Chapter 3
Stability of Unforced Linear Systems

In this chapter we continue our investigation of the properties of linear unforced differential systems

$$\dot{x} = Ax \tag{3.1}$$

where A is a square $n \times n$ matrix with real entries, and $x \in \mathbf{R}^n$. We focus in particular on the stability problem.

3.1 Equilibrium Positions

The equilibrium positions of system (3.1) coincide with the solutions of the algebraic equation $Ax = 0$. Hence, a system of the form (3.1) always have an equilibrium position for $x = 0$. Such an equilibrium position is unique (and hence isolated) if and only if $\det A \neq 0$. Otherwise, there are infinitely many equilibrium positions (none of which isolated): more precisely, the set of all the equilibrium positions of (3.1) constitutes a subspace of $\mathbf{R}^n$.

Remark 3.1 Assume that there is a point $\bar{x} \neq 0$ such that $A\bar{x} = 0$. Then, $\bar{x}$ is a stable equilibrium position for system (3.1) if and only if the origin is a stable equilibrium position for system (3.1). Indeed, setting $y = x - \bar{x}$, we have

$$\dot{y} = \dot{x} = Ax = Ax - A\bar{x} = Ay.$$

∎

The displacements of x with respect to $\bar{x}$, that is the displacement of y with respect to $y = 0$, are determined by the same system which determines the displacements of x with respect to $x = 0$.

Remark 3.2 If an equilibrium point is attractive for system (3.1), then it must be isolated. Hence, if A is singular, there exist no attractive equilibrium positions. In other words, if system (3.1) possesses an attractive equilibrium position $\bar{x}$, then $\bar{x} = 0$, and there are no other equilibria $\bar{x} \neq 0$.   ■

According to the previous remarks, when studying stability and asymptotic stability of linear systems, it is not restrictive to limit ourselves to the origin.

Proposition 3.1 *If the origin is stable [respectively, asymptotically stable] for system (3.1), then the origin is stable [respectively, asymptotically stable] for all the systems linearly equivalent to (3.1).*

Proof Let $B = P^{-1}AP$ and let $\psi(t)$ be any solution of the system $\dot{y} = By$. Let us fix $\varepsilon > 0$, and let $\varepsilon' = \varepsilon/\|P^{-1}\|$. Since (3.1) is stable at the origin, there exists $\delta' > 0$ such that $\|\varphi(0)\| < \delta' \Longrightarrow \|\varphi(t)\| < \varepsilon'$ for each $t \geq 0$ and each solution $\varphi(t)$ of (3.1). Let $\delta = \delta'/\|P\|$ and $\varphi(t) = P\psi(t)$. Then,

$$\|\psi(0)\| < \delta \Longrightarrow \|\varphi(0)\| = \|P\psi(0)\| \leq \|P\| \cdot \|\psi(0)\| < \delta'$$

so that

$$\|\psi(t)\| = \|P^{-1}\varphi(t)\| \leq \|P^{-1}\| \cdot \|\varphi(t)\| < \varepsilon .$$

Finally, assume that $\lim_{t \to +\infty} \varphi(t) = 0$ for a given solution of (3.1). Then, for each $\sigma > 0$ there exists $T > 0$ such that

$$t > T \Longrightarrow \|\varphi(t)\| < \sigma'$$

where $\sigma' = \sigma/\|P^{-1}\|$, and this implies that $\|\psi(t)\| \leq \|P^{-1}\| \cdot \|\varphi(t)\| < \sigma$. The reasoning is easily completed.   ■

3.2 Conditions for Stability

For linear time invariant systems, the analysis of the stability properties can be carried on by means of purely algebraic tools.

Definition 3.1 We say that a real square matrix A possesses the *Hurwitz property* if all the eigenvalues of A have (strictly) negative real part.

In short, when A possesses the Hurwitz property we shall also say that A is a *Hurwitz matrix*. Note that every Hurwitz matrix is nonsingular.

Theorem 3.1 *If A is a Hurwitz matrix then the origin is a globally and exponentially stable equilibrium point for system (3.1). If the origin is a locally attractive equilibrium point for system (3.1), then A is a* Hurwitz matrix.

Proof Assume that all the eigenvalues of A have negative real part. Then we can choose α and k_0 in (2.15) in such a way that $\alpha_0 < \alpha < 0$. The global and exponential attraction of the origin trivially follows. As far as the stability property is concerned, we may use again (2.15). As already noticed, it is not restrictive to take $\alpha < 0$; for $t \geq 0$, we have therefore $e^{\alpha t} \leq 1$. Hence, for each $\varepsilon > 0$, it is sufficient to take $\delta = \varepsilon/k_0$.

