Chapter 7
Stabilization

As already pointed out in Chap. 1, the behavior of a system can be regulated, without need of radical changes in its internal plant, by the construction of a suitable device which interacts with the system by means of a feedback connection. The action of such a device may have a *static* nature (and, in this case, it can be mathematically represented as a function) or a *dynamic* one (and so being interpreted as an auxiliary system). The feedback connection allows us to exert the control action in an automatic way (i.e., without need of the presence of a human operator), and requires the installation of sensors and actuators.

When all the state variables can be monitored and measured at each time, and all the information about their evolution can be used by the control device, we speak about *state* feedback. On the contrary, when the information about the state is only partially available (since they are, for instance, obtained by means of an observation function) we speak about *output* feedback.

7.1 Static State Feedback

In the static state feedback stabilization problem, the observation function is not involved. Hence, in this section we can limit ourselves to systems of the form

$$\dot{x} = Ax + Bu \ , \quad x \in \mathbf{R}^n \ , \ u \in \mathbf{R}^m \ . \tag{7.1}$$

First of all, we try to understand what happens when the feedback connection is implemented. Let $v(t)$ be an external signal, injected into the system through the input channel u, and let $x(t)$ be the solution representing the resulting state evolution. The feedback map $k(x) : \mathbf{R}^n \to \mathbf{R}^m$ generates another signal $w(t) = k(x(t))$. The signal actually received by the system is the sum of $v(t)$ and $w(t)$, that is $u = w(t) + v(t)$.

A. Bacciotti, *Stability and Control of Linear Systems*, Studies in Systems, Decision and Control 185, https://doi.org/10.1007/978-3-030-02405-5_7

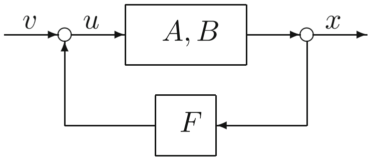
A static state feedback is said to be *linear* if $k(x) = Fx$, F being a $m \times n$ matrix. The implementation of a linear feedback can be mathematically interpreted as a substitution

$$u = Fx + v \tag{7.2}$$

which gives rise to a transformation of the system. Indeed, replacing (7.2) in (7.1), we obtain

$$\dot{x} = (A + BF)x + Bv \ , \tag{7.3}$$

the so-called *closed loop* system.



We stress that by virtue of the particular structure of the control (7.2), the transformed system is still of the form (7.1), with the matrix B unchanged, and the matrix A replaced by the new matrix $\tilde{A} = A + BF$. We also notice that the transformation induced by (7.2) is invertible; indeed, if we apply the feedback law $v = -Fx + u$ to (7.3) we recover the form (7.1) of the system. Thus, the transformation (7.2) defines an equivalence relation on the set of all the systems of the form (7.1); this fact can be formalized by the following definition.

Definition 7.1 We say that systems (7.1) and

$$\dot{x} = \tilde{A}x + Bu$$

are *feedback equivalent* if there exists a matrix F such that $\tilde{A} = A + BF$.

In this perspective, we can formulate the following problem pattern: assume that we are interested in a certain property, and that this property is not satisfied by the given system (7.1). We wonder whether the property is satisfied by system (7.3), for a suitable choice of the matrix F. More precisely, we want to find conditions under which the qualitative behavior of the given system can be modified in the desired way by means of a convenient feedback connection.

7.1.1 Controllability

As a first example, we ask whether a system can achieve the complete controllability property by means of a feedback transformation (diversely stated, whether in the same feedback equivalence class there may exist systems whose reachable spaces have different dimensions). The answer is negative; indeed, the following theorem holds.

Theorem 7.1 *For each matrix F, the reachable spaces of the systems (7.1) and (7.3), denoted here respectively by* $\mathrm{R}_{(7.1)}$ *and* $\mathrm{R}_{(7.3)}$*, coincide. As a consequence, we have*

$$\text{rank}\,(B|AB|\dots|A^{n-1}B) = \text{rank}\,(B|(A+BF)B|\dots|(A+BF)^{n-1}B)\,.$$

Proof According to Theorem 5.3

$$\mathrm{R}_{(7.1)} = \text{span}\,\{b^1,\dots,b^m;\; Ab^1,\dots,Ab^m;\;\dots;\; A^{n-1}b^1,\dots,A^{n-1}b^m\}\,.$$

Thus, $v \in \mathrm{R}_{(7.1)}$ if and only if v is a linear combination of the vectors

$$b^1,\dots,b^m;\; Ab^1,\dots,Ab^m;\;\dots;\; A^{n-1}b^1,\dots,A^{n-1}b^m\,.$$

Analogously,

$$\begin{aligned}\mathrm{R}_{(7.3)} = \text{span}\,\{&b^1,\dots,b^m;\; (A+BF)b^1,\dots,(A+BF)b^m;\\ &\dots;(A+BF)^{n-1}b^1,\dots,(A+BF)^{n-1}b^m\}\,.\end{aligned}$$

Notice that $(A+BF)b^j = Ab^j + BFb^j$. The vector Ab^j belongs to $\mathrm{R}_{(7.1)}$ and the vector

$$BFb^j = (b^1|\dots|b^m)Fb^j$$

belongs to $\mathrm{R}_{(7.1)}$ as well, since it is a linear combination of $b^1,\dots,b^m$. Continuing in this way, we notice that

$$(A+BF)^2b^j = (A+BF)(A+BF)b^j = A^2b^j + ABFb^j + BFAb^j + BFBFb^j\,.$$

The first term is in $\mathrm{R}_{(7.1)}$ by construction; the second because it is a linear combination of $Ab^1,\dots,Ab^m$; the third and the fourth term because they are linear combination of $b^1,\dots,b^m$. The same reasoning applies to each term of the form $(A+BF)^kb^j$.

In conclusion, $\mathrm{R}_{(7.3)} \subseteq \mathrm{R}_{(7.1)}$, since all the vectors of $\mathrm{R}_{(7.3)}$ are linear combinations of vectors of $\mathrm{R}_{(7.1)}$.

The opposite inclusion can be achieved by exchanging the roles of the systems (recall that (7.1) can be recovered from (7.3) by the inverse feedback transformation $v = -Fx + u$). ■

In other words, Theorem 7.1 states that the complete controllability property is invariant under feedback equivalence.

7.1.2 Stability

In the previous chapter we tried to characterize those systems of the form (7.1) which enjoy the external stability property. We noticed that this property is intimately linked to the internal stability properties of the system (Hurwitz property). This motivates the effort to elaborate models for which the eigenvalues of the system matrix A lie in the open left half of the complex plane and, in case this condition is not fulfilled, the interest in devising appropriate corrections.

The main purpose of this chapter is to show that feedback connections represent a convenient tool in order to improve the internal stability properties of a system.

7.1.3 Systems with Scalar Input

Consider first the case of a system with scalar input (i.e., with $m = 1$ and B reduced to a column vector b). Our approach is based on the following theorem.

Theorem 7.2 *Assume that $m = 1$, and that system (7.1) is completely controllable. Then, there exists a change of coordinates $x = P\zeta$ for which the system takes the form*

$$\dot{\zeta} = A_0\zeta + ub_0 \tag{7.4}$$

where A_0 is the companion matrix

$$\begin{pmatrix} 0 & 1 & 0 \dots & 0 \\ 0 & 0 & 1 \dots & 0 \\ \dots & \dots & \dots\dots & \dots \\ 0 & 0 & 0 \dots & 1 \\ -a_n & -a_{n-1} & \dots\dots & -a_1 \end{pmatrix} \quad \textit{and} \quad b_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Proof In Sects. 2.10 and 2.11 we saw that if there exists a cyclic vector for A (that is a vector v such that

$$v, Av, \dots, A^{n-1}v$$

are linearly independent), then A is similar to the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ 0 & 1 & \dots & 0 & -a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & -a_1 \end{pmatrix}$$

where the numbers $a_1, \ldots, a_n$ are the coefficients of the characteristic polynomial of A, apart for a possible change of sign. Such a matrix is the transpose of the companion form.

