Chapter 8
Frequency Domain Approach

As illustrated in the last section of Chap. 4, one possible approach to the analysis of linear differential equations of any order with forcing term makes use of the Laplace transform. This approach reveals a great significance when the forcing term is interpreted as a control. In this chapter, we first generalize the Laplace transform approach to MIMO systems with external inputs. For the particular case of SISO systems which are completely controllable and completely observable, we show how, by this approach, we can obtain a simple, alternative solution of the *synthesis problem*, which means the explicit construction of static output stabilizing feedback laws.

8.1 The Transfer Matrix

Consider a finite dimensional, time invariant linear system represented by the equations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \ , \end{cases} \tag{8.1}$$

where, as usual, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$, with n, m and p positive integers. Making use of the variation of constants formula, the output corresponding to an initial state x_0 and an input $u(\cdot)$ can be written

$$y(t) = Cx(t) \qquad \text{where} \qquad x(t) = \int_0^t e^{(t-\tau)A} Bu(\tau)\, d\tau + e^{tA} x_0 \ . \tag{8.2}$$

An alternative representation can be obtained by applying the vector Laplace transform (Sect. B.4) to both side of the first equation in (8.1). To this end, we

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assume, as we did in Sect. 4.4, that $u(\cdot)$, and hence also $x(\cdot)$ and $y(\cdot)$, are defined for $t \geq 0$. Moreover, we restrict the set of admissible inputs to subexponential functions of the class $\mathcal{PC}([0, +\infty), \mathbf{R}^m)$. By virtue of (8.2), this implies in turn that also $x(t)$ and $y(t)$ are subexponential.

Let $X(s) = \mathcal{L}[x(t)]$, $U(s) = \mathcal{L}[u(t)]$ and $Y(s) = \mathcal{L}[y(t)]$. We have

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x_0 = \mathcal{L}[Ax(t) + Bu(t)] = AX(s) + BU(s)$$

which yields

$$-(A - sI)X(s) = x_0 + BU(s) \ .$$

The matrix $(A - sI)$ is invertible, except for those values of $s \in \mathbf{C}$ which coincides with some eigenvalue of A. Let us denote by σ_0 the maximal real part of the eigenvalues of A. Therefore, if $\mathrm{Re}\, s > \sigma_0$, we may write

$$X(s) = -(A - sI)^{-1}x_0 - (A - sI)^{-1}BU(s)$$

and

$$Y(s) = CX(s) = T(s)U(s) + C(sI - A)^{-1}x_0 \tag{8.3}$$

where we set

$$T(s) = C(sI - A)^{-1}B \ . \tag{8.4}$$

The analogy between (8.2) and (8.3) is not surprising: both contain the sum of two terms; one of them depends on the initial state, the other depends on the input function. As a matter of fact, (8.3) can be alternatively obtained by applying the Laplace transform to (8.2) for $s > \sigma_0$, and making use of Proposition B.4.

Comparing (8.3) with (4.24) of Chap. 4, we see that the role of the polynomial $p_{ch}(s)$ is now played by the matrix $(A - sI)$.

Remark 8.1 Summing up, we have at our disposal two ways in order to represent a physical system with input and output: the matrix (8.4) and the Eq. (8.1) identified in short, in what follows, by the triplet of matrices (A, B, C). When (8.4) is used, we say that the system is represented according to the *frequency domain* approach. This terminology comes from the classical problem of frequency response analysis illustrated in Sect. 4.4.2. When (8.1) is used, we say that the system is represented according to the *time domain* approach. In principle, we should expect that both representations supply the same information about the behavior of the system but, as we shall see later, this is only partially true.

Notice also that in the frequency domain approach, the notion of "state" of the system is not explicitly involved. On the other hand, (8.3) requires purely algebraic computations, while in order to solve (8.1) integral calculus is needed. ■

The matrix $T(s)$ given by (8.4) is called the *transfer matrix*. Notice that $T(s)$ is independent of the initial conditions, so that in order to compute it, we may assume $x_0 = 0$. In the case of a SISO system i.e., when $p = m = 1$, the transfer matrix reduces to a unique element. When in addition the system is defined by a single linear differential equation of order n, it coincides with the transfer function already introduced in Remark 4.7.

In principle, it is possible to compute explicitly the transfer matrix making use of the formula (8.4): the main difficulty rests on the computation of the inverse of $(sI - A)$. Such inverse matrix is sometimes called the *resolvent* of A. For our purposes, the following proposition is sufficient.

Proposition 8.1 *If $s \in \mathbf{C}$ is not an eigenvalue of A, then*

$$(sI - A)^{-1} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n} M(s) \tag{8.5}$$

where $s^n + a_1 s^{n-1} + \cdots + a_n = (-1)^n p_A(s)$, and $M(s)$ is a matrix, whose entries are polynomials of degree less than or equal to $n - 1$ (recall that $p_A(s)$ denotes the characteristic polynomial of A).

The proof of (8.5) is easily obtained, having in mind the construction of the inverse of a matrix based on cofactors. A more precise formula for $(sI - A)^{-1}$ can be found for instance in [25], p. 12.

Example 8.1 Consider the system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 + u_1 \\ \dot{x}_2 = x_1 + u_2 \end{cases}$$

with observation map $y = (x_1 + x_2)/2$. The input, state and output spaces have respectively dimension equal to 2, 2, 1. The matrices which define the system are

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} \frac{1}{2} \, , \, \frac{1}{2} \end{pmatrix} .$$

The system is completely controllable and completely observable. The matrix

$$sI - A = \begin{pmatrix} s - 1 & 1 \\ -1 & s \end{pmatrix}$$

is invertible for $s \neq (1 \pm \mathrm{i}\sqrt{3})/2$, and

$$(sI - A)^{-1} = \frac{1}{s^2 - s + 1} \begin{pmatrix} s & -1 \\ 1 & s - 1 \end{pmatrix} .$$

The matrix $T(s)$ coincides with the row-vector

$$T(s) = \frac{1}{2(s^2 - s + 1)} \big(s + 1\ ,\ s - 2\big)$$

■

8.2 Properties of the Transfer Matrix

Given a system of the form (8.1), we now list some important properties of its transfer matrix $T(s)$ defined in (8.4).

Property 1 *Each element of $T(s)$ is a proper rational function of $s \in \mathbf{C}$. Moreover, the denominator of each element of $T(s)$ is a polynomial of degree less than or equal to n.*

Property 1 is a straightforward consequence of Proposition 8.1.

Remark 8.2 Property 1 points out that rational functions play a relevant role in the frequency domain approach. Thus, it is convenient to fix some terminology. Let $R_1(s)$, $R_2(s)$ be two rational functions of the complex variable s. We agree that the expression "$R_1(s)$ and $R_2(s)$ are *equal* (or *coincide*, or are the *same function*)" means that there exists a real number r_0 such that

$$R_1(s) = R_2(s) \quad \forall s \in \mathbf{C} \quad \text{with } \mathrm{Re}\, s > r_0\ . \tag{8.6}$$

To understand the meaning of this definition, we may look at the following example. Let

$$R_1(s) = \frac{s}{s(s-1)}\ , \qquad R_2(s) = \frac{1}{s-1}\ .$$

These functions cannot be considered "the same function" in the usual sense, since they do not have the same domain. However, they are "equal" in the aforementioned sense. Notice that if the numerator and the denominator of a rational function are polynomial of high degree, it may be very hard to recognize the existence of possible common factors and to get rid of them.

Let $N(s)$, $D(s)$ be two polynomials. We say that $N(s)$ and $D(s)$ are *coprime polynomials* if they do not have common factors. In this case, we also say that the rational function $N(s)/D(s)$ is written in *lowest terms*.

