Chapter 8
Special Topics in Rigid-Body Kinematics

8.1 Introduction

The motivation for this chapter is twofold. On the one hand, the determination of the angular velocity and angular acceleration of a rigid body from point-velocity measurements is a fundamental problem in kinematics. On the other hand, the solution of this problem is becoming increasingly relevant in the kinematics of parallel manipulators, to be studied in Chap. 10. Moreover, the estimation of the attitude of a rigid body from knowledge of the Cartesian coordinates of some of its points is sometimes accomplished by time-integration of the velocity data. Likewise, the use of accelerometers in the area of motion control readily leads to estimates of the acceleration of a sample of points of a rigid body, which can be used to estimate the angular acceleration of the body, and hence, to better control its motion.

In order to keep the discussion at the level of fundamentals, we assume throughout this chapter that the information available on point velocity and point acceleration is error-free, a rather daring assumption, but useful for understanding the underlying concepts at this level. Once the fundamentals are well understood, devising algorithms that yield the best estimates of angular velocity and acceleration in the presence of noisy measurements becomes an easier task. For the sake of conciseness, the problem of motion estimation will not be discussed in this book.

8.2 Computation of Angular Velocity from Point-Velocity Data

The twist of a rigid body, as introduced in Eq. (3.72), defines completely the velocity field of a rigid body under arbitrary motion. Notice that the twist involves two vector quantities, the angular velocity and the velocity of a point of the rigid body. Since we are assuming that point-velocity data are available, the only item to be computed

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is the angular velocity of the body under study, which is the subject of this section. Once the angular velocity is known and the velocities of a set of body points are available, other relevant motion parameters, such as the location of the ISA—see Sect. 3.4—-can be readily determined.

If the twist of a rigid body is known, the computation of the velocity of an arbitrary point of the body, of a given position vector, is straightforward. However, the inverse problem, namely, the computation of the twist of the motion under study given the velocities of a set of points of known position vectors, is a more difficult task. A solution to this problem is now outlined.

First and foremost, we acknowledge that the velocities of a minimum of three noncollinear points are needed in order to determine the angular velocity of the rigid body under study. Indeed, if the velocity of a single body point is known, we have no information on the angular motion of the body; if the velocities of two points are known, we can calculate two components of the angular-velocity vector of the body, namely, those that are orthogonal to the line joining the two given points, thereby leaving one component indeterminate, the one along that line. Therefore, in order to know the angular velocity of a rigid body in motion, we need at least the velocities of three noncollinear points of the body—obviously, knowing only the velocities of any number of points along one line yields no more information than knowing only the velocities of two points along that line. We thus assume henceforth that we have three noncollinear points and that we know *perfectly* their velocities.

Let the three noncollinear points of the body under study be denoted by $\{ P_i \}_1^3$ and let $\{ \mathbf{p}_i \}_1^3$ be their corresponding position vectors. The centroid C of the foregoing set has a position vector $\mathbf{c}$ that is the mean value of the three given position vectors, namely,

$$\mathbf{c} \equiv \frac{1}{3} \sum_1^3 \mathbf{p}_i \tag{8.1}$$

Likewise, if the velocities of the three points are denoted by $\dot{\mathbf{p}}_i$, and that of their centroid by $\dot{\mathbf{c}}$, one has

$$\dot{\mathbf{c}} \equiv \frac{1}{3} \sum_1^3 \dot{\mathbf{p}}_i \tag{8.2}$$

From Eq. (3.49), the velocity of the three given points can be expressed as

$$\dot{\mathbf{p}}_i = \dot{\mathbf{c}} + \mathbf{\Omega}(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3 \tag{8.3a}$$

or

$$\dot{\mathbf{p}}_i - \dot{\mathbf{c}} = \mathbf{\Omega}(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3 \tag{8.3b}$$

Now, we define a 3×3 matrix $\mathbf{P}$ as

$$\mathbf{P} \equiv \left[\mathbf{p}_1 - \mathbf{c}\ \mathbf{p}_2 - \mathbf{c}\ \mathbf{p}_3 - \mathbf{c}\right] \tag{8.4}$$

Upon differentiation of both sides of Eq. (8.4) with respect to time, one has

$$\dot{\mathbf{P}} \equiv \left[\dot{\mathbf{p}}_1 - \dot{\mathbf{c}}\ \dot{\mathbf{p}}_2 - \dot{\mathbf{c}}\ \dot{\mathbf{p}}_3 - \dot{\mathbf{c}}\right] \tag{8.5}$$

It is noteworthy that $\mathbf{P}$ and $\dot{\mathbf{P}}$ are *immutable* under a pure translation of the coordinate frame of reference. However, under a pure rotation of the frame, given by a proper orthogonal matrix $\mathbf{Q}$, both $\mathbf{P}$ and $\dot{\mathbf{P}}$ transform as $\mathbf{QP}$ and $\mathbf{Q}\dot{\mathbf{P}}$, respectively. As a consequence, $\mathbf{P}$ and $\dot{\mathbf{P}}$ *are not frame invariant*.

Further, Eqs. (8.3b) can be written in matrix form as

$$\dot{\mathbf{P}} = \mathbf{\Omega}\mathbf{P} \tag{8.6}$$

from which we want to solve for $\mathbf{\Omega}$, or equivalently, for $\boldsymbol{\omega}$. This cannot be done by simply multiplying by the inverse of $\mathbf{P}$, because the latter is a singular matrix. In fact, as the reader can readily verify, any vector having three identical components lies in the null space of $\mathbf{P}$, thereby showing that $\mathbf{P}$ is singular, its null space being spanned by that vector. Furthermore, notice that from Eq. (8.3b), it is apparent that

$$(\dot{\mathbf{p}}_i - \dot{\mathbf{c}})^T\boldsymbol{\omega} = 0, \quad i = 1, 2, 3 \tag{8.7a}$$

Upon assembling all three scalar equations above in one single vector equation, we obtain

$$\dot{\mathbf{P}}^T\boldsymbol{\omega} = \mathbf{0} \tag{8.7b}$$

a result that is summarized below:

Theorem 8.2.1. *The angular-velocity vector lies in the null space of matrix* $\dot{\mathbf{P}}^T$, *with* $\dot{\mathbf{P}}$ *defined in Eq. (8.5).*

