Chapter 1
Introduction

Many phenomena observed in the real world, regardless to their different nature, involve several physical quantities and result from the interaction of various components: for these reasons, in these situations the term "system" is generally used.

The experimental information obtained by studying a physical system gives often rise to the construction of a mathematical model. In this way, it can be easily communicated and elaborated qualitatively or numerically, and possibly employed to control the evolution of the system. In this book, the term *system* will be often referred to the mathematical model, rather than the represented real phenomenon.

Without any pretence of giving an axiomatic definition, the present introductory chapter aims to describe informally the main features of the notion of system, and the way we can take advantages of them.

1.1 The Abstract Notion of System

Direct experience shows that a system is often subject to time evolution. This means that the numerical values of the physical quantities characterizing the state of the system change while time passes. For this reason, they will be treated as variables. The changes are due, in general, to the action of internal forces and constraints, as well as of possible external forces or signals.

1.1.1 The Input-Output Operator

In order to provide an abstract description of the evolution of a system, we need to assign the following objects:

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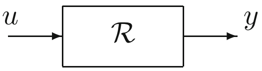
(1) a set $\mathcal{T}$ representing the time;
(2) a set $\mathcal{U}$, whose members represent the numerical values of the possible external variables (forces or signals);
(3) a set $\mathcal{X}$, whose members represent the numerical values of the internal state variables;
(4) a set $\mathcal{Y}$, whose members represent the numerical values of the variables carrying the information provided by the system about its internal state.

The action exerted on the system by the external world during the evolution is therefore represented by a function $u(\cdot) \in \mathcal{F}(\mathcal{T}, \mathcal{U})$: it is called the *input map*. The response of the system, that is the available information about the state of the system during the evolution, is represented by a function $y(\cdot) \in \mathcal{F}(\mathcal{T}, \mathcal{Y})$: it is called the *output map*. Finally, the internal state of the system, during the evolution, is represented by a function $x(\cdot) \in \mathcal{F}(\mathcal{T}, \mathcal{X})$, called the *state evolution map*. The sets $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{X}$ are respectively called the *input set*, the *output set* and the *state set*.

The system acts as on operator $\mathcal{R}$ transforming elements $u(\cdot) \in \mathcal{F}(\mathcal{T}, \mathcal{U})$ to elements $y(\cdot) \in \mathcal{F}(\mathcal{T}, \mathcal{Y})$. We will write

$$y(\cdot) = \mathcal{R}(u(\cdot)) \ , \quad \text{with} \quad \mathcal{R} : \mathcal{F}(\mathcal{T}, \mathcal{U}) \to \mathcal{F}(\mathcal{T}, \mathcal{Y}) \ .$$

The operator $\mathcal{R}$ is called the *input-output operator*. In system theory, the action of an input-output operator $\mathcal{R}$ is often represented graphically by means of a flow chart, as the following figure shows.



An input-output operator is not necessarily defined for each element $u(\cdot) \in \mathcal{F}(\mathcal{T}, \mathcal{U})$. For instance, in some applications the input maps need to satisfy some constraints, imposed by the nature of the problem either on their values or on their functional character. The subset of $\mathcal{F}(\mathcal{T}, \mathcal{U})$ formed by all the maps satisfying the required conditions constitutes the domain of $\mathcal{R}$. It is called the *set of admissible inputs*.

The following subsections aim to specify the nature of the sets $\mathcal{T}, \mathcal{U}, \mathcal{X}, \mathcal{Y}$.

Remark 1.1 When we assume that the output map can be uniquely and exactly determined by applying the operator $\mathcal{R}$, we are implicitly assuming that we have a full knowledge about the structure of the system and the physical laws governing its evolution. But in real situations this is not always true. In practice, it might happen that repeated experiments (with the same input map) give rise to different outputs, or that the output is affected by imprecisions, due to one or more of the following reasons: neglecting or simplifying some details during the modeling process; measurement errors; uncertainty in parameters identification; random phenomena. To face these or similar situations, suitable extensions of the theory need to be developed. But these will not be considered in this book.

Remark 1.2 A comment about terminology is in order. The word "system" is often used with lightly different meanings in the common language, and sometimes also in the technical literature. For instance, in Mathematics, "system" classically means "set of coupled equations". A system corresponding to an input-output operator as above, should be more properly called an *input-output system*. However, throughout this book, we prefer to use for simplicity the term "system" also in this case. The ambiguity is not serious. The right sense can be easily understood every time from the context.

1.1.2 Discrete Time and Continuous Time

The time can be represented by any totally ordered set $\mathcal{T}$, endowed with a group structure. In practice, we have two possible choices: either $\mathcal{T} = \mathbf{Z}$ or $\mathcal{T} = \mathbf{R}$. In the former case we speak about *discrete time systems*: the functions representing the input, the state and the output are actually sequences. In the latter case we speak about *continuous time systems*. It may happens that a physical system can be modeled both as a discrete time system and as a continuous time system. This may depends on the purposes of the search, on the measure scales and on the measure devices. Sometimes, different representations of the same physical system provide complementary information.

1.1.3 Input Space and Output Space

We can distinguish several types of inputs variables. A *disturbance* is a undesired signal, which cannot be governed, and sometimes not even measured in real time. A *reference signal* is an input representing the ideal evolution, to be tracked by the real evolution of the system. A *control* is an input completely determined by the decisions of a supervisor, which can be used to modify the behavior of the system during the evolution.