This terminology extends in the obvious way to matrices whose elements are rational functions. For instance, we say that a matrix $T(s)$ is written in lowest terms if the common factors have been canceled in all its entries. ■

Property 2 *Let σ_0 be the maximum of the real parts of the eigenvalues of A, and let $W(\tau) = Ce^{\tau A}B$. Then*

$$T(s) = \mathcal{L}[W(t)] \tag{8.7}$$

for each $s \in \mathbf{C}$ such that $s > \sigma_0$.

Formula (8.7) is an easy consequence of (B.26). The matrix $W(t)$ has been already encountered in Chap. 6 (Lemma 6.2 and subsequent comments). It can be interpreted as the matrix of the *impulse response* (Sect. 1.2): indeed, the columns of $W(t)$ coincide with the output functions of the system corresponding to the initial state $x_0 = 0$ and the impulsive inputs $\delta(t)e_1, \ldots, \delta(t)e_m$.

Property 1 could be also recovered from Property 2 taking into account the rules of the Laplace transform. Indeed, the elements of e^{tA} are obtained as sum of terms of the type $q_1(t)e^{\mu t}\cos\omega t$ and $q_2(t)e^{\mu t}\sin\omega t$, where $q_1(t), q_2(t)$ are polynomials and $\lambda = \mu + \mathrm{i}\,\omega$ is an eigenvalue of A.

Notice that $T(s)$ coincides with the Laplace transform of $W(t)$ only if $s > \sigma_0$ but its natural domain, as a rational function, contains all the points of the complex plane, with finitely many exceptions.

Property 3 *Let the system*

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \\ y = \tilde{C}\tilde{x} \end{cases} \tag{8.8}$$

be linearly equivalent to (8.1), according to the definition of Sect. 5.3. Then, (8.1) and (8.8) have the same transfer matrix.

Indeed, if (8.1) and (8.8) are linearly equivalent, then there exists a nonsingular matrix P such that $\tilde{A} = P^{-1}AP$, $\tilde{B} = P^{-1}B$, and $\tilde{C} = CP$. To compute the transfer matrix of system (8.8) we may apply the usual procedure, starting with the differential part of the system. Without loss of generality, we assume a vanishing initial state. We have

$$s\tilde{X}(s) = \tilde{A}\tilde{X}(s) + \tilde{B}U(s) = P^{-1}AP\tilde{X}(s) + P^{-1}BU(s)$$

which yields

$$(sI - P^{-1}AP)\tilde{X}(s) = P^{-1}BU(s)$$

or

$$P^{-1}(sI - A)P\tilde{X}(s) = P^{-1}BU(s)\ .$$

Then we proceed in the following way: we multiply both sides on the left first by P, then by $(sI - A)^{-1}$, and finally by P^{-1}. We obtain

$$\tilde{X}(s) = P^{-1}(sI - A)^{-1}BU(s)\ .$$

In conclusion,

$$Y(s) = \tilde{C}\tilde{X}(s) = CPP^{-1}(sI - A)^{-1}BU(s) = C(sI - A)^{-1}BU(s) .$$

Property 4 *The transfer matrix $T(s)$ depends only on the completely controllable and completely observable part of system (8.1).*

The proof of the statement above makes use of the decomposition of Sect. 5.3, which can be recovered by linear equivalence. Recall that if the matrix A has a triangular block structure, the exponential matrix e^{tA} has an analogous triangular block structure, too. Then, it is not difficult to see that $W(t) = Ce^{tA}B = C_2e^{tA_{22}}B_2$ (see Sect. 5.3 for the notation).

Definition 8.1 Assume that every element of $T(s)$ has been reduced to lowest terms. We say that the complex number s_0 is a *pole* of the system with multiplicity $\mu \geq 1$ if:

1. the denominator of at least one element of $T(s)$ can be exactly divided by $(s - s_0)^{\mu}$;
2. there exists no element of $T(s)$ whose denominator can be exactly divided by $(s - s_0)^{\mu+1}$.

In other words, the poles of a system are the points of the complex plane where at least one of the elements of the matrix $T(s)$ is not defined.

Property 5 *If s_0 is a pole of $T(s)$ with multiplicity μ, then s_0 is an eigenvalue of A with algebraic multiplicity greater than or equal to μ.*

Property 5 follows directly from Proposition 8.1. On the contrary, it may happens that A possesses some eigenvalue s_0 which is not a pole $T(s)$. We shall come on this point very soon.

Remark 8.3 Properties 4 and 5 imply that if s_0 is a pole of $T(s)$ with multiplicity μ, then s_0 is an eigenvalue of the matrix A_{22} (i.e., the matrix of the completely controllable and completely observable part of system) with algebraic multiplicity greater than or equal to μ. ■

8.3 The Realization Problem

In the previous section we saw how to determine the transfer matrix of a system given under the form (8.1). Now we address the inverse problem. Namely, we want to know if (and how) it is possible to recover the representation (8.1), when the system is assigned by means of its transfer matrix.

Definition 8.2 Let $T(s)$ be a matrix with p rows and m columns, whose elements are proper rational functions of the variable $s \in \mathbf{C}$. The triplet of matrices (A, B, C) whose dimensions are respectively $n \times n$, $n \times m$, $p \times n$, is said to be a *realization* of $T(s)$ if $T(s)$ coincides with the transfer function of the system (8.1) defined by means of the matrices A, B, C. The number n is said to be the *dimension* of the realization.

In order to illustrate some difficulties of the problem, we propose two examples.

Example 8.2 Let us consider the SISO system defined by the equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 + 3x_2 + u \\ y = x_1 - x_2 \,. \end{cases}$$

The differential part of the system is equivalent to the scalar equation $\xi'' - 3\xi' + 2\xi = u$, where we set $\xi = x_1$. The transfer function is

$$T(s) = \frac{1-s}{s^2 - 3s + 2} = \frac{1-s}{(s-1)(s-2)} = \frac{1}{2-s} \,.$$

The eigenvalues of the system matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

are 1 and 2, while the unique pole of the transfer function is 2. This example shows that the number of the poles can be actually less than the number of the eigenvalues: this is related to the cancelation of common factors appearing at the numerator and at the denominator of the transfer function. Notice that the system at hand is completely controllable but not completely observable. We also notice that the given system and the system represented by the single equation

$$\begin{cases} \dot{\xi} = 2\xi - u \\ y = \xi \end{cases}$$

have the same transfer function. ■

Example 8.3 Consider the SISO system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_2 \\ y = x_1 + x_2 \,. \end{cases}$$

Denoting by A the system matrix, we readily obtain

$$sI - A = \begin{pmatrix} s & 0 \\ 0 & s-1 \end{pmatrix} .$$

This matrix is invertible for $s \notin \{0, 1\}$, and

$$(sI - A)^{-1} = \frac{1}{s(s-1)} \begin{pmatrix} s-1 & 0 \\ 0 & s \end{pmatrix} .$$

It follows that $T(s) = 1/s$. This time, we notice that the system is completely observable, but not completely controllable. The transfer matrix is the same as the transfer matrix of the system

$$\begin{cases} \dot{\xi} = u \\ y = \xi . \end{cases}$$

■

Examples 8.2 and 8.3 point out that systems with different time domain representation may have the same transfer function. In other words, the realization problem does not have, in general, a unique solution. Of course, taking into account that the transfer function depends only on the completely controllable and completely observable part of the system (Property 4 above), this is not surprising. Examples 8.2 and 8.3 suggest also that the presence of uncontrollable or unobservable parts may lead to the cancelation of common factors at the numerator and the denominator of some element of the transfer matrix: this implies in turn a loss of information about the evolution of the internal state of the system. To distinguish among different realizations of the matrix $T(s)$, the following definitions are useful.

