Chapter 5
Controllability and Observability of Linear Systems

In this chapter we begin to study differential systems with inputs and outputs. We focus in particular on the so-called structural properties of finite-dimensional, time invariant linear systems, that is systems of the form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{5.1}$$

where $x \in \mathbf{R}^n$ represents the state of the system, $u \in \mathbf{R}^m$ represents the input and $y \in \mathbf{R}^p$ represents the output (n, m and p arbitrary integers greater than or equal to 1). Throughout this chapter, the admissible inputs are functions $u(\cdot) \in \mathcal{PC}([0, +\infty), \mathbf{R}^m)$. Indeed, the qualifier "structural" identifies properties which depend only on the matrices A, B, C, and so are not affected by possible restrictions on the inputs variables.

5.1 The Reachable Sets

For each admissible input $u(\cdot) \in \mathcal{PC}([0, +\infty), \mathbf{R}^m)$ and for each initial state $x(0) = x_0$, there is a unique solution of the system

$$\dot{x} = Ax + Bu(t) \tag{5.2}$$

denoted by $x(t, x_0, u(\cdot))$, and defined for $t \geq 0$. System (5.2) can be thought of as a linear nonhomogeneous system with forcing term $b(t) = Bu(t)$. Hence, the variation of constants formula applies and we can represent the solution as

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$$x(t, x_0, u(\cdot)) = e^{tA}\left(x_0 + \int_0^t e^{-\tau A} Bu(\tau)\, d\tau\right). \tag{5.3}$$

As already mentioned (Chap. 4), it is natural to think of (5.3) as the sum of

$$x(t, x_0, 0) = e^{tA} x_0 \tag{5.4}$$

also called the *free* (or *unforced*) *solution*, and

$$x(t, 0, u(\cdot)) = \int_0^t e^{(t-\tau)A} Bu(\tau) d\tau\,. \tag{5.5}$$

We emphasize that (5.4) represents the solution corresponding to the input $u = 0$, while (5.5) represents the solution corresponding to the actual input but with zeroed initial state. To this respect, there is an analogue of Proposition A.1.

Proposition 5.1 *For each pair of real numbers $t, \tau \in [0, +\infty)$, for each admissible input $u(\cdot) : [0, +\infty) \to \mathbf{R}^m$, and for each initial state x_0, we have*

$$x(0, x_0, u(\cdot)) = x_0$$

and

$$x(t + \tau, x_0, u(\cdot)) = x(t, x(\tau, x_0, u(\cdot)), w(\cdot))$$

where we set $w(t) = u(t + \tau)$ for $t \in [0, +\infty)$. ■

We now introduce the first important notion of this chapter.

Definition 5.1 Let $x_0, \eta_0 \in \mathbf{R}^n$. We say that η_0 is *reachable* from x_0 at time $T > 0$ (or also that x_0 is *controllable* to η_0 at time T) if there exists an admissible input $u(\cdot) : [0, T] \to \mathbf{R}^m$ such that

$$\eta_0 = x(T, x_0, u(\cdot))\,. \tag{5.6}$$

For fixed x_0 and T, the set of points reachable from x_0 at time T is denoted by $\mathrm{R}(T, x_0)$ and it is called the *reachable set*.

Intuitively, the "size" of the set $\mathrm{R}(T, x_0)$ provides a measure of our ability to control the performances of the system. We are in particular interested in the following definitions.

Definition 5.2 A system of the form (5.1) is said to be:

- *globally reachable* from x_0 at time T if $\mathrm{R}(T, x_0) = \mathbf{R}^n$;
- *globally reachable* at time T if $\mathrm{R}(T, x_0) = \mathbf{R}^n$ for each x_0.

In fact, the two notions introduced in the previous definition are equivalent.

Proposition 5.2 *If there exists a state x_0 such that the system is globally reachable from x_0 at time T, then the system is globally reachable at time T from the origin, as well. If the system is globally reachable from the origin at time T, then it is globally reachable at time T.*

Proof Assume that there exists a point x_0 such that the system is globally reachable at time T from x_0, and let η_0 be an arbitrary point of $\mathbf{R}^n$. Let $\bar{\eta} = \eta_0 + e^{TA}x_0$. By assumption, there exists an input function $u(\cdot)$ such that

$$\bar{\eta} = \eta_0 + e^{TA}x_0 = e^{TA}x_0 + \int_0^T e^{(T-\tau)A}Bu(\tau)\,d\tau\,.$$

This yields

$$\eta_0 = \int_0^T e^{(T-\tau)A}Bu(\tau)\,d\tau\,,$$

meaning that η_0 is reachable from the origin at time T.

Vice versa, assume that the system is globally reachable at time T from the origin. Let x_0 and η_0 be two arbitrary points of $\mathbf{R}^n$. Setting $\bar{\eta} = \eta_0 - e^{TA}x_0$, we can find an input function $u(\cdot)$ such that

$$\bar{\eta} = \eta_0 - e^{TA}x_0 = \int_0^T e^{(T-\tau)A}Bu(\tau)\,d\tau$$

that is

$$\eta_0 = e^{TA}x_0 + \int_0^T e^{(T-\tau)A}Bu(\tau)\,d\tau$$

and this means that η_0 is reachable from x_0 at time T. ■

Remark 5.1 Analogously, we may fix η_0 and T and then we may consider the set of points x_0 for which there exists an admissible input $u(\cdot) : [0, T] \to \mathbf{R}^m$ such that (5.6) holds. This is called the *controllable set* and it is denoted by $\mathrm{C}(T, \eta_0)$. Clearly, $\mathrm{C}(T, \eta_0)$ is nothing else that $\mathrm{R}(T, \eta_0)$ for the reversed time system, obtained replacing A, B by $-A$, $-B$ in (5.1). Indeed, multiplying by e^{-TA} both sides of the equality

$$\eta_0 = e^{TA}x_0 + \int_0^T e^{(T-\tau)A}Bu(\tau)\,d\tau$$

and transforming the integral by the substitution $\tau = T - \theta$, we get

$$x_0 = e^{T(-A)}\eta_0 + \int_0^T e^{(T-\theta)(-A)}(-B)u(T-\theta)\,d\theta\,.$$

■

5.1.1 Structure of the Reachable Sets

The reachability property introduced in the previous section involves uniquely the input variables and the state variables. It does not depend on the matrix C, and therefore it is natural to argue that it can be characterized only in terms of the matrices A and B. Moreover, Proposition 5.2 suggests that our attention can be focused on the set of points reachable from the origin.

Theorem 5.1 *Let the linear system (5.1) be given. The map which associates to each* $u(\cdot) \in \mathcal{PC}([0, +\infty), \mathbf{R}^m)$ *the function*

$$t \mapsto x(t, 0, u(\cdot)) = \int_0^t e^{(t-\tau)A} Bu(\tau)\, d\tau \ \in \ \mathcal{C}([0, +\infty), \mathbf{R}^n) \tag{5.7}$$

is linear.

Proof If $u_1(\cdot), u_2(\cdot) \in \mathcal{PC}([0, +\infty), \mathbf{R}^m)$ and $\alpha, \beta \in \mathbf{R}$, then also $\alpha u_1(\cdot) + \beta u_2(\cdot) \in \mathcal{PC}([0, +\infty), \mathbf{R}^m)$ and, according to the basic properties of the integral,

$$\int_0^t e^{(t-\tau)A} B(\alpha u_1(\tau) + \beta u_2(\tau))\, d\tau$$
$$= \alpha \int_0^t e^{(t-\tau)A} Bu_1(\tau)\, d\tau + \beta \int_0^t e^{(t-\tau)A} Bu_2(\tau)\, d\tau\,.$$

■

Fix now $T > 0$. We can reinterpret (5.7) as a map Λ which associates to each input function $u(\cdot) \in \mathcal{PC}([0, T], \mathbf{R}^m)$ the element of $\mathbf{R}^n$

$$x = \Lambda(u(\cdot)) = x(T, 0, u(\cdot)) = \int_0^T e^{(T-\tau)A} Bu(\tau)\, d\tau \in \mathbf{R}^n\,. \tag{5.8}$$

Corollary 5.1 *The map* $\Lambda : \mathcal{PC}([0, T], \mathbf{R}^m) \to \mathbf{R}^n$ *is linear.*

Corollary 5.2 *For each fixed* $T > 0$*, the set* $\mathrm{R}(T, 0)$ *is a linear subspace of* $\mathbf{R}^n$*. For each* $T > 0$ *and each* $x_0 \neq 0$*, the set* $\mathrm{R}(T, x_0)$ *is a linear manifold of* $\mathbf{R}^n$*.*

Proof For each fixed $T > 0$, the set $\mathrm{R}(T, 0)$ coincides with the image of the operator Λ and hence it is a linear subspace of $\mathbf{R}^n$. As far as the second statement is concerned, it is sufficient to remark that $\mathrm{R}(T, x_0)$ is the translation of $\mathrm{R}(T, 0)$ by means of the vector $v = e^{TA}x_0$. ■

According to these conclusions, it is natural to assume as a measure of the "size" of the set $\mathrm{R}(T, x_0)$ the dimension of $\mathrm{R}(T, x_0)$ as a linear manifold of $\mathbf{R}^n$. Moreover, $\mathrm{R}(T, 0)$ will be often referred to as the *reachable space*.