In order to find the desired expression for $\boldsymbol{\omega}$ from the above equation, we recall Theorem A.1, which is proven in Appendix A: Let $\mathbf{S}$ be a skew-symmetric 3×3 matrix and $\mathbf{A}$ be an arbitrary 3×3 matrix. Then,

$$\text{vect}(\mathbf{SA}) = \frac{1}{2}\left[\text{tr}(\mathbf{A})\mathbf{1} - \mathbf{A}\right]\text{vect}(\mathbf{S}) \tag{8.8}$$

Upon application of the foregoing result, Eq. (8.6) leads to

$$\mathbf{D}\boldsymbol{\omega} = \text{vect}(\dot{\mathbf{P}}) \tag{8.9}$$

where $\mathbf{D}$ is defined below and vect($\boldsymbol{\Omega}$) is nothing but $\boldsymbol{\omega}$:

$$\mathbf{D} \equiv \frac{1}{2}[\mathrm{tr}(\mathbf{P})\mathbf{1} - \mathbf{P}] \tag{8.10}$$

Thus, Eq. (8.9) can be solved for $\boldsymbol{\omega}$ as long as $\mathbf{D}$ is invertible. It is to be expected that, if the three points are collinear, then $\mathbf{D}$ is invertible, but, given that $\mathbf{P}$ is not frame-invariant, neither is tr($\mathbf{P}$). Hence, it is not apparent from Eq. (8.10) that $\mathbf{D}$ is singular when the three given points are collinear. We will discuss this singularity presently.

Now, if tr($\mathbf{P}$) vanishes, $\mathbf{D}$ becomes just one-half the negative of $\mathbf{P}$, which, as we saw above, is singular. Moreover, if the three given points are noncollinear and we assume that the trace of $\mathbf{P}$ does not vanish, then the inverse of $\mathbf{D}$ can be proven to be

$$\mathbf{D}^{-1} = \alpha\mathbf{1} - \beta\mathbf{P}^2 \tag{8.11}$$

where coefficients α and β are given below:

$$\alpha \equiv \frac{2}{\mathrm{tr}(\mathbf{P})}, \quad \beta \equiv \frac{4}{\mathrm{tr}(\mathbf{P})[\mathrm{tr}(\mathbf{P}^2) - \mathrm{tr}^2(\mathbf{P})]} \tag{8.12}$$

From expressions (8.12) it is apparent that $\mathbf{D}$ fails to be invertible not only when tr($\mathbf{P}$) vanishes, but also when the term in brackets in the denominator of β does. In Exercise 8.3, the reader is asked to prove that the foregoing term vanishes whenever the three points are collinear.

From the foregoing discussion, it is apparent that given the velocities and the position vectors of three noncollinear points of a rigid body, the angular velocity of the body can always be determined. However, the data, i.e., the velocities of the three given points, cannot be arbitrary, for they must conform to Eq. (8.6) or, equivalently, to Theorem 8.2.1. Equation (8.6) states that the columns of matrix $\dot{\mathbf{P}}$ must lie in the range of $\boldsymbol{\Omega}$, while Theorem 8.2.1 states that $\boldsymbol{\omega}$ lies in the null space of $\dot{\mathbf{P}}$. However, prior to the computation of $\boldsymbol{\omega}$, or equivalently, of $\boldsymbol{\Omega}$, it is not possible to verify this condition. An alternative approach to verifying the compatibility of the data follows: Since lines P_iC belong to a rigid body, vectors $\mathbf{p}_i - \mathbf{c}$ must remain of the same magnitude throughout a rigid-body motion. Moreover, the angles between any two of the said lines must be preserved throughout the motion as well. This means that the conditions below must hold:

$$(\mathbf{p}_i - \mathbf{c})^T(\mathbf{p}_j - \mathbf{c}) = c_{ij}, \quad i, j = 1, 2, 3 \tag{8.13}$$

or in compact form,

$$\mathbf{P}^T\mathbf{P} = \mathbf{C} \tag{8.14}$$

where the (i, j) entry of the constant matrix $\mathbf{C}$ is c_{ij}, as defined in Eq. (8.13) above. Upon differentiation of both sides of Eq. (8.14) with respect to time, we obtain:

Theorem 8.2.2 (Velocity Compatibility). *The velocities of three points of a rigid body satisfy the compatibility condition:*

$$\dot{\mathbf{P}}^T\mathbf{P} + \mathbf{P}^T\dot{\mathbf{P}} = \mathbf{O} \tag{8.15}$$

with matrices $\mathbf{P}$ *and* $\dot{\mathbf{P}}$ *defined in Eqs. (8.4) and (8.5) and* $\mathbf{O}$ *denoting the* 3×3 *zero matrix.*

The above equation, then, states that for the given velocities of three points of a rigid body to be compatible, the product $\mathbf{P}^T\dot{\mathbf{P}}$ must be skew-symmetric. Note that the above matrix compatibility equation represents six independent scalar equations that the data of the problem at hand must satisfy. There is a tendency to neglect the foregoing six independent scalar compatibility conditions and to focus only on the three scalar conditions drawn from the diagonal entries of the above matrix equation. This is, however, a mistake, for these three conditions do not suffice to guarantee data compatibility in this context; all these three conditions guarantee is that the distance between any pair of points of the set remains constant, but they say nothing about the angles between the pairs of lines formed by each pair of points.

Note, on the other hand, that the product $\mathbf{P}\mathbf{P}^T$ has no direct geometric interpretation, although the difference $\text{tr}(\mathbf{P}\mathbf{P}^T)\mathbf{1} - \mathbf{P}\mathbf{P}^T$ does, as discussed in Exercise 8.9. Furthermore, while Theorem 8.2.2 states that matrix $\mathbf{P}^T\dot{\mathbf{P}}$ is skew-symmetric, it says nothing about the product $\mathbf{P}\dot{\mathbf{P}}^T$. All we can say about this product is stated in the result below:

Theorem 8.2.3. *With matrices* $\mathbf{P}$ *and* $\dot{\mathbf{P}}$ *defined in Eqs. (8.4) and (8.5), the product* $\mathbf{P}\dot{\mathbf{P}}^T$ *obeys the constraint*

$$\text{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 0 \tag{8.16}$$

If $m \times n$ matrices are regarded as forming a vector space, then an *inner product* of two such matrices $\mathbf{A}$ and $\mathbf{B}$, denoted by $(\mathbf{A}, \mathbf{B})$, can be defined as

$$(\mathbf{A}, \mathbf{B}) \equiv \text{tr}(\mathbf{A}\mathbf{B}^T) \tag{8.17}$$

the two matrices being said to be *orthogonal* when the foregoing inner product vanishes. We thus have that Theorem 8.2.3 states that matrices $\dot{\mathbf{P}}$ and $\mathbf{P}$ are orthogonal, a result that parallels that about the orthogonality of the relative velocity of two points and the line joining them, as stated in Eq. (3.51) and summarized in the ensuing theorem. The proof of Theorem 8.2.3 is left as an exercise.