Definition 8.3 Let $T(s)$ be a matrix with p rows and m columns, whose elements are proper rational functions of the variable $s \in \mathbf{C}$. A realization (A, B, C) of $T(s)$ is said to be *minimal* if, denoting by n its state space dimension, the state space dimension of any other realization of $T(s)$ is greater than (or equal to) n.

A realization (A, B, C) is said to be *canonical* if the system (8.1) defined by the triplet A, B, C is completely controllable and completely observable.

Theorem 8.1 *Let $T(s)$ be a matrix with p rows and m columns, whose elements are proper rational functions of the variable $s \in \mathbf{C}$. A realization of $T(s)$ is minimal if and only if it is canonical.*

Proof If the realization (A, B, C) is minimal then it must be completely controllable and completely observable. Indeed, assume the contrary. Then according to Property 4, the completely controllable and completely observable part of the system defined by the triplet (A, B, C) provides another realization of $T(s)$, whose state space dimension is strictly less than the previous one.

To prove the converse, let us first recall (see Sects. 5.1.3 and 5.2.3) that a system of the form (8.1) is:

(1) completely controllable if and only if the matrix

$$\Gamma(T) = \int_0^T e^{-\sigma A} B B^{\mathbf{t}} e^{-\sigma A^{\mathbf{t}}}\, d\sigma$$

is nonsingular for some (and hence for each) $T > 0$;

(2) completely observable if and only if the matrix

$$E(T) = \int_0^T e^{\sigma A^{\mathbf{t}}} C^{\mathbf{t}} C e^{\sigma A}\, d\sigma$$

is nonsingular for some (and hence for each) $T > 0$.

Even in this case we may argue by contradiction. Let (A, B, C) be a canonical realization of dimension n and let us assume that for some system

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \\ y = \tilde{C}\tilde{x} \end{cases}$$

where $\tilde{A}$ is a square matrix of dimensions $\nu \times \nu$ with $\nu < n$, we have

$$T(s) = \mathcal{L}[Ce^{tA}B] = \mathcal{L}[\tilde{C}e^{t\tilde{A}}\tilde{B}]\,.$$

By applying $\mathcal{L}^{-1}$ to both sides, we get $Ce^{tA}B = \tilde{C}e^{t\tilde{A}}\tilde{B}$ for each $t \geq 0$, and so also for each $t \in \mathbf{R}$. Now let τ, σ be two arbitrary real numbers. We have:

$$Ce^{\tau A}e^{-\sigma A}B = Ce^{(\tau-\sigma)A}B = \tilde{C}e^{(\tau-\sigma)\tilde{A}}\tilde{B} = \tilde{C}e^{\tau\tilde{A}}e^{-\sigma\tilde{A}}\tilde{B}\,.$$

Multiplying by $e^{\tau A^{\mathbf{t}}}C^{\mathbf{t}}$ to the left and by $B^{\mathbf{t}}e^{-\sigma A^{\mathbf{t}}}$ to the right we obtain

$$e^{\tau A^{\mathbf{t}}}C^{\mathbf{t}}Ce^{\tau A}e^{-\sigma A}BB^{\mathbf{t}}e^{-\sigma A^{\mathbf{t}}} = e^{\tau A^{\mathbf{t}}}C^{\mathbf{t}}\tilde{C}e^{\tau\tilde{A}}e^{-\sigma\tilde{A}}\tilde{B}B^{\mathbf{t}}e^{-\sigma A^{\mathbf{t}}}\,.$$

Next we integrate both sides on the square $0 \leq \tau \leq T$, $0 \leq \sigma \leq T$. The result can be written as

$$E(T) \cdot \Gamma(T) = \tilde{E}(T) \cdot \tilde{\Gamma}(T) \tag{8.9}$$

where $\tilde{E}(T) = \int_0^T e^{\tau A^{\mathbf{t}}}C^{\mathbf{t}}\tilde{C}e^{\tau\tilde{A}}\, d\tau$ and $\tilde{\Gamma}(T) = \int_0^T e^{-\sigma\tilde{A}}\tilde{B}B^{\mathbf{t}}e^{-\sigma A^{\mathbf{t}}}\, d\sigma$.

The matrix $E(T) \cdot \Gamma(T)$ is, by hypothesis, the product of two nonsingular matrices. Hence, it has maximal rank equal to n. On the other hand, $\tilde{E}(T)$ has only ν columns, and $\tilde{\Gamma}(T)$ only ν rows. Hence, the rank of their product cannot be greater than $\nu < n$. Therefore, we conclude that the identity (8.9) is false and the statement is proved. ■

The realization problem has not a unique solution, not even if we limit ourselves to minimal realizations. The reason is clear, having in mind Property 3. Indeed, the first essential step in the construction of a realization is the choice of a real vector space to serve as state space. But, in order to write the Eq. (8.1), one needs also to fix a system of coordinates in this space. This choice is, of course, absolutely arbitrary: for different choices of the system of coordinates the system equations will look formally different, although of the same dimension. The following results clear up the situation.

Theorem 8.2 *Let $T(s)$ be a matrix with p rows and m columns, whose elements are proper rational functions of the variable $s \in \mathbf{C}$. Then there exists at least one realization of $T(s)$.*

Of course, if there exists one realization of $T(s)$, then there exists also a minimal (and canonical) realization.

Proposition 8.2 *Under the same assumptions of Theorem 8.2, if (A, B, C) (with state space dimension equal to n) and $(\tilde{A}, \tilde{B}, \tilde{C})$ (with state space dimension equal to $\tilde{n}$) are two canonical realizations of $T(s)$, then $n = \tilde{n}$ and, moreover, the systems respectively defined by the triplets (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are linearly equivalent.*

For the proofs of these results we refer to [3, 6, 23].

8.4 SISO Systems

In this section we focus our attention on SISO systems. Thus in what follows, we always have $m = p = 1$. Moreover, the transfer matrix reduces to a unique element represented by a proper rational function, referred to as the *transfer function*.

8.4.1 The Realization Problem for SISO Systems

Consider the rational function of the variable $s \in \mathbf{C}$

$$T(s) = \frac{N(s)}{D(s)} = \frac{c_0 + c_1 s + \cdots + c_k s^k}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{8.10}$$

where $N(s)$ and $D(s)$ may possibly have common factors. If $k < n$, $T(s)$ is proper, and so according to Theorem 8.2 there must exist a realization. Next proposition provides a simple, explicit construction for such a realization, and so it provides also a proof of Theorem 8.2 for the particular case of SISO systems.

Proposition 8.3 *If $k < n$, the system*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - \dots - a_1 x_n + u(t) \end{cases} \tag{8.11}$$

with the observation function

$$y = c_0 x_1 + c_1 x_2 + \dots + c_k x_{k+1} \tag{8.12}$$

represents a realization of (8.10).

Remark 8.4 We emphasize that the dimension of realization (8.11) is equal to the degree of the polynomial $D(s)$. ■

Proof of Proposition 8.3 Let C be the matrix of system (8.11). Since C is in companion form, system (8.11) is equivalent to the differential equation of order n

$$\xi^{(n)} + a_1 \xi^{(n-1)} + \dots + a_{n-1}\xi' + a_n \xi = u(t) \tag{8.13}$$

where we set $\xi = x_1$. Taking the Laplace transform of both sides, with the usual notation, we get

$$\Xi(s) = \frac{1}{p_{ch}(s)} U(s)$$

where $p_{ch}(s) = s^n + a_1 s^{n-1} + \dots + a_n = D(s)$ coincides with the characteristic polynomial of (8.13) and, by a possible change of sign, with the characteristic polynomial $p_C(s)$ of the matrix C. Taking into account (8.12), we easily get

$$Y(s) = \frac{c_0 + c_1 s + \dots + c_k s^k}{p_{ch}(s)} U(s) = \frac{N(s)}{D(s)} U(s) = T(s)U(s) \ .$$

Thus we see that $T(s)$ coincides with the transfer function of system (8.11), (8.12). ■

Clearly, the realization (8.11), (8.12) provided by Proposition 8.3 is completely controllable, but not necessarily completely observable. The following proposition concludes the reasoning.

