Chapter 4
Linear Systems with Forcing Term

The simplest way to model an external input is to introduce an additive term in the system equations. In this chapter we shall see how the solutions of a system of differential equations, whose right-hand side is the sum of a linear part and a time-varying term, can be explicitly found.

4.1 Nonhomogeneous Systems

A linear nonhomogeneous system[1] of differential equations has the general form

$$\dot{x} = Ax + b(t) \tag{4.1}$$

where $b(t)$, frequently referred to as the *forcing term*, belongs to the space $\mathcal{PC}(I, \mathbf{R})$. Here, I denotes in general any interval of $\mathbf{R}$ with nonempty interior, although for our purposes, the relevant cases are $I = \mathbf{R}$ and $I = [0, +\infty)$. We report below some basic facts.

Fact 1. *For each initial instant $t_0 \in I$ and each initial state $x_0 \in \mathbf{R}^n$ there exists a unique solution $x = \psi(t)$ of (4.1) such that $\psi(t_0) = x_0$. Moreover, $\psi(t)$ is defined for each $t \in I$.*

Fact 2. *If $\psi_1(t)$, $\psi_2(t)$ are solutions of (4.1) defined on I, then $\psi_1(t) - \psi_2(t)$ is a solution of the so-called* associated homogeneous system

$$\dot{x} = Ax. \tag{4.2}$$

[1]According to a more correct terminology, a system of the form (4.1) should be called an "affine" system; however, the term "linear nonhomogeneous" is very frequent in the literature.

A. Bacciotti, *Stability and Control of Linear Systems*, Studies in Systems, Decision and Control 185, https://doi.org/10.1007/978-3-030-02405-5_4

Fact 3. *If $\varphi(t)$ is any solution of the associated homogeneous system* (4.2) *and* $\psi^*(t)$ *is any solution of the nonhomogeneous system* (4.1), *then* $\varphi(t) + \psi^*(t)$ *is a solution of the nonhomogeneous system* (4.1).

Fact 4. (Superposition principle) If $\psi_1(t)$ *is a solution of system* (4.1) *with* $b(t) = b_1(t)$ *and* $\psi_2(t)$ *is a solution of system* (4.1) *with* $b(t) = b_2(t)$, *then* $\psi_1(t) + \psi_2(t)$ *is a solution of system* (4.1) *with* $b(t) = b_1(t) + b_2(t)$.

From Facts 2 and 3 it follows that in order to determine the set of all the solutions of system (4.1), we need to find:

(a) a fundamental matrix $\Phi(t)$ of (4.2);
(b) a particular solution $\psi^*(t)$ of (4.1).

The set of all the solutions of system (4.1) can be therefore represented by the formula

$$x = \psi(t) = \Phi(t)c + \psi^*(t) \tag{4.3}$$

where c is a vector of arbitrary constants. It is called the *general integral* of system (4.1). The particular solution corresponding to a given initial condition (t_0, x_0) can be obtained solving the algebraic system

$$x_0 - \psi^*(t_0) = \Phi(t_0)c.$$

If $\Phi(t) = e^{(t-t_0)A}$, then $c = x_0 - \psi^*(t_0)$.

4.1.1 The Variation of Constants Method

The problem of determining a fundamental matrix of system (4.2) has been solved in Chap. 2. As far as point (b) is concerned, we have the following general result.

Proposition 4.1 *The function*

$$\psi_0^*(t) = \int_{t_0}^{t} e^{(t-\tau)A} b(\tau)\, d\tau \tag{4.4}$$

provides the solution of (4.1) such that $\psi_0^*(t_0) = 0$.

Taking into account this result, we can write the solution corresponding to the initial state (t_0, x_0) as

$$\psi(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-\tau)A} b(\tau) d\tau = e^{(t-t_0)A}(x_0 + \int_{t_0}^{t} e^{(t_0-\tau)A} b(\tau) d\tau). \tag{4.5}$$

This is called *Lagrange formula* or *variation of constants formula*. This formula is very well suited for theoretical purposes but sometimes not so convenient in practice, because of the presence of the integral that, for certain functions $b(t)$, might be hard or even impossible to compute explicitly.