Example 8.2.1. The rigid cube shown in Fig. 8.1 moves in such a way that vertices P_1, P_2, and P_3 undergo the velocities shown in that figure, for three different possible motions. The length of the sides of the cube is 1, and the velocities all have magnitude $\sqrt{2}$ in Fig. 8.1a, c; these velocities are of unit magnitude in Fig. 8.1b.

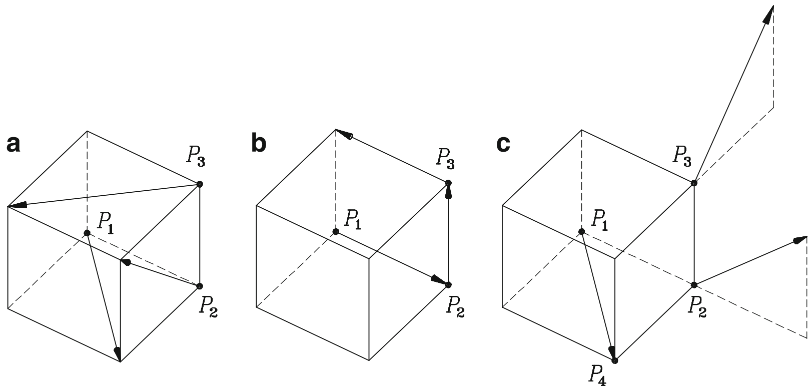

Fig. 8.1 A rigid cube undergoing a motion determined by the velocities of three of its points

Furthermore, in the motion depicted in Fig. 8.1c, the velocity of P_3 is parallel to line P_4P_3, whereas that of P_2 is parallel to line P_1P_3. Out of the three different motions, it is known that at least one is compatible. Identify the compatible motion and compute its angular velocity.

Solution: Let $\dot{\mathbf{p}}_i$ denote the velocity of P_i, of position vector $\mathbf{p}_i$. Each proposed motion is then analyzed: (a) The projection of $\dot{\mathbf{p}}_1$ onto P_1P_2 is 1, but that of $\dot{\mathbf{p}}_2$ onto the same line is 0, and hence, this motion is incompatible; (b) Again, the projection of $\dot{\mathbf{p}}_1$ onto P_1P_2 is 1, but that of $\dot{\mathbf{p}}_2$ onto the same line vanishes, and hence, this motion is also incompatible. Thus, the only possibility is (c), which is now analyzed more formally: Use a dextrous—right-handed—rectangular coordinate frame with origin at P_1, axis Y along P_1P_2, and axis Z parallel to P_2P_3. All vectors and matrices are now represented in this coordinate frame, and hence,

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\dot{\mathbf{p}}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \dot{\mathbf{p}}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{p}}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\mathbf{c} = \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{c}} = \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Now matrices $\mathbf{P}$ and $\dot{\mathbf{P}}$ are constructed:

$$\mathbf{P} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}, \quad \dot{\mathbf{P}} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

Furthermore,

$$\mathbf{P}^T\dot{\mathbf{P}} = \frac{1}{3}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

which is skew-symmetric, and hence, the motion is compatible. Now, matrix $\mathbf{D}$ is computed:

$$\mathbf{D} \equiv \frac{1}{2}[\mathbf{1}\mathrm{tr}(\mathbf{P}) - \mathbf{P}] = \frac{1}{6}\begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

The angular velocity $\boldsymbol{\omega}$ is computed as the solution to

$$\mathbf{D}\boldsymbol{\omega} = \mathrm{vect}(\dot{\mathbf{P}})$$

where

$$\mathrm{vect}(\dot{\mathbf{P}}) = \frac{1}{6}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Equations (8.9) are thus

$$\begin{aligned} 3\omega_1 &= 3 \\ 2\omega_1 + 2\omega_2 - \omega_3 &= -1 \\ \omega_1 + \omega_2 + \omega_3 &= 1 \end{aligned}$$

The first of the foregoing equations leads to

$$\omega_1 = 1$$

whereas the second and the third lead to

$$\begin{aligned} 2\omega_2 - \omega_3 &= -3 \\ \omega_2 + \omega_3 &= 0 \end{aligned}$$

and hence,

$$\omega_2 = -1, \quad \omega_3 = 1$$

Now, as a verification, $\boldsymbol{\omega}$ should be normal to the three columns of $\dot{\mathbf{P}}$ as defined in Eq. (8.7b); in other words, $\boldsymbol{\omega}$ should lie in the null space of $\dot{\mathbf{P}}^T$. But this is so, because

$$\dot{\mathbf{P}}^T\boldsymbol{\omega} = \frac{1}{3}\begin{bmatrix} 3 & 1 & -2 \\ 0 & 1 & 1 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

thereby verifying that $\boldsymbol{\omega}$ lies, in fact, in the null space of $\dot{\mathbf{P}}^{\mathbf{T}}$.

8.2.1 A Robust Formulation

The foregoing formulation fails when matrix $\mathbf{D}$ of Eq. (8.10) becomes singular. While it is not surprising that the matrix becomes singular in the presence of three collinear points, it is a bit frustrating that, even if the points are noncollinear, $\mathbf{D}$ becomes singular when $\text{tr}(\mathbf{P}) = 0$. Moreover, in light of the lack of frame-invariance of $\mathbf{P}$, it is not possible to predict *geometrically* under which conditions $\text{tr}(\mathbf{P})$ vanishes. It is thus imperative to look for an alternative, robust approach, which is the aim of this subsection.