Corollary 5.3 *System (5.1) is globally reachable at time T if and only if* $\mathrm{R}(T, 0) = \mathbf{R}^n$*, that is if and only if the dimension of* $\mathrm{R}(T, 0)$ *is maximal.*

5.1.2 The Input-Output Map

The previous results enable us to prove Proposition 1.10. From (5.3), given any admissible input $u(\cdot) : [0, +\infty) \to \mathbf{R}^m$ and any initial state x_0, the following representation for the output function of system (5.1)

$$y(t, x_0, u(\cdot)) = Cx(t, x_0, u(\cdot)) = Ce^{tA}\big(x_0 + \int_0^t e^{-\tau A} Bu(\tau)\, d\tau\big) \tag{5.9}$$

can be readily deduced. Of course, $y(0, x_0, u(\cdot)) = Cx_0$ and

$$y(t, x_0, u(\cdot)) = y(t, x_0, 0) + y(t, 0, u(\cdot))\,. \tag{5.10}$$

Proof of Proposition 1.10 The map which associates to each $u(\cdot) \in \mathcal{PC}([0, +\infty), \mathbf{R}^m)$ the function $x(t, 0, u(\cdot))$ is linear, by virtue of Theorem 5.1. Hence, the map which associates to $u(\cdot)$ the function $y(t, 0, u(\cdot)) = Cx(t, 0, u(\cdot))$ is linear, as well. On the other hand, also the map which associates to x_0 the function $y(t, x_0, 0) = Ce^{tA}x_0$ is linear. To finish, it is sufficient to take into account (5.10) and the fact that if $f_1 : V_1 \to W$, $f_2 : V_2 \to W$ are linear maps, then $f_1 + f_2 : V_1 \times V_2 \to W$ is a linear map. ■

5.1.3 Solution of the Reachability Problem

Next theorem provides a first necessary and sufficient condition for the global reachability of a linear system.

Theorem 5.2 *System (5.1) is globally reachable at time $T > 0$ if and only if the matrix*

$$\Gamma(T) = \int_0^T e^{-\tau A} BB^{\mathbf{t}} e^{-\tau A^{\mathbf{t}}}\, d\tau$$

is nonsingular.

Proof First we show that if $\Gamma(T)$ is nonsingular, then for each pair of states $x_0, \eta_0 \in \mathbf{R}^n$ there exists an input function $u(\cdot)$ for which (5.6) holds. Let, for $\tau \in [0, T]$,

$$u(\tau) = -B^{\mathbf{t}} e^{-\tau A^{\mathbf{t}}} \Gamma^{-1}(T)[x_0 - e^{-TA}\eta_0] \tag{5.11}$$

and compute

$$\begin{aligned} e^{TA}x_0 + \int_0^T e^{(T-\tau)A} Bu(\tau)\, d\tau \qquad (5.12) \\ = e^{TA}x_0 - e^{TA}\left[\int_0^T e^{-\tau A} BB^{\mathbf{t}} e^{-\tau A^{\mathbf{t}}}\, d\tau\right] \Gamma^{-1}(T)[x_0 - e^{-TA}\eta_0] \\ = e^{TA}x_0 - e^{TA}\Gamma(T)\Gamma^{-1}(T)[x_0 - e^{-TA}\eta_0] = e^{TA}x_0 - e^{TA}x_0 + \eta_0\, . \end{aligned}$$

In conclusion,

$$e^{TA}x_0 + \int_0^T e^{(T-\tau)A} Bu(\tau)\, d\tau = \eta_0\, .$$

In order to prove the converse, we need some preliminary remarks. Clearly, $\Gamma(T)$ is symmetric, and the quadratic form

$$\xi^{\mathbf{t}}\Gamma(T)\xi = \int_0^T ||B^{\mathbf{t}} e^{-\tau A^{\mathbf{t}}}\xi||^2\, d\tau \qquad (5.13)$$

is, in general, positive semidefinite. If $\Gamma(T)$ is singular, then there exists a point $x_0 \in \mathbf{R}^n$ ($x_0 \neq 0$) such that ${x_0}^{\mathbf{t}}\Gamma(T)x_0 = 0$. Therefore, taking into account (5.13), we have $B^{\mathbf{t}}e^{-\tau A^{\mathbf{t}}}x_0 = 0$, identically for $\tau \in [0, T]$. The global reachability assumption implies that starting from x_0 it is possible to reach the origin at time T. This yields

$$e^{TA}x_0 = -\int_0^T e^{(T-\tau)A} Bu(\tau)\, d\tau \qquad (5.14)$$

for some admissible input $u(\cdot)$. From (5.14) it follows

$$x_0 = -\int_0^T e^{-\tau A} Bu(\tau)\, d\tau$$

and so

$$\begin{aligned} ||x_0||^2 = {x_0}^{\mathbf{t}}x_0 = -\left(\int_0^T e^{-\tau A} Bu(\tau)\, d\tau\right)^{\mathbf{t}} x_0 \\ = -\int_0^T u^{\mathbf{t}}(\tau) B^{\mathbf{t}} e^{-\tau A^{\mathbf{t}}} x_0\, d\tau = 0\, . \end{aligned}$$

This contradicts the assumption $x_0 \neq 0$. ■

Remark 5.2 Formula (5.11) provides an answer to the problem of determining a control function which allows us to steer the system from the state x_0 to the state η_0. ■

5.1.4 The Controllability Matrix

The condition stated in Theorem 5.2 is useful for theoretical developments, but not easy to apply in practice. From this point of view, the criterion we are going to present in this section is more convenient, since it amounts to purely algebraic computations involving only the matrices A and B which define the system.

Theorem 5.3 *For a system of the form (5.1), the set* $\mathrm{R}(T, 0)$ *is independent of* T. *Moreover, for each* $T > 0$ *we have*

$$\mathrm{R}(T, 0) = V \tag{5.15}$$

where

$$V = \text{span}\left\{b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, \ldots, A^{n-1}b_1, \ldots, A^{n-1}b_m\right\} \tag{5.16}$$

and $b_1, \ldots, b_m$ *denote the columns of* B.

Proof Since both sides of (5.15) are subspaces of $\mathbf{R}^n$, it is sufficient to prove that the respective orthogonal spaces coincide. First we prove that $V \subset \mathrm{R}(T, 0)$. Let $\mu \neq 0$ be a vector orthogonal to $\mathrm{R}(T, 0)$. Then we have, for each admissible input,

$$\begin{aligned} 0 &= \mu^{\mathbf{t}} \int_0^T e^{(T-\tau)A} Bu(\tau)\, d\tau \\ &= \int_0^T \mu^{\mathbf{t}} e^{(T-\tau)A} Bu(\tau)\, d\tau = \int_0^T \mu^{\mathbf{t}} e^{\theta A} Bu(T-\theta)\, d\theta\,. \end{aligned} \tag{5.17}$$

Taking into account (5.17), now we show that

$$\mu^{\mathbf{t}} e^{\theta A} Bu = 0 \tag{5.18}$$

for each $\theta \in (0, T)$ and each $u \in \mathbf{R}^m$. Assume that this is false. Then we can find $\bar{\theta} \in (0, T)$ and $\bar{u} \in \mathbf{R}^m$ such that $\mu^{\mathbf{t}} e^{\bar{\theta} A} B\bar{u} \neq 0$ (say for instance, $\mu^{\mathbf{t}} e^{\bar{\theta} A} B\bar{u} > 0$). Then, by continuity, there exists $\delta > 0$ such that $(\bar{\theta} - \delta, \bar{\theta} + \delta) \subset (0, T)$ and the function

$$\theta \mapsto \mu^{\mathbf{t}} e^{\theta A} B\bar{u}$$

takes positive values for $\bar{\theta} - \delta < \theta < \bar{\theta} + \delta$. Setting $\bar{\tau} = T - \bar{\theta}$, we can therefore define

$$u(\tau) = \begin{cases} \bar{u} & \text{for } \bar{\tau} - \delta < \tau < \bar{\tau} + \delta \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.19}$$

