CHAPTER 4

Statue of Persian mathematician and poet Omar Khayyám (1048–1131), whose algebra book written in 1070 contained the first serious study of cubic polynomials.

Polynomials

This short chapter contains material on polynomials that we will need to understand operators. Many of the results in this chapter will already be familiar to you from other courses; they are included here for completeness.

Because this chapter is not about linear algebra, your instructor may go through it rapidly. You may not be asked to scrutinize all the proofs. Make sure, however, that you at least read and understand the statements of all the results in this chapter—they will be used in later chapters.

The standing assumption we need for this chapter is as follows:

4.1 Notation $\mathbf{F}$

$\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

LEARNING OBJECTIVES FOR THIS CHAPTER

- Division Algorithm for Polynomials
- factorization of polynomials over $\mathbf{C}$
- factorization of polynomials over $\mathbf{R}$

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Complex Conjugate and Absolute Value

Before discussing polynomials with complex or real coefficients, we need to learn a bit more about the complex numbers.

4.2 Definition $\operatorname{Re} z$, $\operatorname{Im} z$

Suppose $z = a + bi$, where a and b are real numbers.

- The ***real part*** of z, denoted $\operatorname{Re} z$, is defined by $\operatorname{Re} z = a$.
- The ***imaginary part*** of z, denoted $\operatorname{Im} z$, is defined by $\operatorname{Im} z = b$.

Thus for every complex number z, we have

$$z = \operatorname{Re} z + (\operatorname{Im} z)i.$$

4.3 Definition ***complex conjugate, $\bar{z}$, absolute value, $|z|$***

Suppose $z \in \mathbf{C}$.

- The ***complex conjugate*** of $z \in \mathbf{C}$, denoted $\bar{z}$, is defined by
$$\bar{z} = \operatorname{Re} z - (\operatorname{Im} z)i.$$
- The ***absolute value*** of a complex number z, denoted $|z|$, is defined by
$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

4.4 Example Suppose $z = 3 + 2i$. Then

- $\operatorname{Re} z = 3$ and $\operatorname{Im} z = 2$;
- $\bar{z} = 3 - 2i$;
- $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$.

Note that $|z|$ is a nonnegative number for every $z \in \mathbf{C}$.

You should verify that $z = \bar{z}$ if and only if z is a real number.

The real and imaginary parts, complex conjugate, and absolute value have the following properties:

4.5 Properties of complex numbers

Suppose $w, z \in \mathbf{C}$. Then

sum of z and $\bar{z}$

$$z + \bar{z} = 2\operatorname{Re} z;$$

difference of z and $\bar{z}$

$$z - \bar{z} = 2(\operatorname{Im} z)i;$$

product of z and $\bar{z}$

$$z\bar{z} = |z|^2;$$

additivity and multiplicativity of complex conjugate

$$\overline{w + z} = \bar{w} + \bar{z} \text{ and } \overline{wz} = \bar{w}\bar{z};$$

conjugate of conjugate

$$\bar{\bar{z}} = z;$$

real and imaginary parts are bounded by $|z|$

$$|\operatorname{Re} z| \le |z| \text{ and } |\operatorname{Im} z| \le |z|$$

absolute value of the complex conjugate

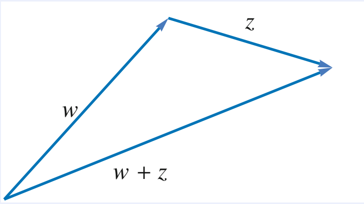
$$|\bar{z}| = |z|;$$

multiplicativity of absolute value

$$|wz| = |w|\,|z|;$$

Triangle Inequality

$$|w + z| \le |w| + |z|.$$



Proof Except for the last item, the routine verifications of the assertions above are left to the reader. To verify the last item, we have

$$\begin{aligned}
|w + z|^2 &= (w + z)(\bar{w} + \bar{z}) \\
&= w\bar{w} + z\bar{z} + w\bar{z} + z\bar{w} \\
&= |w|^2 + |z|^2 + w\bar{z} + \overline{w\bar{z}} \\
&= |w|^2 + |z|^2 + 2\operatorname{Re}(w\bar{z}) \\
&\le |w|^2 + |z|^2 + 2|w\bar{z}| \\
&= |w|^2 + |z|^2 + 2|w|\,|z| \\
&= (|w| + |z|)^2.
\end{aligned}$$

Taking the square root of both sides of the inequality $|w + z|^2 \le (|w| + |z|)^2$ now gives the desired inequality. ■

Uniqueness of Coefficients for Polynomials

Recall that a function $p\colon \mathbf{F} \to \mathbf{F}$ is called a polynomial with coefficients in $\mathbf{F}$ if there exist $a_0, \dots, a_m \in \mathbf{F}$ such that

4.6 $$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$.