Proposition 8.4 *Let the rational function (8.10) be given, and let $k < n$. The polynomials $N(s)$ and $D(s)$ are coprime if and only if system (8.11) with the observation function (8.12) represents a minimal realization of (8.10).*

Proof Let us assume that numerator and denominator of $T(s)$ do not have common factors. We already know that the system (8.11) with observation function (8.12) is a completely controllable realization of (8.10). If it is not completely observable, taking the observable part we can obtain another realization for which the dimension of the state space is $\tilde{n} < n$. Because of Property 1, we could rewrite $T(s)$ as $T(s) = \tilde{N}(s)/\tilde{D}(s)$, where $\tilde{D}(s)$ is a polynomial of degree not greater than $\tilde{n}$. This is impossible, since by assumption there is no common factors to cancel in $N(s)$ and $D(s)$.

Vice versa, let us assume that the system (8.11), (8.12) satisfies the complete observability condition, in addition to the complete controllability one. By Proposition 8.3, the transfer function of system (8.11), (8.12) coincides with $T(s) = N(s)/D(s)$. Recall that the dimension of the state space of (8.11) is, by construction, equal to the degree of $D(s)$.

If there are common factors to cancel in $N(s)$ and $D(s)$, we could rewrite $T(s)$ as $\tilde{N}(s)/\tilde{D}(s)$, where $\tilde{D}(s)$ is a polynomial of degree $\tilde{n} < n$. But then, using again Proposition 8.3, we can construct another realization of dimension $\tilde{n} < n$. This would imply that the realization (8.11), (8.12) is not minimal, and hence not canonical. This contradicts the assumption. ■

Remark 8.5 In particular, if in (8.10) we have $k = 0$ and $c_0 = 1$, (8.12) reduces to $y = x_1$ and the realization provided by Proposition 8.3 is canonical. ■

Finally, we show that in the case of SISO systems, Property 5 of the previous section admits a partial converse.

Proposition 8.5 *Let a system (8.1) with $m = p = 1$ be given, and let $T(s)$ be its transfer function. Assume that the system is completely controllable and completely observable. If λ is an eigenvalue of A of algebraic multiplicity μ, then λ is a pole of $T(s)$ of multiplicity μ.*

Proof According to Property 1, the transfer function can be written in the form

$$T(s) = \frac{N(s)}{D(s)}$$

where the degree of the polynomial $D(s)$ is not greater than n. If the degree of $D(s)$ is strictly less than n, then $T(s)$ would admit realizations of dimension strictly less than n, a contradiction to the complete controllability and complete observability assumptions. The unique possibility is therefore that the degree of $D(s)$ is exactly equal to n.

It follows that $T(s)$ has exactly n poles (counting possible multiplicities). Let us denote by $s_1, \ldots, s_k$ the distinct poles of $T(s)$, and by $\mu_1, \ldots, \mu_k$ their multiplicities, so that $\mu_1 + \cdots + \mu_k = n$. By Property 5, every s_i is an eigenvalue of A and its algebraic multiplicity is greater than or equal to μ_i. But the eigenvalues of A (counting multiplicities) cannot be more than n. Then, if there is some eigenvalue of A different from $s_1, \ldots, s_k$, or if for some index i, s_i regarded as an eigenvalue of A would have multiplicity strictly greater than μ_i, we get a contradiction. ■

Example 8.4 Consider a system for which the set of the admissible inputs is restricted to the set of functions $u(t)$ of class C^k on the interval $[0, +\infty)$. Assume that the evolution of the system is determined by a linear differential equation of order n

$$\xi^{(n)} + a_1\xi^{(n-1)} + \cdots + a_{n-1}\xi' + a_n\xi = c_k u^{(k)} + \cdots + c_0 u \tag{8.14}$$

where $a_1, \ldots, a_n, c_0, \ldots, c_k$ are real numbers, with $n > k$. Moreover, we assume that the output y coincides with ξ. Such a model, because of the presence of the derivative of the input, seems not to be covered by the form (8.1).

Let us apply the Laplace transform to both sides of (8.14). Assuming $\xi(0) = \xi'(0) = \cdots = \xi^{(n-1)}(0) = 0$ and $u(0) = u'(0) = \cdots = u^{(k)}(0) = 0$, we obtain

$$(s^n + a_1 s^{n-1} + \cdots + a_n)\Xi(s) = (c_k s^k + \cdots + c_0)U(s)$$

that is

$$Y(s) = \Xi(s) = \frac{c_k s^k + \cdots + c_0}{s^n + a_1 s^{n-1} + \cdots + a_n} U(s)$$

where, according to the assumption $m = p = 1$, the transfer matrix reduces to the scalar function

$$T(s) = \frac{c_k s^k + \cdots + c_0}{s^n + a_1 s^{n-1} + \cdots + a_n} \, . \tag{8.15}$$

This is a proper rational function, possibly with some common factors in the numerator and the denominator. Note that the denominator coincides with the characteristic polynomial $p_{ch}(s)$ of the homogeneous equation associated to (8.14). Therefore, system (8.14) can be realized by means of (8.11) with observation function (8.12). Recall that such a realization is completely controllable, but could be not completely observable.

In the modeling of physical systems, it is not rare the case where the derivative of the input appears explicitly in the equations; this happens for instance when a nonholonomic constraint is modeled as an input. ■

8.4.2 External Stability

As a consequence of Proposition 8.5, and recalling the conclusions of Chaps. 3 and 6 (in particular, Theorem 3.1, Theorem 6.2 and Proposition 6.5), we can immediately state the following theorem.

Theorem 8.3 *Let a system (8.1), with $m = p = 1$, be given. Let $T(s)$ be its transfer function. Assume that (8.1) is completely controllable and completely observable. Then the following statements are equivalent:*

(1) all the eigenvalues of A have negative real part;
(2) all the poles of the transfer function have negative real part;

(3) the system is internally stable;
(4) the system is BIBO-stable.

Remark 8.6 Recall that for systems which are not completely controllable or not completely observable, the previous statements are no more equivalent, in general. For instance, the system

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = x_2 \\ y = x_1 + x_2 \end{cases}$$

is not BIBO-stable, but its transfer function $T(s) = 1/(s+1)$ has the unique pole $s = -1$. ■

Remark 8.7 Theorem 8.3 applies in particular to BIBO systems of the special form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = u(t) \tag{8.16}$$

where y is taken as the output variable. Indeed, systems of this form are recognized to be completely controllable and completely observable (Remarks 5.4 and 5.7). Hence, for such systems, external stability and internal stability are equivalent properties. ■

The equivalence (2) $\Longleftrightarrow$ (4) of Theorem 8.3 can be proved by using only frequency domain methods. Consider, for simplicity, the case of a system defined by the second order equation

$$y'' + a_1 y' + a_2 y = u \tag{8.17}$$

with output variable y, under the assumption that the characteristic roots s_1, s_2 of the associated unforced equation are real and distinct. Recall that the solutions of (8.17) can be put in the form

$$y(t) = \varphi(t) + \chi(t) \tag{8.18}$$

where $\varphi(t)$ and $\chi(t)$ represent respectively the general solution of the associated homogeneous equation i.e., the unforced solution, and $\chi(t)$ represents the inverse Laplace transform of the function $U(s)/p_{ch}(s)$ i.e., the solution corresponding to the zeroed initial state (compare with (4.24)). Here, with the usual notation, $U(s)$ denotes the Laplace transform of the input $u(t)$. The following lemma exploits the linearity of the system; the argument is similar to that used in the proof of Theorem 6.1.