Then,

$$u(T-\theta) = \begin{cases} \bar{u} & \text{for } \bar{\theta} - \delta < \theta < \bar{\theta} + \delta \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.20}$$

This yields

$$\int_0^T \mu^{\mathbf{t}} e^{\theta A} Bu(T-\theta)\, d\theta = \int_{\bar\theta-\delta}^{\bar\theta+\delta} \mu^{\mathbf{t}} e^{\theta A} B\bar u\, d\theta > 0$$

and we have a contradiction to (5.17). Hence, (5.18) is true. Taking the limit for $\theta \to 0^+$, we get

$$\mu^{\mathbf{t}} Bu = 0 \quad \forall u \in \mathbf{R}^m .$$

Choosing respectively $u = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, this last expression indicates that μ is orthogonal to $b_1, \dots, b_m$. Moreover, taking the derivative of (5.18) with respect to θ, we have

$$\mu^{\mathbf{t}} e^{\theta A} ABu = 0 \quad \forall \theta \in (0, T) \quad \text{and} \quad \forall u \in \mathbf{R}^m$$

which in turn implies, for $\theta \to 0^+$,

$$\mu^{\mathbf{t}} ABu = 0 .$$

Repeating the previous reasoning, we see that μ is orthogonal to the vectors $Ab_1, \dots, Ab_m$, as well. The procedure can be iterated, until the conclusion is achieved.

Now we prove the opposite inclusion. Let μ be orthogonal to

$$b_1, \dots, b_m, Ab_1, \dots, Ab_m, \dots, A^{n-1}b_1, \dots, A^{n-1}b_m .$$

For each $u \in \mathbf{R}^m$, we have

$$\mu^{\mathbf{t}} Bu = \cdots = \mu^{\mathbf{t}} A^{n-1} Bu = 0 .$$

Moreover

$$\begin{aligned} \mu^{\mathbf{t}} e^{\theta A} Bu &= \mu^{\mathbf{t}} \sum_{i=0}^{n-1} \frac{\theta^i A^i}{i!} Bu + \mu^{\mathbf{t}} \sum_{i=n}^{\infty} \frac{\theta^i A^i}{i!} Bu \\ &= \sum_{i=0}^{n-1} \frac{\theta^i}{i!} \mu^{\mathbf{t}} A^i Bu + \sum_{i=n}^{\infty} \frac{\theta^i}{i!} \mu^{\mathbf{t}} A^i Bu . \end{aligned}$$

Clearly, the terms of the first sum vanish. But also the terms of the second sum vanish since, by Cayley-Hamilton Theorem, for each $i \geq n$, the vector $A^i Bu$ is a linear combination of the vectors $A^i Bu$ with $i < n$. In conclusion, $\mu^{\mathbf{t}} e^{\theta A} Bu = 0$, for all $\theta \in [0, T]$ and all $u \in \mathbf{R}^m$. But then also

$$\mu^{\mathbf{t}} \int_0^T e^{(T-\tau)A} Bu(\tau) d\tau = 0$$

for every $u(\cdot) \in \mathcal{PC}([0, T], \mathbf{R}^m)$. The theorem is finally proved. ■

Definition 5.3 System (5.1) is said to be *completely controllable* when

$$\text{rank } (B|AB|\ldots|A^{n-1}B) = n \tag{5.21}$$

where $(B|AB|\ldots|A^{n-1}B)$ is the matrix with n rows and nm columns formed by the columns of the matrices $B, AB, \ldots, A^{n-1}B$.

The matrix $(B|AB|\ldots|A^{n-1}B)$ is called the *controllability matrix* of system (5.1). The following corollary is a straightforward consequence of Theorem 5.3.

Corollary 5.4 *System (5.1) is completely controllable if and only if it is globally controllable for some (and hence for each)* $T > 0$.

Remark 5.3 The vectors $v_1, \ldots, v_n$ of $\mathbf{R}^n$ form a linearly independent set if and only if

$$\det (v_1|\ldots|v_n) \neq 0\,.$$

Since the determinant depends continuously on the entries of the matrix, replacing the vectors $v_1, \ldots, v_n$ by some other vectors $\tilde{v}_1, \ldots, \tilde{v}_n$ such that $\tilde{v}_k$ is sufficiently close to v_k (for every $k = 1, \ldots, n$), then also the vectors $\tilde{v}_1, \ldots, \tilde{v}_n$ form a linearly independent set.

From this remark it follows that if system (5.1) is completely controllable and if the matrices $\tilde{A}, \tilde{B}$ are sufficiently close to, respectively, A and B, then the system defined by the matrices $\tilde{A}, \tilde{B}$ is completely controllable, as well. It is also clear that if system (5.1) is not completely controllable, then there exist pairs of matrices $\tilde{A}, \tilde{B}$ arbitrarily close to A, B, such that the system defined by $\tilde{A}, \tilde{B}$ is completely controllable. In other words, we can say that "generically", any linear system is completely controllable, in the sense that:

- the complete controllability property is preserved under arbitrary small perturbations of the coefficients;
- the complete controllability property can be achieved by means of suitable small perturbations of the coefficients.

These considerations can be also resumed by saying that complete controllability is an open-dense property. ■

Remark 5.4 If the input of system (5.1) is scalar i.e., $m = 1$, matrix B reduces to a single column b and the controllability matrix is square. Checking the complete controllability condition reduces to compute the determinant of the controllability matrix. It is easily seen that such a system is completely controllable if and only if b is cyclic for A (see Sect. 2.10).

Consider in particular a SISO system defined by a linear nonhomogeneous equation of order n

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = u(t) \tag{5.22}$$

where the forcing term plays the role of a (scalar) input, and y is reviewed as a (scalar) output. According to the procedure illustrated in Sects. 2.10 and 4.3, (5.22) can be rewritten in the form (5.1) with A a companion matrix,

$$b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and C reduced to the row $(1\ 0 \ldots 0)$. The state variable coincides with the vector $x = (y, y', \ldots, y^{(n-1)})$.

We can easily check that every system of the form (5.22) is completely controllable regardless the choice of the coefficients $a_1, \ldots, a_n$. ■

In force of the conclusions of Theorem 5.3, we can slightly simplify our notation: from now on, we write R instead of R$(T, 0)$.

5.1.5 Hautus' Criterion

Conditions equivalent to complete controllability of the system (5.1) can be given in several different forms. In this section we present a criterion which will be sometimes recalled in our future developments.

Theorem 5.4 (Hautus' criterion) *System (5.1) is completely controllable if and only if*

$$\forall \lambda \in \mathbf{C}, \qquad \text{rank}\,(A - \lambda I | B) = n\,. \tag{5.23}$$

We remark that (5.23) is trivially fulfilled if λ is not an eigenvalue of A. Note also that in general, $(A - \lambda I | B)$ is a matrix with complex entries. In order to prove the theorem, it is therefore advisable to interpret also A and B as operators acting on complex spaces.

Definition 5.4 A subspace V of $\mathbf{C}^n$ is called a *(complex) algebraic invariant* for A if $AV \subseteq V$.