Upon multiplying both sides of Eq. (8.6) by $\mathbf{P}^T$ from the right, we obtain

$$\dot{\mathbf{P}}\mathbf{P}^T = \mathbf{\Omega}\mathbf{R}, \quad \mathbf{R} \equiv \mathbf{P}\mathbf{P}^T \tag{8.18}$$

Further, if we take the vector of both sides of Eq. (8.18), we obtain

$$\frac{1}{2}\mathbf{J}\boldsymbol{\omega} = \text{vect}(\dot{\mathbf{P}}\mathbf{P}^T) \tag{8.19a}$$

where, by application of Theorem A.1, as done above, $\mathbf{J}$ is defined as

$$\mathbf{J} \equiv \text{tr}(\mathbf{R})\mathbf{1} - \mathbf{R} \tag{8.19b}$$

which, as the reader is invited to prove in Exercise 8.9, is nothing but the inertia tensor of a system of three unit-mass particles located at points $\{ P_i \}_1^3$ with respect to their mass center, which coincides with the centroid C of the three given points. As such, matrix $\mathbf{J}$ is, in general, positive-definite, becoming semidefinite only in the special case in which the three masses are collinear. Hence, the *formulation singularity* brought about by the vanishing of $\text{tr}(\mathbf{P})$ is eliminated, which is the reason why this formulation is billed as *robust*. Hence, as long as the three given points are noncollinear, Eq. (8.19a) can always be solved for $\boldsymbol{\omega}$, thus obtaining

$$\boldsymbol{\omega} = 2\mathbf{J}^{-1}\text{vect}(\dot{\mathbf{P}}\mathbf{P}^T) \tag{8.20}$$

Example 8.2.2. Solve Example 8.2.1 with the robust formulation introduced above.

Solution: We need both matrix $\mathbf{J}$ and the right-hand side of Eq. (8.19a). To this end, we compute first

$$\mathbf{R} = \mathbf{P}\mathbf{P}^T = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

which is apparently a simple, positive-semidefinite matrix.[1] Hence,

$$\mathbf{J} = \frac{1}{6}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

whose inverse is readily calculated as

$$\mathbf{J}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

Further,

$$\text{vect}(\dot{\mathbf{P}}\mathbf{P}^T) = \text{vect}\left(\frac{1}{3}\begin{bmatrix} 0 & -3 & -3 \\ 0 & -1 & -2 \\ 0 & 2 & 1 \end{bmatrix}\right) = \frac{1}{6}\begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix}$$

Therefore,

$$\boldsymbol{\omega} = \frac{1}{2}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

thereby completing the calculations, and verifying the result obtained with the non-robust formulation.

[1] $\mathbf{R}$ is apparently singular because it has one row and one column of zeros; it has two positive eigenvalues because its trace is $4/3 > 0$ and the determinant of its 2×2 lower-right block is $1/3 > 0$.

8.3 Computation of Angular Acceleration from Point-Acceleration Data

The angular acceleration of a rigid body under general motion is determined in this section from knowledge of the position, velocity, and acceleration vectors of three noncollinear points of the body. The underlying procedure parallels that of Sect. 8.2. Indeed, recalling the notation introduced in that section, and letting vectors $\ddot{\mathbf{p}}_i$, for $i = 1, 2, 3$, denote the acceleration of the given points, one can rewrite Eq. (3.85) for each point in the form

$$\ddot{\mathbf{p}}_i = \ddot{\mathbf{c}} + (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3 \tag{8.21a}$$

or

$$\ddot{\mathbf{p}}_i - \ddot{\mathbf{c}} = (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)(\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3 \tag{8.21b}$$

where $\mathbf{c}$ was defined in Eq. (8.1), and $\ddot{\mathbf{c}}$ is the acceleration of the centroid, i.e.,

$$\ddot{\mathbf{c}} \equiv \frac{1}{3}\sum_{1}^{3} \ddot{\mathbf{p}}_i \tag{8.21c}$$

Furthermore, matrix $\ddot{\mathbf{P}}$ is defined as

$$\ddot{\mathbf{P}} \equiv \left[\ddot{\mathbf{p}}_1 - \ddot{\mathbf{c}} \;\; \ddot{\mathbf{p}}_2 - \ddot{\mathbf{c}} \;\; \ddot{\mathbf{p}}_3 - \ddot{\mathbf{c}}\right] \tag{8.22}$$

Thus, Eqs. (8.21b) can be written in compact form as

$$\ddot{\mathbf{P}} = (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)\mathbf{P} \tag{8.23}$$

from which one is interested in computing $\dot{\mathbf{\Omega}}$, or correspondingly, $\dot{\boldsymbol{\omega}}$. To this end, Eq. (8.23) is rewritten as

$$\dot{\mathbf{\Omega}}\mathbf{P} = \mathbf{W} \tag{8.24a}$$

with matrix $\mathbf{W}$ defined as

$$\mathbf{W} \equiv \ddot{\mathbf{P}} - \mathbf{\Omega}^2\mathbf{P} \tag{8.24b}$$

The counterpart of Theorem 8.2.1 is now derived from Eqs. (8.21b). First, these equations are cast in the form

$$\ddot{\mathbf{p}}_i - \ddot{\mathbf{c}} - \mathbf{\Omega}^2(\mathbf{p}_i - \mathbf{c}) = \dot{\boldsymbol{\omega}} \times (\mathbf{p}_i - \mathbf{c}), \quad i = 1, 2, 3$$

It is now apparent that if we dot-multiply the above equations by $\dot{\boldsymbol{\omega}}$, we obtain

$$[\ddot{\mathbf{p}}_i - \ddot{\mathbf{c}} - \boldsymbol{\Omega}^2(\mathbf{p}_i - \mathbf{c})] \cdot \dot{\boldsymbol{\omega}} = 0, \quad i = 1, 2, 3 \tag{8.25a}$$

Upon assembling the three foregoing equations in one single vector equation, we derive the counterpart of Eq. (8.7b), namely,

$$(\ddot{\mathbf{P}} - \boldsymbol{\Omega}^2\mathbf{P})^T \dot{\boldsymbol{\omega}} = \mathbf{0} \tag{8.25b}$$

a result that is summarized below in theorem form:

Theorem 8.3.1. *The angular-acceleration vector $\dot{\boldsymbol{\omega}}$ lies in the null space of matrix $\mathbf{W}^T$, with $\mathbf{W}$ defined in Eq. (8.24b).*