The matrix associated to the change of coordinates is formed by the columns $v, Av, \ldots, A^{n-1}v$. The complete controllability hypothesis states that the rank of the matrix

$$R = (b|Ab|\ldots|A^{n-1}b)$$

is n. Hence, b is cyclic for A and $R^{-1}AR = A_0{}^{\mathbf{t}}$. Moreover,

$$R\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = b \quad \text{that is} \quad R^{-1}b = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}.$$

We have so proved that our system is linearly equivalent to

$$\dot{w} = A_0^{\mathbf{t}}w + u\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}. \tag{7.5}$$

On the other hand, the matrix $Q = \big(b_0|A_0b_0|\ldots|A_0^{n-1}b_0\big)$ has the form

$$\begin{pmatrix} 0 & \ldots\ldots & 0 & 1 \\ 0 & \ldots\ 0 & 1 & * \\ \ldots & \ldots\ldots & \ldots & \ldots \\ 0 & 1 & * \ldots & * \\ 1 & * & \ldots\ldots & * \end{pmatrix}$$

and so it is nonsingular (the stars stand for some numbers whose explicit expression is unessential). By the same arguments as before, we must have $Q^{-1}A_0Q = A_0^{\mathbf{t}}$ as well. Moreover,

$$Q\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = b_0 \quad \text{that is} \quad Q^{-1}b_0 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}.$$

Thus, the system (7.4) is linearly equivalent to (7.5), as well. Finally, (7.4) and the given system, being both linearly equivalent to (7.5), are equivalent each other. ■

Recall that the companion form characterizes the system representation of scalar linear differential equations. Theorem 7.2 states therefore that any completely controllable linear system with single input and state space dimension n is linearly equivalent to a system represented by a single linear differential equation of order n. We emphasize that the proof of Theorem 7.2 supplies an explicit expression for the matrix P which determines the similarity between A and its companion form A_0. Indeed, it is immediately seen that $P = RQ^{-1}$.

We rewrite for convenience system (7.4) as

$$\begin{cases} \dot{\zeta}_1 = \zeta_2 \\ \quad \vdots \\ \dot{\zeta}_{n-1} = \zeta_n \\ \\ \dot{\zeta}_n = -a_n\zeta_1 - \cdots - a_1\zeta_n + u \; . \end{cases} \tag{7.6}$$

Definition 7.2 A $2k$-tuple $\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}$ is said to be *consistent* if:

(1) $1 \leq k \leq n$;
(2) $\lambda_1, \ldots, \lambda_k$ are distinct complex numbers;
(3) $\mu_1, \ldots, \mu_k$ are (not necessarily distinct) positive integers such that $\mu_1 + \cdots + \mu_k = n$;
(4) for each $i \in \{1, \ldots, k\}$ there exists $j \in \{1, \ldots, k\}$ such that $\lambda_j = \overline{\lambda_i}$ (the conjugate of λ_i) and $\mu_i = \mu_j$.

Given any consistent $2k$-tuple, it is easy to construct a monic polynomial with real coefficients

$$\lambda^n + b_1\lambda^{n-1} + \cdots + b_n$$

whose roots are exactly $\lambda_1, \ldots, \lambda_k$, with respective multiplicities $\mu_1, \ldots, \mu_k$. Now, let us apply to system (7.6) the feedback

$$u = (-b_n + a_n)\zeta_1 + \cdots + (-b_1 + a_1)\zeta_n + v \; . \tag{7.7}$$

The resulting closed-loop system is

$$\begin{cases} \dot{\zeta}_1 = \zeta_2 \\ \quad \vdots \\ \dot{\zeta}_n = -b_n\zeta_1 - \cdots - b_1\zeta_n + v \; . \end{cases}$$

Setting finally $v = 0$, we obtain an unforced linear system whose characteristic equation has exactly the roots $\lambda_1, \ldots, \lambda_k$. Of course, the roots of the characteristic equation coincide with the eigenvalues of the matrix. If the numbers λ_i have been chosen in such a way that $\mathrm{Re}\lambda_i < 0$ for each i, we have obtained, by means of the feedback (7.7), a system for which the origin is asymptotically stable.

If fact, we have proven something more. For any preassigned real $n \times n$ matrix M, a completely controllable system with scalar input can be always transformed in a new system, such that the eigenvalues of the matrix of the new system coincide exactly with those of M.

7.1.3.1 System with Multiple Inputs

The discussion of the previous section motivates the following general definitions.

Definition 7.3 We say that (7.1) is *stabilizable* if there exists a static state feedback $u = Fx$ such that all the eigenvalues of the matrix $(A + BF)$ have negative real part.

We say that (7.1) is *superstabilizable* if for each $\alpha > 0$ there exists a static state feedback $u = Fx$ (with F dependent on α) such that the real part of each eigenvalue of the matrix $(A + BF)$ is less than $-\alpha$.

We say that (7.1) has the *pole assignment* property if for each given consistent $2k$-tuple there exists a static state feedback $u = Fx$ such that the eigenvalues of $A + BF$ are exactly the numbers $\lambda_1, \dots, \lambda_k$, with respective multiplicities $\mu_1, \dots, \mu_k$.

Systems which are superstabilizable are particularly interesting for applications. Indeed for these systems, it is not only possible to construct stabilizing feedback laws, but also to assign an arbitrary decay rate.

We already know that any completely controllable system with a scalar input possesses the pole assignment property, and hence it is stabilizable and superstabilizable. This result can be extended, with some technical complications in the proof, to systems with multiple input.

Theorem 7.3 *For any system of the form (7.1), the following properties are equivalent:*

(i) complete controllability
(ii) pole assignment
(iii) superstabilizability.

The reader interested in the full proof of Theorem 7.3 is referred, for instance, to [11], p. 145 or [28], p. 58. It follows in particular from Theorem 7.3 that for any system in the general form (7.1), complete controllability implies stabilizability by static state feedback. We give below an independent and direct proof of this fact.

Proposition 7.1 *If (7.1) is completely controllable, then it is stabilizable.*

Proof The completely controllability assumption amounts to say that for each $T > 0$ the matrix

$$\Gamma(T) = \int_0^T e^{-\tau A} B B^{\mathbf{t}} e^{-\tau A^{\mathbf{t}}} \, d\tau$$

is positive definite (Theorem 5.2). Write for simplicity $\Gamma = \Gamma(1)$. Let us show that the feedback law $u = Fx = -B^{\mathbf{t}}\Gamma^{-1}x$ actually stabilizes the system. Compute the derivative

$$\frac{d}{dt}\left(e^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}}\right) = -Ae^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}} - e^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}}A^{\mathbf{t}}$$

which yields

$$\int_0^1 \frac{d}{dt}\Big(e^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}}\Big)\,dt = \int_0^1 \Big(- Ae^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}} - e^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}}A^{\mathbf{t}}\Big)\,dt\ . \tag{7.8}$$

Clearly, the expression on the left hand side can be rewritten as

$$e^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}}\Big|_0^1 = e^{-A}BB^{\mathbf{t}}e^{-A^{\mathbf{t}}} - BB^{\mathbf{t}}\ .$$

By the definition of Γ, the right hand side of (7.8) is equal to

$$-A\int_0^1 e^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}}\,dt - \int_0^1 e^{-tA}BB^{\mathbf{t}}e^{-tA^{\mathbf{t}}}\,dt A^{\mathbf{t}} = -A\Gamma - \Gamma A^{\mathbf{t}}\ .$$