Lemma 5.1 *If a subspace V is an algebraic invariant for A, then there exists an eigenvector $v \neq 0$ of A such that $v \in V$.*

Proof Let $q = \dim V$ and let $v_1, \ldots, v_n$ be a basis of $\mathbf{C}^n$, such that its first q elements $v_1, \ldots, v_q$ constitute a basis of V. With respect to this basis, A takes the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

by virtue of the invariance assumption. The operator acting from V to V and defined by the matrix A_{11} will necessarily have at least one eigenvector $v \in V$. It is not difficult to check that the same vector v, reinterpreted as an element of $\mathbf{C}^n$, is an eigenvector of A corresponding to the same eigenvalue λ. ■

Proof of Theorem 5.4 We show that (5.21) implies (5.23). Assume by contradiction that for some $\lambda \in \mathbf{C}$, the n rows of matrix $(A - \lambda I|B)$ are linearly independent. Then, there exists a vector $\eta \in \mathbf{C}^n$ $(\eta \neq 0)$ such that

$$\eta^{\mathbf{t}} A = \lambda \eta^{\mathbf{t}} \quad \text{and} \quad \eta^{\mathbf{t}} B = 0\,.$$

In particular, the function

$$\varphi(t) = e^{\lambda t} \eta^{\mathbf{t}} B = (e^{\lambda t} \eta)^{\mathbf{t}} B = \eta^{\mathbf{t}} B + \lambda t \eta^{\mathbf{t}} B + \frac{\lambda^2 t^2}{2} \eta^{\mathbf{t}} B + \cdots$$

must vanish. Thus we will have

$$\varphi(0) = \varphi'(0) = \varphi''(0) = \cdots = 0\,.$$

By applying the theorem about the derivative of a power series, and taking into account

$$\eta^{\mathbf{t}} A = \lambda \eta^{\mathbf{t}} \Longrightarrow \eta^{\mathbf{t}} A^2 = \lambda \eta^{\mathbf{t}} A = \lambda^2 \eta^{\mathbf{t}} \quad \text{etc.}$$

we finally obtain

$$\eta^{\mathbf{t}} B = \eta^{\mathbf{t}} AB = \eta^{\mathbf{t}} A^2 B = \cdots = 0\,.$$

This implies that the n rows of matrix $(B|AB|\ldots|A^{n-1}B)$ are linearly dependent, so that its rank is not equal to n.

Finally, we show that (5.23) implies (5.21). According to Cayley-Hamilton Theorem, if (5.21) is false then the rows of all the matrices of the form $A^j B$ $(j = 0, 1, \ldots)$ will belong to a same proper subspace of $\mathbf{C}^n$. In other words, we could find a vector $v \neq 0$ such that

$$v^{\mathbf{t}} B = v^{\mathbf{t}} AB = v^{\mathbf{t}} A^2 B = \cdots = 0\,. \tag{5.24}$$

Setting $w = A^{\mathbf{t}} v$, we have

$$w^{\mathbf{t}} B = (A^{\mathbf{t}} v)^{\mathbf{t}} B = v^{\mathbf{t}} AB = 0, \quad w^{\mathbf{t}} AB = (A^{\mathbf{t}} v)^{\mathbf{t}} AB = v^{\mathbf{t}} A^2 B = 0, \quad \text{etc.}$$

Denoted by V the subspace of $\mathbf{C}^n$ constituted by all vectors v for which (5.24) holds, we have so proved that if $v \in V$, also $w = A^{\mathbf{t}} v \in V$, and this in turn implies that V is an algebraic invariant with respect to the linear operator associated to matrix $A^{\mathbf{t}}$. Then by Lemma 5.1, it must exists a nonzero vector $\eta \in V$ and a complex number $\lambda \in \mathbf{C}$ such that

$$A^{\mathbf{t}}\eta = \lambda\eta \quad \text{that is} \quad \eta^{\mathbf{t}} A = \lambda\eta^{\mathbf{t}} .$$

As a consequence of the definition of V, we have in particular that $\eta^{\mathbf{t}} B = 0$. In conclusion, the rows of $(A - \lambda I | B)$ are linearly independent and (5.23) does not hold. ■

5.2 Observability

In common applications, the output variable does not coincide with the state variable. In these cases, the observability function plays an essential role.

Definition 5.5 We say that two points $x_0, \eta_0 \in \mathbf{R}^n$ are *indistinguishable* at time T if for each admissible input $u(\cdot) : [0, T] \to \mathbf{R}^n$ one has

$$y(t, x_0, u(\cdot)) = y(t, \eta_0, u(\cdot)) \qquad \forall t \in [0, T] .$$

The previous definition is inspired by the following idea: for each fixed input function $u(\cdot)$, if the initial state x_0 is replaced by η_0, then the system response remains unchanged. In other words, it is not possible in general to reconstruct exactly the initial state on the base of information obtained uniquely by monitoring the output corresponding to a known input.

Example 5.1 Consider the system

$$\begin{cases} \dot{x}_1 = x_1 + u \\ \dot{x}_2 = x_2 \end{cases}$$

with $y = x_1 - x_2$. The solution corresponding to an initial state of the form (a, a) is easily found:

$$x_1 = e^t \left(a + \int_0^t e^{-\tau} u(\tau) d\tau \right), \quad x_2 = a e^t .$$

Hence, we see that $y(t) = e^t \int_0^t e^{-\tau} u(\tau) d\tau$ is independent of a. In other words, two distinct arbitrary points on the line $x_1 = x_2$ are indistinguishable. ■

Let us emphasize that in practical applications, the knowledge of the initial state is an important issue. Assume that we have a physical system, and that a mathematical model has been constructed. In principle, the mathematical model should be used to simulate the evolution of the physical system and to predict the future behavior. To this end, we need to integrate analytically or numerically the system equations. But this is impossible, if we do not known how to set the initial state of the model, which should be the same as the initial state of the physical system.

5.2.1 The Unobservability Space

Our aim now is to characterize those systems for which there exist no pairs of indistinguishable points. First of all, we remark that in Definition 5.5, the role of the input function is unessential, in the sense explained by the following proposition.

Proposition 5.3 *The points x_0 and η_0 are indistinguishable at time T for the system (5.1) if and only if they are indistinguishable at time T for the unforced system*

$$\begin{cases} \dot{x} = Ax \\ y = Cx\,. \end{cases} \tag{5.25}$$

Proof If x_0 and η_0 are indistinguishable at time T for the system (5.1), then for each $u(\cdot) \in \mathcal{PC}([0,T], \mathbf{R}^m)$ and each $t \in [0,T]$ we have

$$y(t, x_0, u(\cdot)) = y(t, \eta_0, u(\cdot))$$

that is

$$C[e^{tA}(x_0 + \int_0^t e^{-\tau A} Bu(\tau)\,d\tau)] = C[e^{tA}(\eta_0 + \int_0^t e^{-\tau A} Bu(\tau)\,d\tau)]\,.$$

Getting rid of the common term, we obtain the identity

$$Ce^{tA}x_0 = Ce^{tA}\eta_0$$

for each $t \in [0,T]$. This actually means that x_0 and η_0 are indistinguishable with respect to the system (5.25). The reverse argument proves the vice versa. ■

Next proposition points out that in order to characterize the set of points which are indistinguishable from a fixed $x \in \mathbf{R}^n$, it is sufficient to characterize the set of points which are indistinguishable from the origin.

Proposition 5.4 *If x_0, η_0 are indistinguishable at the time $T > 0$, then $\xi = x_0 - \eta_0$ is indistinguishable from the origin at time T. Vice versa, if ξ is indistinguishable from the origin at the time T and if x_0 is any vector of $\mathbf{R}^n$, then x_0 and $\eta_0 = x_0 + \xi$ are indistinguishable each other at time T.*

Proof By virtue of Proposition 5.3, we can refer to system (5.25). From the assumption that x_0 and η_0 are indistinguishable at time T, we deduce that

$$Ce^{tA}x_0 = Ce^{tA}\eta_0$$

for each $t \in [0, T]$. This last equality rewrites

$$Ce^{tA}(\eta_0 - x_0) = 0 = Ce^{tA}0$$

for each $t \in [0, T]$. The statement follows, setting $\xi = x_0 - \eta_0$. Vice versa, if ξ is indistinguishable from the origin (which means that $Ce^{tA}\xi = 0$ for each $t \in [0, T]$), for each $x_0 \in \mathbf{R}^n$ the equality

$$Ce^{tA}(\xi + x_0) = Ce^{tA}\xi + Ce^{tA}x_0 = Ce^{tA}x_0$$

holds for each $t \in [0, T]$. This means that $\xi + x_0$ and x_0 are indistinguishable at time T for system (5.25), and so also for system (5.1). ■

We denote by $\mathrm{N}(T, 0)$ the set of the states indistinguishable from the origin at time T. A first characterization of $\mathrm{N}(T, 0)$ is provided by the following Theorem.