In general, we assume that the value of each single physical quantity involved in the model can be expressed by a real number. Moreover, it is convenient to order the input variables (and, separately, the output variables), and to rewrite them as the components of a vector. It is therefore natural to assume that the input set $\mathcal{U}$ and the output set $\mathcal{Y}$ are endowed with the structure of a real vector spaces.

1.1.4 State Space

In common applications, the number of the state variables is usually greater than the number of the input and output variables. Moreover, the state variables are difficult to identify, since in general they are not directly available to the observation. Sometimes, one should think of the state variables as mathematical idealizations, inherent to the model. We will assume that also the state set $\mathcal{X}$ has the structure of a real vector space.

1.1.5 Finite Dimensional Systems

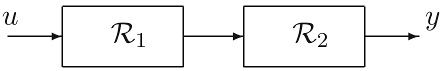
We say that a system is *finite dimensional* when the input variables, the output variables and the state variables can be represented as vectors with finitely many real components. Thus, for a finite dimensional system, it is natural to assume $\mathcal{X} = \mathbf{R}^n$, $\mathcal{U} = \mathbf{R}^m$, $\mathcal{Y} = \mathbf{R}^p$, where n, m, p are given integers, greater than or equal to 1. The sets of functions representing the input, the output and the state maps will be therefore respectively denoted by $\mathcal{F}(\mathbf{R}, \mathbf{R}^m)$, $\mathcal{F}(\mathbf{R}, \mathbf{R}^p)$, $\mathcal{F}(\mathbf{R}, \mathbf{R}^n)$. In particular, the system is said to be SISO (single-input-single-output) when $m = p = 1$; otherwise, the system is said to be MIMO (multi-input-multi-output).

Remark 1.3 From now on, by the term *system* we mean a finite dimensional, time continuous system.

1.1.6 Connection of Systems

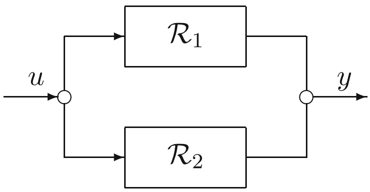
In some applications, it is necessary to manage simultaneously several systems, and to enable connections among them. The result of these manipulations may be often reviewed as a new system. On the contrary, it may be sometimes convenient to decompose a given system as the connection of certain subsystems. Let two systems, whose representing operators are denoted respectively by $\mathcal{R}_1$ and $\mathcal{R}_2$, be given. We describe below three basic types of connections.

(1) *Cascade connection*. The input of the second system coincides with the output of the first system.



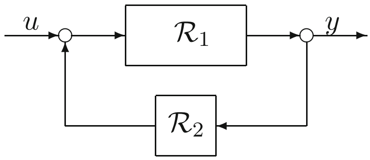
If we denote by $\mathcal{R}$ the operator representing respectively the new resulting system, we have $\mathcal{R} = \mathcal{R}_2 \circ \mathcal{R}_1$, where $\circ$ denotes the composition of maps.

(2) *Parallel connection*. The first and the second system have the same input and both contribute to determine the output.



If we denote as before by $\mathcal{R}$ the operators representing the resulting system, a typical parallel connection is obtained by taking $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$.

(3) *Feedback connection*. The output of the first system is conveyed into the second system; it is elaborated, possibly combined with other external inputs and finally re-injected into the input channel of the first system.



In this case, the operator $\mathcal{R}$ representing the result of the connection of $\mathcal{R}_1$ and $\mathcal{R}_2$, is implicitly defined by the relation $y(\cdot) = \mathcal{R}_1(\mathcal{R}_2(y(\cdot)) + u(\cdot))$.

These types of connection can be combined, to obtain very general patterns. As a rule, the complexity of a system becomes greater and greater, as the number of connections increases.

1.1.7 System Analysis

The purpose of the *analysis of a system* is the study of the properties of the input-output operator. For instance, it is interesting to estimate how the energy carried by the output signal depends on the energy carried by the input signal. To this end, it is necessary to assume that the spaces of the input maps and of the output maps are endowed with a structure of normed vector space. For the moment we do not need to chose a specific norm, which may depend on the particular application. For simplicity, we continue to use the notation $\mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ and $\mathcal{F}(\mathbf{R}, \mathbf{R}^p)$ for the space of the input maps and the space of the output maps, but remember that from now on they are normed space. The norms on these spaces are respectively denoted by $||\cdot||_{\mathcal{F}(\mathbf{R},\mathbf{R}^m)}$ and $||\cdot||_{\mathcal{F}(\mathbf{R},\mathbf{R}^p)}$.

Informally, it is used to say that a system is *externally stable* when each bounded input map generates a bounded output map. More precisely, we give the following definition.

Definition 1.1 A system, or its representing operator $\mathcal{R}$, is said to be *BIBO-stable* (i.e., *bounded-input-bounded-output-stable*) with respect to the norms $||\cdot||_{\mathcal{F}(\mathbf{R},\mathbf{R}^m)}$ and $||\cdot||_{\mathcal{F}(\mathbf{R},\mathbf{R}^p)}$ if for each real number $R > 0$ there exists a real number $S > 0$ such that for each input map $u(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ one has

$$||u(\cdot)||_{\mathcal{F}(\mathbf{R},\mathbf{R}^m)} \le R \implies ||y(\cdot)||_{\mathcal{F}(\mathbf{R},\mathbf{R}^m)} \le S$$

where $y(\cdot)$ is the output map of the system corresponding to the input map $u(\cdot)$.