Remark 4.1 Formula (4.5) is often used with $t_0 = 0$ (provided of course that $0 \in I$), that is in the form

$$\psi(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}b(\tau)\,d\tau. \tag{4.6}$$

There is an interesting interpretation of (4.6): it shows that each solution is the sum of two contributions. The first one depends on the initial state but not on the forcing term. On the contrary, the second one depends on the forcing term but not on the initial state. This suggests that the analysis of the dynamical behavior of a linear system can be carried out by investigating separately the effect of the initial conditions (with zeroed forcing term) and the effect of the forcing term (with zeroed initial state).[2] ■

Nonhomogeneous equations arise frequently in applications, both in classical physics and in system theory. In any case, it is natural to presume that $b(t)$ represents a signal generated by an *exosystem*, that is an external system connected to the main plant by a cascade connection. Exosystems are often simple linear devices without forcing terms. Hence it is reasonable to focus on forcing terms of the form

$$b(t) = \sum_{h=1}^{H} P_h(t)e^{\gamma_h t}$$

where each $P_h(t)$ is a polynomial with vector coefficients and $\gamma_h \in \mathbf{C}$. As illustrated in the next section, in such cases the computation of the integral in (4.5) can be avoided by virtue of the superposition principle and the use of some practical rules which allows us to find a particular solution in a more direct way. These rules are presented in the next section.

4.1.2 The Method of Undetermined Coefficients

We can limit ourselves to assume $b(t) = P(t)e^{\gamma t}$, where $P(t)$ is a polynomial with vector coefficients. We distinguish two cases.

Case 1: γ is not an eigenvalue of A. Then, there exists a particular solution of (4.1) with the following structure: $\psi^*(t) = Q(t)e^{\gamma t}$ where Q is a polynomial with vector coefficients and the same degree of P.

Case 2: γ is an eigenvalue of A, with algebraic multiplicity $\mu \geq 1$. Then, there exists a particular solution of (4.1) with the following structure: $\psi^*(t) = Q(t)e^{\gamma t}$ where Q

[2] This agrees with the conclusions of Chap. 1 (Sect. 1.3.5) provided that the forcing term is interpreted as an input and taking into account Proposition 1.10.

is a polynomial with vector coefficients, and $\deg Q = \deg P + \mu$: in this case, we say that the system exhibits *resonance*.

In both cases, the coefficients of Q depend on A, γ and the coefficients of $P(t)$; they can be determined by exploiting the identity $\dot{\psi}^*(t) = A\psi^*(t) + P(t)e^{\gamma t}$, which leads to a system of merely algebraic equations. This is the reason why this procedure is called the *method of undetermined coefficients*. We emphasize that these rules hold even if γ is a complex number. By virtue of the formulæ

$$\cos \omega t = \frac{e^{\mathrm{i}\,\omega t} + e^{-\mathrm{i}\,\omega t}}{2}\,, \qquad \sin \omega t = \frac{e^{\mathrm{i}\,\omega t} - e^{-\mathrm{i}\,\omega t}}{2\mathrm{i}}$$

the method of undetermined coefficients can be therefore extended to forcing terms of the form $b(t) = P_1(t)\cos \omega t + P_2(t)\sin \omega t$ ($P_1(t)$ and $P_2(t)$ being polynomials with real vector coefficients).

Example 4.1 We are especially interested in the case where the forcing term is a periodic function of the form

$$b(t) = (\cos \omega t)u + (\sin \omega t)v \tag{4.7}$$

where $u, v \in \mathbf{R}^n$ are constant vectors. This case occurs frequently in applications, and it will be further developed later in this chapter, in different situations. Let us apply the method of undetermined coefficients separately to the systems

$$\dot{x} = Ax + \frac{e^{\mathrm{i}\,\omega t}}{2}(u - \mathrm{i}\,v) \quad \text{and} \quad \dot{x} = Ax + \frac{e^{-\mathrm{i}\,\omega t}}{2}(u + \mathrm{i}\,v).$$

Assuming for simplicity that resonance does not occur and taking into account that the elements of A and u are real, we find respectively particular solutions of the type

$$\psi_1(t) = e^{\mathrm{i}\,\omega t}c \quad \text{and} \quad \psi_2(t) = e^{-\mathrm{i}\,\omega t}\bar{c}$$

for some constant vector $c \in \mathbf{C}^n$. According to the superposition principle, a particular solution of the system will be found of the form

$$\psi^*(t) = e^{\mathrm{i}\,\omega t}c + e^{-\mathrm{i}\,\omega t}\bar{c}.$$

This solution is actually real, since it is the sum of two conjugate terms. It can be rewritten as

$$\psi^*(t) = (\cos \omega t)a + (\sin \omega t)b \tag{4.8}$$

for some vector constants $a, b \in \mathbf{R}^n$. ■

Remark 4.2 It is important to notice that (4.8) is a periodic solution, with the same frequency as the forcing term (4.7). It should be also noticed that in (4.8) a and b may be both nonzero, even if in (4.7) one between u and v is zero. ■

Remark 4.3 Notice that in general it is not possible to preassign the initial state of the particular solution obtained by the method of undetermined coefficients. In general, we will have $\psi^*(t_0) \neq 0$, so that it does not coincide with the solution introduced in Proposition 4.1. More precisely, let $\psi^*(t)$ be a particular solution obtained by the method of undetermined coefficients, and let for simplicity $t_0 = 0$. We may rewrite (4.3), as

$$x = e^{tA}(x_0 - \psi^*(0)) + \psi^*(t) \tag{4.9}$$

where x_0 stands for the desired initial state. The particular solution provided by the method of variation of constants can be recovered as

$$\psi_0^*(t) = \psi^*(t) - e^{tA}\psi^*(0). \tag{4.10}$$

■

4.2 Transient and Steady State

Throughout this section, we assume that $I = [0, +\infty)$ and $t_0 = 0$. In addition, we assume that the matrix A in (4.1) possesses the Hurwitz property (Definition 3.1).