Hence,

$$e^{-A}BB^{\mathbf{t}}e^{-A^{\mathbf{t}}} - BB^{\mathbf{t}} = -A\Gamma - \Gamma A^{\mathbf{t}}\ . \tag{7.9}$$

On the other hand, Γ being a symmetric matrix,

$$(A - BB^{\mathbf{t}}\Gamma^{-1})\Gamma + \Gamma(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}} = A\Gamma + \Gamma A^{\mathbf{t}} - 2BB^{\mathbf{t}}\ . \tag{7.10}$$

From (7.9) and (7.10) we infer

$$(A - BB^{\mathbf{t}}\Gamma^{-1})\Gamma + \Gamma(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}} = -e^{-A}BB^{\mathbf{t}}e^{-A^{\mathbf{t}}} - BB^{\mathbf{t}}\ . \tag{7.11}$$

The matrix at the right hand side is (at least) negative semidefinite. According to Theorem 3.4, we can conclude that the origin is stable for the system

$$\dot{x} = (A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}}x \tag{7.12}$$

and so also for the system

$$\dot{x} = (A - BB^{\mathbf{t}}\Gamma^{-1})x \tag{7.13}$$

since any square matrix has the same eigenvalues as its transpose. However, on the base of (7.11), we are not able to conclude that (7.13) is asymptotically stable: there are indeed simple examples of completely controllable linear systems for which the matrix at the right hand side of (7.11) is actually not positive definite.[1] In other words, we cannot be sure that $V(x) = x^{\mathbf{t}}\Gamma x$ is a strict Lyapunov function for (7.12).

To finish the proof, we need therefore to try another way. We will resort directly to Theorem 3.1. More precisely, we will show that all the eigenvalues of $(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}}$ have strictly negative real part. To this end, we take advantage of the previous computations.

Let λ be an eigenvalue (real or complex) of $(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}}$, and let $v \neq 0$ be a corresponding eigenvector. We have

$$(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}} v = \lambda v \ . \tag{7.14}$$

From (7.11) we obtain

$$\bar{v}^{\mathbf{t}}[(A - BB^{\mathbf{t}}\Gamma^{-1})\Gamma + \Gamma(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}}]v = -\bar{v}^{\mathbf{t}}[e^{-A}BB^{\mathbf{t}}e^{-A^{\mathbf{t}}} + BB^{\mathbf{t}}]v \ . \tag{7.15}$$

On the other hand,

$$\begin{aligned} &\bar{v}^{\mathbf{t}}[(A - BB^{\mathbf{t}}\Gamma^{-1})\Gamma + \Gamma(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}}]v \\ &= \bar{\lambda}\bar{v}^{\mathbf{t}}\Gamma v + \lambda\bar{v}^{\mathbf{t}}\Gamma v = (\bar{\lambda} + \lambda)\bar{v}^{\mathbf{t}}\Gamma v = 2\,\mathrm{Re}\,\lambda(\bar{v}^{\mathbf{t}}\Gamma v) \ . \end{aligned} \tag{7.16}$$

Hence,

$$\bar{v}^{\mathbf{t}}[e^{-A}BB^{\mathbf{t}}e^{-A^{\mathbf{t}}} + BB^{\mathbf{t}}]v = -2\,\mathrm{Re}\,\lambda(\bar{v}^{\mathbf{t}}\Gamma v) \ . \tag{7.17}$$

Since Γ is real and positive definite, we can easily check that $\bar{v}^{\mathbf{t}}\Gamma v > 0$ (notice the analogies between this argument and the computation in the proof of Theorem 3.3). We recover in this way the previous conclusion that $\mathrm{Re}\,\lambda \leq 0$. Now, if it happens that $\mathrm{Re}\,\lambda = 0$ for some λ, then we should also have that

$$\bar{v}^{\mathbf{t}}[e^{-A}BB^{\mathbf{t}}e^{-A^{\mathbf{t}}} + BB^{\mathbf{t}}]v = 0$$

and so in particular

$$\bar{v}^{\mathbf{t}}BB^{\mathbf{t}}v = \bar{v}^{\mathbf{t}}\bar{B}B^{\mathbf{t}}v = 0 \ .$$

This implies that $B^{\mathbf{t}}v = 0$. It follows

$$(A - BB^{\mathbf{t}}\Gamma^{-1})^{\mathbf{t}}v = A^{\mathbf{t}}v - (\Gamma^{-1})^{\mathbf{t}}BB^{\mathbf{t}}v = A^{\mathbf{t}}v \ . \tag{7.18}$$

Comparing (7.14) and (7.18), we conclude that v is also an eigenvector of $A^{\mathbf{t}}$ corresponding to the same eigenvalue λ. But then $e^{-tA^{\mathbf{t}}}v = e^{-\lambda t}v$. Finally we get

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[1] One such example can be obtained taking $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\bar{v}^{\mathbf{t}}\Gamma v = \int_0^1 \bar{v}^{\mathbf{t}} e^{-tA} B B^{\mathbf{t}} e^{-tA^{\mathbf{t}}} v\, dt = \int_0^1 e^{-\lambda t} \bar{v}^{\mathbf{t}} e^{-tA} B B^{\mathbf{t}} v\, dt = 0$$

which is impossible since Γ is positive definite and $v \neq 0$. ■

7.1.4 Stabilizability

The stabilizability property is actually weaker than complete controllability; this can be easily realized looking at a system for which A is Hurwitz and $B = 0$. In this section we aim to characterize the stabilizability property by means of suitable and easy-to-check conditions.

To this end, it is convenient to apply a preliminary change of coordinates in order to put the system in the controllability canonical form. In other words, without loss of generality, we can assume for our system the form

$$\begin{cases} \dot{z}_1 = A_{11} z_1 + A_{12} z_2 + B_1 u \\ \dot{z}_2 = \qquad\qquad A_{22} z_2 \end{cases} \tag{7.19}$$

where the subsystem corresponding to the block of coordinates z_1 is completely controllable.

Theorem 7.4 *System (7.19) is stabilizable if and only if the matrix A_{22} is Hurwitz.*

Proof The set of the eigenvalues of a matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ is the union of the sets of the eigenvalues of the matrices A_{11} and A_{22}. By means of a feedback, there is no way to modify the eigenvalues of the matrix A_{22}. Hence, if the system is stabilizable, A_{22} must be Hurwitz.

Vice versa, since the subsystem corresponding to the components z_1 is completely controllable, we can construct a feedback $u = F_1 z_1$ (for instance, by the method illustrated in the proof of Proposition 7.1) in such a way that matrix $A_{11} + B_1 F_1$ is Hurwitz. The matrix A_{22} is Hurwitz by hypothesis. Hence, the matrix A is Hurwitz, as well. ■

Next we present (without proof) other necessary and sufficient conditions for stabilization.

Theorem 7.5 *Let V be the subspace of $\mathbf{C}^n$ generated by all the eigenvectors (including the generalized ones) associated to all the eigenvalues λ of A, having nonnegative real part. Moreover, let U be the subspace of $\mathbf{R}^n$ generated by all the vectors of the form* $\mathrm{Re}\, v$ *and* $\mathrm{im}\, v$*, with $v \in V$. System (7.1) is stabilizable if and only if U is contained in its reachable space* R.

Theorem 7.6 *System (7.1) is stabilizable if and only if*

$$\text{rank}\,(A - \lambda I \mid B) = n$$

for every complex number λ with nonnegative real part.

It is interesting to compare Theorem 7.6 and Hautus' controllability criterion (Theorem 5.4).

