# Chapter 18
# Analytic Continuation; The Gamma and Zeta Functions

## Introduction

Suppose we are given a function $f$ which is analytic in a region $D$. We will say that $f$ can be continued analytically to a region $D_1$ that intersects $D$ if there exists a function $g$, analytic in $D_1$ and such that $g = f$ throughout $D_1 \cap D$. By the Uniqueness Theorem (6.9) any such continuation of $f$ is uniquely determined. (It is possible, however, to have two analytic continuations $g_1$ and $g_2$ of a function $f$ to regions $D_1$ and $D_2$ respectively with $g_1 \neq g_2$ throughout $D_1 \cap D_2$. See Exercise 1.)

The Schwarz Reflection Principle (7.8) is an example of how, in some cases, an analytic function can be continued beyond its original domain of analyticity. In this chapter, we first examine the possibility of such "extensions" for functions given by power series. We then consider the classical Gamma and Zeta functions, defined originally by a definite integral and a Dirichlet series, respectively.

## 18.1 Power Series

As we have seen in Chapter 2, a power series, $\sum_{n=0}^{\infty} a_n z^n$, may converge at some or all or even none of the points on its circle of convergence. As the examples below indicate, the convergence or divergence of the power series at a point does not determine whether the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, can or cannot be continued beyond that point.

i.
$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for} \quad |z| < 1.$$

Although the power series diverges at every point on the unit circle, $f$ is analytic throughout the punctured plane $z \neq 1$.

ii. $\sum_{n=1}^{\infty} (z^n/n^2)$ converges at all points on the unit circle; however, $g(z)$ cannot be continued analytically to a domain including $z = 1$ since

$$g''(z) = \sum_{n=0}^{\infty} \frac{(n+1)z^n}{n+2} \to \infty \quad \text{as } z \to 1^-.$$

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## 18.1 Definition

Suppose that $f$ is analytic in a disc $D$ and that $z_0 \in \partial D$. Then $f$ is said to be *regular* at $z_0$ if $f$ can be continued analytically to a region $D_1$ with $z_0 \in D_1$. Otherwise, $f$ is said to have a *singularity* at $z_0$.

## 18.2 Theorem

*If $\sum_{n=0}^{\infty} a_n z^n$ has a positive radius of convergence $R$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has at least one singularity on the circle $|z| = R$.*

**Proof**

If $f$ were regular at every point on the circle of convergence, then for each $z$ with $|z| = R$, there would exist some maximal $\epsilon_z$ such that $f$ could be continued to a region containing $D(z; \epsilon_z)$. Clearly $\epsilon_z$ would depend continuously on $z$ so that, since the circle $|z| = R$ is compact,

$$\min_{|z|=R} \epsilon_z = \epsilon > 0.$$

Hence, a function $g$ would exist, analytic in $D(0; R + \epsilon)$ and such that $g = f$ in $D(0; R)$. But then $g$ must have a power series representation $\sum_{n=0}^{\infty} b_n z^n$ convergent for $|z| < R+\epsilon$. Yet since $g(z) = f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$, by the Uniqueness Theorem for Power Series (2.12), $a_n \equiv b_n$. Thus the radius of convergence would be $R$, and we have arrived at a contradiction. □

In general, it is difficult to determine when a function has a singularity at a particular point on the circle of convergence of its power series. The following theorem is one of the few results we have in this direction.

## 18.3 Theorem

*Suppose that $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $R < \infty$ and that $a_n \geq 0$ for all $n$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a singularity at $z = R$.*

**Proof**

By Theorem 18.2, $f$ has a singularity at some point $Re^{i\alpha}$. If we consider the power series for $f$ about a point $\rho e^{i\alpha}$, with $0 < \rho < R$:

$$f(z) = \sum_{n=0}^{\infty} b_n (z - \rho e^{i\alpha})^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(\rho e^{i\alpha})}{n!} (z - \rho e^{i\alpha})^n$$

we see that the radius of convergence of this series is $R - \rho$. (If it were larger, the power series would define an analytic extension of $f$ beyond $Re^{i\alpha}$). Note, however, that for any non-negative integer $j$,

$$f^{(j)}(\rho e^{i\alpha}) = \sum_{n=j}^{\infty} n(n-1)\dots(n-j+1)a_n(\rho e^{i\alpha})^{n-j}$$

so that, since $a_n \geq 0$,

$$|f^{(j)}(\rho e^{i\alpha})| \leq f^{(j)}(\rho).$$

Hence the power series expansion of $f$ about $\rho$,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\rho)}{n!}(z-\rho)^n,$$

must have radius of convergence $R-\rho$. On the other hand, if $f$ were regular at $z=R$, the above power series would converge in a disc of radius greater than $R-\rho$; therefore, $f$ is singular at $z=R$. □

## 18.4 Definition

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a singularity at every point on its circle of convergence, then that circle is called a *natural boundary* of $f$.