Lemma 8.1 *The solution $y(t)$ is bounded on the interval $[0, +\infty)$ for each initial condition and each bounded input $u(t)$ if and only if both the following conditions are met:*

(i) the function $\varphi(t)$ is bounded on $[0, +\infty)$ for each initial condition;
(ii) the function $\chi(t)$ is bounded on $[0, +\infty)$ for each bounded input $u(t)$.

Proof The sufficient part is straightforward, by virtue of the triangular inequality

$$|y(t)| \leq |\varphi(t)| + |\chi(t)| .$$

The necessary part can be proved by contradiction. Indeed, assume that there is a choice of the initial conditions y_0, y_1, for which $\varphi(t)$ is not bounded. Let us apply the zero input and let the system evolve with these initial conditions. We have $y(t) = \varphi(t)$ for each $t \geq 0$, so that $y(t)$ is unbounded, as well. To prove that also $\chi(t)$ must be bounded, we can argue in a similar way. ■

The poles of the transfer function coincide with the characteristic roots of the unforced equation associated to (8.17), and the condition that all the characteristic roots have negative real part is necessary and sufficient for internal stability. Therefore, under this condition, the function $\varphi(t)$ is bounded in $[0, +\infty)$.

Let us show that the same condition implies the boundedness of $\chi(t)$ as well, provided that the inputs are bounded. Recalling that $\chi(t)$ can be written in the form (4.25) we recall Proposition 1.5, which states that proving the boundedness of $\chi(t)$ is equivalent to proving that the integral of the function $h(\rho)$ is absolutely convergent. To this end, we need an explicit expression of $h(\rho)$. Since in this case $p_{ch}(s) = s^2 + a_1 s + a_2 = (s - s_2)(s - s_1)$, the Laplace transform of $\chi(t)$ is

$$\frac{1}{s - s_1} \cdot \frac{1}{s - s_2} \cdot U(s) . \tag{8.19}$$

This last expression tells us that the system acts as a cascade connection of two systems of the first order

$$\dot{y} = s_2 y + v \qquad \text{and} \qquad \dot{v} = s_1 v + u .$$

Solving independently these two systems, we obtain

$$y(t) = \int_0^t e^{(t-\tau_2)s_2} v(\tau_2)\, d\tau_2 \qquad \text{and} \qquad v(t) = \int_0^t e^{(t-\tau_1)s_1} u(\tau_1)\, d\tau_1 .$$

Combining these two expressions leads to

$$y(t) = \int_0^t e^{(t-\tau_2)s_2} \left(\int_0^{\tau_2} e^{(\tau_2-\tau_1)s_1} u(\tau_1)\, d\tau_1 \right) d\tau_2 .$$

We can eliminate the variable τ_2 by changing the order of integration and applying the substitution $\tau_2 - \tau_1 = r$. We find

$$\begin{aligned} y(t) &= \int_0^t \left(\int_{\tau_1}^t e^{(t-\tau_2)s_2} e^{(\tau_2-\tau_1)s_1}\, d\tau_2 \right) u(\tau_1)\, d\tau_1 \\ &= \int_0^t \left(\int_0^{t-\tau_1} e^{(t-\tau_1-r)s_2} e^{rs_1}\, dr \right) u(\tau_1)\, d\tau_1 . \end{aligned}$$

Comparing this last expression with (4.25) and applying further the substitution $\rho = t - \tau_1$, we finally get[1]

$$h(\rho) = \int_0^\rho e^{(\rho - r)s_2} e^{r s_1}\, dr \, . \tag{8.20}$$

Now it is straightforward to see that the integral $\int_0^\infty h(\rho)\, d\rho$ is absolutely convergent, by virtue of the assumption $s_1 < 0$, $s_2 < 0$. To finish the proof, we show that the negativity of the real part of the characteristic roots of the unforced system associated to (8.17) is also necessary for the boundedness of $\chi(t)$. Computing the integral in (8.20), we immediately find

$$h(\rho) = \frac{e^{s_1\rho} - e^{s_2\rho}}{s_1 - s_2} \, . \tag{8.21}$$

If both s_1 and s_2 (that are distinct by assumption) are not zero, we have with a further integration

$$\int_0^\infty h(\rho)\, d\rho = \lim_{T\to\infty} \frac{1}{s_1 - s_2} \left(\frac{e^{s_1 T}}{s_1} - \frac{e^{s_2 T}}{s_2} - \frac{1}{s_1} + \frac{1}{s_2} \right) \, .$$

Hence, if at least one is positive, the integral is not convergent. If one is equal to zero (say, s_1), we have:

$$\int_0^\infty h(\rho)\, d\rho = \lim_{T\to\infty} \frac{1}{s_1 - s_2} \left(T - \frac{e^{s_2 T}}{s_2} + \frac{1}{s_2} \right) \, .$$

and also in this case the integral $\int_0^\infty h(\rho)\, d\rho$ does not converge, regardless the sign of s_2.

[1] Alternatively, performing a decomposition to partial fractions, we can rewrite (8.19) as

$$\frac{1}{s_2 - s_1} \left(\frac{U(s)}{s - s_2} - \frac{U(s)}{s - s_1} \right) \, .$$

Taking the inverse transform we obtain

$$\begin{aligned} &\frac{1}{s_2 - s_1} \left(\int_0^t e^{s_2(t-\tau)} u(\tau)\, d\tau - \int_0^t e^{s_1(t-\tau)} u(\tau)\, d\tau \right) \\ &= \frac{1}{s_2 - s_1} \left(\int_0^t [e^{s_2(t-\tau)} - e^{s_1(t-\tau)}] u(\tau)\, d\tau \right) \end{aligned}$$

and we can recover (8.20), comparing again with (4.25).

8.4.3 Nyquist Diagram

The so-called Nyquist diagram is a graphic criterion which, applied to the transfer function, allows us to recognize whether a given system possesses the BIBO stability property.

Assume that the system is given by means of its state equations (8.1), where $m = p = 1$, and that it is completely controllable and completely observable, so that the conclusions of Theorem 8.3 hold. Moreover, by virtue of Proposition 8.4, under the same conditions as before, there is no loss of generality assuming that the numerator and the denominator of the transfer function are coprime.

Now let $T(s)$ be a proper rational function of the variable $s \in \mathbf{C}$, with no common factors. A number $s_0 \in \mathbf{C}$ is said to be a *zero*[2] of $T(s)$ if $T(s_0) = 0$.

Let us denote by $w = T(s) \in \mathbf{C}$ the dependent variable. Any complex number s can be thought of as a point of a plane, where a system of coordinates has been fixed $(\mathrm{Re}\, s, \mathrm{Im} s)$. Analogously, any complex number w will be thought of as a point in a plane referred to the coordinates $(\mathrm{Re}\, w, \mathrm{Im} w)$. A continuous map $s = \gamma(t)$ from $\mathbf{R}$ to $\mathbf{C}$ can be interpreted as a planar curve. Analogously, the image of $\gamma(t)$ throughout T can be interpreted as a planar curve $w = \delta(t) = T(\gamma(t))$.

If $s = \gamma(t)$ is simple and closed, it surrounds an open and bounded region $\Gamma \subset \mathbf{C}$. Of course, if $s = \gamma(t)$ is simple and closed, $\delta(t)$ is closed, but it is not necessarily simple.