Theorem 7.7 *System (7.1) is stabilizable if and only if there exists a symmetric, positive definite matrix P such that*

$$A^{\mathbf{t}}P + PA - PBB^{\mathbf{t}}P = -I\ . \tag{7.20}$$

In this case, a stabilizing feedback can be found of the form $u = Fx$, with $F = -\alpha B^{\mathbf{t}}P$ and $\alpha \geq \frac{1}{2}$.

It is not difficult to see that (7.20) is sufficient for stabilizability. Indeed, replacing the feedback $F = -\alpha B^{\mathbf{t}}P$ into the equation we obtain the closed-loop system in the form

$$\dot{x} = Ax - \alpha BB^{\mathbf{t}}Px\ . \tag{7.21}$$

Taking into account (7.20), we have

$$\begin{aligned}(A - \alpha BB^{\mathbf{t}}P)^{\mathbf{t}}P + P(A - \alpha BB^{\mathbf{t}}P) &= A^{\mathbf{t}}P + PA - 2\alpha PBB^{\mathbf{t}}P \\ &= -I + (1 - 2\alpha)PBB^{\mathbf{t}}P\ .\end{aligned}$$

The matrix $PBB^{\mathbf{t}}P = (B^{\mathbf{t}}P)^{\mathbf{t}}B^{\mathbf{t}}P$ corresponds to a positive semidefinite quadratic form. Hence, if $\alpha \geq \frac{1}{2}$, P solves a Lyapunov matrix equation for system (7.21). The conclusion follows by Corollary 3.1.

On the contrary, proving that the same condition is also necessary for stabilizability is more difficult (a proof can be found in [11], p. 133).

Equation (7.20) is called the *algebraic Riccati matrix equation* associated to system (7.1). We emphasize that (7.20) is nonlinear with respect to the entries of the unknown matrix P. We emphasize also that, once a solution of (7.20) has been found, the feedback law provided by Theorem 7.7 is explicit and simpler than the feedback provided in the proof of Proposition 7.1.

Corollary 7.1 *If there exists a symmetric, positive definite matrix Q such that the matrix equation*

$$A^{\mathbf{t}}P + PA - PBB^{\mathbf{t}}P = -Q \tag{7.22}$$

admits a symmetric, positive definite solution P, then system (7.1) is stabilizable.

On the other hand, if system (7.1) is stabilizable, then for each symmetric, positive definite matrix Q there exists a symmetric, positive definite solution P of the matrix equation (7.22).

Proof The proof of the first statement is similar to the proof of the sufficient part of Theorem 7.7. In order to prove the second statement, we write $Q = R^{\mathbf{t}} R$, with R nonsingular and symmetric. If the feedback $u = Fx$ stabilizes the given system, then the system

$$\dot{x} = \tilde{A}x + \tilde{B}u$$

where $\tilde{A} = RAR^{-1}$ and $\tilde{B} = RB$, is stabilized by the feedback $u = FR^{-1}$. Hence, according to the necessary part of Theorem 7.7, there must exists a symmetric, positive definite matrix $\tilde{P}$ such that

$$\tilde{A}^{\mathbf{t}} \tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}\tilde{B}^{\mathbf{t}}\tilde{P} = -I \ .$$

The remaining part of the proof is similar to that of Corollary 3.1. ■

7.1.5 Asymptotic Controllability

If system (7.1) is stabilizable by means of a feedback $u = Fx$, then for each initial state $x_0 \in \mathbf{R}^n$ we can consider the solution $x(t, x_0)$ of the problem

$$\begin{cases} \dot{x} = (A + BF)x \\ x(0) = x_0 \ . \end{cases}$$

This solution $x(t, x_0)$ obviously coincides with the solution $x(t)$ of the problem

$$\begin{cases} \dot{x} = Ax + Bu_{x_0}(t) \\ x(0) = x_0 \end{cases} \tag{7.23}$$

where $u_{x_0}(t) = Fx(t, x_0)$.

Definition 7.4 We say that the system (7.1) is *asymptotically controllable* if for each $x_0 \in \mathbf{R}^n$ there exists an input map $u_{x_0}(t)$ such that the corresponding solution $x(t)$ of the problem (7.23) approaches the origin for $t \to +\infty$.

The previous reasoning shows that a stabilizable system is asymptotically controllable. But also the converse is true. Indeed, by an argument similar to that used in the proof of Theorem 7.4, it is not difficult to see that if system (7.1) is asymptotically controllable, then the uncontrollable part of the associated canonical controllability form must be asymptotically stable. Then, the conclusion follows in force of Theorem 7.4. We can therefore state a further necessary and sufficient condition for stabilizability.

Theorem 7.8 *System (7.1) is stabilizable if and only if it is asymptotically controllable.*

7.2 Static Output Feedback

In the previous section we studied the problem of stabilizability by means of a static state feedback. Obviously, this way is not feasible when the whole state of the system is not available. This usually happens when we deal with a system with an observation function

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{7.24}$$

($x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$), and the requested feedback law has of the form $u = Ky$. The problem of stabilization by means of output feedback is more natural in view of the applications, but also much more difficult to study.

To become familiar with these new difficulties, we examine some simple examples.

Example 7.1 The two dimensional system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \tag{7.25}$$

is completely controllable, and hence stabilizable by means of a feedback of the form $u = k_1 x_1 + k_2 x_2$. In fact, provided that the parameters k_1 and k_2 can chosen freely and independently each other, the system is superstabilizable.

However, it is not possible to stabilize the system if the choice is limited to feedback laws of the form $u = kx_1$. Indeed, by applying such a feedback, the system becomes

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = kx_1 \end{cases}$$

whose eigenvalues are both either on the imaginary axis, or on the real axis and, in this second case, they have opposite sign. In other words, since now only one parameter can be arbitrarily chosen, we do not have the degrees of freedom necessary to solve the problem.

As already suggested, the impossibility of implementing a feedback which uses all the state variables typically arises when we have an observation function. In this example, the feedback $u = kx_1$ can be interpreted as an output feedback, if we assume an observation function $y = c^{\mathbf{t}} x$ with $c = (1\ 0)$. We emphasize that the system, with respect to this observation function, is completely observable, as well; nevertheless, the system is not stabilizable by an output feedback. ■

Example 7.2 Consider again the system (7.25), but this time with the observation function $y = x_1 + x_2$. By applying the feedback $u = -ky = -k(x_1 + x_2)$, we obtain the system

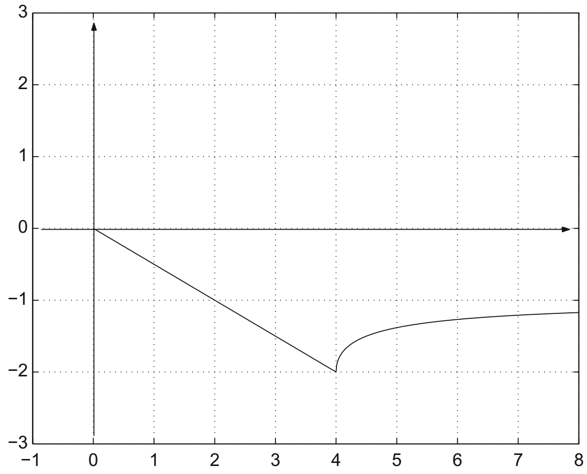


Fig. 7.1 Graph of $\mathrm{Re}\,(\lambda_2(k))$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -kx_1 - kx_2 \end{cases}$$

whose characteristic equation is $\lambda^2 + k\lambda + k = 0$. Thus, the system is stabilized if $k > 0$. However, the system is not superstabilizable. Indeed, if $0 < k < 4$ the characteristic roots are not real. Their real part is $\mathrm{Re}\,\lambda = -k/2$; it decreases when k increases, and

$$\lim_{k \to 4^-} \mathrm{Re}\,\lambda = -2\ .$$

If $k = 4$ we have $\lambda_1 = \lambda_2 = -2$. If $k > 4$, the characteristic roots are real and can be represented as

$$\lambda_1 = \frac{-k - \sqrt{k^2 - 4k}}{2} \quad \text{and} \quad \lambda_2 = \frac{-k + \sqrt{k^2 - 4k}}{2}\ .$$

Clearly $\lambda_2 > \lambda_1$: moreover, λ_2 increases when k increases, and

$$\lim_{k \to 4^+} \lambda_2 = -2\ , \qquad \lim_{k \to +\infty} \lambda_2 = -1\ .$$

In conclusion, $\mathrm{Re}\,\lambda_2(k) \geq -2$ for each $k \geq 0$, and it attains the minimum for $k = -2$ (the graph of the real part of λ_2 as a function of k is shown in Fig. 7.1).

