$

(a) Show that the measurements are compatible, based on invariance arguments.

(b) Determine the relative orientation of the two frames, i.e., find the rotation matrix transforming $\mathcal{F}_2$-coordinates into $\mathcal{F}_1$-coordinates.
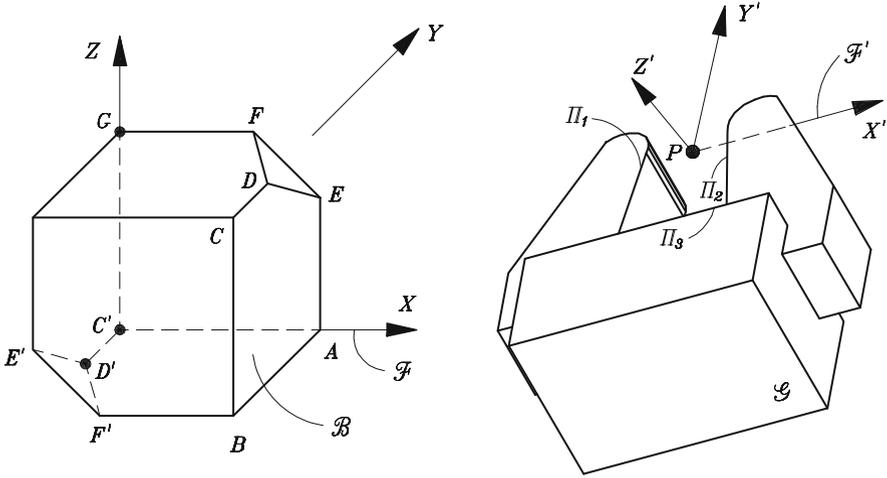
Fig. 3.11 A workpiece $\mathcal{B}$ to be grasped by a gripper $\mathcal{G}$

3.5 A robot-calibration method has been proposed that allows us to determine the location of a joint axis, $\mathcal{L}$, via the Plücker coordinates of the axis in a coordinate frame fixed to the gripper. The Plücker coordinates are given as $\boldsymbol{\pi}_{\mathcal{L}} = [\mathbf{e}^T, \mathbf{n}^T]^T$.

(a) Show that the distance of the axis to the origin of the gripper-fixed coordinate frame, d, can be determined as $d = \|\mathbf{n}\|$.

(b) Show that the point P^* on the axis, which lies closest to the above-mentioned origin, has a position vector $\mathbf{p}^*$ given as

$$\mathbf{p}^* = \mathbf{e} \times \mathbf{n}$$

(c) From measurements on a robot, the Plücker coordinates were estimated, in a gripper-fixed frame $\mathcal{G}$, as

$$[\boldsymbol{\pi}_{\mathcal{L}}]_{\mathcal{G}} = [-\sqrt{2}/2, \, 0, \, \sqrt{2}/2, \, 0, \, -\sqrt{2}, \, 0]^T$$

Find d and $\mathbf{p}^*$ in gripper coordinates

3.6 The gripper $\mathcal{G}$ of a robot is approaching a workpiece $\mathcal{B}$, as indicated in Fig. 3.11, with planes Π_1 and Π_2 parallel to each other and normal to Π_3. The workpiece is made out of a cube of unit length from which two vertices have been removed, thereby producing the equilateral triangular faces DEF and $D'E'F'$. Moreover, two coordinate frames, $\mathcal{F}(X, Y, Z)$ and $\mathcal{F}'(X', Y', Z')$, are defined as indicated in the figure, in which Y is, apparently, parallel to line $D'C'$.

It is required to grasp $\mathcal{B}$ with $\mathcal{G}$ in such a way that planes Π_1 and Π_2 coincide with the triangular faces, while carrying the Y' axis to an orientation perpendicular to the diagonal CC' of $\mathcal{B}$. More concretely, in the grasping configuration, frame $\mathcal{F}'$ is carried into $\mathcal{F}''$ (X'', Y'', and Z''), not shown in the figure, in such a way that unit vectors $\mathbf{i}''$, $\mathbf{j}''$, $\mathbf{k}''$, parallel to X'', Y'', Z'', respectively, are oriented so that $\mathbf{i}''$ has all three of its $\mathcal{F}$-components positive, while $\mathbf{j}''$ has its Z-component positive.