Notice that according to Definition 1.1, the output is allowed to be different from zero, even if the input vanishes.

Proposition 1.1 *A system, or its representing operator* $\mathcal{R}$*, is* BIBO-stable *if and only if there exists a continuous and non-decreasing function*

$$\alpha(r) : [0, +\infty) \to [0, +\infty)$$

such that for each input map $u(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ *one has:*

$$||y(\cdot)||_{\mathcal{F}(\mathbf{R},\mathbf{R}^p)} \leq \alpha(||u(\cdot)||_{\mathcal{F}(\mathbf{R},\mathbf{R}^m)}). \tag{1.1}$$

The meaning of (1.1) can be explained as follows: if the energy carried by the input signal is bounded, then the energy of the output signal can be estimated in terms of the energy of the input signal. The value of $\alpha(0)$ is sometimes called the *bias* term, while the function $\alpha(r) - \alpha(0)$ is called the *gain function*.

In system analysis, it is very important to know the conditions under which a system is BIBO-stable and, in the positive case, to give information about the shape of the function α.

1.1.8 Control System Design

The *control system design* consists in the development of a control strategy, to be exerted throughout the input channel, in such a way that the output matches a reference signal as well as possible. Roughly speaking, we can distinguish two kinds of control strategies.

(1) *Open loop control*. The control is realized as a function of time $u(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$, and directly injected into the system.
(2) *Closed loop control*. The control is implemented by constructing a second system and establishing a feedback connection.

The closed loop control strategy is also called *automatic control*. It provides some advantages. Indeed, it enables the system to self-regulate, also in presence of unpredictable perturbations, without the need of intervention of a human supervisor. Let us use the term *plant* to denote the system to be controlled, and let us denote by $\mathcal{R}_P : \mathcal{F}(\mathbf{R}, \mathbf{R}^m) \to \mathcal{F}(\mathbf{R}, \mathbf{R}^p)$ the corresponding operator. Let us call *compensator* or *controller* the system to be designed, and let us denote by $\mathcal{R}_C : \mathcal{F}(\mathbf{R}, \mathbf{R}^p) \to \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ the representing operator. The closed loop control strategy consists basically in the following procedure. The output of the plant is monitored and compared with the reference signal; when a unacceptable difference between the two signals is detected, the compensator is activated and the necessary corrections are sent to the plant.

When it is possible to observe directly the state of the system, the compensator can be realized as an operator $\mathcal{R}_C : \mathcal{F}(\mathbf{R}, \mathbf{R}^n) \to \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$. We will use the terms *output feedback* or *state feedback* when we need to distinguish the two situations.

1.1.9 Properties of Systems

In this section we aim to discuss the properties that a generic operator $\mathcal{R} : \mathcal{F}(\mathbf{R}, \mathbf{R}^m) \to \mathcal{F}(\mathbf{R}, \mathbf{R}^p)$ is expected to satisfy in order to represent a real physical system.

1.1.9.1 Causal Systems

Usually, systems encountered in applications are *causal* (or *non anticipative*). This means that for each $t \in \mathbf{R}$ and for each pair of input maps $u_1(\cdot), u_2(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$, if

$$u_1(\tau) = u_2(\tau) \text{ for each } \tau \leq t \ ,$$

then

$$y_1(t) = y_2(t)$$

where $y_1(\cdot) = \mathcal{R}(u_1(\cdot))$ and $y_2(\cdot) = \mathcal{R}(u_2(\cdot))$. In other words, the value of the output at any instant t is determined only by the values that the input map takes at the interval $(-\infty, t]$.

1.1.9.2 Time Invariant Systems

We say that a system, or its representing operator, is *time invariant* if for each $t, T \in \mathbf{R}$ and for each input map $u(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$, one has

$$z(t) = y(t - T)$$

where

$$v(t) = u(t - T), \quad y(\cdot) = \mathcal{R}(u(\cdot)), \quad z(\cdot) = \mathcal{R}(v(\cdot)).$$

In other words, if the input signal is delayed (or anticipated) of a fixed duration, also the output signal is delayed (or anticipated) of the same duration, but its shape is unchanged. Time invariant systems are also called *stationary*, or *autonomous*.

1.1.9.3 Linear Systems

A system is said to be *linear* when its input-output operator is linear, that is

$$a_1\mathcal{R}(u_1(\cdot)) + a_2\mathcal{R}(u_2(\cdot)) = \mathcal{R}(a_1u_1(\cdot) + a_2u_2(\cdot))$$

for each pair of input maps $u_1(\cdot), u_2(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ and each pair of scalars a_1, a_2.

Notice that this definition makes sense, since the input and the output sets are vector spaces.

1.2 Impulse Response Systems

In this section we try to make more concrete the description of a continuous time, finite dimensional system. More precisely, here we assume the existence of a matrix $h(t)$ with p rows and m columns, whose elements are continuous functions defined for each $t \in \mathbf{R}$, such that the response $y(\cdot) = \mathcal{R}(u(\cdot))$ corresponding to an input map $u(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ admits the representation

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)u(\tau)\, d\tau. \tag{1.2}$$

Of course, here we are implicitly assuming that the integral is absolutely convergent.[1] A system for which such a matrix exists is called an *impulse response system*, and the matrix $h(t)$ is called an *impulse response matrix*. This terminology can be explained in the following way.