We now pass to the second statement. Being the origin locally attractive, there exists a neighborhood Ω of the origin such that all the solutions issuing from a point of Ω asymptotically approach zero when $t \to +\infty$. We proceed by distinguishing several cases.

Assume first that there is an eigenvalue λ with strictly positive real part. If λ is real and if v is a corresponding (real) eigenvector, then we can construct a solution of the form $e^{\lambda t}v$. Note that the norm of v can be taken arbitrarily small. Instead, if $\lambda = \alpha + i\beta$ is not real, then we can construct a solution of the form

$$e^{\alpha t}[(\cos \beta t)u + (\sin \beta t)w] \,,$$

where u, w are certain real vectors, whose norm can be taken arbitrarily small, and $\alpha > 0$. In both cases, these solutions are unbounded for $t \geq 0$. This contradicts the assumptions.

In similar way we exclude the existence of eigenvalues λ with zero real part. Indeed, in this case either $\lambda = 0$, so that there is a nonzero constant solution, or λ is purely imaginary, so that we can construct a periodic solution $(\cos \beta t)u + (\sin \beta t)w$, which is bounded but does not approach zero. ∎

From Theorem 3.1 and its proof we can infer other information, which can be resumed in the following way.

- For the linear system (3.1), the condition that A possesses the Hurwitz property is necessary and sufficient for the asymptotic stability of the origin.
- If the origin is locally attractive for the linear system (3.1), then it is globally and exponentially attractive, as well.
- For a linear system, if the origin is locally attractive then it is also stable.

Instead, even in the case of a linear system it may happen that the origin is stable but not attractive; very simple examples are given by the system $\dot{x} = 0$ with $x \in \mathbf{R}$, whose solutions are constant, and by the system in Example 2.1, whose solutions are periodic.

From the proof of Theorem 3.1, we also immediately see that if there exists an eigenvalue of A with strictly positive real part then the origin is unstable. Thus, it remains to discuss the case where all the eigenvalues of A have non-positive real part, and at least one among them has a real part exactly equal to zero.

Theorem 3.2 *The following statements are equivalent.*

(i) *All the eigenvalues of A have non-positive real part, and for each possible eigenvalue with zero real part, the algebraic multiplicity and the geometric multiplicity coincide.*

(ii) *The norm of the exponential matrix e^{tA} is bounded for $t \geq 0$.*
(iii) *The origin is stable for system (3.1).*

Proof $(i) \Longrightarrow (ii)$. If all the eigenvalues of A have nonpositive real part and the possible eigenvalues with zero real part have the same algebraic and geometric multiplicity, then we can apply (2.14) with $\alpha = \alpha_0 = 0$. The conclusion is straightforward.

$(ii) \Longrightarrow (iii)$. If there is a constant $k_0 > 0$ such that $||e^{tA}|| \leq k_0$ for $t \geq 0$, then for every $x_0 \in \mathbf{R}^n$ we have

$$\|e^{tA}x_0\| \leq k_0\|x_0\| . \tag{3.2}$$

The definition of stability is recovered taking $\delta = \varepsilon / k_0$.

Finally we prove by contradiction that $(iii) \Longrightarrow (i)$. We already know that if the origin is stable, there exist no eigenvalues with strictly positive real part. Assume that there is an eigenvalue λ with zero real part and whose geometric multiplicity is less than its algebraic multiplicity.

If $\lambda = i\beta$ con $\beta \neq 0$, we could construct a complex solution of the form $(\cos \beta t + i \sin \beta t)(tu + v)$, where v is an eigenvector corresponding to λ, and u is a generalized eigenvector; both v and u can be chosen of arbitrarily small norm. But then, we could also find a real solution

$$(\cos \beta t)[tu_1 + v_1] - (\sin \beta t)[tu_2 + v_2]$$

where v_1, v_2, u_1, u_2 are some real vectors. This solution corresponds to the initial state $x_0 = v_1$. Since u_1 and u_2 cannot be both zero, the solution exhibits an oscillatory behavior and the amplitude of the oscillations increases as t increases. This solution is not bounded for $t \geq 0$, so that the stability assumption is contradicted.

The case $\lambda = 0$ can be ruled out in similar way.   ∎

We may also prove the following proposition by analogous arguments.