Theorem 5.5 *The following statements are equivalent.*

(i) *ξ is indistinguishable from the origin at time T for system (5.1);*
(ii) *$\xi \in \ker\ Ce^{tA}$, $\forall t \in [0, T]$;*
(iii) *the output function of system (5.1) corresponding to the input $u(t) = 0$ for each $t \in [0, T)$ and to the initial state ξ, vanishes on $[0, T]$.*

Proof The equivalence between *(i)* and *(ii)* follows from Proposition 5.3. The equivalence between *(ii)* and *(iii)* is straightforward. ■

Theorem 5.5 *(ii)* implies in particular that the set $\mathrm{N}(T, 0)$ coincides with

$$\bigcap_{t\in[0,T]} \ker Ce^{tA}\,. \tag{5.26}$$

But (5.26) is a subspace of $\mathbf{R}^n$. Hence $\mathrm{N}(T, 0)$ is a subspace of $\mathbf{R}^n$. It is called the *unobservability space*.

5.2.2 *The Observability Matrix*

Now, consider the matrices

$$C^{\mathbf{t}}, A^{\mathbf{t}}C^{\mathbf{t}}, \ldots, (A^{\mathbf{t}})^{n-1}C^{\mathbf{t}}\,.$$

Their columns can be interpreted as vectors of $\mathbf{R}^n$. Let $V \subset \mathbf{R}^n$ be the space engendered by these vectors.

Theorem 5.6 *Given the system (5.1), the unobservability space is independent of T. In fact, for each $T > 0$ one has* $\mathrm{N}(T, 0) = V^{\perp}$.

Proof We limit ourselves to sketch the main steps, since the proof is similar to that of Theorem 5.3. Let $v \in \mathrm{N}(T, 0)$. Then for all $\theta \in [0, T]$ and all $\mu \in \mathbf{R}^p$

$$(Ce^{\theta A}v)^{\mathbf{t}}\mu = 0$$

or

$$v^{\mathbf{t}}e^{\theta A^{\mathbf{t}}}C^{\mathbf{t}}\mu = 0\,.$$

For $\theta = 0$, we find $v^{\mathbf{t}}C^{\mathbf{t}}\mu = 0$ and, being μ arbitrary, v is orthogonal to all the columns of $C^{\mathbf{t}}$. Next step is to compute iteratively the derivatives of any order with respect to θ. Each derivative is evaluated at $\theta = 0$. Vice versa, if $v \in V^{\perp}$, then for each $\mu \in \mathbf{R}^p$

$$v^{\mathbf{t}}C^{\mathbf{t}}\mu = \cdots = v^{\mathbf{t}}(A^{\mathbf{t}})^{n-1}C^{\mathbf{t}}\mu = 0\,.$$

Making use, as in Theorem 5.3, of the series expansion of the exponential and of Cayley-Hamilton Theorem, this implies in turn that $v^{\mathbf{t}}e^{\theta A^{\mathbf{t}}}C^{\mathbf{t}}\mu = 0$ for each θ. Finally,

$$(Ce^{\theta A}v)^{\mathbf{t}}\mu = 0 \qquad \forall\theta \in \mathbf{R}$$

which implies $Ce^{\theta A}v = 0$, for each $\theta \in \mathbf{R}$. ■

Matrix $(C^{\mathbf{t}}|A^{\mathbf{t}}C^{\mathbf{t}}|\ldots|(A^{\mathbf{t}})^{n-1}C^{\mathbf{t}})$ is called the *observability matrix* of system (5.1). From now on, since $\mathrm{N}(T, 0)$ is independent of T, we write simply N.

Definition 5.6 The system is said to be *completely observable* when

$$\operatorname{rank}\,(C^{\mathbf{t}}|A^{\mathbf{t}}C^{\mathbf{t}}|\ldots|(A^{\mathbf{t}})^{n-1}C^{\mathbf{t}}) = n\,. \tag{5.27}$$

Corollary 5.5 *System (5.1) is completely observable if and only if for each pair of indistinguishable states x_0, η_0 we have $x_0 = \eta_0$, or, equivalently, when* $\mathrm{N} = \{0\}$.

Remark 5.5 Neither *(ii)* of Theorem 5.5 nor (5.27) depend on matrix B. This is not surprising, if we have in mind Proposition 5.3. ■

Remark 5.6 Since the unobservability space N does not depend on T, we may say that a point $x_0 \in \mathrm{N}$ if and only if $Ce^{\theta A}x_0 = 0$ for each $\theta \geq 0$. On the other hand, from Theorem 5.6 it follows that if we replace A by $-A$, the space N does not change. Hence, the previous statement can be strengthened, writing that $x_0 \in \mathrm{N}$ if and only if $Ce^{\theta A}x_0 = 0$ for each $\theta \in \mathbf{R}$. This implies in turn that N is dynamically invariant (compare with Definition A.4) with respect to the unforced system. Indeed, if $x_0 \in \mathrm{N}$ and $\eta = e^{tA}x_0$ for $t \in \mathbf{R}$, we have

$$Ce^{\theta A}\eta = Ce^{\theta A}e^{tA}x_0 = Ce^{(\theta+t)A}x_0 = 0$$

for each $\theta \in \mathbf{R}$, and so $\eta \in$ N. ■

Remark 5.7 It is easily checked that a system (5.1) defined by means of a scalar linear differential equation of order n,

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = u(t)$$

where u is interpreted as the input and y as the output, is completely observable, regardless the choice of the coefficients $a_1, \ldots, a_n$. ■

5.2.3 Reconstruction of the Initial State

For a fixed input function $u(\cdot) : [0, +\infty) \to \mathbf{R}^m$, let us look at the map which associates the output function $y(t)$ to a given initial state $x(0) = x_0$. Complete observability of system (5.1) implies that such a map is injective. Hence, we expect that monitoring $y(t)$ on the interval $[0, T]$ (for some $T > 0$) provides sufficient information in order to recover the exact value of x_0. Next we show how this can be actually done. Assume that $u(t)$ and $y(t)$ are known for $t \in [0, T]$. Recall that

$$y(t) = C\left[e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau\right].$$

Multiplying both sides by $e^{tA^{\mathbf{t}}}C^{\mathbf{t}}$ we get

$$e^{tA^{\mathbf{t}}}C^{\mathbf{t}}y(t) = e^{tA^{\mathbf{t}}}C^{\mathbf{t}}Ce^{tA}x_0 + e^{tA^{\mathbf{t}}}C^{\mathbf{t}}C\int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

and integrating from 0 to T:

$$E(T)x_0 = \int_0^T e^{tA^{\mathbf{t}}}C^{\mathbf{t}}y(t)\,dt - \int_0^T e^{tA^{\mathbf{t}}}C^{\mathbf{t}}C\left(\int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau\right)dt\,, \quad (5.28)$$

where we set $E(T) = \int_0^T e^{tA^{\mathbf{t}}}C^{\mathbf{t}}Ce^{tA}\,dt$.

Theorem 5.7 *The following properties are equivalent.*

(i) System (5.1) is completely observable.
(ii) Matrix $E(T)$ is positive definite for each $T > 0$, and so invertible.

Proof A simple computation shows that

$$\xi^{\mathbf{t}}E(T)\xi = \int_0^T ||Ce^{tA}\xi||^2\,dt \geq 0\,.$$

The integral on the right-hand side is zero if and only if the integrand vanishes, that is if and only if $\xi \in \ker C e^{tA}$ for each $t \in [0, T]$. If the system is completely observable, this may happen only if $\xi = 0$. The opposite statement can be easily proven by contradiction. ■

Therefore, if (5.1) is completely observable, from (5.28) it is possible to exactly compute x_0 taking the inverse of the matrix $E(T)$. We emphasize that in this procedure the input is absolutely arbitrary: the more natural choice is, of course, $u(t) = 0$ for each $t \in [0, T]$.

The approach to the observability problem described in this section has a drawback: indeed, (5.28) may be sometimes hard to apply, because of the need of computing the exponential matrix, some integrals and an inverse matrix.

5.2.4 Duality

The analogies between the notions of controllability and observability are evident. We point out that the observability matrix of system (5.1) coincides with the controllability matrix of system

$$\begin{cases} \dot{x} = A^{\mathbf{t}} x + C^{\mathbf{t}} u \\ y = B^{\mathbf{t}} x \end{cases} \tag{5.29}$$

where $u \in \mathbf{R}^p$ and $y \in \mathbf{R}^m$. Note that with respect to (5.1), the roles of B and C are exchanged. Thus, (5.1) is completely controllable if and only if (5.29) is completely observable and vice versa. System (5.29) is called the *dual* of (5.1). The properties of complete controllability and complete observability are also said to be dual properties.

We emphasize also the analogies (and the differences) between Theorems 5.2 and 5.7.