Let $e_1, \ldots, e_m$ be the canonical basis of $\mathbf{R}^m$, and let $u(t) = \delta(t)e_i$ (for some $i \in \{1, \ldots, m\}$), where $\delta(t)$ represents the *Dirac delta* function (see Appendix B). We have:

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)u(\tau)\, d\tau = \int_{-\infty}^{+\infty} h(t-\tau)\delta(\tau)e_i\, d\tau = h(t)e_i\ .$$

This shows that the response of the system to the unit impulse in the direction of the vector e_i coincides with the i-th column of the matrix $h(t)$. Notice that for SISO systems (i.e., with $p = m = 1$), $h(t)$ is simply a real function of one real variable. The proof of the following proposition is straightforward.

Proposition 1.2 *For any impulse response system, the associated input-output operator $\mathcal{R}$ is linear.*

In particular, it follows from Proposition 1.2 that for an impulse response system with a vanishing input map, the output is zero for each t.

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[1]This may require some restrictions on the nature of the system and the set of admissible inputs.

Proposition 1.3 *Each impulse response system is time invariant.*

Proof Let $u(t)$ be an input map, and let $y(t)$ be the corresponding output map. Let moreover $T \in \mathbf{R}$, $v(t) = u(t - T)$, and let $z(t)$ be the output corresponding to the input $v(t)$. We have:

$$z(t) = \int_{-\infty}^{+\infty} h(t - \tau)v(\tau)\, d\tau = \int_{-\infty}^{+\infty} h(t - \tau)u(\tau - T)\, d\tau.$$

Setting $\tau - T = \theta$, we finally obtain:

$$z(t) = \int_{-\infty}^{+\infty} h(t - T - \theta)u(\theta)\, d\theta = y(t - T).$$

■

Remark 1.4 The two previous propositions provide a complete characterization of the class of impulse response systems. Indeed, it is possible to prove that each time invariant, linear system is an impulse response system (see for instance [32], pp. 152–154).

■

However, in general, an impulse response system is not causal.

Proposition 1.4 *Let an impulse response system be given, and let $h(t)$ be its impulse response matrix. The following properties are equivalent.*

(1) The system is causal.
(2) $h(t) = 0$ for each $t < 0$.
(3) For each input map $u(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ and each $t \in \mathbf{R}$

$$y(t) = \int_{-\infty}^{t} h(t - \tau)u(\tau)\, d\tau.$$

Proof We start to prove that (1) $\Longrightarrow$ (2). For convenience of exposition, we first discuss the case $m = p = 1$. Let $t > 0$ be fixed. Let

$$u_1(\tau) = 0 \quad \text{and} \quad u_2(\tau) = \begin{cases} 0 & \text{if } \tau < t \\ \operatorname{sgn} h(t - \tau) & \text{if } \tau \geq t \end{cases}$$

for $\tau \in \mathbf{R}$, be two input maps (note that u_2 depends on t).

Since $u_1(\tau) = u_2(\tau)$ for $\tau < t$ and the system is causal, we have $y_1(t) = y_2(t)$. On the other hand, it is evident that $y_1(t) = 0$, while

$$y_2(t) = \int_{-\infty}^{+\infty} h(t - \tau)u_2(\tau)\, d\tau = \int_{t}^{+\infty} |h(t - \tau)|\, d\tau.$$

We are so led to conclude that $h(t-\tau)=0$ for each $\tau>t$, that is $h(r)=0$ for $r<0$.

If m or p (or both) are not equal to 1, the proof is technically more complicated, but the basic idea is the same. One starts by fixing a pair of indices i, j, $(i=1,\dots,p,\ j=1,\dots,m)$. As before, we chose $u_1(\tau)=0$ for each $\tau\in\mathbf{R}$, which implies that the corresponding output vanishes identically. Next we define $u_2(\tau)$ component-wise, according to the following rule: if $l\neq j$, then $(u_2)_l(\tau)=0$ for each $\tau\in\mathbf{R}$, while

$$(u_2)_j(\tau)=\begin{cases}0 & \text{if } \tau<t\\ \operatorname{sgn} h_{ij}(t-\tau) & \text{if } \tau\geq t.\end{cases}$$

The causality assumption implies that the output corresponding to $u_2(\tau)$ must be identically zero, as well. On the other hand, the i-th component of the output corresponding to $u_2(\tau)$ is

$$\begin{aligned}(y_2)_i(t)&=\int_{-\infty}^{+\infty} h_{ij}(t-\tau)\cdot(u_2)_j(\tau)\,d\tau\\ &=\int_{t}^{+\infty} h_{ij}(t-\tau)\cdot\operatorname{sgn} h_{ij}(t-\tau)\,d\tau=\int_{t}^{+\infty}|h_{ij}(t-\tau)|\,d\tau\end{aligned}$$

which is zero only if $h_{ij}(r)$ vanishes for each $r\in(-\infty,t)$. The conclusion is achieved, by repeating the argument for each choice of i, j.