Using (2.14), we may give an asymptotic estimation of the solutions also for nonhomogeneous systems of type (4.1).

Proposition 4.2 *If A is a Hurwitz matrix and $b(t)$ is bounded on the interval $[0, +\infty)$, then for each solution $\psi(t)$ of (4.1) we have*

$$\|\psi(t)\| \leq k_0\|x_0\|e^{\alpha t} + k_1 \cdot b_0, \quad t \geq 0,$$

where k_0 and k_1 are positive constants, $\alpha < 0$, $b_0 = \sup_{\tau \geq 0} \|b(\tau)\|$, and $x_0 = \psi(0)$.

Proof The assumptions imply the existence of constants $\alpha < 0$ and $k_0 > 0$ such that for each t and each $\tau \in [0, t]$

$$\|e^{(t-\tau)A}b(\tau)\| \leq k_0\|b(\tau)\|e^{(t-\tau)\alpha}.$$

Since the initial state x_0 is assigned for $t_0 = 0$, we may use the version (4.6) of the variation of constants formula. We have:

$$\|\psi(t)\| \leq k_0\|x_0\|e^{\alpha t} + b_0k_0 \int_0^t e^{(t-\tau)\alpha}\, d\tau = k_0\|x_0\|e^{\alpha t} - \frac{b_0k_0}{\alpha}\left[e^{(t-\tau)\alpha}\right]_0^t$$
$$= k_0\|x_0\|e^{\alpha t} + \frac{b_0k_0}{\alpha}\left[e^{\alpha t} - 1\right].$$

Being $\alpha < 0$, we have $e^{\alpha t} \leq 1$ for $t > 0$. Setting $k_1 = \frac{k_0}{|\alpha|}$, the previous inequality reduces to the desired one. ■

Proposition 4.2 implies in particular that if the matrix A is Hurwitz and the forcing term is bounded, then every solution is bounded on $[0, +\infty)$. We want to focus on the following two particular cases:

(1) the forcing term $b(t)$ is constant;
(2) the forcing term $b(t)$ is a periodic function of the form (4.7).

Note that in force of the Hurwitz property, resonance does not occur neither in case (1) nor in case (2). As a consequence, the system admits a unique constant solution in case (1) and, respectively, a unique periodic solution[3] in case (2). One such solution $\psi^*(t)$ can be used in (4.9), in order to represent a generic solution. Recalling again that A is Hurwitz and using (2.14) with $\alpha < 0$, we have that

$$\lim_{t\to+\infty} e^{tA}c = 0$$

for each $c \in \mathbf{R}^n$. This means that in (4.9), for sufficiently large t, the term $e^{tA}(x_0 - \psi^*(0))$ can be neglected and the evolution of the system "becomes independent" of the initial state x_0. It is approximately constant or periodic, and it is essentially determined by the forcing term. It is customary to distinguish two stages in the time evolution of the system. The first stage, where the evolution is appreciably affected by the initial state x_0, is called the *transient*. The subsequent stage, where the effect of the initial state is no more perceptible, is called the *steady state*. Of course, the distinction between the transient and the steady state is not rigorous, since the term $e^{tA}(x_0 - \psi^*(0))$ in (4.9) will never be exactly equal to zero. Distinguishing the two stages depends on the admitted error margins and on the precision of the measurements, but it is very impressive and convenient, at least from the heuristic point of view.

Remark 4.4 The steady state solution is be more correctly thought of as a "limit" solution, asymptotically approached by all the solutions of system (4.1). As already noticed, such a limit solution does not necessarily vanish for $t = 0$, and so it does not coincide, in general, with the particular solution appearing in the variation of constants formula (4.6). Indeed, as we can understand from (4.10), further terms vanishing when $t \to +\infty$, could be hidden in the particular solution appearing in (4.6). These terms compensate for the gap between the assigned initial state and the initial state of the steady state solution. The steady state solution is found in a natural way when the method of undetermined coefficients is adopted. ■

Example 4.2 Consider the system represented by the scalar differential equation

$$\dot{x} = -x + 2 \tag{4.11}$$

with the initial condition $x(0) = 1$. The general integral of the associated homogeneous system is $x = e^{-t}c$, with c an arbitrary constant. A solution of the

[3] On the other hand, it is easy to check that there exists a constant or periodic solution only if the forcing term is, respectively, constant or periodic.