4.7 If a polynomial is the zero function, then all coefficients are 0

Suppose $a_0, \dots, a_m \in \mathbf{F}$. If

$$a_0 + a_1 z + \cdots + a_m z^m = 0$$

for every $z \in \mathbf{F}$, then $a_0 = \cdots = a_m = 0$.

Proof We will prove the contrapositive. If not all the coefficients are 0, then by changing m we can assume $a_m \neq 0$. Let

$$z = \frac{|a_0| + |a_1| + \cdots + |a_{m-1}|}{|a_m|} + 1.$$

Note that $z \geq 1$, and thus $z^j \leq z^{m-1}$ for $j = 0, 1, \dots, m-1$. Using the Triangle Inequality, we have

$$\begin{aligned} |a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}| &\leq (|a_0| + |a_1| + \cdots + |a_{m-1}|) z^{m-1} \\ &< |a_m z^m|. \end{aligned}$$

Thus $a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} \neq -a_m z^m$. Hence we conclude that $a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + a_m z^m \neq 0$. ■

The result above implies that the coefficients of a polynomial are uniquely determined (because if a polynomial had two different sets of coefficients, then subtracting the two representations of the polynomial would give a contradiction to the result above).

Recall that if a polynomial p can be written in the form 4.6 with $a_m \neq 0$, then we say that p has degree m and we write $\deg p = m$.

The 0 polynomial is declared to have degree $-\infty$ so that exceptions are not needed for various reasonable results. For example, $\deg(pq) = \deg p + \deg q$ even if $p = 0$.

The degree of the 0 polynomial is defined to be $-\infty$. When necessary, use the obvious arithmetic with $-\infty$. For example, $-\infty < m$ and $-\infty + m = -\infty$ for every integer m.

The Division Algorithm for Polynomials

If p and s are nonnegative integers, with $s \neq 0$, then there exist nonnegative integers q and r such that

$$p = sq + r$$

and $r < s$. Think of dividing p by s, getting quotient q with remainder r. Our next task is to prove an analogous result for polynomials.

The result below is often called the Division Algorithm for Polynomials, although as stated here it is not really an algorithm, just a useful result.

Think of the Division Algorithm for Polynomials as giving the remainder r when p is divided by s.

Recall that $\mathcal{P}(\mathbf{F})$ denotes the vector space of all polynomials with coefficients in $\mathbf{F}$ and that $\mathcal{P}_m(\mathbf{F})$ is the subspace of $\mathcal{P}(\mathbf{F})$ consisting of the polynomials with coefficients in $\mathbf{F}$ and degree at most m.

The next result can be proved without linear algebra, but the proof given here using linear algebra is appropriate for a linear algebra textbook.

4.8 Division Algorithm for Polynomials

Suppose that $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbf{F})$ such that

$$p = sq + r$$

and $\deg r < \deg s$.

Proof Let $n = \deg p$ and $m = \deg s$. If $n < m$, then take $q = 0$ and $r = p$ to get the desired result. Thus we can assume that $n \geq m$.

Define $T : \mathcal{P}_{n-m}(\mathbf{F}) \times \mathcal{P}_{m-1}(\mathbf{F}) \to \mathcal{P}_n(\mathbf{F})$ by

$$T(q, r) = sq + r.$$

The reader can easily verify that T is a linear map. If $(q, r) \in \operatorname{null} T$, then $sq + r = 0$, which implies that $q = 0$ and $r = 0$ [because otherwise $\deg sq \geq m$ and thus sq cannot equal $-r$]. Thus $\dim \operatorname{null} T = 0$ (proving the "unique" part of the result).