Just as we did in Sect. 8.2 when solving for $\boldsymbol{\omega}$ from Eq. (8.9), we apply the result already invoked in connection with Eq. (8.9), thereby deriving an alternative form of Eq. (8.24a), namely,

$$\mathbf{D}\dot{\boldsymbol{\omega}} = \text{vect}(\ddot{\mathbf{P}} - \boldsymbol{\Omega}^2\mathbf{P}) \tag{8.26}$$

where $\mathbf{D}$ is defined as in Eq. (8.10). Thus,

$$\dot{\boldsymbol{\omega}} = \mathbf{D}^{-1}\text{vect}(\ddot{\mathbf{P}} - \boldsymbol{\Omega}^2\mathbf{P}) \tag{8.27}$$

with $\mathbf{D}^{-1}$ given as in Eqs. (8.11) and (8.12). As in Sect. 8.2, then, given the position, velocity, and acceleration vectors of three noncollinear points of a rigid body, it is always possible to compute the associated angular acceleration. However, as discussed in that section, the data cannot be given arbitrarily, for they must comply with Eq. (8.24a), or correspondingly, with Eq. (8.25b). The former implies that the three columns of matrix $\mathbf{W}$ lie in the range of matrix $\dot{\boldsymbol{\Omega}}$; alternatively, Eq. (8.25b) implies that $\dot{\boldsymbol{\Omega}}$ lies in the null space of $\mathbf{W}^T$. Again, prior to the determination of $\dot{\boldsymbol{\Omega}}$, it is impossible to verify this condition, for which reason an alternative approach is taken to verifying compatibility. The obvious one is to differentiate both sides of Eq. (8.15), which produces

$$\ddot{\mathbf{P}}^T\mathbf{P} + 2\dot{\mathbf{P}}^T\dot{\mathbf{P}} + \mathbf{P}^T\ddot{\mathbf{P}} = \mathbf{0} \tag{8.28}$$

thereby deriving the *compatibility conditions* that the acceleration measurements should satisfy.

Finally, upon differentiation of both sides of Eq. (8.16) with respect to time, and while doing this, resorting to Lemma A.2 of Appendix A, we have

$$\text{tr}(\mathbf{P}\ddot{\mathbf{P}}^T + \dot{\mathbf{P}}\dot{\mathbf{P}}^T) = 0 \tag{8.29}$$

which is the counterpart of Eq. (8.16).

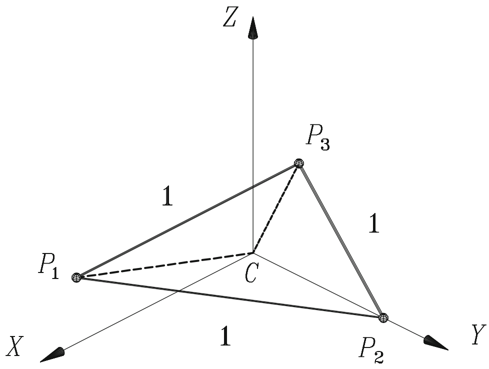

Fig. 8.2 A rigid triangular plate undergoing a motion given by the velocity and acceleration of its vertices

Example 8.3.1. The three vertices of the equilateral triangular plate of Fig. 8.2, which lies in the X–Y plane, are labeled P_1, P_2, and P_3, their position vectors being $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. Moreover, the velocities of the foregoing points are denoted by $\dot{\mathbf{p}}_i$, for $i = 1, 2, 3$. The origin of the coordinate frame X, Y, Z lies at the centroid C of the triangle, the velocities of the vertices, in this coordinate frame, being given as

$$\dot{\mathbf{p}}_1 = \frac{4-\sqrt{2}}{4}\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \dot{\mathbf{p}}_2 = \frac{4-\sqrt{3}}{4}\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \dot{\mathbf{p}}_3 = \frac{4+\sqrt{2}}{4}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Likewise, $\ddot{\mathbf{p}}_1$, $\ddot{\mathbf{p}}_2$, and $\ddot{\mathbf{p}}_3$ denote the accelerations of the three vertices of the plate, given below in the same coordinate frame:

$$\ddot{\mathbf{p}}_1 = \frac{1}{24}\begin{bmatrix}-6+4\sqrt{3}\\12-3\sqrt{2}\\0\end{bmatrix}, \quad \ddot{\mathbf{p}}_2 = -\frac{1}{24}\begin{bmatrix}8\sqrt{3}+3\sqrt{6}\\3\sqrt{3}\\0\end{bmatrix},$$

$$\ddot{\mathbf{p}}_3 = \frac{1}{24}\begin{bmatrix}6+4\sqrt{3}\\-12+3\sqrt{2}\\0\end{bmatrix}$$

With the foregoing information,

(a) show that the three given velocities are compatible;
(b) compute the angular velocity of the plate;
(c) determine the set of points of the plate that undergo a velocity of minimum magnitude;
(d) show that the given accelerations are compatible;
(e) compute the angular acceleration of the plate.

Solution:

(a) Since the centroid of the triangle coincides with that of the three given points, we have $\mathbf{c} = \mathbf{0}$. Moreover,

$$\mathbf{p}_1 = \begin{bmatrix} 1/2 \\ -\sqrt{3}/6 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ \sqrt{3}/3 \\ 0 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/6 \\ 0 \end{bmatrix}$$

Thus,

$$\mathbf{P} = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ -\sqrt{3} & 2\sqrt{3} & -\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Furthermore,

$$\dot{\mathbf{c}} = \begin{bmatrix} 0 \\ 0 \\ (12 - \sqrt{3})/12 \end{bmatrix}$$

and hence,

$$\dot{\mathbf{P}} = \frac{1}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{3} - 3\sqrt{2} & -2\sqrt{3} & \sqrt{3} + 3\sqrt{2} \end{bmatrix}$$

We can readily show from the above results that

$$\mathbf{P}^T\dot{\mathbf{P}} = \mathbf{O}$$

with $\mathbf{O}$ denoting the 3×3 zero matrix. Hence, matrix $\mathbf{P}^T\dot{\mathbf{P}}$ is skew-symmetric and the velocities are compatible

(b) Next, we have

$$\mathbf{D} \equiv \frac{1}{2}[\mathrm{tr}(\mathbf{P})\mathbf{1} - \mathbf{P}] \equiv \frac{1}{12}\begin{bmatrix} 2\sqrt{3} & 0 & 3 \\ \sqrt{3} & 3 & \sqrt{3} \\ 0 & 0 & 3 + 2\sqrt{3} \end{bmatrix}$$

and

$$\mathrm{vect}(\dot{\mathbf{P}}) = \frac{1}{24}\begin{bmatrix} -2\sqrt{3} \\ -\sqrt{3} + 3\sqrt{2} \\ 0 \end{bmatrix}$$