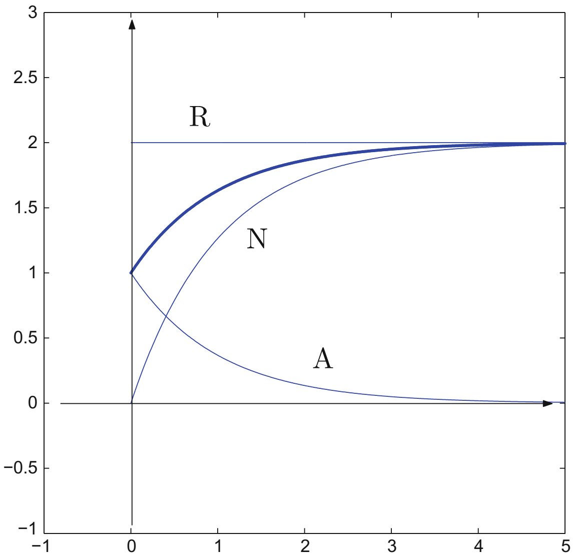

Fig. 4.1 The curve in bold represents the graph of the solution of (4.11) such that $x(0) = 1$; the curve marked by A represents the graph of the solution of the associated homogeneous equation with the same initial condition $x(0) = 1$; the curve marked by R represents the graph of the steady state solution; the curve marked by N represents the graph of the solution of (4.11) such that $x(0) = 0$

nonhomogeneous equation (4.11) can be found by applying Proposition 4.1: we obtain $x = 2\int_0^t e^{-(t-\tau)}\,d\tau = 2 - 2e^{-t}$. Since it vanishes for $t = 0$, we set $c = x(0) = 1$. According to (4.6), the required solution writes

$$x = e^{-t} - 2e^{-t} + 2. \tag{4.12}$$

Alternatively, we can use the method of undetermined coefficients. In this way we find directly the steady state solution $x = 2$. The general integral of (4.11) takes therefore the form

$$x = e^{-t}c + 2$$

and imposing the condition $x(0) = 1$, now we find $c = -1$. Of course, the two approaches lead to the same result. The graphs of the various components of the sum (4.12) are shown in Fig. 4.1. ■

4.3 The Nonhomogeneous Scalar Equation of Order *n*

By the same procedure illustrated in Sect. 2.10, the nonhomogeneous scalar equation of order n (with constant coefficients)

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{4.13}$$

can be rewritten as a system of the form (4.1) with a matrix A in companion form, and

$$b(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}.$$

Thus, (4.13) can be considered a particular case of (4.1), the function $g(t)$ playing the role of the *forcing term*. It follows that for each function $g(\cdot) \in \mathcal{PC}(I, \mathbf{R})$ and for each set of initial conditions

$$y(t_0) = y_0, \; y'(t_0) = y_1, \dots, y^{(n-1)}(t_0) = y_{n-1} \tag{4.14}$$

($t_0 \in I$) there is a unique solution defined for $t \in I$. We emphasize that by this procedure, we are led to identify the state of the system with the vector whose components are $(y, y', \dots, y^{(n-1)})$.

In order to determine the solutions of (4.13), the methods described in the previous sections can be applied. However, if $g(t) = p(t)e^{\gamma t}$ where $\gamma \in \mathbf{C}$ and $p(t)$ is a polynomial with real or complex coefficients, it is more convenient to work directly with (4.13). Indeed, we can write the general integral as

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + \chi^*(t) \tag{4.15}$$

where $y_1(t), \dots, y_n(t)$ are linearly independent solutions of the *associated homogeneous (or unforced) equation*

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \tag{4.16}$$

$c_1, \dots, c_n$ are arbitrary constants, and $\chi^*(t)$ is a particular solution of (4.13). Now, the method of undetermined coefficients gives rise to the following simplified rules: a particular solution $\chi^*(t)$ can be sought of the form

1. $\chi^*(t) = q(t)e^{\gamma t}$ provided that γ is not a characteristic root of (4.16);
2. $\chi^*(t) = t^{\mu} q(t)e^{\gamma t}$ provided that γ is a characteristic root of (4.16) with algebraic multiplicity μ (case of resonance).

In both cases, $q(t)$ represents a polynomial of the same degree as $p(t)$. Recall that the solution $\chi^*(t)$ obtained by the method of undetermined coefficients does not coincide, in general, with the solution of (4.13) vanishing at $t = t_0$.