Notice that again in this example, the system is completely controllable and completely observable. ■

Of course, if a system is stabilizable by a static output feedback, it is stabilizable also by a static state feedback. Hence, all the static state feedback stabilizability conditions listed in the previous section can be reviewed as necessary (but no more sufficient) conditions for static output feedback stabilizability.

7.2.1 Reduction of Dimension

In this section we present a theorem which allows us to simplify, under particular conditions, the study of the static output feedback stabilization problem.

Given a system (7.24), we begin by applying a linear change of coordinates, with the purpose of rewriting the matrices A, B, C in Kalman's canonical form (5.35), that is

$$\begin{cases} \dot{z}_1 = A_{11}z_1 + A_{12}z_2 + A_{13}z_3 + A_{14}z_4 + B_1 u \\ \dot{z}_2 = \qquad A_{22}z_2 \qquad + A_{24}z_4 + B_2 u \\ \dot{z}_3 = \qquad\qquad A_{33}z_3 + A_{34}z_4 \\ \dot{z}_4 = \qquad\qquad\qquad A_{44}z_4 \\ y = C_2 z_2 + C_4 z_4 \,. \end{cases} \tag{7.26}$$

Recall that according to the usual notation, the controllable part of the system is identified by the indices 1 and 2, while the completely observable part is identified by the indices 2 and 4.

Theorem 7.9 *The overall system (7.24) is stabilizable by static output feedback if and only if the following conditions are both satisfied:*

(1) the matrices A_{11}, A_{33} and A_{44} have all the eigenvalues with negative real part;
(2) the completely controllable and completely observable part of the system, that is the part corresponding to the subsystem

$$\begin{cases} \dot{z}_2 = A_{22}z_2 + B_2 u \\ y = C_2 z_2 \end{cases} \tag{7.27}$$

is stabilizable by static output feedback.

Proof We start by a preliminary remark. Let K be a matrix with p columns and m rows. If we apply the feedback $u = Ky = KC_2z_2 + KC_4z_4$ to system (7.26), the system matrix becomes

$$\begin{pmatrix} A_{11} & \tilde{A}_{12} & A_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{22} & 0 & \tilde{A}_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix} \tag{7.28}$$

where $\tilde{A}_{12} = A_{12} + B_1KC_2$, $\tilde{A}_{14} = A_{14} + B_1KC_4$, $\tilde{A}_{22} = A_{22} + B_2KC_2$, $\tilde{A}_{24} = A_{24} + B_2KC_4$.

By virtue of the triangular block form of (7.28), it is clear that the feedback $u = Ky$ stabilizes the system if and only if the matrices A_{11}, $\tilde{A}_{22}$, A_{33} and A_{44} have all their eigenvalues with negative real part. Taking into account of condition (1), this actually happens if and only if the feedback $u = Ky = KC_2z_2$ stabilizes the reduced order system (7.27). ■

In view of Theorem 7.9, as far as we are interested in the static output feedback stabilization problem, it is not restrictive to assume that the system at hand is completely controllable as well as completely observable. Then, the following sufficient condition may be of some help.

Proposition 7.2 *Let the system (7.24) be given. Assume that it is completely controllable, and that the matrix C is invertible. Then, the system is stabilizable by a static output feedback.*

Proof By the complete controllability hypothesis, there exists a matrix K such that the system is stabilizable by a static state feedback $u = Kx$. We can write $u = KC^{-1}Cx = KC^{-1}y$. We obtain in this way a static output feedback $u = Fy$ with $F = KC^{-1}$ whose effect on the system is the desired one. ■

In the previous statement, the assumption that C is invertible implies of course that $p = n$ and that the system is completely observable, as well.

Example 7.3 Consider the system

$$\begin{cases} \dot{x}_1 = x_1 + 4x_2 \\ \dot{x}_2 = 2x_1 - 6x_2 + u \\ y = 2x_1 + x_2 \ . \end{cases}$$

It is clear that it is completely controllable, but not completely observable. Using the change of coordinates $x = Pz$ where P is defined as

$$P = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

(according to the method explained in Sect. 5.3) we recover the observability canonical form

$$\begin{cases} \dot{z}_1 = -7z_1 - 2z_2 + \frac{2}{5}u \\ \dot{z}_2 = 2z_2 + \frac{1}{5}u \\ y = 5z_2 \ . \end{cases}$$

The unobservable part (first equation), when we set $u = 0$, is asymptotically stable. By Theorem 7.9, now it is clear that the system, in the new coordinates, is stabilizable by means of a static output feedback $u = ky$. Convenient values for the parameter k can be determined by direct computation: we find $k < -2$. Coming back to the original coordinates, the feedback to be applied is $u = ky = k(2x_1 + x_2)$ (again, $k < -2$). ■

As suggested by the previous example, once the reduction of dimension has been performed, if the dimension of the completely controllable and observable part turns out to be small, the existence of static output stabilizers can be checked by direct computation. An other example is given below.

Example 7.4 Consider a two-dimensional, completely controllable and completely observable system (in controllability canonical form)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a_2 x_1 - a_1 x_2 + u \\ y = c_0 x_1 + c_1 x_2 \ . \end{cases} \tag{7.29}$$

The complete observability assumption amount to say that $c_0^2 - c_0 c_1 a_1 + c_1^2 a_2 \neq 0$, which in turn implies that c_0 and c_1 cannot be both zero. By the substitution $u = ky$, the system matrix takes the form

$$\begin{pmatrix} 0 & 1 \\ -a_2 + kc_0 & -a_1 + kc_1 \end{pmatrix}$$

whose characteristic equation is

$$\lambda^2 - \lambda(-a_1 + kc_1) + a_2 - kc_0 = 0 \ .$$

By the Routh-Hurwitz criterion, it is easy to see that the system is stabilizable with static output feedback if and only if there exists $k \in \mathbf{R}$ such that $kc_0 < a_2$, and $kc_1 < a_1$. ■

The following statement is a dual version of Proposition 7.2.