The proof that (2) $\Longrightarrow$ (3) is straightforward. Thus, it remains to prove that (3) $\Longrightarrow$ (1). Let $t\in\mathbf{R}$ be fixed. If u_1, u_2 are input maps such that $u_1(\tau)=u_2(\tau)$ for each $\tau\leq t$, then the corresponding output maps y_1, y_2 satisfy

$$y_1(t)=\int_{-\infty}^{t} h(t-\tau)u_1(\tau)\,d\tau=\int_{-\infty}^{t} h(t-\tau)u_2(\tau)\,d\tau=y_2(t).$$

Hence, the system is causal. ■

A further simplification in the representation of impulse response systems is possible, if we limit ourselves to input maps $u(\cdot)\in\mathcal{F}(\mathbf{R},\mathbf{R}^m)$ satisfying the following condition: there exists $t_0\in\mathbf{R}$ such that $u(\tau)=0$ for each $\tau<t_0$. Indeed, in such a case, we will have $y(t_0)=0$ and, for each $t>t_0$,

$$y(t)=\int_{t_0}^{t} h(t-\tau)u(\tau)\,d\tau.$$

For impulse response systems, there is also a simple characterization of the external stability property. Recall that the Frobenius norm of a real matrix $M=(m_{ij})_{i=1,\dots,p,j=1,\dots,m}$ satisfies the following inequality:

$$||M||\leq\sum_{i,j}|m_{ij}|. \tag{1.3}$$

Proposition 1.5 *Let an impulse response system be given, and let $h(t)$ be its impulse response matrix. Let us assume in addition that the system is causal. Let $\mathcal{B}(\mathbf{R}, \mathbf{R}^m)$ and $\mathcal{B}(\mathbf{R}, \mathbf{R}^p)$ be respectively, the input maps and the output maps space, both endowed with the uniform convergence norm. The system is BIBO-stable if and only if the integral*

$$\int_0^{+\infty} ||h(r)||\, dr$$

is convergent or, equivalently, if and only if the function $\int_0^t ||h(r)||\, dr$ is bounded for $t \in [0, +\infty)$.

Proof Since the system is causal, for each $t \in \mathbf{R}$ we have:

$$\begin{aligned} ||y(t)|| = ||\int_{-\infty}^{t} h(t-\tau)u(\tau)\, d\tau|| &\leq \int_{-\infty}^{t} ||h(t-\tau)u(\tau)||\, d\tau \\ &\leq \int_{-\infty}^{t} ||h(t-\tau)|| \cdot ||u(\tau)||\, d\tau \\ &\leq \int_{-\infty}^{t} ||h(t-\tau)||\, d\tau \cdot ||u(\cdot)||_\infty . \end{aligned}$$

By the substitution $t - \tau = r$, we get

$$\int_{-\infty}^{t} ||h(t-\tau)||\, d\tau = \int_0^{+\infty} ||h(r)||\, dr.$$

Hence, if $\int_0^{+\infty} ||h(r)||\, dr = \ell < \infty$, from the previous computation we obtain

$$||y(t)|| \leq \ell ||u(\cdot)||_\infty$$

for each $t \in \mathbf{R}$ and, finally, $||y(\cdot)||_\infty \leq \ell ||u(\cdot)||_\infty$. The BIBO-stability condition will be therefore satisfied taking, for each $R > 0$, $S = \ell R$.

As far as the reverse implication is concerned, let us consider first the case $m = p = 1$. Assuming that the system is BIBO-stable, let us fix $t > 0$ and define the input map

$$\tilde{u}(\tau) = \begin{cases} 0 & \text{if } \tau < 0 \\ \operatorname{sgn} h(t-\tau) & \text{if } \tau \in [0, t] \end{cases}$$

(notice that $\tilde{u}(\tau)$ depends on t). Let $\tilde{y}(t)$ be the corresponding output. Invoking again the causality assumption, we have:

$$\tilde{y}(t) = \int_{-\infty}^{t} h(t-\tau)\tilde{u}(\tau)\, d\tau = \int_0^t |h(t-\tau)|\, d\tau = \int_0^t |h(r)|\, dr. \tag{1.4}$$

Since $|\tilde{u}(\tau)| \leq 1$ for each t and τ, applying the BIBO-stability condition with $R = 1$ we find a constant $S > 0$ such that

$$|\tilde{y}(t)| \leq ||\tilde{y}(\cdot)||_\infty \leq S \tag{1.5}$$

for each $t > 0$. From (1.4) and (1.5) we infer that the integral $\int_0^{+\infty} |h(r)|\, dr$ is convergent.

With some notational complication, the proof extends to the general case where m or p (or both) are greater than 1. Since there are similarities with the proof of Proposition 1.4, we limit ourselves to sketch the reasoning. Let $h_{ij}(t)$ $(i = 1, \ldots, p,\ j = 1, \ldots, m)$ be the elements of the matrix $h(t)$, and let $t > 0$. Let a pair of indices i, j be fixed, and define the input map $\tilde{u}(\tau) = (\tilde{u}_1(\tau), \ldots, \tilde{u}_m(\tau))$ by taking $\tilde{u}_l(\tau) \equiv 0$ if $l \neq j$ and

$$\tilde{u}_j(\tau) = \begin{cases} 0 & \text{if } \tau < 0 \\ \operatorname{sgn} h_{ij}(t - \tau) & \text{if } \tau \in [0, t]. \end{cases}$$