Remark 4.5 If all the characteristic roots have strictly negative real part, then all the solutions of the associated homogeneous system (4.16) (and all their derivatives) go to zero when $t \to +\infty$. Hence, if the forcing term $g(t)$ is constant or periodic, the particular solution $\chi^*(t)$ can be interpreted, also in this case, as the steady state solution. ■

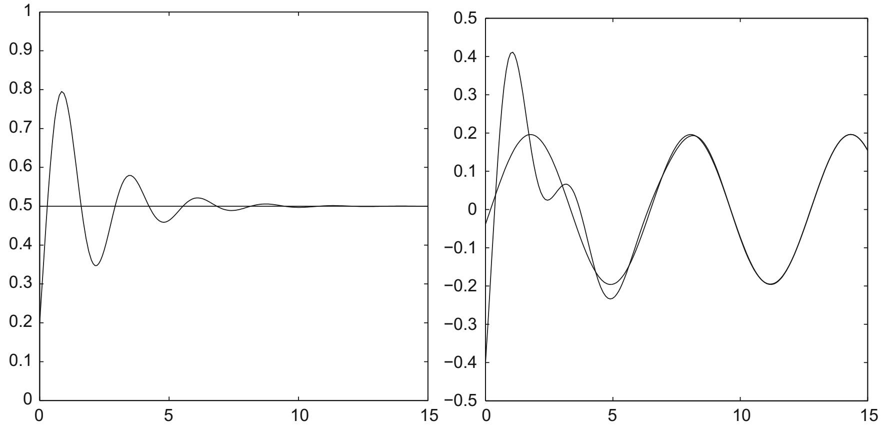

Fig. 4.2 Examples 4.3 and 4.4: steady state solution and transient

Example 4.3 The linear equation of order 2 in general form

$$y'' + ay' + by = g(t) \tag{4.17}$$

constitutes a model for a large variety of physical problems, and it is appropriate to illustrate the transient and the steady state phenomena. To this end, we assume that the characteristic polynomial of the associated homogeneous equation has a pair of complex conjugate roots $\alpha \pm \mathrm{i}\,\beta$ with $\alpha = -a/2 < 0$ and $\beta \neq 0$ (which necessarily yields $b \neq 0$).

If the forcing term is constant, say $g(t) = g_0$, the unique constant solution is $\chi^*(t) = g_0/b$. Its graph is drawn in Fig. 4.2 (left), for the case $a = 1$, $b = 6$, $g_0 = 3$. The figure shows also the graph of a solution corresponding to different initial conditions. The transient stage can be recognized in the interval where the two graphs can be clearly distinguished. ■

Example 4.4 Considered again the general second order equation (4.17) under the same assumptions about the coefficients a, b, but now with a periodic forcing term

$$g(t) = p_1 \cos \omega t + p_2 \sin \omega t, \quad p_1, p_2 \in \mathbf{R}.$$

By the same procedure of Example 4.1, we can find a particular solution of the form

$$\chi^*(t) = q_1 \cos \omega t + q_2 \sin \omega t, \quad q_1, q_2 \in \mathbf{R} \tag{4.18}$$

which can be recognized as the steady state solution. The general integral can be written as

$$y(t) = (c_1 \cos \beta t + c_2 \sin \beta t)e^{\alpha t} + q_1 \cos \omega t + q_2 \sin \omega t. \tag{4.19}$$

The transient will be shorter and shorter, as the absolute value of α becomes larger and larger.

The coefficients q_1 and q_2 in (4.18) depend on p_1, p_2 and ω (as well as on a and b) and can be computed by direct substitution.[4] Sometimes, it may be convenient to rewrite (4.18) as

$$\chi^*(t) = \rho \cos(\omega t + \theta) \tag{4.20}$$

where $q_1 = \rho \cos\theta$, $q_2 = \rho \sin\theta$. The quantities ρ and θ represent the amplitude and, respectively, the phase of the periodic function at hand (compare with Example 2.2). Also the forcing term can be rewritten in a similar way. Note that the initial conditions contribute to determine the values of c_1 and c_2 in (4.19), but not the values of q_1, q_2 (equivalently, ρ, θ) characterizing the shape of (4.18).

Consistently with Remark 4.2, we see that the frequency of the steady state solution is unchanged, when compared with the frequency of the forcing term. On the contrary, while the signal goes through the system, the phase and the amplitude may undergo a variation.

A simulation is presented in Fig. 4.2 (right), for the case $a = 1, b = 6, g_0 = \sin t$. The periodic steady state solution is $\chi^*(t) = (-\cos t + 5 \sin t)/26$. ■

Example 4.5 Let us consider again the Eq. (4.17), with the same forcing term but now with $a = 0$. If $\omega^2 = b$ then $\mathrm{i}\,\omega$ is a solution of the characteristic equation. The system resonates. The form of the general integral is

$$\chi^*(t) = (c_1 + tq_1)\cos\omega t + (c_2 + tq_2)\sin\omega t. \tag{4.21}$$

The constants q_1, q_2 characterizing the particular solution can be easily determined by direct substitution. The solutions exhibit an oscillatory behavior, and the amplitude of the oscillations goes to $+\infty$ when $t \to \infty$. ■

4.4 The Laplace Transform Method

In this section we discuss a different approach to the problem of determining the solutions of (4.13), based on the Laplace transform (see Appendix B for notation and properties of the Laplace transform).