Proposition 3.2 *The following statements are equivalent.*

(i) *All the eigenvalues of A have nonpositive real part, and for each possible eigenvalue with zero real part, the algebraic multiplicity and the geometric multiplicity coincide.*
(ii) *All the solutions of the system are bounded on $t \geq 0$.*

Remark 3.3 If the system at hand is defined by a scalar differential equation of order n like (2.16), the stability conditions of the equilibrium position $y = y' = \ldots = y^{(n-1)} = 0$ can be stated in terms of its characteristic roots.   ∎

3.3 Lyapunov Matrix Equation

In this section we present a different characterization of stable linear systems. Recall that a real symmetric matrix P is said to be:

- *positive definite* if for each $0 \neq x \in \mathbf{R}^n$ we have $x^t P x > 0$.
- *positive semidefinite* if $x^t P x \geq 0$ for each $x \in \mathbf{R}^n$.

If the conditions above are fulfilled by the matrix $-P$, then we say that, respectively, P is *negative definite, negative semidefinite*. Finally, we say that P is *indefinite* if $x^t P x$ takes both strictly positive and strictly negative values.

Theorem 3.3 *The following statements are equivalent.*

(i)   *A possesses the Hurwitz property.*
(ii) *There exists a positive definite, real symmetric matrix P such that for each nontrivial solution $\varphi(t)$ of (3.1), we have*

$$\left. \frac{d}{dt} g(t) \right|_{t=0} = -||\varphi(0)||^2 \tag{3.3}$$

where $V(x) = x^t P x$ and $g(t) = V(\varphi(t))$.
(iii) *There exists a positive definite, real symmetric matrix P such that*

$$A^t P + P A = -I . \tag{3.4}$$

Proof $(i) \implies (ii)$. Assume that (i) holds and denote by p_{ij} the unknown elements of the matrix P. Let $\psi_1(t), \ldots, \psi_n(t)$ be the columns of the exponential matrix e^{tA}, and let us define

$$p_{ij} = \int_0^{+\infty} \psi_i^t(s) \psi_j(s) ds .$$

The numbers p_{ij} are well defined: indeed, if the eigenvalues of A have negative real part, all the entries of e^{tA} go to zero exponentially and so, the integral converges. Let us check that the matrix P fulfils the required properties. Clearly, P is symmetric. The solution $\varphi(t)$ corresponding to the initial state $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, can be written as

$$\varphi(t) = e^{tA} x = \sum_{i=1}^n \psi_i(t) x_i .$$

Thus,

$$V(x) = x^t P x = \sum_{i,j=1}^n p_{ij} x_i x_j = \sum_{i,j=1}^n \left(\int_0^{+\infty} \psi_i^t(s) \psi_j(s) ds \right) x_i x_j$$

$$= \int_0^{+\infty} \sum_{i,j=1}^n \left(\psi_i^t(s) \psi_j(s) \right) x_i x_j \, ds$$

$$= \int_0^{+\infty} \left(\sum_{i=1}^n \psi_i(s) x_i \right)^t \left(\sum_{j=1}^n \psi_j(s) x_j \right) ds$$

$$= \int_0^{+\infty} \|\varphi(s)\|^2 \, ds \ .$$

The properties $V(0) = 0$ and $V(x) > 0$ for each $x \neq 0$ are easily checked. It remains to prove (3.3). Since

$$V(\varphi(t)) = \int_0^{+\infty} \|\varphi(t+s)\|^2 \, ds = \int_t^{+\infty} \|\varphi(\sigma)\|^2 \, d\sigma$$

we have, for each $x \in \mathbf{R}^n$,

$$\frac{d}{dt} V(\varphi(t))\big|_{t=0} = \frac{d}{dt} \int_t^{+\infty} \|\varphi(\sigma)\|^2 d\sigma\big|_{t=0}$$
$$= -\|\varphi(t)\|^2\big|_{t=0} = -\|x\|^2 \ .$$

Now we prove that $(ii) \Longrightarrow (iii)$. We reconsider the already defined function $V(\varphi(t))$ and we compute its derivative in a different way. We have

$$\frac{d}{dt} V(\varphi(t)) = \frac{d}{dt} \big((\varphi(t))^{\mathbf{t}} P \varphi(t) \big) = (\dot\varphi(t))^{\mathbf{t}} P \varphi(t) + (\varphi(t))^{\mathbf{t}} P \dot\varphi(t)$$
$$= (\varphi(t))^{\mathbf{t}} A^{\mathbf{t}} P \varphi(t) + (\varphi(t))^{\mathbf{t}} P A \varphi(t) \ .$$

Setting $t = 0$ and $\varphi(0) = x$, and taking the assumption into account, the identity above yields

$$x^{\mathbf{t}} [A^{\mathbf{t}} P + P A] x = -x^{\mathbf{t}} x \ .$$

Since the solution $\varphi(t)$ is arbitrary, we obtain $A^{\mathbf{t}} P + P A = -I$, as required.