(a) Compute the angle of rotation of the motion undergone by $\mathcal{G}$ from a pose in which $\mathcal{F}'$ and $\mathcal{F}$ have identical orientations, termed the *reference pose*, and find the unit vector parallel to the axis of rotation, in frame $\mathcal{F}$.

(b) The position vector of point P of $\mathcal{G}$ is known to be, in the reference pose,

$$[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ -1 \\ 0.25 \end{bmatrix}$$

Determine the set of points of $\mathcal{G}$ undergoing a displacement of minimum magnitude, under the condition that P, in the displaced configuration of $\mathcal{G}$, coincides with C'.

3.7 In calibrating a robot, the Plücker coordinates of one of its axes are to be determined in a given coordinate frame. To this end, the moment of this axis is measured with respect to two points, A and B, of position vectors $[\mathbf{a}] = [1, 0, 0]^T$ and $[\mathbf{b}] = [0, 1, 1]^T$, respectively. The said moments, $\mathbf{n}_A$ and $\mathbf{n}_B$, respectively, are measured as

$$[\mathbf{n}_A] = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [\mathbf{n}_B] = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

with all entries given in meters.

(a) Determine the unit vector $\mathbf{e}$ defining the direction of the axis under discussion.

(b) Find the coordinates of the point P^* of the axis that lies closest to the origin

(c) Find the Plücker coordinates of the axis about the origin, i.e., the Plücker coordinates of the axis in which the moment is defined with respect to the origin.

3.8 Prove that for any three-dimensional vectors $\boldsymbol{\omega}$ and $\mathbf{p}$,

$$\underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \cdots (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p})) \cdots)}_{2k \ \text{factors}} = (-1)^k (\|\boldsymbol{\omega}\|^{2k} \mathbf{1} - \|\boldsymbol{\omega}\|^{2(k-1)} \boldsymbol{\omega}\boldsymbol{\omega}^T)\mathbf{p}$$

$$\underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \cdots (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p})) \cdots)}_{2k+1 \ \text{factors}} = (-1)^k (\|\boldsymbol{\omega}\|^{2k} \boldsymbol{\omega}) \times \mathbf{p}$$

3.9 A "small" rotation is defined as that about an arbitrary axis parallel to the unit vector $\mathbf{e}$, through a "small" angle ϕ, so that $\phi \ll 1$. Prove that the angular-velocity vector, in the special case of "small" rotations, turns out to be a time-derivative. What is the vector whose time-derivative yields the angular-velocity vector?

3.10 Derive an expression for the angular velocity $\boldsymbol{\omega}$ in terms of Euler angles, which were introduced in Exercise 2.19. More specifically, if we store the Euler angles in array $\boldsymbol{\eta} = [\,\theta, \phi, \psi\,]^T$, then, find the matrix $\mathbf{W}$ such that

$$\boldsymbol{\omega} = \mathbf{W}\dot{\boldsymbol{\eta}}$$

Notice that, given $\boldsymbol{\eta}$ and $\boldsymbol{\omega}$, an expression for $\dot{\boldsymbol{\eta}}$ can be obtained upon inverting $\mathbf{W}$. However, $\mathbf{W}$ is not always invertible. Find under which conditions $\mathbf{W}$ becomes singular. *Notice: The use of computer algebra is strongly recommended to solve this exercise.*

3.11 A rectangular prism with regular hexagonal bases whose sides are 25 mm long and whose height is 150 mm is to undergo a pick-and-place operation— see Chap. 6 to understand what this means—that requires knowledge of its centroid location and its moment-of-inertia matrix. Find the centroidal principal axes and moments of inertia under the assumption that the prism is made from a homogeneous material.

3.12 The prism of Exercise 3.11 now undergoes a machining process cutting it into two parts, which are separated by a plane that contains one of the edges of the base and makes an angle of 45° with the axis of the prism. Find the centroidal principal axes and moments of inertia of each of the two parts.

3.13 In Exercise 2.23 assume that a mass m is located at every point P_i of position vector $\mathbf{p}_i$. Give a mechanical interpretation of the matrix $m[\mathrm{tr}(\mathbf{PP}^T)\mathbf{1} - \mathbf{PP}^T]$, with $\mathbf{P}$ defined in that exercise.