4.4.1 Transfer Function

Let us assume that the forcing term $g(\cdot)$ is defined for $t \geq 0$, and that it belongs to the set of subexponential functions of class $\mathcal{PC}([0, +\infty), \mathbf{R}^m)$. According to what

[4] Recall that $p_i = 0$ $(i = 1, 2)$ does not imply in general $q_i = 0$.

exposed in the previous sections, we know that the solution $y(t)$ is defined for $t \geq 0$, and that it is a subexponential function, as well. This justifies the use of the Laplace transform.

Let us apply the Laplace transform to both side of (4.13). Recalling (4.14) and (B.10), from

$$\mathcal{L}[y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y] = \mathcal{L}[g(t)]$$

we have

$$\begin{aligned}
&(s^n + a_1 s^{n-1} + \cdots + a_n) Y(s) \\
&-\big\{ s^{n-1} y_0 + s^{n-2} y_1 + s^{n-3} y_2 + \cdots + y_{n-1} \\
&\quad + \big[s^{n-2} y_0 + s^{n-3} y_1 + \cdots + y_{n-2} \big] a_1 \\
&\quad + \big[s^{n-3} y_0 + \cdots + y_{n-3} \big] a_2 \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + y_0 a_{n-1} \big\} = G(s).
\end{aligned}$$

We recognize that $s^n + a_1 s^{n-1} + \cdots + a_n$ is nothing else but the characteristic polynomial $p_{ch}(s)$ of the homogeneous equation (4.16) associated to (4.13). Thus we can write

$$p_{ch}(s) Y(s) - P_0(s) = G(s) \tag{4.22}$$

where

$$P_0(s) = A_0 s^{n-1} + A_1 s^{n-2} + \cdots + A_{n-1} \tag{4.23}$$

with

$$A_0 = y_0, \;\; A_1 = y_1 + a_1 y_0, \ldots, \;\; A_{n-1} = y_{n-1} + a_1 y_{n-2} + \cdots + a_{n-1} y_0.$$

We remark that:

(i) $p_{ch}(s)$ is independent of both the forcing term and the initial conditions;
(ii) $\deg P_0(s) < \deg p_{ch}(s)$;
(iii) $P_0(s)$ vanishes if and only if $y_0 = \cdots = y_{n-1} = 0$.

From (4.22) we obtain formally

$$Y(s) = \frac{P_0(s)}{p_{ch}(s)} + \frac{G(s)}{p_{ch}(s)}. \tag{4.24}$$

Formula (4.24) is well defined provided that s is not a solution of the characteristic equation $p_{ch}(s) = 0$. Since the characteristic equation has finitely many solutions, there exists a real number σ_0 such that (4.24) holds in the half plane $\{s \in \mathbf{C} : \operatorname{Re} s > \sigma_0\}$.

Formula (4.24) provides in a purely algebraic way the Laplace transform of the solution $y(t)$ corresponding to the given initial conditions. Therefore, the solution $y(t)$ can be now determined for $t \geq 0$ by applying the inverse of the Laplace transform $\mathcal{L}^{-1}$. It is convenient to set

$$\varphi(t) = \mathcal{L}^{-1}\left[\frac{P_0(s)}{p_{ch}(s)}\right] \qquad \text{and} \qquad \chi(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{p_{ch}(s)}\right]$$

so that $y(t) = \varphi(t) + \chi(t)$. The following remarks point out the analogy between the structures of (4.24) and of (4.15).

Remark 4.6 The first summand of (4.24) contains the information about the initial conditions: it coincides with the solution of the homogeneous equation (4.16) associated to (4.13), with the same initial conditions. This term is a proper rational function: once it has been decomposed as a sum of partial fractions, we may easily go back to $\varphi(t)$ by means of the table of inverse Laplace transforms.