Hence, if the components of $\boldsymbol{\omega}$ in the given coordinate frame are denoted by ω_i, for $i = 1, 2, 3$, then we obtain

$$2\sqrt{3}\omega_1 + 3\omega_3 = -\sqrt{3}$$
$$\sqrt{3}\omega_1 + 3\omega_2 + \sqrt{3}\omega_3 = \frac{-\sqrt{3} + 3\sqrt{2}}{2}$$
$$(3 + 2\sqrt{3})\omega_3 = 0$$

From the third equation,

$$\omega_3 = 0$$

Substitution of the foregoing value into the first of the above equations yields $\omega_1 = -1/2$. Further, upon substitution of the values of ω_1 and ω_3 into the second of the above equations, we obtain $\omega_2 = \sqrt{2}/2$ and hence,

$$\boldsymbol{\omega} = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

(c) Let $\mathbf{p}_0'$ be the position vector of the point P_0' on the instantaneous screw axis lying closest to the origin. Now, in order to find $\mathbf{p}_0'$, we can resort to Eq. (3.70), using point C as a reference, i.e., with $\mathbf{c}$ and $\dot{\mathbf{c}}$ playing the roles of $\mathbf{a}$ and $\dot{\mathbf{a}}$ in that equation. Moreover, since $\mathbf{c} = \mathbf{0}$, the expression for $\mathbf{p}_0'$ reduces to

$$\mathbf{p}_0' = \frac{1}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\Omega}\dot{\mathbf{c}}$$

where from item (b),

$$\|\boldsymbol{\omega}\|^2 = \frac{3}{4}$$

while

$$\boldsymbol{\Omega}\dot{\mathbf{c}} = \frac{12 - \sqrt{3}}{24}\begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

and hence,

$$\mathbf{p}_0' = \frac{12 - \sqrt{3}}{18}\begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

As a verification, $\mathbf{p}_0'$ should be perpendicular to the ISA, as it is, for the product $\boldsymbol{\omega}^T\mathbf{p}_0'$ to vanish. Next, the vector representing the direction of the screw axis is obtained simply as

$$\mathbf{e} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} = \frac{\sqrt{3}}{3}\left[-1\ \sqrt{2}\ 0\right]^T$$

thereby defining completely the instant screw axis.

(d) The acceleration of the centroid of the three given points is given as follows:

$$\ddot{\mathbf{c}} = [-\frac{\sqrt{6}}{24},\ -\frac{\sqrt{3}}{24},\ 0]^T$$

Then, matrices $\ddot{\mathbf{P}}$, $\mathbf{P}^T\ddot{\mathbf{P}}$, $\ddot{\mathbf{P}}^T\mathbf{P}$, and $\dot{\mathbf{P}}^T\dot{\mathbf{P}}$ are readily computed as

$$\ddot{\mathbf{P}} = \frac{1}{24}\begin{bmatrix} -6+4\sqrt{3}+\sqrt{6} & -8\sqrt{3}-2\sqrt{6} & 6+4\sqrt{3}+\sqrt{6} \\ 12-3\sqrt{2}+\sqrt{3} & -2\sqrt{3} & -12+3\sqrt{2}+\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}^T\ddot{\mathbf{P}} = \frac{1}{144}\begin{bmatrix} -21+6\sqrt{6} & 6-24\sqrt{3}-6\sqrt{6} & 15+24\sqrt{3} \\ 6+24\sqrt{3}-6\sqrt{6} & -12 & 6-24\sqrt{3}+6\sqrt{6} \\ 15-24\sqrt{3} & 6+24\sqrt{3}+6\sqrt{6} & -21-6\sqrt{6} \end{bmatrix}$$

$$\ddot{\mathbf{P}}^T\mathbf{P} = \frac{1}{144}\begin{bmatrix} -21+6\sqrt{6} & 6+24\sqrt{3}-6\sqrt{6} & 15-24\sqrt{3} \\ 6-24\sqrt{3}-6\sqrt{6} & -12 & 6+24\sqrt{3}+6\sqrt{6} \\ 15+24\sqrt{3} & 6-24\sqrt{3}+6\sqrt{6} & -21-6\sqrt{6} \end{bmatrix}$$

$$\dot{\mathbf{P}}^T\dot{\mathbf{P}} = \frac{1}{144}\begin{bmatrix} 21-6\sqrt{6} & -6+6\sqrt{6} & -15 \\ -6+6\sqrt{6} & 12 & -6-6\sqrt{6} \\ -15 & -6-6\sqrt{6} & 21+6\sqrt{6} \end{bmatrix}$$

Now, it is a simple matter to verify that

$$\ddot{\mathbf{P}}^T\mathbf{P} + 2\dot{\mathbf{P}}^T\dot{\mathbf{P}} + \mathbf{P}^T\ddot{\mathbf{P}} = \mathbf{O}$$

and hence, the given accelerations are compatible.

(e) $\boldsymbol{\Omega}$ is defined as the unique skew-symmetric matrix whose vector is $\boldsymbol{\omega}$, the latter having been computed in item (b). Thus,

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 1 \\ -\sqrt{2} & -1 & 0 \end{bmatrix},\quad \boldsymbol{\Omega}^2 = \frac{1}{4}\begin{bmatrix} -2 & -\sqrt{2} & 0 \\ -\sqrt{2} & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

$$\mathbf{\Omega}^2\mathbf{P} = \frac{1}{24}\begin{bmatrix} -6+\sqrt{6} & -2\sqrt{6} & 6+\sqrt{6} \\ -3\sqrt{2}+\sqrt{3} & -2\sqrt{3} & 3\sqrt{2}+\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\ddot{\mathbf{P}} - \mathbf{\Omega}^2\mathbf{P} = \frac{1}{24}\begin{bmatrix} 4\sqrt{3} & -8\sqrt{3} & 4\sqrt{3} \\ 12 & 0 & -12 \\ 0 & 0 & 0 \end{bmatrix}$$

The angular-acceleration vector is thus computed from

$$\mathbf{D}\dot{\boldsymbol{\omega}} = \text{vect}(\ddot{\mathbf{P}} - \mathbf{\Omega}^2\mathbf{P})$$

where **D** was computed in item (b), while

$$\text{vect}(\ddot{\mathbf{P}} - \mathbf{\Omega}^2\mathbf{P}) = \frac{1}{12}\begin{bmatrix} 3 \\ \sqrt{3} \\ 3+2\sqrt{3} \end{bmatrix}$$

and hence, letting $\dot{\omega}_i$ denote the ith component of $\dot{\boldsymbol{\omega}}$ in the given coordinate frame, we obtain

$$\frac{1}{12}(2\sqrt{3}\dot{\omega}_1 + 3\dot{\omega}_3) = \frac{1}{4}$$
$$\frac{1}{12}(\sqrt{3}\dot{\omega}_1 + 3\dot{\omega}_2 + \sqrt{3}\dot{\omega}_3) = \frac{\sqrt{3}}{12}$$
$$\frac{1}{12}(3+2\sqrt{3})\dot{\omega}_3 = \frac{3+2\sqrt{3}}{12}$$

which yields

$$\dot{\boldsymbol{\omega}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

thereby completing the solution. Note that $\dot{\boldsymbol{\omega}}$ lies, in fact, in the null space of matrix $(\ddot{\mathbf{P}} - \mathbf{\Omega}^2\dot{\mathbf{P}})^T$.