Finally, we prove that $(iii) \Longrightarrow (i)$. Let λ be a (real or complex) eigenvalue of A, and let $Av = \lambda v$ ($v \neq 0$). we have

$$-\bar{v}^{\mathbf{t}} v = \bar{v}^{\mathbf{t}} (A^{\mathbf{t}} P + P A) v = (A\bar{v})^{\mathbf{t}} P v + \bar{v}^{\mathbf{t}} P A v$$
$$= \bar\lambda \bar{v}^{\mathbf{t}} P v + \lambda \bar{v}^{\mathbf{t}} P v = (\bar\lambda + \lambda) \bar{v}^{\mathbf{t}} P v = 2\alpha \, \bar{v}^{\mathbf{t}} P v$$

where α denotes the real part of λ. Now, it is not difficult to check that if P is any positive definite, real symmetric matrix and if v is any (real or complex) nonzero vector, then $\bar{v}^{\mathbf{t}} P v > 0$. Hence we must have $\alpha < 0$. ∎

In what follows, we refer to (3.4) as the *Lyapunov matrix equation*. Indeed, it can be conveniently interpreted as an equation in the unknown P. It is equivalent to a system with $n(n+1)/2$ algebraic linear equations, whose unknowns are the elements of P; Theorem 3.3 states in particular that if A is a Hurwitz matrix, then such a system admits a solution. More precisely, under this condition there exists a unique positive definite solution of (3.4).

A function of the form $V(x) = x^{\mathbf{t}} P x$ where P is a positive definite, real symmetric matrix enjoying one of the properties (ii) or (iii) listed in Theorem 3.3, is called a

quadratic Lyapunov function. It is indeed a homogeneous real polynomial of degree 2 with n variables.

The following corollary provides a generalized version of (3.4). It enlightens in particular the flexibility of the matrix Lyapunov equation.

Corollary 3.1 *If there exists a positive definite, real symmetric matrix Q such that the matrix equation*

$$A^t P + P A = -Q \tag{3.5}$$

admits a (positive definite, real symmetric) solution P, then A is a Hurwitz matrix.

If A is a Hurwitz matrix, then for each positive definite, real symmetric matrix Q, there exists a unique (positive definite, real symmetric) solution P of the matrix equation (3.5).

Proof The proof of the first statement is a slight modification of the proof that $(iii) \Longrightarrow (i)$ of Theorem 3.3. As far as the second statement is concerned, we start by writing $Q = R^t R$, where R is some nonsingular symmetric matrix (see [21] Chap. 11, or [6]). Since the eigenvalues of A have strictly negative real part, the same is true for the matrix $\tilde{A} = RAR^{-1}$. According to Theorem 3.3 (iii), there exists a matrix $\tilde{P}$ such that

$$\tilde{A}^t \tilde{P} + \tilde{P} \tilde{A} = -I .$$

This implies

$$(R^{-1})^t A^t R^t \tilde{P} + \tilde{P} R A R^{-1} = -I .$$

The conclusion follows, multiplying both sides of this equation by R^t and R (respectively, on the left and on the right), and setting $P = R^t \tilde{P} R$. ∎

An elegant representation of the solution of (3.5) is (see [31]) $P = \int_0^\infty e^{tA^t} Q e^{tA} dt$. In order to characterize the stability property, we may use a weakened version of Theorem 3.3.

Theorem 3.4 *The following properties are equivalent.*

(i) *All the eigenvalues of A have nonpositive real part, and for each possible eigenvalue with zero real part the algebraic multiplicity and the geometric multiplicity coincide.*

(ii) *There exists a positive definite, real symmetric matrix P such that the matrix $A^t P + P A$ is negative semidefinite.*

A function of the form $V(x) = x^t P x$ with P positive definite, real symmetric, enjoying one of the properties listed in Theorem 3.4 is called a *quadratic weak Lyapunov function*.

3.4   Routh-Hurwitz Criterion

The results presented in this chapter emphasize the interest of criteria which enable us to predict the sign of the roots of a polynomial, without need of computing them explicitly. Recall that the eigenvalues of a matrix A coincide with the roots of the characteristic polynomial of A. Let

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

be a monic polynomial of degree n with real coefficients.