Of course, in this way we recover the well known conclusions about the structure of the form of the general integral of a linear homogeneous differential equation. Indeed, the inverse transform of the rational function $P_0(s)/p_{ch}(s)$ is given by the sum of functions of the form $Q_1(t)e^{\alpha t}\cos\beta t$ and $Q_2(t)e^{\alpha t}\sin\beta t$ where $Q_1(t)$, $Q_2(t)$ are polynomials of degree less than n, whose coefficients depend on the initial conditions. ■

Remark 4.7 The second summand of (4.24) depends on the forcing term. It coincides with the solution obtained solving (4.13) with zero initial state (instead of the conditions (4.14)). It is written as a product $H(s)G(s)$, where the function $H(s) = 1/p_{ch}(s)$ (defined on the half plane $\{s \in \mathbf{C} : \operatorname{Re} s > \sigma_0\}$) is called the *transfer function*. Let $h(t)$ be the function which coincides with the inverse Laplace transform of $H(s)$ for $t \geq 0$, and vanishes for $t < 0$. Then the solution of (4.13) corresponding to the initial conditions $y_0 = \cdots = y_{n-1} = 0$ can be represented by the formula

$$\chi(t) = \int_0^t h(t-\tau)g(\tau)\,d\tau \qquad \text{per } \ t \geq 0 \tag{4.25}$$

(recall (B.12)). In particular, if we interpret $g(t)$ as an input and we agree that it vanishes for $t < 0$, then $h(t)$ can be reviewed as the impulse response function of the system defined by (4.13).

Formula (4.25) can be considered as an extension of the variation of constants formula to the differential equation (4.13). ■

Example 4.6 We want to find the solution of the system defined by the linear differential equation of second order

$$y'' + 3y' + 2y = 1 \tag{4.26}$$

with initial conditions $y(0) = 1$, $y'(0) = 0$. We apply to both sides of (4.26) the operator $\mathcal{L}$. We have

$$\begin{aligned}\mathcal{L}\left[y''\right] + 3\mathcal{L}\left[y'\right] + 2\mathcal{L}\left[y\right] &= -y'(0) + s\mathcal{L}\left[y'\right] + 3\mathcal{L}\left[y'\right] + 2\mathcal{L}\left[y\right] \\ &= -y'(0) + (s+3)(sY(s) - y(0)) + 2Y(s) \\ &= (s^2 + 3s + 2)Y(s) - y(0)(s+3) - y'(0) \\ &= \frac{1}{s}.\end{aligned}$$

The Laplace transform of the forcing term requires the restriction $\operatorname{Re} s > 0$. In this region of the complex plane there is no solutions of the characteristic equation

$$s^2 + 3s + 2 = 0$$

which are both real and negative. So we obtain

$$\begin{aligned}Y(s) &= \frac{s+3}{s^2+3s+2} + \frac{1}{s(s^2+3s+2)} = \frac{s^2+3s+1}{s(s+2)(s+1)} \\ &= \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2} + \frac{2}{s+1}\right).\end{aligned}$$

By applying the inverse transform $\mathcal{L}^{-1}$, we easily get

$$y(t) = \frac{1}{2}\left(1 - e^{-2t} + 2e^{-t}\right)$$

for $t \geq 0$. We recognize in this last expression the sum of a particular solution of (4.26) and a particular solution of the associated homogeneous equation. The computations above deserve some comments. In particular, we remark that $Y(s)$ was obtained as the sum of two terms: then we passed to a single rational expression and finally we performed the partial fraction decomposition. This approach is the most natural and convenient for practical purposes. However, we may also rearrange the computation in a different way. Consistently with the previous analysis (Remarks 4.6 and 4.7), we now maintain separate the term carrying the information about the initial conditions and the term carrying the information about the forcing term. We have

$$Y(s) = \left(-\frac{1}{s+2} + \frac{2}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right)$$

which yields

$$y(t) = \left(-e^{-2t} + 2e^{-t}\right) + \left[\frac{1}{2}\left(e^{-2t} - 2e^{-t}\right) + \frac{1}{2}\right].$$

Now it is easier to interpret the structure of $y(t)$. The first summand represents the solution corresponding to the zero input (that is, the solution of the associated homogeneous equation) and the same initial conditions. Since the roots of the characteristic polynomial are negative, this part affects only the transient.

The second summand represents the solution corresponding to zero initial conditions. In turn, it is formed by a constant term (the steady state solution) plus other terms whose effect can be appreciated only in the transient. As already mentioned, the presence of these terms is due to the need of compensating the difference between the initial data of the actual solution and the steady state solution.

When the forcing term is not constant, the problem of the factorization of a polynomial of higher degree arises, For instance, if we take an input signal $g(t) = \sin t$, we have:

$$\begin{aligned} Y(s) &= \frac{s+3}{(s+2)(s+1)} + \frac{1}{(s+2)(s+1)(s^2+1)} \\ &= \frac{s^2+3s^2+s+4}{(s+2)(s+1)(s^2+1)} = \frac{1}{10}\left(\frac{-12}{s+2} + \frac{25}{s+1} + \frac{1-3s}{s^2+1}\right) \end{aligned}$$

and so

$$y(t) = \frac{1}{10}\left(-12e^{-2t} + 25e^{-t} + \sin t - 3\cos t\right).$$

■

4.4.2 Frequency Response Analysis

In this section we present some further developments about the study of a linear differential equation (4.13), with a periodic forcing term of the form

$$g(t) = p_1 \cos \omega t + p_2 \sin \omega t \tag{4.27}$$

under the assumption that all the solutions of the characteristic equation $p_{ch}(s) = 0$ have strictly negative real part. As well known, under these conditions the steady state solution is periodic, with the same frequency as the forcing term (4.27). One of the classical problems at the origin of system theory is the analysis of the solution (response) corresponding to a periodic forcing term (input) of this form.