8.3.1 A Robust Formulation

In computing the angular acceleration vector from point-velocity and -acceleration data, we face exactly the same singularity we faced when computing the angular-velocity vector. We thus follow the same robust approach introduced in Sect. 8.2.1. To this end, we multiply both sides of Eq. (8.24a) by $\mathbf{P}^T$ from the right, thereby obtaining

$$\dot{\mathbf{\Omega}}\mathbf{R} = \mathbf{W}\mathbf{P}^T \tag{8.30}$$

with $\mathbf{R}$ defined already in Eq. (8.18). Moreover, from Eq. (8.24b),

$$\mathbf{W}\mathbf{P}^T = \ddot{\mathbf{P}}\mathbf{P}^T - \mathbf{\Omega}^2\mathbf{R} \tag{8.31}$$

Now, the angular-acceleration vector is computed from Eq. (8.30) upon taking the vector of both sides of this equation, namely,

$$\frac{1}{2}\mathbf{J}\dot{\boldsymbol{\omega}} = \mathrm{vect}(\ddot{\mathbf{P}}\mathbf{P}^T - \mathbf{\Omega}^2\mathbf{R}) \tag{8.32}$$

whence, as long as the three given points are not collinear, $\dot{\boldsymbol{\omega}}$ is computed as

$$\dot{\boldsymbol{\omega}} = 2\mathbf{J}^{-1}\mathrm{vect}(\ddot{\mathbf{P}}\mathbf{P}^T - \mathbf{\Omega}^2\mathbf{R}) \tag{8.33}$$

thereby completing the intended computation.

Example 8.3.2. Using the foregoing robust approach, compute the angular-acceleration vector of the motion undergone by the plate of Fig. 8.2, for the point-velocity and -acceleration data given in Example 8.3.1. Use the value of $\boldsymbol{\omega}$ computed in that example.

Solution: All we need now is $\mathbf{J}$ and the right-hand side of Eq. (8.32). We thus have

$$\mathbf{R} = \mathbf{P}\mathbf{P}^T = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

whence $\mathrm{tr}(\mathbf{R}) = 1$; therefore,

$$\mathbf{J} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Furthermore,

$$\ddot{\mathbf{P}}\mathbf{P}^T = \frac{1}{8}\begin{bmatrix} -2 & -4-\sqrt{2} & 0 \\ 4-\sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

while $\boldsymbol{\Omega}$ is obtained simply as CPM($\boldsymbol{\omega}$), i.e.,

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 1 \\ -\sqrt{2} & -1 & 0 \end{bmatrix}$$

and hence,

$$\boldsymbol{\Omega}^2\mathbf{R} = \frac{1}{8}\begin{bmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

thereby obtaining

$$\ddot{\mathbf{P}}\mathbf{P}^T - \boldsymbol{\Omega}^2\mathbf{R} = \frac{1}{2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

whence,

$$\text{vect}(\ddot{\mathbf{P}}\mathbf{P}^T - \boldsymbol{\Omega}^2\mathbf{R}) = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

which thus yields

$$\dot{\boldsymbol{\omega}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

thereby completing the required computation.

8.4 Exercises

8.1 The regular tetrahedron of Fig. 3.10, of unit-length edges, moves in such a way that vertex P_1 has a velocity of unit magnitude directed from P_1 to P_4, whereas the velocity of P_2 is parallel to edge P_2P_3. Define a coordinate frame X, Y, Z with origin at P_1, Y axis directed from P_1 to the midpoint M of P_2P_3, and X axis in the plane of P_1, P_2, P_3, as shown in that figure. With the above information,

(a) find the velocity of P_2;
(b) show that the velocity of P_3 cannot be zero;
(c) if the velocity of P_3 lies in the $P_1P_2P_3$ plane, find that velocity;
(d) find the angular velocity of the tetrahedron;
(e) find the set of points of the tetrahedron undergoing a velocity of minimum magnitude.

8.2 The position vectors of three points of a rigid body, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, as well as their velocities, $\dot{\mathbf{p}}_1$, $\dot{\mathbf{p}}_2$, and $\dot{\mathbf{p}}_3$, are given below:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\dot{\mathbf{p}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \dot{\mathbf{p}}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad \dot{\mathbf{p}}_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

(a) Is the motion possible?
(b) If the motion is possible, find its angular velocity.

8.3 For matrix $\mathbf{P}$ defined as in Eq. (8.4), i.e., as

$$\mathbf{P} \equiv \begin{bmatrix} \mathbf{p}_1 - \mathbf{c} & \mathbf{p}_2 - \mathbf{c} & \mathbf{p}_3 - \mathbf{c} \end{bmatrix}$$

where $\{\mathbf{p}_k\}_1^3$ are the position vectors of three points of a rigid body, while $\mathbf{c}$ is that of their centroid, prove that $\mathrm{tr}(\mathbf{P}^2) = \mathrm{tr}^2(\mathbf{P})$ whenever the three given points are collinear. Is the converse true?

8.4 With matrix $\mathbf{P}$ defined as in Exercise 8.3 above, prove Theorem 8.2.3. That is, prove that

$$\mathrm{tr}(\mathbf{P}\dot{\mathbf{P}}^T) = 0$$

8.5 With the notation of Sect. 8.3, prove that

$$\mathrm{vect}(\mathbf{\Omega}^2\mathbf{P}) = \dot{\mathbf{D}}\boldsymbol{\omega}$$

8.6 Derive the velocity and acceleration compatibility conditions for a body that is known to undergo spherical motion, i.e., a motion under which one point of the body remains fixed.