Let finally $\tilde{y}(t) = (\tilde{y}_1(t), \ldots, \tilde{y}_p(t))$ the corresponding output map. Using the causality hypothesis we have

$$\tilde{y}_i(t) = \int_0^t h_{ij}(t - \tau)\tilde{u}_j(\tau)\, d\tau = \int_0^t |h_{ij}(r)|\, dr. \tag{1.6}$$

Since $\|\tilde{u}(\cdot)\|_\infty \leq 1$, the BIBO-stability condition allows us to determine a real number S such that

$$||\tilde{y}(t)|| \leq S \tag{1.7}$$

for each $t > 0$. Clearly, $\tilde{y}_i(t) = |\tilde{y}_i(t)| \leq \|\tilde{y}(t)\|$. As a consequence of (1.6) and (1.7) we conclude that

$$\int_0^t |h_{ij}(r)|\, dr \leq S$$

for each $t > 0$. Finally, by virtue of (1.3), we have

$$\int_0^t \|h(r)\|\, dr \leq \sum_{i,j} \int_0^t |h_{ij}(r)|\, dr \leq pmS$$

for each $t > 0$. The conclusion easily follows. ■

1.3 Initial Conditions

Representing a system as an operator $\mathcal{R} : \mathcal{F}(\mathbf{R}, \mathbf{R}^m) \to \mathcal{F}(\mathbf{R}, \mathbf{R}^p)$ is a very simple and attractive idea, but it is not realistic. In common applications indeed, the inputs are not known on the whole time axis, but only starting from some instant $t_0 \in \mathbf{R}$, assumed as the *initial instant*. Moreover, we are interested to study the behavior of the system in the future, that is for $t \geq t_0$. In these cases, in order to compensate the loss of information about the inputs for $t < t_0$, we need to assume the assignment of the *initial state*, that is the value $x_0 \in \mathbf{R}^n$ assumed by the state variable at the initial instant t_0.

1.3.1 Deterministic Systems

We may image that the *initial condition* i.e., the pair $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$, summarizes the past history of the system. It is also reasonable to presume that the assignment of the initial data, together with the assignment of the input map for $t \geq t_0$, is sufficient to determine uniquely the future evolution of the system. This is actually an assumption, similar to the causality assumption, but more appropriate to the new point of view.

Definition 1.2 We say that a system, or its representing operator, is *deterministic* if for each $t_0 \in \mathbf{R}$,

$$u_1(t) = u_2(t) \ \ \forall t \geq t_0 \textit{ and } x_1(t_0) = x_2(t_0) \implies y_1(t) = y_2(t) \ \forall t \geq t_0$$

where $x_i(t)$, $y_i(t)$ are respectively the state evolution map and the output map corresponding to the input map $u_i(t)$, $i = 1, 2$.

Note that the deterministic hypothesis basically differs from the causality assumption, since it explicitly involves the state of the system. When a system is deterministic, it is convenient to interpret the input-output operator as an "initialized" operator $\mathcal{R}(t_0, x_0)(u(\cdot))$, mapping functions[2]

$$u(\cdot) \in \mathcal{F}([t_0, +\infty), \mathbf{R}^m)$$

to functions

$$y(\cdot) \in \mathcal{F}([t_0, +\infty), \mathbf{R}^p).$$

We write also $y(\cdot) = \mathcal{R}(t_0, x_0)(u(\cdot))$.

Remark 1.5 According to Definition 1.2, the so-called *delayed systems* (systems whose behavior for $t \geq t_0$ depends not only on the state of the system at the initial instant t_0, but also on the values assumed by the state variable on some interval $[t_0 - \theta, t_0]$, $(\theta > 0)$ and, more generally, systems with memory, cannot be considered deterministic. ■

When we want to make use of the notion of initialized operator, the definitions of time invariant system and of linear system need to be appropriately modified.

[2] Alternatively, we may agree that the admissible inputs are restricted to functions $u(\cdot) \in \mathcal{F}(\mathbf{R}, \mathbf{R}^m)$ vanishing for $t < t_0$.

1.3.2 Time Invariant Systems

A system represented by a deterministic initialized operator is *time invariant* if for each $t_0, T \in \mathbf{R}$, each $x_0 \in \mathbf{R}^n$, and each input map $u(\cdot)$, denoting

$$v(t) = u(t - T), \quad y(\cdot) = \mathcal{R}(t_0, x_0)(u(\cdot)), \quad z(\cdot) = \mathcal{R}(t_0 + T, x_0)(v(\cdot))$$

one has

$$z(t) = y(t - T).$$

Proposition 1.6 *Let $\mathcal{R}$ be a time invariant, deterministic initialized operator, and let $y(\cdot) = \mathcal{R}(t_0, x_0)(u(\cdot))$. Then,*

$$y(t) = z(t - t_0)$$

where $z(\cdot) = \mathcal{R}(0, x_0)(v(\cdot))$, and $v(t) = u(t + t_0)$.

In other words, dealing with a time invariant operator, we may assume, without loss of generality, that the initial instant coincides with the origin of the time axis.