From 3.76 we have

$$\dim\big(\mathcal{P}_{n-m}(\mathbf{F}) \times \mathcal{P}_{m-1}(\mathbf{F})\big) = (n - m + 1) + (m - 1 + 1) = n + 1.$$

The Fundamental Theorem of Linear Maps (3.22) and the equation displayed above now imply that $\dim \operatorname{range} T = n + 1$, which equals $\dim \mathcal{P}_n(\mathbf{F})$. Thus $\operatorname{range} T = \mathcal{P}_n(\mathbf{F})$, and hence there exist $q \in \mathcal{P}_{n-m}(\mathbf{F})$ and $r \in \mathcal{P}_{m-1}(\mathbf{F})$ such that $p = T(q, r) = sq + r$. ■

Zeros of Polynomials

The solutions to the equation $p(z) = 0$ play a crucial role in the study of a polynomial $p \in \mathcal{P}(\mathbf{F})$. Thus these solutions have a special name.

4.9 Definition ***zero of a polynomial***

A number $\lambda \in \mathbf{F}$ is called a ***zero*** (or ***root***) of a polynomial $p \in \mathcal{P}(\mathbf{F})$ if

$$p(\lambda) = 0.$$

4.10 Definition ***factor***

A polynomial $s \in \mathcal{P}(\mathbf{F})$ is called a ***factor*** of $p \in \mathcal{P}(\mathbf{F})$ if there exists a polynomial $q \in \mathcal{P}(\mathbf{F})$ such that $p = sq$.

We begin by showing that λ is a zero of a polynomial $p \in \mathcal{P}(\mathbf{F})$ if and only if $z - \lambda$ is a factor of p.

4.11 Each zero of a polynomial corresponds to a degree-1 factor

Suppose $p \in \mathcal{P}(\mathbf{F})$ and $\lambda \in \mathbf{F}$. Then $p(\lambda) = 0$ if and only if there is a polynomial $q \in \mathcal{P}(\mathbf{F})$ such that

$$p(z) = (z - \lambda)q(z)$$

for every $z \in \mathbf{F}$.

Proof One direction is obvious. Namely, suppose there is a polynomial $q \in \mathcal{P}(\mathbf{F})$ such that $p(z) = (z - \lambda)q(z)$ for all $z \in \mathbf{F}$. Then

$$p(\lambda) = (\lambda - \lambda)q(\lambda) = 0,$$

as desired.

To prove the other direction, suppose $p(\lambda) = 0$. The polynomial $z - \lambda$ has degree 1. Because a polynomial with degree less than 1 is a constant function, the Division Algorithm for Polynomials (4.8) implies that there exist a polynomial $q \in \mathcal{P}(\mathbf{F})$ and a number $r \in \mathbf{F}$ such that

$$p(z) = (z - \lambda)q(z) + r$$

for every $z \in \mathbf{F}$. The equation above and the equation $p(\lambda) = 0$ imply that $r = 0$. Thus $p(z) = (z - \lambda)q(z)$ for every $z \in \mathbf{F}$. ■

Now we can prove that polynomials do not have too many zeros.

4.12 A polynomial has at most as many zeros as its degree

Suppose $p \in \mathcal{P}(\mathbf{F})$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zeros in $\mathbf{F}$.

Proof If $m = 0$, then $p(z) = a_0 \neq 0$ and so p has no zeros.

If $m = 1$, then $p(z) = a_0 + a_1 z$, with $a_1 \neq 0$, and thus p has exactly one zero, namely, $-a_0/a_1$.

Now suppose $m > 1$. We use induction on m, assuming that every polynomial with degree $m - 1$ has at most $m - 1$ distinct zeros. If p has no zeros in $\mathbf{F}$, then we are done. If p has a zero $\lambda \in \mathbf{F}$, then by 4.11 there is a polynomial q such that

$$p(z) = (z - \lambda)q(z)$$

for all $z \in \mathbf{F}$. Clearly $\deg q = m - 1$. The equation above shows that if $p(z) = 0$, then either $z = \lambda$ or $q(z) = 0$. In other words, the zeros of p consist of λ and the zeros of q. By our induction hypothesis, q has at most $m - 1$ distinct zeros in $\mathbf{F}$. Thus p has at most m distinct zeros in $\mathbf{F}$. ■

Factorization of Polynomials over C

So far we have been handling polynomials with complex coefficients and polynomials with real coefficients simultaneously through our convention that $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$. Now we will see some differences between these two cases. First we treat polynomials with complex coefficients. Then we will use our results about polynomials with complex coefficients to prove corresponding results for polynomials with real coefficients.