8.7 The position vectors of three points of a rigid body, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, are given as in Exercise 8.2, and repeated below for quick reference:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Now, the velocities of these points are all zero, while their accelerations are given as

$$\ddot{\mathbf{p}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \ddot{\mathbf{p}}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad \ddot{\mathbf{p}}_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

(a) Show that the motion is compatible.
(b) Find the angular acceleration of the body.

8.8 With reference to Example 8.2.1, compute the angular acceleration of the cube of Fig. 8.1c if $\ddot{\mathbf{p}}_i = \mathbf{0}$, for $i = 1, 2, 3$.

8.9 With the notation of Sect. 8.2, let

$$\mathbf{R} \equiv \mathbf{P}\mathbf{P}^T$$

(a) Show that the moment of inertia $\mathbf{J}$ of the three given points, which is identical to that of a system of unit masses located at these points, with respect to the centroid C of the given points, is

$$\mathbf{J} = \text{tr}(\mathbf{R})\mathbf{1} - \mathbf{R}$$

(b) Show that if the three given points move as points of a rigid body undergoing an angular velocity $\boldsymbol{\omega}$ whose cross-product matrix is $\boldsymbol{\Omega}$, then

$$\dot{\mathbf{J}} = \mathbf{R}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{R}$$

(c) Furthermore, show that if under the conditions of item (b) above, the set of points undergoes an angular acceleration $\dot{\boldsymbol{\omega}}$ of cross-product matrix $\dot{\boldsymbol{\Omega}}$, then

$$\ddot{\mathbf{J}} = \mathbf{R}\dot{\boldsymbol{\Omega}} - \dot{\boldsymbol{\Omega}}\mathbf{R} - \boldsymbol{\Omega}^2\mathbf{R} - \mathbf{R}\boldsymbol{\Omega}^2 + 2\boldsymbol{\Omega}\mathbf{R}\boldsymbol{\Omega}$$

8.10 A wrench of unknown force $\mathbf{f}$ is applied to a rigid body. In order to find this force, its moment with respect to a set of points $\{ P_k \}_1^3$, of position vectors $\{ \mathbf{p}_k \}_1^3$, is measured and stored in the set $\{ \mathbf{n}_k \}_1^3$. Show that $\mathbf{f}$ can be calculated from the relation

$$\mathbf{Df} = -\text{vect}(\mathbf{M})$$

with $\mathbf{D}$ defined as in Sect. 8.2, i.e., as

$$\mathbf{D} \equiv \frac{1}{2}[\text{tr}(\mathbf{P})\mathbf{1} - \mathbf{P}]$$

and $\mathbf{M}$ given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{n}_1 - \mathbf{n} & \mathbf{n}_2 - \mathbf{n} & \mathbf{n}_3 - \mathbf{n} \end{bmatrix}, \quad \mathbf{n} \equiv \frac{1}{3}\sum_1^3 \mathbf{n}_k$$

Note that $\mathbf{P}$ is defined in Exercise 8.3.

8.11 A wrench is applied to the tetrahedron of Fig. 3.10. When the force of this wrench acts at point P_k, the resulting moment is $\mathbf{n}_k$, for $k = 1, 2, 3$. For the data displayed below, in frame $\mathcal{F}$ of that figure, find the resultant force $\mathbf{f}$, as well as the line of action of this force that will lead to a moment of minimum magnitude. Determine this moment.

$$\mathbf{n}_1 = -\frac{\sqrt{2}}{4}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{n}_2 = \frac{1}{12}\begin{bmatrix} 3\sqrt{2} \\ -2\sqrt{6} \\ 2\sqrt{3} \end{bmatrix}, \quad \mathbf{n}_3 = \frac{1}{12}\begin{bmatrix} 3\sqrt{2} \\ 2\sqrt{6} \\ -2\sqrt{3} \end{bmatrix}$$

8.12 Matrix $\mathbf{D}$, as defined from Eq. (8.6) and displayed in Eq. (8.10), was found to involve frequent singularities, even in the presence of noncollinear points. This weakness stems from its lack of frame-invariance, and can be readily fixed if both sides of Eq. (8.6) are multiplied by $\mathbf{P}^T$ from the right. Show that, under these conditions, an equation similar to (8.9) is derived, but with $\mathbf{D}$ replaced by $(1/2)\mathbf{J}$, with $\mathbf{J}$ defined as in Exercise 8.9. Now show that $\mathbf{J}$ is frame-invariant in the sense of Sect. 2.7, and becomes singular if and only if the three given points are collinear.

8.13 A ball-wheel is used to drive a mobile robot. For feedback control, its angular velocity must be estimated using information on the velocities of two of its points, P_1 and P_2, under the assumption that the ball rolls without slipping on a horizontal, rigid floor. The radius of the wheel is 30 mm, and the two above points lie on a horizontal diameter. Now, define a coordinate frame with origin at the contact point, its Y-axis in the direction from P_1 to P_2 and its Z-axis vertical, as sketched in Fig. 8.3. Off-board sensors provide reliable estimates, in mm/s, of $\dot{\mathbf{p}}_1$ and $\dot{\mathbf{p}}_2$ as displayed below.

$$\dot{\mathbf{p}}_1 = \begin{bmatrix} 120 \\ -60 \\ -60 \end{bmatrix}, \quad \dot{\mathbf{p}}_2 = \begin{bmatrix} 0 \\ -60 \\ 60 \end{bmatrix}$$

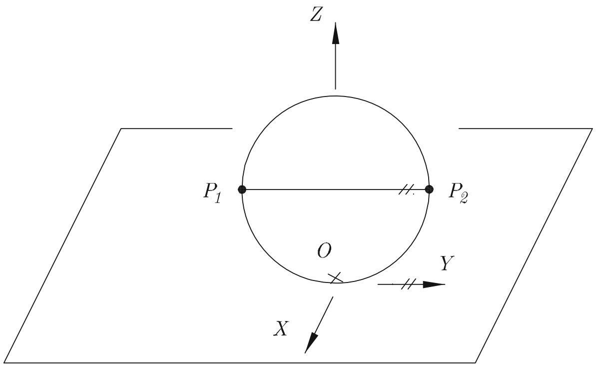

Fig. 8.3 A ball-wheel

(a) Show that the given estimates of $\dot{\mathbf{p}}_1$ and $\dot{\mathbf{p}}_2$ are compatible.
(b) Find the angular velocity of the ball.