1.3.3 Linear Systems

A system represented by means of a deterministic initialized operator $\mathcal{R}$, is *linear* if for each $t_0 \in \mathbf{R}$, $\mathcal{R}$ is linear as a map from $\mathbf{R}^n \times \mathcal{F}([t_0, +\infty), \mathbf{R}^m)$ to $\mathcal{F}([t_0, +\infty), \mathbf{R}^p)$, that is if for each $t_0 \in \mathbf{R}$, and for each choice of the pairs $x_1, x_2 \in \mathbf{R}^n$, $u_1(\cdot), u_2(\cdot) \in \mathcal{F}([t_0, +\infty), \mathbf{R}^m)$, and $a_1, a_2 \in \mathbf{R}$ one has

$$\begin{aligned}&\mathcal{R}\,(t_0, a_1x_1 + a_2x_2)(a_1u_1(\cdot) + a_2u_2(\cdot))\\&= a_1\mathcal{R}(t_0, x_1)(u_1(\cdot)) + a_2\mathcal{R}(t_0, x_2)(u_2(\cdot)).\end{aligned}$$

Proposition 1.7 *Let $\mathcal{R}$ be a linear, time invariant, deterministic initialized operator. For each $t_0 \in \mathbf{R}$, each $x_0 \in \mathbf{R}^n$ and each $u(\cdot) \in \mathcal{F}([t_0, +\infty), \mathbf{R}^m)$ one has:*

$$y(\cdot) = \mathcal{R}(t_0, x_0)(0) + \mathcal{R}(t_0, 0)(u(\cdot)).$$

Proof By applying the definition of linear initialized system with $x_1 = x_0$, $x_2 = 0$, $u_1(\cdot) = 0$, $u_2(\cdot) = u(\cdot)$, $a_1 = a_2 = 1$, we immediately have

$$y(\cdot) = \mathcal{R}(t_0, x_0)(u(\cdot)) = \mathcal{R}(t_0, x_0)(0) + \mathcal{R}(t_0, 0)(u(\cdot))$$

as required. ■

1.3.4 External Stability

We now update Definition 1.1, for the case of systems represented by initialized operators.

Definition 1.3 A system represented by a deterministic initialized operator $\mathcal{R}$, is *BIBO-stable* (uniformly with respect to the initial instant) if for each real number $R > 0$ there exists a real number $S > 0$ such that for each $t_0 \in \mathbf{R}$ and each input map $u(\cdot) \in \mathcal{B}([t_0, +\infty), \mathbf{R}^m)$ we have

$$||x_0|| \leq R, \;\; ||u(\cdot)||_\infty \leq R \implies ||y(\cdot)||_\infty \leq S$$

where $y(\cdot) = \mathcal{R}(t_0, x_0)(u(\cdot))$.

1.3.5 Zero-Initialized Systems and Unforced Systems

We may interpret Proposition 1.7 by saying that the response of a linear system corresponding to some initial state x_0 and some input map $u(\cdot)$ can be decomposed as the sum of

- the response corresponding to the initial state x_0 when the the input is set to be zero;
- the response corresponding to the input $u(\cdot)$ when the initial state is set to be zero.

In other words, when analyzing the behavior of a linear system, and more precisely when we are interested in the study of the external stability, the way the response is affected by the initial data and the way it is affected by the inputs can be analyzed separately. We will frequently refer to this principle in this book.

Therefore, in the study of linear systems we may conveniently distinguish two different steps. In the first step we may assume that the input vanishes, while in the second step we may assume that the initial state vanishes. In this way, we will be also able to recover some analogies with the theory of the impulse response systems.

We say that a deterministic system represented by a time invariant initialized operator is *zero-initialized* (or *initialized at zero*) if the initial state x_0 at the instant $t_0 = 0$ is set to be zero. We say that a deterministic system represented by a time invariant initialized operator is *unforced* if the input map is set to be equal to zero for each $t \geq 0$.

Unforced systems may present a non-zero evolution in time: indeed, because of the energy stored in the system at the initial instant, the initial state does not coincide, in general, with a rest point. In these circumstances, we expect that the unforced system evolves in such a way that the initial energy is dissipated, by approaching a rest point asymptotically. If this really happens, we will say informally that the system is *internally stable*. A more precise and formal definition of internal stability will be given later.

In the analysis of the qualitative properties of a system, the study of the behavior when the forcing terms are provisionally suppressed, is an essential preliminary step. As we shall see, the properties of internal stability and external stability are intimately related.

1.4 Differential Systems

In this section we focus on systems which are modeled by means of ordinary differential equations; they will be called *differential systems*. This class of systems is very important, because of the variety and the large amount of applications. Moreover, a well developed and complete theory is available, for these systems. However, its introduction requires some restrictions.

1.4.1 Admissible Inputs

Dealing with differential systems, by *admissible input map* we mean a function $u(\cdot) \in \mathcal{PC}([t_0, +\infty), \mathbf{R}^m)$, for some $t_0 \in \mathbf{R}$. For some applications, it is necessary to limit further the admissible inputs: a typical choice is $u(\cdot) \in \mathcal{PC}([t_0, +\infty), U)$, where U is a given nonempty, bounded subset of $\mathbf{R}^m$. The role of U is to represent possible limitations on the energy available in order to exert the control. Notice that if U is bounded, $\mathcal{PC}([t_0, +\infty), U)$ is not a vector space.