Proposition 7.3 *Let the system (7.24) with $m = n$ be given. Assume that it is completely observable, and that the matrix B is invertible. Then, the system is stabilizable by a static output feedback.*

Proof The observability assumption about (7.24) implies that the dual system

$$\begin{cases} \dot{x} = A^{\mathbf{t}} x + C^{\mathbf{t}} v \\ y = B^{\mathbf{t}} x \end{cases}$$

is completely controllable. Hence, there is a matrix K such that $A^{\mathbf{t}} + C^{\mathbf{t}} K$ is Hurwitz. The matrix $A + K^{\mathbf{t}} C$ is Hurwitz, as well. Thus, if we apply to (7.24) the feedback law $u = B^{-1} K^{\mathbf{t}} y$, then the matrix of the closed loop system is $A + BB^{-1} K^{\mathbf{t}} C = A + K^{\mathbf{t}} C$. The statement is proven. ■

7.2.2 Systems with Stable Zero Dynamics

Concerning systems represented by Eq. (7.26), there is a further interesting remark. For a moment, let us limit ourselves to look at the differential part of the system, operated in open loop. The solution corresponding to an initial state of the form $\bar{z} = (\bar{z}_1, 0, \bar{z}_3, 0)$ and a vanishing input $u = 0$ evolves inside the subspace of equations $z_2 = z_4 = 0$, which is therefore invariant (in the sense of Definition A.4) for the unforced system. We point out that each solution lying in this subspace gives rise to a vanishing output. For this reason, the subspace of equations $z_2 = z_4 = 0$ is called the *space of the zero dynamics*, and the system

$$\begin{cases} \dot{z}_1 = A_{11}z_1 + A_{13}z_3 \\ \dot{z}_3 = \qquad\quad A_{33}z_3 \end{cases}$$

is called the *system of the zero dynamics*. The following statement is a straightforward consequence of Theorem 7.9 and Proposition 7.2.

Corollary 7.2 *Let the dimension of the observable but not controllable part of the system (7.24) be zero. Assume in addition that the matrix C_2 is invertible. Then, the system is stabilizable by static output feedback if and only if the origin is asymptotically stable for the system of the zero dynamics.*

7.2.3 A Generalized Matrix Equation

Other sufficient conditions for static output stabilization can be obtained by suitable generalizations of the Riccati matrix equation (7.20). Next we present one such generalization.

Let C be a matrix with p rows and n columns. A *generalized inverse* (or *pseudoinverse*) of C is any matrix $C^\dagger$ with n rows and p columns such that

$$CC^\dagger C = C \quad \text{and} \quad C^\dagger CC^\dagger = C^\dagger \ .$$

If in addition we require that $CC^\dagger$ is symmetric, then $C^\dagger$ is uniquely determined and it is called the *Moore-Penrose generalized inverse* of C (see [4] for properties of generalized inverse matrices). The space $\mathbf{R}^n$ can be decomposed as $\text{im}\,(C^{\mathbf{t}}) \oplus \text{ker}\,(C)$. Moreover, the subspaces $\text{im}\,(C^{\mathbf{t}})$ and $\text{ker}\,(C)$ are orthogonal each other. The square $n \times n$ matrix $E_{\text{im}} = C^\dagger C$ represents the orthogonal projection on $\text{im}\,(C^{\mathbf{t}})$, while the orthogonal projection on $\text{ker}\,(C)$ is given by $E_{\text{ker}} = I - E_{\text{im}}$. The matrices E_{im} and E_{ker} are uniquely determined and symmetric.

Theorem 7.10 *Consider the system (7.24), and assume that there exist symmetric and positive definite matrices P and Q such that*

$$A^{\mathbf{t}}P + PA - E_{im}PBB^{\mathbf{t}}PE_{im} + E_{\text{ker}}PBB^{\mathbf{t}}PE_{\text{ker}} = -Q \ . \qquad (7.30)$$

Then, the system is stabilizable by the static output feedback $u = Ky$*, with* $K = -B^{\mathbf{t}}PC^{\dagger}$.

Before proving the theorem, we point out the following matrix identity:

$$\begin{aligned} &E_{\text{im}}PBB^{\mathbf{t}}PE_{\text{im}} - E_{\text{ker}}PBB^{\mathbf{t}}PE_{\text{ker}} + PBB^{\mathbf{t}}P \\ &= E_{\text{im}}PBB^{\mathbf{t}}P + PBB^{\mathbf{t}}PE_{\text{im}} \end{aligned} \tag{7.31}$$

which can be easily checked taking into account that $E_{\text{ker}} = I - E_{\text{im}}$.

Proof of Theorem 7.10 Applying the static output feedback $u = -B^{\mathbf{t}}PC^{\dagger}y$, the closed-loop system takes the form

$$\dot{x} = (A - BB^{\mathbf{t}}PC^{\dagger}C)x\ .$$

We show that, by virtue of (7.30) and (7.31), P solves the Lyapunov matrix equation for this system. Indeed, we have:

$$\begin{aligned} &(A - BB^{\mathbf{t}}PC^{\dagger}C)^{\mathbf{t}}P + P(A - BB^{\mathbf{t}}PC^{\dagger}C) \\ &= A^{\mathbf{t}}P + PA - [E_{\text{im}}PBB^{\mathbf{t}}P + PBB^{\mathbf{t}}PE_{\text{im}}] \\ &= A^{\mathbf{t}}P + PA - E_{\text{im}}PBB^{\mathbf{t}}PE_{\text{im}} + E_{\text{ker}}PBB^{\mathbf{t}}PE_{\text{ker}} - PBB^{\mathbf{t}}P \\ &= -Q - PBB^{\mathbf{t}}P\ . \end{aligned}$$

The conclusion follows by Theorem 3.3, since the matrix $PBB^{\mathbf{t}}P$ is clearly positive semidefinite. ■

Notice that (7.30) reduces to (7.20) when $C = I$.

Example 7.5 The condition of Theorem 7.10 works, for the case considered in the previous Example 7.2. Since $C = (1\ 1)$, as a generalized inverse we can take for instance

$$C^{\dagger} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

so that

$$E_{\text{im}} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad E_{\text{ker}} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}\ .$$

Writing

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \tag{7.32}$$

the left-hand side of (7.30) takes the form

$$\begin{pmatrix} -p_{12}p_{22} & p_{11} - \dfrac{p_{12}^2 + p_{22}^2}{2} \\ p_{11} - \dfrac{p_{12}^2 + p_{22}^2}{2} & 2p_{12} - p_{12}p_{22} \end{pmatrix}\ .$$

A solution of (7.30) with the required properties is obtained taking $p_{11} = 6$, $p_{22} = 3$, $p_{12} = 2$. The corresponding output feedback is $u = -\frac{5}{2}y$. ■

Note that assuming $Q = I$ in (7.30) would be restrictive, contrary to what happens in the case of the Lyapunov matrix equation (see Theorem 3.3 and Corollary 3.1) and in the case of the Riccati matrix equation. For instance, it is not difficult to check that in the previous example, there is no solutions for (7.30) if we set $Q = I$.

Example 7.6 The condition of Theorem 7.10 is not necessary for the existence of an output feedback stabilizer. Consider the two-dimensional system with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = (1\ 0) .$$

Clearly, this time we have

$$E_{\mathrm{im}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_{\mathrm{ker}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

Assume that (7.30) has a positive definite solution P, that we write again in the form (7.32). Then it would be possible to construct an output stabilizer of the form $u = -B^{\mathbf{t}} P C^{\dagger} y$, which reduces in this case to $u = -p_{12} y$. By direct substitution, it is easily seen that the system can be actually stabilized by an output feedback of this form, provided that $p_{12} > 2$.

On the other hand, the left-hand side of (7.30) takes now the form

$$\begin{pmatrix} 2p_{11} - p_{12}^2 & p_{11} - p_{12} \\ p_{11} - p_{12} & 2p_{12} - 4p_{22} + p_{22}^2 \end{pmatrix} .$$

This matrix is definite negative only if the term $2p_{12} - 4p_{22} + p_{22}^2$ is negative: as easily seen, this requires that $p_{12} < 2$.

In conclusion, the system is stabilizable by an output feedback, but the coefficient of the feedback cannot be determined on the base of Theorem 7.10. ■

7.2.4 A Necessary and Sufficient Condition

The static output feedback stabilization problem is sometimes referred to in the literature as an unsolved problem. From a practical point of view, a numerical solution of a nonlinear matrix equation like (7.30) can be indeed very hard to find. On the contrary, theoretical characterizations of systems admitting static output stabilizing feedbacks expressed in the form of nonlinear generalized Riccati equations can be actually found in the existing literature. For instance, the following theorem appears in [29].