5.3 Canonical Decompositions

In the analysis of a system, it is important to find out certain canonical forms; they are particular representations which make possible to understand at a first glance the main structural properties of the system. This requires the search for suitable changes of coordinates.

5.3.1 Linear Equivalence

The systems

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{z} = \tilde{A}z + \tilde{B}u \\ y = \tilde{C}z \end{cases}$$

are called *linearly equivalent* if there exists a linear change of coordinates $x = Pz$ ($\det P \neq 0$) such that $\tilde{A} = P^{-1}AP$, $\tilde{B} = P^{-1}B$, and $\tilde{C} = CP$. We recognize in this definition a generalization of a notion already introduced for linear unforced system, and fruitfully applied in Chap. 2.

We remark that such a transformation does not affect the controllability properties of the system. Indeed, one has

$$\begin{aligned}(\tilde{B}|\,\tilde{A}\tilde{B}|\ldots|\tilde{A}^{n-1}\tilde{B}) &= (P^{-1}B|P^{-1}AB|\ldots|P^{-1}A^{n-1}B) \\ &= P^{-1}(B|AB|\ldots|A^{n-1}B)\end{aligned}$$

so that the controllability matrices of any pair of linearly equivalent systems have the same rank. Moreover, the subspaces engendered by the columns of these matrices (that is, the controllability spaces of these systems) are consistently transformed each other by the change of coordinates.

Similar conclusions can be achieved, of course, about the observability property and the unobservability space.

5.3.2 Controlled Invariance

Before to introduce the first important canonical form, we still need a definition. A subspace $W \subset \mathbf{R}^n$ is said to be a *controlled invariant* for system (5.1) if for each $x_0 \in W$ and for each admissible input we have:

$$x(t, x_0, u(\cdot)) \in W \qquad \forall t > 0\,.$$

We recognize in this definition an extension of Definition A.4. The space R is an example of controlled invariant. Indeed, assume by contradiction that there exist a point $x_0 \in$ R and an input function $u(\cdot) : [0, T] \to \mathbf{R}^m$ such that $x(T, x_0, u(\cdot)) = \eta \notin$ R. If $u_0(\cdot) : [0, t_0] \to \mathbf{R}^m$ is an input function for which $x(t_0, 0, u_0(\cdot)) = x_0$ (at least one such input function exists by hypothesis) we can take the new input

$$\tilde{u}(\tau) = \begin{cases} u_0(\tau) & \text{for} \quad 0 \le \tau < t_0 \\ u(\tau - t_0) & \text{for} \quad t_0 \le \tau \le t_0 + T\,. \end{cases}$$

Clearly $x(t_0 + T, 0, \tilde{u}(\cdot)) = \eta$, and this is a contradiction.

Lemma 5.2 *Let the subspace $W \subset \mathbf{R}^n$ be a controlled invariant for system (5.1) and let*

$$x(t, x_0, u(\cdot))$$

a trajectory with $x_0 \in W$. Then the tangent vector of the curve $t \mapsto x(t, x_0, u(\cdot))$ for $t = 0$ belongs to W, as well.

Proof Let us consider the difference quotient

$$q(t) = \frac{x(t, x_0, u(\cdot)) - x_0}{t}$$

for $t \neq 0$. The tangent vector is defined as $\lim_{t \to 0} q(t)$. Since W is a subspace, $q(t) \in W$ for each $t \neq 0$. Hence, the limit must belong to W, as well, being a subspace a closed set. ■

5.3.3 Controllability Form

Theorem 5.8 *There exist a change of coordinates $x = Pz$ and an integer q $(0 \leq q \leq n)$ such that in the new coordinates z the system (5.1) takes the form*

$$\begin{cases} \dot{z}_1 = A_{11} z_1 + A_{12} z_2 + B_1 u \\ \dot{z}_2 = \qquad\quad A_{22} z_2 \end{cases} \tag{5.30}$$

where $z = (z_1, z_2)$ with $z_1 \in \mathbf{R}^q$, $z_2 \in \mathbf{R}^{n-q}$, and where A_{11}, A_{12}, A_{22}, and B_1 are matrices of suitable dimensions. Moreover, the system

$$\dot{z}_1 = A_{11} z_1 + B_1 u \tag{5.31}$$

with state variable $z_1 \in \mathbf{R}^q$, is completely controllable.

Proof Let $q = \dim \mathrm{R}$. The limit cases $q = 0$ and $q = n$ correspond respectively to the cases where (5.1) is completely uncontrollable (that is $B = 0$) and the case where (5.1) is completely controllable. So, we can limit ourselves to assume $(0 < q < n)$. Consider a basis of the state space, such that the first q vectors form a basis of R. Let z be the coordinates in this new basis, partitioned in such a way that $\mathrm{R} = \{z : z_2 = 0\}$. In general, the representation of the system in these new coordinates can be written

$$\begin{cases} \dot{z}_1 = A_{11} z_1 + A_{12} z_2 + B_1 u \\ \dot{z}_2 = A_{21} z_1 + A_{22} z_2 + B_2 u\,. \end{cases}$$

We show that, according to the particular choice of the basis, $A_{21} = B_2 = 0$. Recall that R is a controlled invariant, and notice that this property does not depend on the choice of the coordinates. Assume that $A_{21} \neq 0$. Let us take an initial state $(z_1, 0) \in \mathrm{R}$ with $z_1 \neq 0$, $A_{21}z_1 \neq 0$. Let moreover $u(t) = 0$ for each $t \geq 0$. The tangent vector to the corresponding solution, evaluated at $t = 0$, is

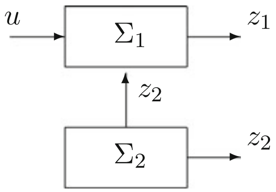
$$\begin{pmatrix} A_{11}z_1 \\ A_{21}z_1 \end{pmatrix} \notin \mathrm{R}\,.$$

This is a contradiction to Lemma 5.2. Hence $A_{21} = 0$. In a similar way, it is possible to show also that $B_2 = 0$. In practice, the matrix P which determines the change of coordinates can be written as

$$P = (v_1|\dots|v_q|v_{q+1}|\dots|v_n)$$

where $v_1, \dots, v_q$ are chosen in such a way that they form a basis of R, and $v_{q+1}, \dots, v_n$ in such a way that they form, together with $v_1, \dots, v_q$, a basis of $\mathbf{R}^n$. Notice that the basis $v_1, \dots, v_n$ is not uniquely determined by this construction. For instance, it is not restrictive (in fact, for future developments, it is strongly recommended) to take the vectors $v_1, \dots, v_n$ pairwise orthogonal.

It remains to prove that (5.31) is completely controllable. Let $z_1 \in \mathbf{R}^q$ be given. By construction, there exists an input function $u(\cdot)$ such that the corresponding solution of system (5.30) steers the origin of $\mathbf{R}^n$ to the state $(z_1, 0) \in \mathrm{R}$. Obviously, the same input applied to system (5.31) steers the origin of $\mathbf{R}^q$ in z_1. Hence, (5.31) is completely controllable. ■



The Eq. (5.31) can be obtained from (5.30) setting $z_2 = 0$. It can be therefore interpreted as a subsystem: it is called the *controllable part* of the overall system (in the figure above, it is denoted by Σ_1). Notice that the evolution of the component z_2 of the state in (5.30) does not depend at all on the action of the input function. It represents the *uncontrollable part* of the system (in the figure above, it is denoted by Σ_2).

The form (5.30) reveals the structure of the system, and in particular it allows us to separate and recognize the controllable and uncontrollable parts.

5.3.4 Observability Form

An analogous construction, based on linear equivalence, allows us to obtain a form of system (5.1) which reveals the observability properties. More precisely, it is possible to prove the existence of an integer $r (0 \le r \le n)$ and a nonsingular matrix P such that the change of coordinates $x = Pz$ gives rise to the form

$$\begin{cases} \dot{z}_1 = A_{11}z_1 + A_{12}z_2 + B_1 u \\ \dot{z}_2 = \qquad\qquad A_{22}z_2 + B_2 u \\ y \ \ = C_2 z_2 \end{cases} \tag{5.32}$$

where $z = (z_1, z_2)$, $z_1 \in \mathbf{R}^r$, $z_2 \in \mathbf{R}^{n-r}$, and the reduced order system

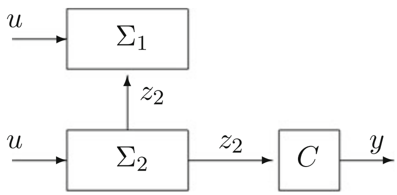
$$\begin{cases} \dot{z}_2 = A_{22}z_2 + B_2 u \\ y \ \ = C_2 z_2 \end{cases} \tag{5.33}$$

with state variable $z_2 \in \mathbf{R}^{n-r}$, is completely observable.