3.14 The centroidal inertia matrix of a rigid body is measured by two observers, who report the two results below:

$$[\mathbf{I}]_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [\mathbf{I}]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 6 & 2 & 2 \\ 2 & 5 & 0 \\ 2 & 0 & 7 \end{bmatrix}$$

Show that the two measurements are acceptable. *Hint: Use invariance arguments.*

3.15 State the conditions under which a point and the center of mass of a rigid body share the same principal axes of inertia. In other words, let $\mathbf{I}_P$ and $\mathbf{I}_C$ be the moment-of-inertia matrices of a rigid body about a point P and its center of mass, C, respectively. State the conditions under which the two matrices have common eigenvectors. Moreover, under these conditions, what are the relationships between the two sets of principal moments of inertia?

3.16 Show that the smallest principal moment of inertia of a rigid body attains its minimum value at the center of mass.

3.17 Show that the time-rate of change of the inertia dyad $\mathbf{M}$ of a rigid body is given by

$$\dot{\mathbf{M}} = \mathbf{W}\mathbf{M} - \mathbf{M}\mathbf{W}$$

Then, recall the momentum screw $\boldsymbol{\mu}$ defined as

$$\boldsymbol{\mu} \equiv \mathbf{M}\mathbf{t}$$

where $\mathbf{t}$ is the twist of the body, defined at its center of mass. Now, with the above expression for $\dot{\mathbf{M}}$, restate the result displayed in Eq. (3.143), i.e., show that

$$\dot{\boldsymbol{\mu}} = \mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t}$$

3.18 A classical result in the dynamics of robots establishes a relation between the time-derivative of the $n \times n$ inertia matrix $\mathbf{I}(\boldsymbol{\theta})$ of a serial robot and what is known as the "Coriolis and centrifugal forces" matrix $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, where $\boldsymbol{\theta}$ is the n-dimensional vector of joint angles—these concepts are the subject of Chap. 7. According to this relation, the difference $\dot{\mathbf{I}}(\boldsymbol{\theta}) - 2\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is skew-symmetric. Show that the same relation holds for one single rigid body moving in space under the action of a given wrench. That is, show that the difference $\dot{\mathbf{M}} - 2\mathbf{W}\mathbf{M}$ is skew-symmetric, with $\mathbf{M}$ and $\mathbf{W}$ introduced in Sect. 3.8.

3.19 A wrench $\mathbf{w} = [\, \mathbf{n}^T \; \mathbf{f}^T \,]^T$, with $\mathbf{f}$ acting at point P of the gripper of Fig. 2.10, is measured by a *six-axis force sensor*, to which a frame $\mathcal{F}_S$ is attached, as indicated in that figure. If points P and S lie a distance of 100 mm apart, find the wrench in $\mathcal{F}_2$, when the readouts of the sensor are

$$[\,\mathbf{n}\,]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{Nm}, \quad [\,\mathbf{f}\,]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{N}$$

3.20 Derive Eq. (3.147). Moreover, show that $\mathbf{M}_A$, as given in that equation, is positive-definite—*Hint: show that* $\mathbf{M}_A$ *can be expressed as* $\mathbf{M}_A = \mathbf{M}_C + \mathbf{P}\mathbf{P}^T$, *with* $\mathbf{M}_C$ *positive-definite and* $\mathbf{P}\mathbf{P}^T$ *positive-semidefinite.* The above expression can be termed the 6×6 form of Steiner's Theorem, a.k.a. the Parallel-Axis Theorem. Now, in trying to mimic Steiner's Theorem, one would like to express $\mathbf{P}$ in the same form as the second term of Eq. (3.133a), as the product $m\mathbf{R}\mathbf{R}^T$, with $\mathbf{R}$ defined as a suitable 6×6 matrix. Can this matrix be found? If yes, give an expression for $\mathbf{R}$; if not, explain why.

3.21 The inner (dot) product of two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ can be defined as $\mathrm{tr}(\mathbf{A}\mathbf{B}^T)$, which leads to the definition of the angle θ between $\mathbf{A}$ and $\mathbf{B}$ as

$\cos \theta = \text{tr}(\mathbf{A}\mathbf{B}^T)/\|\mathbf{A}\|\|\mathbf{B}\|$, where $\|\mathbf{A}\| = \sqrt{\text{tr}(\mathbf{A}\mathbf{A}^T)}$ with a similar definition for $\|\mathbf{B}\|$. Moreover, Eq. (3.86) can be regarded as the *Cartesian decomposition* of the angular-acceleration matrix $\mathbf{W}$. Show that the two components of the foregoing decomposition are *mutually orthogonal*, which is a restatement of the orthogonality of the tangential and the centripetal relative acceleration of two points of a rigid body.