Theorem 7.11 *System (7.24) is stabilizable by a static output linear feedback if and only if there exist symmetric, positive definite $n \times n$ matrices P, Q and a matrix M (with m rows and n columns) such that*

$$A^{\mathbf{t}}P + PA - E_{im}M^{\mathbf{t}}B^{\mathbf{t}}P - PBME_{im} = -Q\ . \tag{7.33}$$

Moreover, when (7.33) holds, a stabilizing feedback can be taken of the form $u = Ky$, with $K = -MC^{\dagger}$.

Proof To prove the necessity of the condition, let us assume that the system is stabilized by a feedback of the form $u = Ky$ for some matrix K. Then we have also the static state stabilizer

$$u = KCx = KCC^{\dagger}Cx = -MC^{\dagger}Cx = -ME_{\text{im}}x$$

where we used the definition of generalized inverse and we set $M = -KC$. Hence, the closed loop system must satisfy the Lyapunov matrix equation (Theorem 3.3) for some symmetric, positive definite matrix P; namely

$$\begin{aligned}&(A - BME_{\text{im}})^{\mathbf{t}}P + P(A - BME_{\text{im}})\\ &= A^{\mathbf{t}}P + PA - E_{\text{im}}M^{\mathbf{t}}B^{\mathbf{t}}P - PBME_{\text{im}} = -I\end{aligned}$$

which is (7.33) with $Q = I$. As far as the sufficiency is concerned, assuming that (7.33) holds for some matrices P, Q, M, we apply the static output feedback $u = -MC^{\dagger}y = -ME_{\text{im}}x$. The conclusion can be easily achieved by repeating the same computation as before and using the Lyapunov matrix equation as a sufficient condition. ■

Remark 7.1 In [29], condition (7.33) is written in a different, but equivalent, way: indeed, the authors do not use the formalism of generalized inverse. ■

Remark 7.2 It is not difficult to see that (7.30) implies (7.33), setting $M = B^{\mathbf{t}}P$, and using the matrix identity (7.31). However, we notice that with respect to (7.30), the matrix equation (7.33) contains the additional unknown M. ■

Remark 7.3 Another necessary and sufficient condition for the existence of static output stabilizing feedbacks was given in [9]. Reformulated in terms of generalized inverse, this condition reads: there exist matrices P, Q, M (of the same dimensions as before) such that

$$A^{\mathbf{t}}P + PA - PBB^{\mathbf{t}}P + (B^{\mathbf{t}}P - ME_{\text{im}})^{\mathbf{t}}(B^{\mathbf{t}}P - ME_{\text{im}}) = -Q\ . \tag{7.34}$$

Of course, (7.34) is equivalent to (7.33), but not with the same P and Q, in general. ■

7.3 Dynamic Output Feedback

The practical difficulties encountered in the static output stabilization problem can be overcome resorting to a different approach, provided that the system is, in principle, stabilizable by means of a static state feedback law and a suitable (but natural) technical condition is met. The new approach we are going to describe in this section is *dynamic output feedback*.

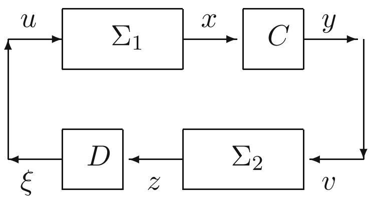
Definition 7.5 We say that system (7.24) is *stabilizable by dynamic output feedback* if there exists a system

$$\begin{cases} \dot{z} = Fz + Gv \\ \xi = Dz \end{cases} \tag{7.35}$$

with $z \in \mathbf{R}^{\nu}$, $v \in \mathbf{R}^{p}$, $\xi \in \mathbf{R}^{m}$ such that the composed system obtained by means of the substitutions $u = \xi$ and $v = y$

$$\begin{cases} \dot{x} = Ax + BDz \\ \dot{z} = Fz + GCx \end{cases}$$

has an asymptotically stable equilibrium position at the origin $(x, z) = (0, 0) \in \mathbf{R}^{n+\nu}$. The system (7.35) represents the *compensator*, or *controller*.



In the figure above, Σ_1 and Σ_2 denote respectively the differential parts of (7.24) and (7.35).

Example 7.7 The system (7.25) (Example 7.1) with the observation function $y = x_1$, can be dynamically stabilized by means of the compensator

$$\begin{cases} \dot{z}_1 = -z_1 + z_2 + v \\ \dot{z}_2 = -2z_1 - z_2 + v \\ \xi = -z_1 - z_2 \,. \end{cases}$$

Indeed, establishing the connection as explained above, we obtain the closed-loop system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -z_1 - z_2 \\ \dot{z}_1 = x_1 - z_1 + z_2 \\ \dot{z}_2 = x_1 - 2z_1 - z_2 \end{cases}$$

whose characteristic equation is $\lambda^4 + 2\lambda^3 + 3\lambda^2 + 2\lambda + 1 = 0$. It is not difficult to check, by the aid of the Routh-Hurwitz criterion, that all the roots have negative real part. ■

The remaining part of this section is devoted to illustrate how the stabilizing compensator can be constructed in practice, for a general system of the form (7.24).

7.3.1 Construction of an Asymptotic Observer

Definition 7.6 We say that system (7.24) has the *detectability* property (or that it is *detectable*) if there exists a matrix K of appropriate dimensions such that the matrix $L^{\mathbf{t}} = A^{\mathbf{t}} - C^{\mathbf{t}} K^{\mathbf{t}}$ is Hurwitz.

A system possesses the detectability property if and only if its dual system is stabilizable by static state feedback. In particular, each completely observable system is detectable.

Proposition 7.4 *Assume that the given system (7.24) is detectable. For each admissible open loop input $u(t) : [0, +\infty) \to \mathbf{R}^m$ and for each pair of vectors $x_0, z_0 \in \mathbf{R}^n$, we denote by $x(t)$ the solution of the system*

$$\dot{x} = Ax + Bu$$

corresponding to the input $u(t)$ and the initial state x_0, and by $z(t)$ the solution of the system

$$\dot{z} = Lz + Ky + Bu \tag{7.36}$$

corresponding to the input $u(t)$ and the initial state z_0. Then we have

$$\lim_{t \to +\infty} (x(t) - z(t)) = 0 \ .$$

Proof Denote by $e(t) = x(t) - z(t)$ the difference between $z(t)$ and the state $x(t)$. We have:

$$\dot{e} = \dot{x} - \dot{z} = Ax + Bu - Lz - KCx - Bu = (L + KC)x - Lz - KCx = Le \ . \tag{7.37}$$

Recall that the eigenvalues of a matrix are the same as the eigenvalues of its transpose. Since by assumption all the eigenvalues of L have negative real part, we conclude that $\lim_{t \to +\infty} e(t) = 0$ as desired. ■

Remark 7.4 Notice that the input of (7.36) is the sum of the same external input received by the given system and the output of the given system. Proposition 7.4 states that, regardless the initialization of the two systems and assuming that the external input is the same, the solutions of (7.36) asymptotically approximate the solutions of the given system. For this reason, system (7.36) is called an *asymptotic observer* and the quantity $e(t)$ introduced in the previous proof is called the *error* between the true state $x(t)$ and the observed state $z(t)$. ■

7.3.2 Construction of the Dynamic Stabilizer

Now assume that system (7.24) is stabilizable by static state feedback, as well as detectable. Under this additional hypothesis, we may find a matrix H such that the matrix $(A + BH)$ is Hurwitz.

If the full state vector is measurable and available for control purposes, we could directly apply the feedback $u = Hx$ and solve in this way the stabilization problem. Otherwise, it is natural to try the control law $u = Hz$, where z is the approximation of x provided by the asymptotic observer (7.36).