Example 8.5 Let $T(s) = 1/(s-1)(s-2)$. Figure 8.1 shows the curve δ obtained by applying T to the circumference

$$\begin{cases} \mathrm{Re}\, s = 2 + 2\cos t \\ \mathrm{Im} s = 2\sin t \ . \end{cases}$$

■

Example 8.6 Let $T(s) = 1/(s-1)^2$, and let δ be now obtained by applying T to the circumference

$$\begin{cases} \mathrm{Re}\, s = 1 + \cos t \\ \mathrm{Im} s = \sin t \ . \end{cases}$$

Figure 8.2 may give the wrong impression of a simple and closed curve. Actually the curve is run twice. ■

Let $s = \gamma(t)$ be a simple and closed curve. Let us denote by Z the number of zeros of $T(s)$ lying in Γ and by P the number of poles of $T(s)$ lying in Γ. For simplicity, we assume that there is neither zeros nor poles on the contour of Γ. We need the

[2] If $T(s)$ is the transfer function of a system, its zeros give useful information about the behavior of the system: the interested reader is referred to [8].

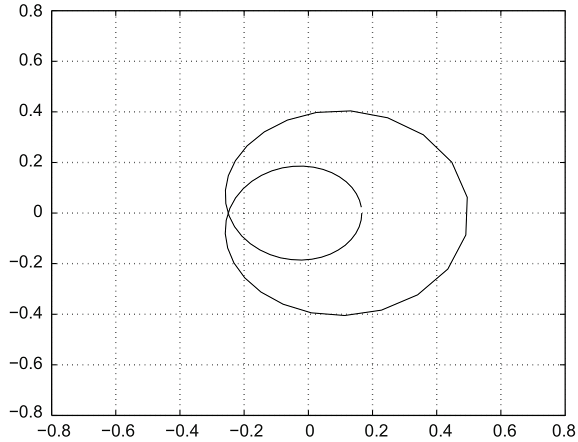


Fig. 8.1 The curve δ of Example 8.5

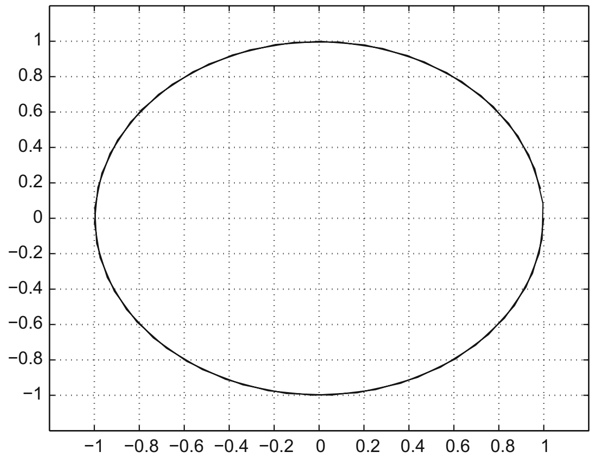


Fig. 8.2 The curve δ of Example 8.6

following, classical result from the theory of functions of a complex variable (see for instance [1]).

Argument principle *Let Q be the integer number denoting how many times the curve $\delta(t)$ encircles the origin in counterclockwise sense, while the contour of Γ is run once in the counterclockwise sense. Then, $Q = Z - P$.*

Definition 8.4 The *Nyquist diagram* of a proper rational function $T(s)$ is the image of the curve $w = T(\gamma(t)) = \delta(t)$, when $\gamma(t) = -\mathrm{i}\, t$ $(t \in \mathbf{R})$.

The curve $\gamma(t) = -\mathrm{i}\, t$ generating the Nyquist diagram is not closed. Nevertheless, the image δ of γ obtained by composition with T, surrounds a bounded region of the complex plane. Indeed, since T is proper, we have

$$\lim_{t\to\pm\infty} \delta(t) = 0 \,.$$

In fact, we may also think of $\gamma(t)$ as a closed curve, by adding to its domain the infinity point: completed in this way, we may imagine that γ surrounds the right half plane of **C** (the contour being run in the counterclockwise sense). Notice that by construction, $\delta(t) = (\mathrm{Re}\, T(-\mathrm{i}\, t), \mathrm{Im} T(-\mathrm{i}\, t))$.

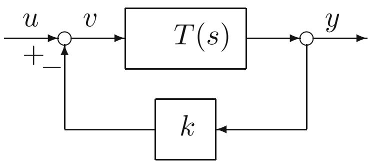
Let $T(s)$ be a proper rational function without zeros or poles on the imaginary axis. Drawing the Nyquist diagram and assuming that Z is known, we can now easily check whether the right half plane of **C** contains some poles of $T(s)$.

By some suitable modifications, these conclusions can be extended to the case where $T(s)$ possesses purely imaginary poles or zeros.

8.4.4 Stabilization by Static Output Feedback

Continuing to deal with a SISO system of the form (8.1) satisfying the complete controllability and the complete observability assumption, in this section we show how to take advantages of the Nyquist criterion in order to determine a static output feedback which stabilizes the given system in the BIBO (and hence also in the internal) sense.

As usual, we denote by $u \in \mathbf{R}$ the input variable and by $y \in \mathbf{R}$ the output variable. First, we examine how the transfer function changes, when a feedback of the form $-ky$ is added to the external input u: here, k is a positive constant, sometimes called the *gain*; the choice of the minus sign is conventional.



Let $T(s)$ be the transfer function of the given system. Let $v = u - ky$. By the aid of the figure above, we easily see that

$$Y(s) = T(s)V(s) = T(s)(U(s) - kY(s))$$

so that

$$Y(s) + kT(s)Y(s) = T(s)U(S) \,.$$

As a consequence, for each $s \in C$ such that $1 + kT(s) \neq 0$,

$$Y(s) = G(s)U(s) = \frac{T(s)}{1 + kT(s)}U(s) = \frac{1}{k} \cdot \frac{T(s)}{\frac{1}{k} + T(s)}U(s)$$

where $G(s)$ denotes the transfer function of the closed loop system. While the value of parameter k varies, the positions of the poles of the resulting transfer function $G(s)$ vary in a continuous way. Thus, to accomplish the desired goal, we need to find a value of k, if any, in such a way that all the poles of the $G(s)$ are moved to the left half of the complex plane.

For simplicity, we assume that $G(s)$ does not have poles on the imaginary axis, and we write

$$H(s) = \frac{1}{k} + T(s) \, .$$

Lemma 8.2 *The poles of $G(s)$ coincide with the zeros of $H(s)$.*

Proof Write $T(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are polynomial. We have

$$H(s) = \frac{1}{k} + \frac{N(s)}{D(s)} = \frac{D(s) + kN(s)}{kD(s)} \, .$$

Hence, s_0 is a zero of $H(s)$ if and only if $D(s_0) + kN(s_0) = 0$. On the other hand

$$G(s) = \frac{N(s)}{D(s)} \cdot \frac{1}{1 + k\dfrac{N(s)}{D(s)}} = \frac{N(s)}{D(s) + kN(s)} \, .$$

Hence, s_0 is a pole of $G(s)$ if and only if $D(s_0) + kN(s_0) = 0$. ■

In addition, it is straightforward to realize that the poles of $H(s)$ coincide with the poles of $T(s)$. Applying the Argument Principle to the rational function $H(s)$ leads to the following conclusions:

$$\begin{aligned} &number\ of\ the\ poles\ of\ G\ with\ positive\ real\ part \\ &= number\ of\ poles\ of\ T\ with\ positive\ real\ part\ +\ Q \end{aligned}$$

where Q denotes the number of times the curve $H(-\mathrm{i}\,t)$ encircles the origin in the counterclockwise sense, while the parameter t moves from $-\infty$ to $+\infty$.