Proposition 3.3 *If all the roots of $P(\lambda)$ have strictly negative real part, then all the coefficients a_i must be strictly grater than zero.*

Proof Let $\lambda_1, \ldots, \lambda_k$ be the (negative) real roots of $P(\lambda)$ and let $\alpha_1 \pm i\,\beta_1, \ldots, \alpha_h \pm i\,\beta_h$ be the complex conjugate roots, with $\alpha_1 < 0, \ldots, \alpha_h < 0$. Then we have

$$\begin{aligned}
P(\lambda) = {}& (\lambda - \lambda_1)^{\mu_1} \cdot \ldots \cdot (\lambda - \lambda_k)^{\mu_k} \\
& \cdot (\lambda - (\alpha_1 + i\,\beta_1))^{\nu_1} \cdot (\lambda - (\alpha_1 - i\,\beta_1))^{\nu_1} \cdot \ldots \\
& \cdot (\lambda - (\alpha_h + i\,\beta_h))^{\nu_h} \cdot (\lambda - (\alpha_h - i\,\beta_h))^{\nu_h} \, .
\end{aligned}$$

Every pair of linear factors where the complex roots appear can be replaced by a unique factor of degree 2

$$\lambda^2 + p_1\lambda + q_1 \, , \quad \ldots \, , \quad \lambda^2 + p_h\lambda + q_h$$

where, being $\alpha_1 < 0, \ldots, \alpha_h < 0$, all the coefficients $p_1, \ldots, p_h, \; q_1, \ldots, q_h$ are positive.

Recovering the expanded form of the polynomial, we find of course that all the coefficients of $P(\lambda)$ are positive.   ■

The necessary condition provided by Proposition 3.3 is also sufficient if the degree of the polynomial is $n = 1$ or $n = 2$, but not in general. For instance,

$$\left[\lambda - \frac{1 + i\sqrt{11}}{2}\right] \cdot \left[\lambda - \frac{1 - i\sqrt{11}}{2}\right] \cdot [\lambda + 2] = \lambda^3 + \lambda^2 + \lambda + 6 \, .$$

There are several necessary and sufficient conditions which allow us to establish when the roots of a polynomial belong to the half plane $\{z \in \mathbf{C} : \quad \mathrm{Re}\,z < 0\}$. They are generally referred to as *Routh* and *Hurwitz* criteria. We state one of these criteria without proof. It is based on the examination of the sign of the determinants of n matrices $\Delta_1, \Delta_2, \ldots, \Delta_n$ of order $1, 2, \ldots, n$, respectively. These matrices are computed starting from the coefficients of $P(\lambda)$, according to the following procedure.

First of all, for sake of convenience, we agree to write $a_j = 0$ for each value of $j > n$, and $a_0 = 1$. We define

$$\Delta_1 = a_1 \, ,$$

$$\Delta_2 = \begin{pmatrix} a_1 & a_3 \\ a_0 & a_2 \end{pmatrix},$$

$$\Delta_3 = \begin{pmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{pmatrix},$$

$$\Delta_4 = \begin{pmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{pmatrix},$$

$$\Delta_5 = \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & a_9 \\ a_0 & a_2 & a_4 & a_6 & a_8 \\ 0 & a_1 & a_3 & a_5 & a_7 \\ 0 & 1 & a_2 & a_4 & a_6 \\ 0 & 0 & a_1 & a_3 & a_5 \end{pmatrix},$$

and so on, finishing with Δ_n. Let us remark that on the "odd" rows of these matrices (the first row, the third row, etc.) we find the coefficients with odd index, displayed in increasing order, while on the "even" rows we find the coefficients of even index. The elements which appear in the first two rows are repeated in the following rows, shifted of one position each time. The free positions at the beginning of any new row are filled with zeros, while the last element on the right of any row is eliminated at a new repetition.

Theorem 3.5 *All the roots of the polynomial $P(\lambda)$ belong to the half plane $\{z \in \mathbf{C} : \mathrm{Re}\, z < 0\}$ if and only if all the determinants of the matrices $\Delta_1, \ldots, \Delta_n$ are positive.*

For instance, in the case $n = 3$ the condition of Theorem 3.5 reduces to

$$a_1 > 0, \ a_3 > 0, \ a_1 a_2 - a_3 > 0 \, .$$

This form of the Routh-Hurwitz criterion can be found in [24] or in [10], where the reader can also find a proof of Theorem 3.5.

Chapter Summary

In this chapter the study of unforced linear systems is continued. We focus in particular on the stability properties of the equilibrium position (the origin). This corresponds to the study of the internal stability properties of a system with input and output. We state and prove the classical Lyapunov Theorem which allows us to reduce the stability analysis to an algebraic problem (computation of the eigenvalues of a matrix). We also introduce the quadratic Lyapunov functions and the Lyapunov matrix equation. The Routh-Hurwitz criterion is given without proof.