The next result, although called the Fundamental Theorem of Algebra, uses analysis its proof. The short proof presented here uses tools from complex analysis. If you have not had a course in complex analysis, this proof will almost certainly be meaningless to you. In that case, just accept the Fundamental Theorem of Algebra as something that we need to use but whose proof requires more advanced tools that you may learn in later courses.

The Fundamental Theorem of Algebra is an existence theorem. Its proof does not lead to a method for finding zeros. The quadratic formula gives the zeros explicitly for polynomials of degree 2. Similar but more complicated formulas exist for polynomials of degree 3 and 4. No such formulas exist for polynomials of degree 5 and above.

4.13 Fundamental Theorem of Algebra

Every nonconstant polynomial with complex coefficients has a zero.

Proof Let p be a nonconstant polynomial with complex coefficients. Suppose p has no zeros. Then $1/p$ is an analytic function on $\mathbf{C}$. Furthermore, $|p(z)| \to \infty$ as $|z| \to \infty$, which implies that $1/p \to 0$ as $|z| \to \infty$. Thus $1/p$ is a bounded analytic function on $\mathbf{C}$. By Liouville's theorem, every such function is constant. But if $1/p$ is constant, then p is constant, contradicting our assumption that p is nonconstant. ■

Although the proof given above is probably the shortest proof of the Fundamental Theorem of Algebra, a web search can lead you to several other proofs that use different techniques. All proofs of the Fundamental Theorem of Algebra need to use some analysis, because the result is not true if $\mathbf{C}$ is replaced, for example, with the set of numbers of the form $c + di$ where c, d are rational numbers.

> *The cubic formula, which was discovered in the 16th century, is presented below for your amusement only. Do not memorize it.*
>
> *Suppose*
>
> $$p(x) = ax^3 + bx^2 + cx + d,$$
>
> *where* $a \neq 0$. *Set*
>
> $$u = \frac{9abc - 2b^3 - 27a^2d}{54a^3}$$
>
> *and then set*
>
> $$v = u^2 + \left(\frac{3ac - b^2}{9a^2}\right)^3.$$
>
> *Suppose* $v \geq 0$. *Then*
>
> $$-\frac{b}{3a} + \sqrt[3]{u + \sqrt{v}} + \sqrt[3]{u - \sqrt{v}}$$
>
> *is a zero of* p.

Remarkably, mathematicians have proved that no formula exists for the zeros of polynomials of degree 5 or higher. But computers and calculators can use clever numerical methods to find good approximations to the zeros of any polynomial, even when exact zeros cannot be found.

For example, no one will ever be able to give an exact formula for a zero of the polynomial p defined by

$$p(x) = x^5 - 5x^4 - 6x^3 + 17x^2 + 4x - 7.$$

However, a computer or symbolic calculator can find approximate zeros of this polynomial.

The Fundamental Theorem of Algebra leads to the following factorization result for polynomials with complex coefficients. Note that in this factorization, the numbers $\lambda_1, \dots, \lambda_m$ are precisely the zeros of p, for these are the only values of z for which the right side of the equation in the next result equals 0.

4.14 Factorization of a polynomial over $\mathbf{C}$

If $p \in \mathcal{P}(\mathbf{C})$ is a nonconstant polynomial, then p has a unique factorization (except for the order of the factors) of the form

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbf{C}$.

Proof Let $p \in \mathcal{P}(\mathbf{C})$ and let $m = \deg p$. We will use induction on m. If $m = 1$, then clearly the desired factorization exists and is unique. So assume that $m > 1$ and that the desired factorization exists and is unique for all polynomials of degree $m - 1$.

First we will show that the desired factorization of p exists. By the Fundamental Theorem of Algebra (4.13), p has a zero λ. By 4.11, there is a polynomial q such that

$$p(z) = (z - \lambda)q(z)$$

for all $z \in \mathbf{C}$. Because $\deg q = m - 1$, our induction hypothesis implies that q has the desired factorization, which when plugged into the equation above gives the desired factorization of p.