On the other hand, it is evident that $-Q$ represents the number of times the curve $T(-\mathrm{i}\,t)$ encircles the point of coordinates $(-\frac{1}{k}, 0)$ of the complex plane in clockwise sense, while the parameter t moves from $-\infty$ to $+\infty$. The following statement resumes the conclusions.

Proposition 8.6 *The static output feedback $-ky$ stabilizes in BIBO (and so also in internal) sense the SISO system (8.1) if the number of times the Nyquist diagram of its transfer function $T(s)$ encircles the point $(-\frac{1}{k}, 0)$ in clockwise sense while the parameter t moves from $-\infty$ to $+\infty$, is equal to the number of poles of the given system lying in the open right half of the complex plane.*

In practical applications, one draws the Nyquist diagram of the given system, and then checks whether there exists a region $\mathcal{D}$ encircled by the diagram the required number of times. If this region exists and intersects the negative real axis, the system is stabilizable. A stabilizing feedback is provided by any value of k such that $(-\frac{1}{k}, 0) \in \mathcal{D}$.

8.5 Disturbance Decoupling

In this last section we discuss an important application which involves both frequency domain and time domain techniques. Consider the system

$$\begin{cases} \dot{x} = Ax + Gd \\ y = Cx \end{cases} \tag{8.22}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$, $d \in \mathbf{R}^q$. The input $d(t) : [0, +\infty) \to \mathbf{R}^q$ is now interpreted as a disturbance. In other words, $d(t)$ is a unknown and undesired input; we just assume that it is piecewise continuous and right continuous, in order to guarantee existence of solutions. For each initial state x_0, the variation of constants formula yields

$$y(t, x_0, d(\cdot)) = Ce^{tA}x_0 + \int_0^t Ce^{(t-\tau)A}Gd(\tau)\,d\tau$$

which reduces to

$$y_0(t) = Ce^{tA}x_0$$

when $d(t) = 0$ for each $t \geq 0$. The function $y_0(t)$ is called the *uncorrupted output signal*. It may happen that $y(t) = y_0(t)$ even for not vanishing disturbances $d(t)$.

Example 8.7 Clearly, the output of the (not completely observable) system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 + d \\ \dot{x}_2 = x_2 \\ \\ y = x_2 \end{cases}$$

is not affected by the disturbance. ∎

Definition 8.5 Let us denote, as before, by $y_0(t)$ the uncorrupted output, that is the output corresponding to some initial state x_0 and the vanishing input $d(t) = 0$. We say that the system is *disturbance decoupled* if we have $y(t, x_0, d(\cdot)) = y_0(t)$ for each $t \geq 0$, each initial state x_0 and each input $d(t)$.

Proposition 8.7 *The following statements are equivalent:*

(i) *the system (8.22) is disturbance decoupled;*
(ii) *the impulse response matrix $W(t) = Ce^{tA}G$ vanishes for $t \geq 0$ (and hence, being a real analytic function, for each $t \in \mathbf{R}$);*
(iii) *the transfer matrix $T(s) = C(sI - A)^{-1}G$ vanishes for $s \in \mathbf{C}$;*
(iv) *for each integer $k \geq 0$, one has $CA^kG = 0$.*

Proof The equivalences *(i)* $\Longleftrightarrow$ *(ii)* $\Longleftrightarrow$ *(iii)* are straightforward. Thus, we focus on the statement *(iv)*, and we will prove that it is equivalent to *(ii)*. Assume first that the identity

$$W(t) = Ce^{tA}G = 0 \tag{8.23}$$

holds for each $t \in \mathbf{R}$. To begin with, the substitution $t = 0$ yields $CG = 0$. Coming back to (8.23) and taking the derivative, we obtain

$$CAe^{tA}G = 0 \tag{8.24}$$

for $t \in \mathbf{R}$, which implies $CAG = 0$ by the substitution $t = 0$. We repeat the procedure, taking now the derivative of (8.24) and letting again $t = 0$. This time we obtain $CA^2G = 0$. Continuing in this way, we conclude finally that $CA^kG = 0$ for each integer $k \geq 0$. The converse implication is immediate, since

$$W(t) = Ce^{tA}G = \sum_{k=0}^{\infty} \frac{t^k}{k!} CA^kG$$

for each $t \in R$. ∎

Remark 8.8 According to the Cayley-Hamilton Theorem, it is sufficient to check condition *(iv)* of Proposition 8.7 for $k = 0, \ldots, n - 1$. ∎

Next we establish a necessary and sufficient condition.

Definition 8.6 Let A be a real matrix of dimensions $n \times n$. A subspace V of $\mathbf{R}^n$ is said to be an *algebraic (o geometric) invariant* for A if $AV \subseteq V$.

The subspace V is said to be a *dynamic invariant* for A if from $x_0 \in V$ it follows $e^{tA}x_0 \in V$ for each $t \geq 0$ (and hence for each $t \in \mathbf{R}$).

Proposition 8.8 *The subspace V is an* algebraic invariant *for A if and only if it is a* dynamic invariant *for A.*

Proof Let V be an algebraic invariant. For each $x_0 \in V$, we clearly have $Ax_0 \in V$, $A^2x_0 \in V$, and so on. Hence, $e^{tA}x_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^kx_0$ belongs to V. On the other hand, let V be a dynamic invariant, and let $x_0 \in V$. Then, for each $t \neq 0$, we also have

$$\frac{e^{tA}x_0 - x_0}{t} \in V \ .$$

Taking the limit for $t \to 0$, we get $Ax_0 \in V$. ∎

Theorem 8.4 *The given system is disturbance decoupled if and only if there exists a subspace V of $\mathbf{R}^n$ which is an algebraic invariant for A, and such that* $\operatorname{im} G \subseteq V \subseteq \ker C$.

Proof Assume that the system is disturbance decoupled. Let us introduce a matrix H, whose columns coincide with the columns of the matrices G, AG, $A^2G, \dots, A^{n-1}G$, in this order. The matrix H can be interpreted as a linear map from $\mathbf{R}^{n\times q}$ in $\mathbf{R}^n$. Let $V = \operatorname{im} H$. By the Cayley-Hamilton Theorem, V is an algebraic invariant. The inclusion $\operatorname{im} G \subseteq V$ is obvious, while the other one $V \subseteq \ker C$ follows from Proposition 8.7, *(iv)*.

To prove the converse, we first remark that if a subspace V is an algebraic invariant and $\operatorname{im} G \subseteq V$, then clearly $\operatorname{im}(A^kG) \subseteq V$ for each positive integer k. As a consequence, since $V \subseteq \ker C$, we also have $CA^kGx = 0$ for each integer $k \geq 0$ and each $x \in \mathbf{R}^n$. The conclusion follows, using again Proposition 8.7, *(iv)*. ∎

There is an other characterization of disturbance decoupled systems. By means of a linear change of coordinates, we can put the system in the observability canonical form

$$\begin{cases} \dot z_1 = A_{11}z_1 + A_{12}z_2 + G_1 d \\ \dot z_2 = A_{22}z_2 + G_2 d \\ \\ y = C_2 z_2 \end{cases} \tag{8.25}$$

where $z_1 \in \mathbf{R}^{n-r}$, $z_2 \in \mathbf{R}^r$ for some nonnegative integer $r \leq n$, and the subsystem

$$\begin{cases} \dot z_2 = A_{22}z_2 + G_2 d \\ \\ y = C_2 z_2 \end{cases} \tag{8.26}$$

is completely observable. The case $r = 0$ is trivial, so we can assume $r > 0$.