1.4.2 State Equations

Let $f(t, x, u) : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ and $h(t, x) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^p$ be given functions. A *differential system* is defined by the equations

$$\dot{x} = \frac{dx}{dt} = f(t, x, u) \tag{1.8}$$

$$y = h(t, x). \tag{1.9}$$

Equation (1.8) is also called the *state equation*, while $h(t, x)$ is called the *observation map*. For each admissible input map $u(t)$, (1.8) becomes a system of ordinary differential equations of the first order in normal form

$$\dot{x} = f(t, x, u(t)) = g(t, x). \tag{1.10}$$

Concerning the functions f and h, it is customary to make the following assumptions:

(**A1**) f is continuous with respect to the pair of variables (x, u); the first partial derivatives of f with respect to all the components x_i of the state vector x exist and are continuous with respect to the pair of variables (x, u); moreover, f is piecewise continuous with respect to t;
(**A2**) h is continuous with respect to x, and piecewise continuous with respect to t;
(**A3**) for each admissible input $u(t)$, there exist continuous, positive real functions $a(t), b(t)$ such that

$$||f(t, x, u(t))|| \leq a(t)||x|| + b(t)$$

for each $(t, x) \in \mathbf{R} \times \mathbf{R}^n$.

Under these assumptions, for each pair of initial values $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ and for each admissible input map $u(t)$ there exists a unique solution of the *Cauchy problem*

$$\begin{cases} \dot{x} = g(t, x) \\ x(t_0) = x_0 \end{cases} \tag{1.11}$$

defined on the whole interval[3] $[t_0, +\infty)$. When we want to emphasize the dependence of the solution of the problem (1.11) on the initial conditions and on the input map, we will use the notation

$$x = x(t; t_0, x_0, u(\cdot)). \tag{1.12}$$

When the dependence on the initial conditions and on the input map is clear from the context, we may also use the simplified notation $x = x(t)$. The initialized input-output operator associated to the differential system (1.8), (1.9)

$$y(\cdot) = \mathcal{R}(t_0, x_0)(u(\cdot)) \tag{1.13}$$

is given by $y(t) = h(t, x(t; t_0, x_0, u(\cdot)))$ for $t \geq t_0$. By analogy with (1.12), sometimes we may use the notation

$$y = y(t; t_0, x_0, u(\cdot)). \tag{1.14}$$

The following proposition summarizes the previous remarks.

Proposition 1.8 *Under the hypotheses* **(A1)**, **(A2)**, **(A3)**, *the differential system (1.8), (1.9) defines a deterministic input-output operator on the set of admissible input maps. Moreover, the output map is continuous.*

[3] Provided that the input is defined for each $t \in \mathbf{R}$, existence and uniqueness of solutions is actually guaranteed on $(-\infty, +\infty)$.

Next proposition characterizes the differential systems which possess the time invariance property.

Proposition 1.9 *Assume that* **(A1)***,* **(A2)***,* **(A3)** *hold. The input-output operator (1.13) defined by the differential system (1.8), (1.9) is time invariant if the functions* f *e* h *do not depend explicitly on* t*, that is* $f(t, x, u) = f(x, u)$ *and* $h(t, x) = h(x)$.

Proof Let $t_0 \in \mathbf{R}$ and let $u(t) \in \mathcal{PC}([t_0, +\infty), \mathbf{R}^m)$. Assume that an initial state x_0 is given; let $x(t)$ be the corresponding solution of (1.8) and let $y(t) = h(x(t))$. Let finally T be a fixed real number. Setting $v(t) = u(t - T)$ and $\xi(t) = x(t - T)$, we have

$$\frac{d}{dt}\xi(t) = \frac{d}{dt}x(t - T) = f(x(t - T), u(t - T)) = f(\xi(t), v(t)).$$

In other words, $\xi(t)$ coincides with the solution corresponding to the translated input map $v(t)$ and to the initial condition $(t_0 + T, x_0)$. Setting finally $z(t) = h(\xi(t))$, it is clear that $z(t) = y(t - T)$. ■

By virtue of Propositions 1.6 and 1.9, if the functions f and h do not depend explicitly on t we may assume $t_0 = 0$ without loss of generality. In this case, the notation (1.12) and (1.14) can be simplified, by avoiding the explicit indication of the initial instant.

1.4.3 Linear Differential Systems

In the mathematical theory of differential systems, a prominent role is played by systems whose state equations are linear. The importance of linear systems is also supported by their interest in applications.

Definition 1.4 A time invariant differential system is said to be *linear* if there exist real matrices A, B, C of respective dimensions $n \times n$, $n \times m$, $p \times n$, such that $f(x, u) = Ax + Bu$ and $h(x) = Cx$.

In other words, a system is linear in the sense of Definition 1.4 when it can be written in the form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx. \end{cases} \tag{1.15}$$

Proposition 1.10 *If a system is linear in the sense of Definition 1.4, then the associated input-output initialized operator (1.13) is linear.*

The proof of Proposition 1.10 will be given later. Beginning with Chap. 2, we focus our attention on the study of linear, time invariant differential systems.

Chapter Summary

The first part of this chapter constitutes a short introduction to systems theory. The basic notions of input, output and state variables are presented in abstract terms, as well as the notion of input-output operator. We discuss the main properties involved in the investigation of a system, and illustrate how distinct systems can be combined to give rise to a new system. In this framework, we also introduce the main concern of this book: how to exploit the input channel in order to control the evolution of a system.

The exposition becomes more concrete in the remaining part of the chapter, where we explain how a system can be represented by certain mathematical models: impulse response, state space equations. The role of initial conditions is emphasized, in a deterministic philosophy, in connection with the notion of state variable.