Now we turn to the question of uniqueness. Clearly c is uniquely determined as the coefficient of z^m in p. So we need only show that except for the order, there is only one way to choose $\lambda_1, \ldots, \lambda_m$. If

$$(z - \lambda_1) \cdots (z - \lambda_m) = (z - \tau_1) \cdots (z - \tau_m)$$

for all $z \in \mathbf{C}$, then because the left side of the equation above equals 0 when $z = \lambda_1$, one of the τ's on the right side equals λ_1. Relabeling, we can assume that $\tau_1 = \lambda_1$. Now for $z \neq \lambda_1$, we can divide both sides of the equation above by $z - \lambda_1$, getting

$$(z - \lambda_2) \cdots (z - \lambda_m) = (z - \tau_2) \cdots (z - \tau_m)$$

for all $z \in \mathbf{C}$ except possibly $z = \lambda_1$. Actually the equation above holds for all $z \in \mathbf{C}$, because otherwise by subtracting the right side from the left side we would get a nonzero polynomial that has infinitely many zeros. The equation above and our induction hypothesis imply that except for the order, the λ's are the same as the τ's, completing the proof of uniqueness. ■

Factorization of Polynomials over R

The failure of the Fundamental Theorem of Algebra for **R** *accounts for the differences between operators on real and complex vector spaces, as we will see in later chapters.*

A polynomial with real coefficients may have no real zeros. For example, the polynomial $1 + x^2$ has no real zeros.

To obtain a factorization theorem over **R**, we will use our factorization theorem over **C**. We begin with the following result.

4.15 Polynomials with real coefficients have zeros in pairs

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbf{C}$ is a zero of p, then so is $\bar{\lambda}$.

Proof Let

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m,$$

where $a_0, \ldots, a_m$ are real numbers. Suppose $\lambda \in \mathbf{C}$ is a zero of p. Then

$$a_0 + a_1 \lambda + \cdots + a_m \lambda^m = 0.$$

Take the complex conjugate of both sides of this equation, obtaining

$$a_0 + a_1 \bar{\lambda} + \cdots + a_m \bar{\lambda}^m = 0,$$

where we have used basic properties of complex conjugation (see 4.5). The equation above shows that $\bar{\lambda}$ is a zero of p. ■

Think about the connection between the quadratic formula and 4.16.

We want a factorization theorem for polynomials with real coefficients. First we need to characterize the polynomials of degree 2 with real coefficients that can be written as the product of two polynomials of degree 1 with real coefficients.

4.16 Factorization of a quadratic polynomial

Suppose $b, c \in \mathbf{R}$. Then there is a polynomial factorization of the form

$$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$$

with $\lambda_1, \lambda_2 \in \mathbf{R}$ if and only if $b^2 \geq 4c$.

Proof Notice that

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

*The equation above is the basis of the technique called **completing the square**.*

First suppose $b^2 < 4c$. Then clearly the right side of the equation above is positive for every $x \in \mathbf{R}$. Hence the polynomial $x^2 + bx + c$ has no real zeros and thus cannot be factored in the form $(x-\lambda_1)(x-\lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbf{R}$.

Conversely, now suppose $b^2 \geq 4c$. Then there is a real number d such that $d^2 = \frac{b^2}{4} - c$. From the displayed equation above, we have

$$\begin{aligned} x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 - d^2 \\ &= \left(x + \frac{b}{2} + d\right)\left(x + \frac{b}{2} - d\right), \end{aligned}$$

which gives the desired factorization. ■

The next result gives a factorization of a polynomial over $\mathbf{R}$. The idea of the proof is to use the factorization 4.14 of p as a polynomial with complex coefficients. Complex but nonreal zeros of p come in pairs; see 4.15. Thus if the factorization of p as an element of $\mathcal{P}(\mathbf{C})$ includes terms of the form $(x - \lambda)$ with λ a nonreal complex number, then $(x - \bar{\lambda})$ is also a term in the factorization. Multiplying together these two terms, we get

$$\left(x^2 - 2(\operatorname{Re}\lambda)x + |\lambda|^2\right),$$

which is a quadratic term of the required form.

The idea sketched in the paragraph above almost provides a proof of the existence of our desired factorization. However, we need to be careful about one point. Suppose λ is a nonreal complex number and $(x - \lambda)$ is a term in the factorization of p as an element of $\mathcal{P}(\mathbf{C})$. We are guaranteed by 4.15 that $(x - \bar{\lambda})$ also appears as a term in the factorization, but 4.15 does not state that these two factors appear the same number of times, as needed to make the idea above work. However, the proof works around this point.