Theorem 8.5 *The system (8.22) is disturbance decoupled if and only if $G_2 = 0$, where G_2 is the matrix appearing in (8.25).*

Proof The sufficient part is evident (to be formal, it can be easily obtained as an application of Theorem 8.4). Let us prove the necessary part.

Assume that the system is disturbance decoupled. Taking into account the form (8.25), for each integer $k \geq 0$, we see that CA^k can be written as a row block matrix $(0 \mid C_2A_{22}^k)$, where 0 denotes here a block of $n - r$ zero columns. From this, it easily follows that $CA^kG = C_2A_{22}^kG_2$ for each integer $k \geq 0$. Since the subsystem (8.26) is completely observable, the matrix

$$M = \begin{pmatrix} C_2 \\ C_2A_{22} \\ \dots \\ C_2A_{22}^{r-1} \end{pmatrix}$$

has a maximal rank i.e., rank $M = r$. Now, assume by contradiction that $v = G_2d \neq 0$ for some $d \in \mathbf{R}^q$ (note that $v \in \mathbf{R}^r$ and that, necessarily, $d \neq 0$). The vector $Mv \in \mathbf{R}^{p\times r}$ is a linear combination of the r linearly independent columns of M, so that being $v \neq 0$, we also have $Mv \neq 0$. But

$$Mv = \begin{pmatrix} C_2G_2d \\ C_2A_{22}G_2d \\ \dots \\ C_2A_{22}^{r-1}G_2d \end{pmatrix} = \begin{pmatrix} C_2G_2 \\ C_2A_{22}G_2 \\ \dots \\ C_2A_{22}^{r-1}G_2 \end{pmatrix} d \tag{8.27}$$

with $d \neq 0$. On the other hand, the disturbance decoupling assumption implies

$$C_2G_2 = C_2A_{22}G_2 = \cdots = C_2A_{22}^{r-1}G_2 = 0 \,. \tag{8.28}$$

Clearly, (8.27) and (8.28) are in contradiction. Therefore, we must have $G_2d = 0$ for each $d \in \mathbf{R}^q$, and this means that $G_2 = 0$. ■

If the given system is not disturbance decoupled, we can try to achieve this property by the use of a suitable feedback law. In other words, we add a control term in the system equation

$$\begin{cases} \dot{x} = Ax + Bu + Gd \\ y = Cx \end{cases} \tag{8.29}$$

where with the usual notation $u \in \mathbf{R}^m$, and we ask whether it is possible to find a static state feedback of the form $u = Fx$ such that the closed-loop system

$$\begin{cases} \dot{x} = (A + BF)x + Gd \\ y = Cx \end{cases}$$

is disturbance decoupled. The conditions for answering this question rest on the introduction of a new notion of invariance, concerning the state equation

$$\dot{x} = Ax + Bu \,. \tag{8.30}$$

Definition 8.7 A subspace $V \subseteq \mathbf{R}^n$ is said to be a *strong controlled invariant* for the system (8.30) if for each $x_0 \in V$ and each admissible input $u(t) : [0, +\infty) \to \mathbf{R}^m$ we have $x(t, x_0, u(\cdot)) \in V$ for each $t \geq 0$.

Apart from the modified terminology, the definition above coincides with the notion already introduced in Sect. 5.3.2.

Definition 8.8 A subspace $V \subseteq \mathbf{R}^n$ is said to be a *weak controlled invariant* for the system (8.30) if for each $x_0 \in V$ there exists an admissible input $u(t) : [0, +\infty) \to \mathbf{R}^m$ such that $x(t, x_0, u(\cdot)) \in V$ for each $t \geq 0$.

Example 8.8 The subspace $V = \{(x_1, x_2) : x_2 = 0\} \subseteq \mathbf{R}^2$ is a weak controlled invariant, but not a strong controlled invariant, for the system

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = u \ . \end{cases}$$

Note that this system is completely controllable. ■

The weak controlled invariant subspaces can be characterized in the following way.

Proposition 8.9 *The following statements are equivalent.*

(i) V *is a weak controlled invariant;*
(ii) $AV \subseteq V + \operatorname{im} B$*;*
(iii) *there exists a matrix* F *with* n *columns and* m *rows such that* $(A + BF)V \subseteq V$.

Proof First we prove that *(i)* $\Longrightarrow$ *(ii)*. Let $x_0 \in V$ and let $u(t) : [0, +\infty) \to \mathbf{R}^m$ be an input such that $x(t, x_0, u(\cdot)) \in V$ for each $t \geq 0$. Without loss of generality, we can extend continuously $u(t)$ on a small interval $(-\varepsilon, 0)$, so that $x(t, x_0, u(\cdot))$ can be considered of class C^1 at $t = 0$. Then

$$\lim_{t \to 0^+} \frac{x(t, x_0, u(\cdot)) - x_0}{t} = \dot{x}(0) \in V$$

that is $Ax_0 + Bu(0) \in V$, or $Ax_0 \in V - Bu(0)$.

Next we prove that *(ii)* $\Longrightarrow$ *(iii)*. Let $\dim V = k \leq n$. Let $e_1, \ldots, e_n$ be a basis of $\mathbf{R}^n$, such that its first k elements $e_1, \ldots, e_k$ constitute a basis of V. Then for each $i = 1, \ldots, k$ one has $Ae_i = g_i + Bu_i$ for some $g_i \in V$ and some $u_i \in \mathbf{R}^m$. Let us chose other vectors $u_{k+1}, \ldots, u_n \in \mathbf{R}^m$ in arbitrary way, and define the matrix F by the relations $Fe_j = -u_j$, for $j = 1, \ldots, n$. Then we have, for $i = 1, \ldots, k$,

$$(A + BF)e_i = Ae_i + BFe_i = g_i + Bu_i - Bu_i = g_i \in V \ .$$

Finally we prove that *(iii)* $\Longrightarrow$ *(i)*. Let $x_0 \in V$ and let $x(t)$ be the solution of the closed loop system

$$\begin{cases} \dot{x} = (A + BF)x \\ x(0) = x_0 \ . \end{cases} \tag{8.31}$$

Of course, $x(t)$ is also a solution of the problem

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ x(0) = x_0 \end{cases}$$

where $u(t) = Fx(t)$. The proof is completed, by noticing that V is a dynamic invariant with respect to system (8.31). ■

We are finally able to state the main result of this section.

Theorem 8.6 *System (8.29) can be rendered disturbance decoupled by means of a linear feedback if and only if there exists a subspace $V \subseteq \mathbf{R}^n$ which is a weak controlled invariant for the system (8.30) and such that* $\operatorname{im} G \subseteq V \subseteq \ker C$.

Proof Let us prove first the necessary part. So let F be a matrix such that the system

$$\begin{cases} \dot{x} = (A + BF)x + Gd \\ y = Cx \end{cases} \tag{8.32}$$

is disturbance decoupled. According to Theorem 8.4, there exists an algebraic invariant subspace V, such that $\operatorname{im} G \subseteq V \subseteq \ker C$. This implies that $(A + BF)V \subseteq V$, and this in turn means that V is a weak controlled invariant, by Proposition 8.9.

Then we prove the sufficient part. If V is weak controlled invariant, then by Proposition 8.9 there exists F such that $(A + BF)V \subseteq V$. Together with the inclusions $\operatorname{im} G \subseteq V \subseteq \ker C$, this implies finally that the system (8.32) is disturbance decoupled by Theorem 8.4. ■

Chapter Summary

The subject of the last chapter is the relationship between two possible approaches to the analysis of a system: the time domain approach (developed in the previous chapters) and the more traditional frequency domain approach based on the Laplace transform. We study in particular the realization problem. For the case of SISO systems, we also give a different solution to the stabilization problem by output feedback. Finally, we illustrate the decoupling problem, whose solution takes advantages of both approaches.