Replacing $u = Hz$ in (7.24) and in (7.36), and recalling that $y = Cx$, we obtain the two systems of differential equations

$$\dot{x} = Ax + BHz \,, \tag{7.38}$$

$$\dot{z} = Lz + KCx + BHz = KCx + (A - KC + BH)z \,. \tag{7.39}$$

Lemma 7.1 *The system composed by (7.38) and (7.39) is asymptotically stable at the origin.*

Proof Let us introduce, as above, the variable $e = x - z$. System (7.38) is equivalent to

$$\dot{x} = (A + BH)x - BHe \tag{7.40}$$

while system (7.39) is equivalent to

$$\begin{aligned} \dot{e} &= Ax + BHz - Az + KCz - BHz - KCx \\ &= Ax - Ax + Ae + KCx - KCe - KCx = Le \,. \end{aligned} \tag{7.41}$$

Systems (7.40) and (7.41) can be reviewed as a unique unforced system, whose matrix is

$$\begin{pmatrix} A + BH & -BH \\ 0 & A - KC \end{pmatrix} . \tag{7.42}$$

The set of the eigenvalues of the matrix (7.42) is the union of the sets of the eigenvalues of the matrices $A + BH$ and $A - KC$ which, by construction, are Hurwitz. The statement is so proven. ■

Theorem 7.12 *If system (7.24) is stabilizable by static state feedback, and if it is detectable, then it is stabilizable by a dynamic output feedback, as well.*

Proof The system composed by (7.38) and (7.39) can be interpreted as the result of the connection of the given system (7.24) and the dynamic compensator

$$\begin{cases} \dot{z} = (A - KC + BH)z + Kv \\ \xi = Hz \, . \end{cases}$$

We have so really constructed a dynamic output feedback: the consistency with the notation of Definition 7.5, is easily recovered setting $F = A - KC + BH$, $G = K$, $D = H$. The argument is finished, thanks to Lemma 7.1. ■

We therefore see that the construction of a stabilizing dynamic output feedback reduces to the construction of a stabilizing static state feedback $u = Hx$ for the given system, and the construction of a stabilizing static state feedback $w = -K^{\mathbf{t}}z$ for the dual system

$$\dot{z} = A^{\mathbf{t}}z + C^{\mathbf{t}}w \, .$$

This conclusion is known as the *separation principle*. We emphasize once more that in order to construct H and K we need to know A, B and C, but in order to practically implement the connection, it is sufficient to dispose of the output y.

At a first glance, Theorem 7.12 seems to suggest that the method of dynamic feedback is more general than the method of static state feedback. As a matter of fact, these two methods are (theoretically but, recall, not practically) equivalent.

Theorem 7.13 *Let the system (7.24) be given, and assume that it is stabilizable by means of a dynamic output feedback (7.35). Then, the system is stabilizable by means of a static state feedback, as well.*

Proof Assume that a stabilizing dynamic output feedback exists. The closed-loop system writes as

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} .$$

Hence, the system in $\mathbf{R}^{n+\nu}$ defined by the matrices

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & G \end{pmatrix}$$

is stabilizable by a static state feedback. Then, according to Theorem 7.6, we must have

$$\text{rank} \begin{pmatrix} A + \lambda I & 0 & | & B & 0 \\ 0 & F + \lambda I & | & 0 & G \end{pmatrix} = n + \nu$$

for each complex number λ with nonnegative real part. This yields

$$\text{rank}\,(A + \lambda I \mid B) = n$$

for each complex number λ with nonnegative real part. ■

7.4 PID Control

As already mentioned, many physical systems of interest in practical applications can be represented by a single linear differential equation

$$\ddot{\xi} + a\dot{\xi} + b\xi = u \; . \tag{7.43}$$

In (7.43) ξ represents the main variable of interest, while u is a scalar input. In the early literature, the control of such systems is based on the following ideas.

1. A feedback proportional to the main variable, that is

$$u = k_0\xi \; . \tag{7.44}$$

2. A feedback proportional to the derivative of the main variable, that is

$$u = k_1\dot{\xi} \; . \tag{7.45}$$

3. An input proportional to the integral of $\xi(t)$, that is

$$u = k_2 \int_0^t \xi(\tau)\, d\tau \; . \tag{7.46}$$

Here, k_0, k_1, k_2 are suitable real constants, often referred to as the *gains*. The feedback (7.44) is called a P control. It can be reviewed as a static output feedback, assuming that (7.43) is associated to the observation function $y = \xi$.

The feedback (7.45) is called a D control. The sum of (7.44) and (7.45), that is the feedback

$$u = k_0\xi + k_1\dot{\xi} \tag{7.47}$$

is called a PD control. Since the full state of (7.43) is the pair $(\xi, \dot{\xi})$, (7.47) can be reviewed as a static state feedback for (7.43).

The function defined in (7.46) is called a I control. Notice that (7.46) can be thought of as a signal generated by the dynamic compensator

$$\dot{z} = k_2\xi \; . \tag{7.48}$$

The combination of (7.44), (7.45), (7.46), that is

$$u = k_0\xi + k_1\dot{\xi} + k_2 \int_0^t \xi(\tau)\, d\tau \tag{7.49}$$

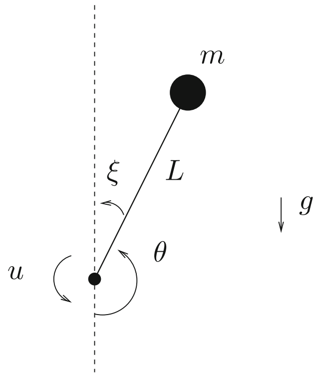
is called a PID control. A PID control can be reviewed as a static state feedback for the system formed by the composition of (7.43) and (7.48). The following example illustrates the use of the PID control.

Consider the equation

$$\ddot{\xi} + \varepsilon\dot{\xi} - \frac{L}{g}\xi = u \quad (0 < \varepsilon << 1)\ . \tag{7.50}$$

It represents the linearized equation of an inverted pendulum with (small) friction. Here, L denotes the length of the bar of the pendulum (of mass m), and g is the gravity constant.

The main variable $\xi = \pi - \theta$ represents the angle with respect to the upward oriented vertical line (see the figure). The control u is exerted by a torque applied to the pivot.



Assume for simplicity that $\frac{L}{g} = 1$. The free system (i.e., with $u = 0$) is clearly unstable. The system can be stabilized by means of a P control, with gain $k_0 < -1$. However, by means of such a control the decay rate cannot be improved, since it depends on the coefficient of the derivative $\dot{\xi}$, which it is not affected by a P control.

Now we try a PI control (that is, a linear combination of P and I controls). To this end, we add the Eq. (7.48) to the system, and write a new system with variables $x_1 = z$, $x_2 = \xi$, $x_3 = \dot{\xi}$. The matrices involved in this three-dimensional representation are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\varepsilon \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad c = (1\ 1\ 0)\ .$$

The choice of c is related to the form $u = k_2x_1 + k_0x_2$ of the desired feedback. The system is completely controllable and observable, and can be actually stabilized by static output feedback. However, as before, the decay rate cannot be improved.

Finally, we can easily see that the system is superstabilizable if a PID control is used. Unfortunately, feedbacks involving a D control are not easy to implement, because measuring the derivative of a variable is usually in practice a hard task. Nevertheless, even today PID control is very popular in industrial applications.

Chapter Summary

In this chapter, the two main topics studied in this book (stability and control) encounter each other. We address the stabilization problem, that is the problem of improving the stability performances of a system by applying a suitable feedback law. We consider several approaches: static state feedback, static output feedback and dynamic feedback. Finally we revisit in this framework the classical PID control method.