In the next result, either m or M may equal 0. The numbers $\lambda_1, \ldots, \lambda_m$ are precisely the real zeros of p, for these are the only real values of x for which the right side of the equation in the next result equals 0.

4.17 Factorization of a polynomial over R

Suppose $p \in \mathcal{P}(\mathbf{R})$ is a nonconstant polynomial. Then p has a unique factorization (except for the order of the factors) of the form

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M),$$

where $c, \lambda_1, \ldots, \lambda_m, b_1, \ldots, b_M, c_1, \ldots, c_M \in \mathbf{R}$, with ${b_j}^2 < 4c_j$ for each j.

Proof Think of p as an element of $\mathcal{P}(\mathbf{C})$. If all the (complex) zeros of p are real, then we are done by 4.14. Thus suppose p has a zero $\lambda \in \mathbf{C}$ with $\lambda \notin \mathbf{R}$. By 4.15, $\bar{\lambda}$ is a zero of p. Thus we can write

$$\begin{aligned} p(x) &= (x - \lambda)(x - \bar{\lambda})q(x) \\ &= \big(x^2 - 2(\operatorname{Re}\lambda)x + |\lambda|^2\big)q(x) \end{aligned}$$

for some polynomial $q \in \mathcal{P}(\mathbf{C})$ with degree two less than the degree of p. If we can prove that q has real coefficients, then by using induction on the degree of p, we can conclude that $(x - \lambda)$ appears in the factorization of p exactly as many times as $(x - \bar{\lambda})$.

To prove that q has real coefficients, we solve the equation above for q, getting

$$q(x) = \frac{p(x)}{x^2 - 2(\operatorname{Re}\lambda)x + |\lambda|^2}$$

for all $x \in \mathbf{R}$. The equation above implies that $q(x) \in \mathbf{R}$ for all $x \in \mathbf{R}$. Writing

$$q(x) = a_0 + a_1 x + \cdots + a_{n-2}x^{n-2},$$

where $n = \deg p$ and $a_0, \ldots, a_{n-2} \in \mathbf{C}$, we thus have

$$0 = \operatorname{Im} q(x) = (\operatorname{Im} a_0) + (\operatorname{Im} a_1)x + \cdots + (\operatorname{Im} a_{n-2})x^{n-2}$$

for all $x \in \mathbf{R}$. This implies that $\operatorname{Im} a_0, \ldots, \operatorname{Im} a_{n-2}$ all equal 0 (by 4.7). Thus all the coefficients of q are real, as desired. Hence the desired factorization exists.

Now we turn to the question of uniqueness of our factorization. A factor of p of the form $x^2 + b_j x + c_j$ with ${b_j}^2 < 4c_j$ can be uniquely written as $(x - \lambda_j)(x - \overline{\lambda_j})$ with $\lambda_j \in \mathbf{C}$. A moment's thought shows that two different factorizations of p as an element of $\mathcal{P}(\mathbf{R})$ would lead to two different factorizations of p as an element of $\mathcal{P}(\mathbf{C})$, contradicting 4.14. ■

EXERCISES 4

1 Verify all the assertions in 4.5 except the last one.

2 Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

3 Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

4 Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

5 Suppose m is a nonnegative integer, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \dots, m + 1$.
[*This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.*]

6 Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

7 Prove that every polynomial of odd degree with real coefficients has a real zero.

8 Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^{\mathbf{R}}$ by

$$Tp = \begin{cases} \dfrac{p - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3. \end{cases}$$

Show that $Tp \in \mathcal{P}(\mathbf{R})$ for every polynomial $p \in \mathcal{P}(\mathbf{R})$ and that T is a linear map.

9 Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q \colon \mathbf{C} \to \mathbf{C}$ by

$$q(z) = p(z)\overline{p(\bar{z})}.$$

Prove that q is a polynomial with real coefficients.

10 Suppose m is a nonnegative integer and $p \in \mathcal{P}_m(\mathbf{C})$ is such that there exist distinct real numbers $x_0, x_1, \dots, x_m$ such that $p(x_j) \in \mathbf{R}$ for $j = 0, 1, \dots, m$. Prove that all the coefficients of p are real.

11 Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.

(a) Show that $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) Find a basis of $\dim \mathcal{P}(\mathbf{F})/U$.