Note that if we put $z_2 = 0$ in the differential part of (5.32), the resulting reduced order system

$$\dot{z}_1 = A_{11}z_1 + B_1 u \tag{5.34}$$

with state variable $z_1 \in \mathbf{R}^r$, does not produce any output. The reduced order systems (5.33) and (5.34) are called, respectively, the *observable part* (denoted by Σ_2 in the figure below) and the *unobservable part* (denoted by Σ_1 in the figure below) of the system.



However, this time the construction of the matrix P which determines the change of coordinates is more delicate. We start by computing the observability matrix. Let $n - r$ be its rank. Choose $n - r$ linearly independent columns of the observability matrix, and let us denote them by $v_{r+1}, \dots, v_n$. Choose finally r linearly independent vectors $v_1, \dots, v_r$ such that the subspace generated by $v_1, \dots, v_r$ is orthogonal[1] to the subspace generated by $v_{r+1}, \dots, v_n$. According to Theorem 5.6, the vector $v_1, \dots, v_r$ constitute a basis of the non-observability subspace N. A possible choice of P is the matrix whose columns are

$$P = (v_1 | \dots | v_r | v_{r+1} | \dots | v_n) .$$

[1]We stress that in general, the construction does not work if the orthogonality requirement is neglected: this is an important difference with respect to the construction of the controllability form. The reason of this fact is implicit in the statement of Theorem 5.6.

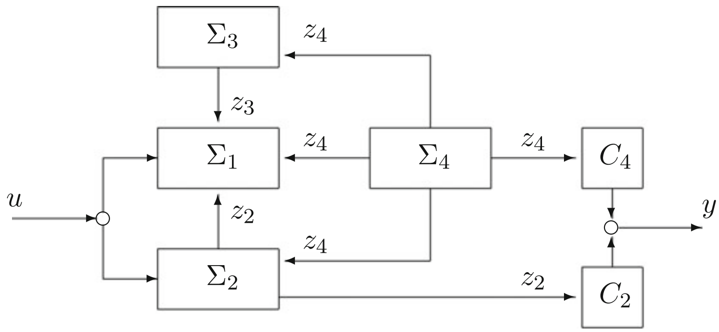
The construction can be easily modified, in order to replace P by a new matrix whose columns are all pairwise orthogonal. To prove that, after the change of coordinates, the block A_{21} becomes zero, we may argue as in the proof of Theorem 5.8, setting $u = 0$ and taking into account Remark 5.6.

5.3.5 Kalman Decomposition

The controllability form and the observability form discussed in the previous sections can be combined, giving rise to the form

$$\begin{cases} \dot z_1 = A_{11}z_1 + A_{12}z_2 + A_{13}z_3 + A_{14}z_4 + B_1u \\ \dot z_2 = \qquad\quad A_{22}z_2 \qquad\qquad\quad + A_{24}z_4 + B_2u \\ \dot z_3 = \qquad\qquad\qquad\qquad A_{33}z_3 + A_{34}z_4 \\ \dot z_4 = \qquad\qquad\qquad\qquad\qquad\quad A_{44}z_4 \\ y \;= C_2z_2 + C_4z_4\,. \end{cases} \tag{5.35}$$

The special structure exhibited by (5.35) corresponds to the connections displayed in the figure below, where by Σ_1, Σ_2, Σ_3, Σ_4 we have denoted the reduced order systems which determine the evolutions of the blocks of coordinates z_1, z_2, z_3, z_4, respectively.



We remark in particular that:

- the block of coordinates (z_1, z_2) identifies the completely controllable part: the form (5.30) is recovered as a particular case, by collecting the blocks of coordinates (z_1, z_2), (z_3, z_4);
- the block of coordinates (z_2, z_4) identifies the completely observable part: the form (5.32) is recovered by collecting the blocks of coordinates (z_1, z_3), (z_2, z_4) and rewriting the equations, after reordering the indices in the following way: 1, 3, 2, 4;
- the block of coordinates z_2 identified the completely controllable and completely observable part;
- the block of coordinates z_3 identifies the uncontrollable and unobservable part.

These remarks are trivial, with the exception of the third one, for which we provide a sketch of proof. Because of the block-triangular form of the matrix, we realize that the controllability matrix for the block of coordinates z_2 can be obtained taking suitable submatrices of the controllability matrix for the block (z_1, z_2). Such a matrix has a maximal rank, since the block (z_1, z_2) corresponds to the completely controllable part of the overall system. But this is possible only if the rank of the controllability matrix for the block z_2 is maximal. The complete observability is proved in similar way.

5.3.6 Some Examples

In order to illustrate the construction of the canonical forms presented in the previous sections we present some examples.

Example 5.2 Consider the system with scalar input defined by the matrices

$$A = \begin{pmatrix} 1 & 4 \\ 2 & -6 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 1 \end{pmatrix} .$$

We do not need to specify C, since in this example we will be interested only in the controllability form. The controllability matrix is $(b|Ab) = \begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}$ and its rank is equal to 1. The system is not completely controllable and we can proceed to the determination of a controllability form. We perform a change of coordinates by means of the matrices

$$P = \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix} \qquad P^{-1} = \frac{1}{17} \begin{pmatrix} 4 & 1 \\ -1 & 4 \end{pmatrix}$$

(notice that the columns of P are orthogonal). We obtain, as desired,

$$P^{-1}AP = \begin{pmatrix} 2 & -2 \\ 0 & -7 \end{pmatrix} \qquad P^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

If we chose a different matrix, with the first column parallel to b and the second column linearly independent (but not necessarily orthogonal) to the first one, we obtain again a controllability form, which in general may differ from the previous one for some unessential details. For instance, with

$$Q = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$

we have

$$Q^{-1}AQ = \begin{pmatrix} 2 & 2 \\ 0 & -7 \end{pmatrix} \qquad Q^{-1}b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

■

Example 5.3 Now consider the system with scalar input and scalar output defined by the matrices

$$A = \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad c = (1 \ \ 1)\,.$$

This system is completely controllable but not completely observable. Indeed, since $A^{\mathbf{t}}c^{\mathbf{t}} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$, the rank of the observability matrix is 1. We can proceed to determine an observability canonical form. Define a change of coordinates by the matrices

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We get

$$P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix} \qquad P^{-1}b = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad cP = (0 \ \ 2)\,.$$

Note that the columns of P are orthogonal. Note also that with this procedure, the unobservability space becomes coincident with the first component. ■

Example 5.4 Consider finally the system with scalar input and scalar output defined by the matrices

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad c = (-1 \ \ 2)\,.$$

In this case, both the controllability and the observability matrices have rank 1. The system is neither completely controllable nor completely observable. It is convenient to start by computing an observability form. However, now it is preferable to proceed in a lightly different way. Making use of the change of coordinates defined by the matrices

$$P = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}, \tag{5.36}$$

we may align, in the new coordinates, the unobservability space with the second component instead of the first one, as we did in Example 5.3. We have:

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} = \hat{A} \quad P^{-1}b = \frac{1}{5}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \hat{b} \quad cP = (5 \ \ 0) = \hat{c}\,.$$

Notice that the columns of P are orthogonal. Notice also that this form does not allow to identify immediately the controllability space. Thus, we need to apply a further change of coordinates, to the purpose of achieving a complete Kalman decomposition. More precisely, we need to find a change of coordinates which, while keeping unchanged the position of the unobservability space, superposes the controllability space to the subspace orthogonal to the unobservability one (that is, with the first component). We can take:

$$Q = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad Q^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

(the first column of Q is parallel to $\hat{b}$, the second generates the unobservability space in the new coordinates). We finally obtain the required form:

$$Q^{-1}\hat{A}Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad Q^{-1}\hat{b} = \frac{1}{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{c}P = (5\ 0)\,.$$

It is now evident, in particular, that the system possesses a completely controllable and completely observable part, and a uncontrollable and unobservable part.