The problem has been already studied in the case where the order of the equation is $n = 2$ (Example 4.4), as an application of the method of undetermined coefficients. For the general case, the method illustrated in this section, based on the Laplace transform, provides a very efficient tool which allows us to obtain further information, and in particular to determine the parameters of the system and of the forcing term in such a way that the solutions have preassigned amplitude and phase. This approach is the so-called *frequency response analysis*. Taking into account (4.27), we may rewrite (4.24) as

$$Y(s) = \frac{P_0(s)}{p_{ch}(s)} + \frac{1}{p_{ch}(s)} \frac{p_1 s + p_2\omega}{s^2 + \omega^2}.$$

Since the solutions of the characteristic equation $p_{ch}(s) = 0$ lies in the negative complex half-plane, we have $p_{ch}(\mathrm{i}\,\omega) \neq 0$; hence $s^2 + \omega^2$ is not a divisor of $p_{ch}(s)$. We can rewrite the right-hand side as

$$Y(s) = \frac{P_0(s)}{p_{ch}(s)} + \frac{P(s)}{p_{ch}(s)} + \frac{q_1 s + q_2}{s^2 + \omega^2}$$

where $P(s)$ is a polynomial, and q_1, q_2 are constants such that

$$P(s)s^2 + q_1 p_{ch}(s)s + P(s)\omega^2 + q_2 p_{ch}(s) = p_1 s + p_2\omega\ . \tag{4.28}$$

We already know (see Remark 4.6) that the inverse transform of the rational function $P_0(s)/p_{ch}(s)$ is the sum of functions of the form $Q_1(t)e^{\alpha t}\cos\beta t$ and $Q_2(t)e^{\alpha t}\sin\beta t$ where $Q_1(t)$, $Q_2(t)$ are polynomials of degree less than n. Our hypothesis that all the characteristic roots have negative real part implies that these terms go to zero when $t \to +\infty$. Formula (4.28) shows in particular that $\deg P < \deg p_{ch}$. Thus, the same reasoning can be repeated about the term $P(s)/p_{ch}(s)$, as well. We finally conclude that the contributions of the terms $P_0(s)/p_{ch}(s)$ and $P(s)/p_{ch}(s)$ can be ultimately neglected, and the steady state response depends essentially on the third summand $(q_1 s + q_2)/(s^2 + \omega^2)$, whose inverse transform is

$$\mathcal{L}^{-1}\left[\frac{q_1 s + q_2}{s^2 + \omega^2}\right] = q_1\cos\omega t + \frac{q_2}{\omega}\sin\omega t = k\sin(\omega t + \theta)$$

being $q_1 = k\sin\theta$ and $q_2 = k\omega\cos\theta$. Recall that the term $P_0(s)/p_{ch}(s)$ represents the solution of the unforced system with the same initial conditions of the given system. The term $P(s)/p_{ch}(s)$ compensates the difference between the assigned initial conditions and the (in general, different) initial conditions of the steady state solution (compare these comments with those in Remark 4.4).

Finally, we show how to compute q_1 and q_2, and hence k and θ. Replacing $s = \mathrm{i}\,\omega$, from (4.28) we find

$$q_2 + \mathrm{i}\,q_1\omega = \frac{\omega}{p_{ch}(\mathrm{i}\,\omega)}(p_2 + \mathrm{i}\,p_1),$$

which yields $q_2 = \mathrm{Re}\,[\frac{\omega}{p_{ch}(\mathrm{i}\,\omega)}(p_2 + \mathrm{i}\,p_1)]$ and $q_1 = \mathrm{Im}\,[\frac{1}{p_{ch}(\mathrm{i}\,\omega)}(p_2 + \mathrm{i}\,p_1)]$. Alternatively, we can compute p_1 and p_2 as functions of some desired values of q_1 and q_2.

Chapter Summary

This chapter constitutes a different development of Chap. 2. We consider the problem of representing the solutions of nonhomogeneous (i.e., with forcing term) systems of linear differential equations. We present the variation of constants formula and the method of undetermined coefficients. Moreover, we illustrate the qualitative notions of transient and steady state solution. Finally we present the Laplace transform method and, as an application, we discuss the frequency response analysis of a system under periodic input.