Of course, the transformation can be accomplished by a unique change of coordinates defined by the matrix

$$PQ = \begin{pmatrix} 5 & 2 \\ 5 & 1 \end{pmatrix}\,.$$

■

Example 5.5 As a last example we take the same matrices A and c as in Example 5.4, but

$$b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\,.$$

By the first change of coordinates given by (5.36), we get

$$P^{-1}b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,.$$

The controllability space is already coincident with the unobservability space. Of course, now it is not possible, by a further change of coordinates, to move the controllability space in such a way that it becomes orthogonal to the unobservability space. The complete Kalman form has been obtained with the first change of coordinates. The system possesses a controllable part which is not observable, and an observable part which is not controllable. ■

5.4 Constrained Controllability

One of the most important developments of control theory is optimization. Typical examples are the minimal time problem [16] and the quadratic regulator problem [6]. In particular, minimal time problems are strictly related to the controllability properties of the system. However, a minimal time problem makes sense only if the admissible control functions are constrained to take values in a bounded set. Although optimization is beyond the purposes of this book, in this section we discuss shortly how the geometric properties of the reachable sets change when the admissible controls are subject to constraints. Thus, in this section we consider linear systems

$$\dot{x} = Ax + Bu \tag{5.37}$$

with $x \in \mathbf{R}^n$ and $u \in U$, where U represents a nonempty, proper (in general, bounded) subset of $\mathbf{R}^m$. To begin with, we need to update the notion of reachability introduced in the previous Sect. 5.1 and the related notation.

Let $x_0, \eta_0 \in \mathbf{R}^n$. We say that η_0 is reachable from x_0 at time $T > 0$ with constrained controls if there exists $u(\cdot) \in \mathcal{PC}([0, +\infty), U)$ such that $x(T, x_0, u(\cdot)) = \eta_0$, and we denote $\mathrm{R}(T, x_0, U)$ the set of such points. This is called the reachable set of (5.37) with constrained controls.

The geometric properties of $\mathrm{R}(T, x_0, U)$ depend not only on the pair of matrices A and B, but also on the set U. For instance, it is no more true in general that $\mathrm{R}(T, 0, U)$ is a subspace of $\mathbf{R}^n$. As a consequence, we should enrich the notion of reachability introducing some new definitions.

Definition 5.7 A system (5.37), with admissible inputs constrained to a set U, is said to be:

- *accessible* from x_0 at time T when $\overset{\circ}{\mathrm{R}}(T, x_0, U) \neq \emptyset$;
- *locally reachable* from x_0 at time T when $x_0 \in \overset{\circ}{\mathrm{R}}(T, x_0, U)$;
- *locally reachable along the free solution* from x_0 at time T if $x(T, x_0, 0) \in \overset{\circ}{\mathrm{R}}(T, x_0, U)$.

Notice that when $x_0 = 0$, the definitions of local reachability and local reachability along the free solution coincide.

Example 5.6 Consider the simple scalar system

$$\dot{x} = x + u, \qquad x, u \in \mathbf{R}.$$

Let $u_0 > 0$ be fixed and assume that the control functions are subject to the constraint $|u(t)| \leq u_0$. Solving the equation, we have for $t \geq 0$

$$x(t) = e^t x_0 + \int_0^t e^{(t-\tau)} u(\tau)\, d\tau$$

that is

$$e^t x_0 - \int_0^t e^{(t-\tau)} u_0\, d\tau \leq x(t) \leq e^t x_0 + \int_0^t e^{(t-\tau)} u_0\, d\tau$$

namely

$$e^t(x_0 - u_0) + u_0 \le x(t) \le e^t(x_0 + u_0) - u_0$$

and finally

$$e^t x_0 - (e^t - 1)u_0 \le x(t) \le e^t x_0 + (e^t - 1)u_0 .$$

This shows that the system is accessible and locally reachable along the free solution, for each choice of x_0 and u_0 (recall that if $t \ge 0$ then $e^t - 1 \ge 0$).

If $-u_0 < x_0 < u_0$, then x_0 belongs to the interval $(e^t(x_0 - u_0) + u_0, e^t(x_0 + u_0) - u_0) = \mathring{\mathrm{R}}(t, x_0, U)$ for each t. Hence, the system is also locally reachable at x_0. Moreover, $x_0 - u_0 < 0 < x_0 + u_0$ so that $\mathrm{R}(x_0, U) = \mathbf{R}$. On the contrary, if $x_0 \ge u_0$ then $\mathrm{R}(x_0, U)$ coincides with the half line $[x_0, +\infty)$. In this case the system is not locally reachable. The conclusion is the same for $x_0 \le -u_0$. ■

The study of the reachability properties of a linear system with constrained input can be actually reduced to the case $x_0 = 0$.

Proposition 5.5 *Let a system of the form (5.37) be given, with admissible control functions constrained to a subset $U \subset \mathbf{R}^m$.*

(i) The system is accessible from x_0 at time T if and only if it is accessible from the origin at the time T.
(ii) The system is locally reachable from x_0 at time T along the free solution if and only if it is locally reachable from the origin at time T or, equivalently, $0 \in \mathring{\mathrm{R}}(T, 0, U)$.

The proof is a straightforward application of the variation of constants formula. Notice that if the constraints are relaxed, system (5.37) is accessible from any initial state if and only if $\mathrm{R}(T, 0, \mathbf{R}^m) = \mathbf{R}^n$, that is if and only if the system is globally reachable at time T.

We already noticed that if U is a proper subset of $\mathbf{R}^m$, then $\mathrm{R}(T, 0, U)$ is no more, in general, a subspace of $\mathbf{R}^n$. However, it preserves an important property.

Proposition 5.6 *Consider the system (5.37). For each $T \ge 0$ and for each nonempty constraint set U, the set $\mathrm{R}(T, 0, U)$ is convex.*

The proof is trivial if U is convex. Otherwise, some advanced results of measure theory are needed (see [23] p. 163, [12] p. 11).

Theorem 5.9 *Assume that system (5.37) is completely controllable. Assume in addition that $0 \in \mathring{U}$. Then, $0 \in \mathring{\mathrm{R}}(T, 0, U)$.*

Proof Let $\mathcal{B}$ be a ball centered at the origin. Let r be radius of $\mathcal{B}$, chosen in such a way $\mathcal{B} \subset U$. Since the system is completely controllable, for each unit vector $\mathbf{e}_i$ of the canonical basis of $\mathbf{R}^n$ there exists a control function $u_i(t) : [0, T] \to \mathbf{R}^m$ which steers the system from the origin to $\mathbf{e}_i$ at time T. The control functions $u_i(t)$ need not to met the prescribed constraints but, since they are piecewise continuous and hence bounded on $[0, T]$, there exists $M > 0$ such that

$$|u_i(t)| < M \qquad \forall t \in [0, T], \forall i = 1, \dots, n\,.$$

Since the system is linear, the controls

$$\tilde{u}_i(t) = \frac{r}{M} u_i(t)$$

steer the system from the origin to some vectors $\tilde{\mathbf{e}}_i$ which still constitute a basis of $\mathbf{R}^n$. These new control functions satisfy the required constraints. Invoking again the linearity of the system, we finally see that the points $-\tilde{\mathbf{e}}_i$ can be reached by means of the control functions $-\tilde{u}_i(t)$. The conclusion follows, since $\mathrm{R}(T, 0, U)$ is convex. ■

Corollary 5.6 *Under the assumptions of Theorem 5.9, the system possesses the property of local reachability along the free solution.*

Corollary 5.7 *Assume that system (5.37) is completely controllable. Moreover, assume that* $\overset{\circ}{U} \neq \emptyset$*. Then, the system possesses the property of accessibility from the origin for each* $T > 0$.

Proof Let $u_0 \in \overset{\circ}{U}$. By assumption, there exists a ball of positive radius centered at u_0, which is contained in U. Replacing U by

$$U - \{u_0\} = \{v : v = u - u_0 \text{ with } u \in U\}$$

we obtain a system which satisfies the assumptions of Theorem 5.9. Thus, it is sufficient to remark that

$$\mathrm{R}(T, 0, U) = \mathrm{R}(T, 0, U - \{u_0\}) + \int_0^T e^{(T-s)A} B u_0 \, ds\,.$$

■

Chapter Summary

In this chapter we deal with the so-called structural properties of a linear system with input and output. These properties depend only on the coefficients of the mathematical model. We study in particular controllability (which provides a measure of our ability to control the system) and observability (which provides a measure of our ability of extracting information about the state of the system). We obtain algebraic characterizations of these properties. We also study canonical forms i.e., linear transformations of the state space which allow us to rewrite the model. This makes more evident, in this way, recognizing the controllability and observability properties.