EXAMPLE

$$\sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + z^8 + \cdots$$

has radius of convergence 1. Yet as $z \to z_0$, where $z_0$ is any $2^n$th root of unity, all the terms of the power series past $z^{2^n}$ approach 1, so that $f(z) \to \infty$. Hence $f$ is singular at every $2^n$th root of unity, $n \geq 1$. Since these are dense on the unit circle, that circle is a natural boundary for the power series. ◊

Similarly, if we set $g(z) = \sum_{k=0}^{\infty}(z^{2^k}/2^k)$ it is clear that $g$ has the unit circle as a natural boundary since $g'(z) = (1/z)\sum_{k=0}^{\infty} z^{2^k} \to \infty$ as $z$ approaches any $2^n$th root of unity. If we set $h(z) = \sum_{k=0}^{\infty}(z^{2^k}/2^{k^2})$ then, while $h$ has radius of convergence 1, all of its derivatives are bounded throughout $|z| < 1$. Nevertheless, according to the following theorem, $h$ too has a natural boundary on the unit circle.

## 18.5 Theorem

*Suppose*

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \quad with \quad \varliminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1.$$

*Then the circle of convergence of the power series is a natural boundary for $f$.*

**Proof**

Since the result is independent of $c_k$, we may assume without loss of generality that the radius of convergence is 1. Also, neglecting finitely many terms if necessary, we will assume that for some $\delta > 0$ and for all $k$, $n_{k+1}/n_k > 1 + \delta$. Finally, it suffices to show that $f$ is singular at the point $z = 1$. For the same result, applied to the series $\sum_{k=0}^{\infty} c_k (ze^{-i\theta})^{n_k}$ shows that $f$ is singular at any point $z = e^{i\theta}$.

Choose an integer $m > 0$ such that $(m+1)/m < 1 + \delta$ and consider the power series $g(w)$ obtained by setting

$$z = \frac{w^m + w^{m+1}}{2}$$

and expanding the terms

$$\left(\frac{w^m + w^{m+1}}{2}\right)^{n_k}$$

in the power series of $f$:

$$\begin{aligned} g(w) = f\left(\frac{w^m + w^{m+1}}{2}\right) &= \frac{c_0 w^{mn_0}}{2^{n_0}} + \frac{c_0 n_0 w^{mn_0+1}}{2^{n_0}} + \cdots + \frac{c_0}{2^{n_0}} w^{mn_0+n_0} \\ &+ \frac{c_1}{2^{n_1}} w^{mn_1} + \frac{c_1 n_1}{2^{n_1}} w^{mn_1+1} + \cdots + \frac{c_1}{2^{n_1}} w^{mn_1+n_1} \\ &+ \cdots . \end{aligned}$$

Note that in this expression no two terms involve the same power of $w$, since

$$mn_{k+1} > mn_k + n_k \text{ holds whenever } \frac{n_{k+1}}{n_k} > \frac{m+1}{m}.$$

If $|w| < 1$, then

$$\frac{|w|^m + |w|^{m+1}}{2} < 1,$$

and since $f(z)$ is absolutely convergent for $|z| < 1$,

$$\sum_{k=0}^{\infty} |c_k| \left(\frac{|w|^m + |w|^{m+1}}{2}\right)^{n_k} \text{ converges.}$$

Hence for $|w| < 1$, $g(w)$ is absolutely convergent. On the other hand, if we take $w$ real and greater than 1, then

$$\frac{w^m + w^{m+1}}{2} > 1$$

so that

$$\sum_{k=0}^{\infty} c_k \left(\frac{w^m + w^{m+1}}{2}\right)^{n_k}$$

diverges. Note, though, that the $j$th partial sums $s_j$ of the above series are exactly the $n_j(m+1)$-st partial sums of the power series for $g$. Hence the series for $g(w)$ diverges and $g$, too, has radius of convergence 1. According to Theorem 18.2, $g$ must have a singularity at some point $w_0$ with $|w_0| = 1$. If $w_0 \neq 1$, then

$$\left| \frac{w_0^m + w_0^{m+1}}{2} \right| < 1$$

and since $f$ is analytic in $|z| < 1$, $g$ is regular at $w_0$. Thus $g$ must have a singularity at $w_0 = 1$ and since

$$g(w) = f\left( \frac{w^m + w^{m+1}}{2} \right),$$

$f(z)$ must have a singularity at $z = 1$. □

*The Method of Moments.* Suppose we are given a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ where the coefficients $c_n$ are the "moments" of a given continuous function. For example, suppose that there exists a continuous function $g$ on [0, 1] such that

$$c_n = \int_0^1 g(t) \cdot t^n \, dt.$$

Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left[ \int_0^1 g(t) t^n \, dt \right] z^n \\ &= \sum_{n=0}^{\infty} \left[ \int_0^1 g(t)(tz)^n \, dt \right], \end{aligned}$$

and, interchanging the order of summation and integration, we find that

$$\begin{aligned} f(z) &= \int_0^1 \left[ \sum_{n=0}^{\infty} g(t)(tz)^n \right] dt \\ &= \int_0^1 \frac{g(t)}{1 - tz} \, dt. \end{aligned}$$

(The interchange of summation and integration is easy to justify if $|z| < 1$.) Moreover, this integral form serves to define an analytic extension of the original power series.

EXAMPLES

i. Consider

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1}, \quad |z| < 1.$$

Since

$$\frac{1}{n+1} = \int_0^1 t^n \, dt,$$

$g(t) = 1$ and

$$f(z) = \int_0^1 \frac{dt}{1 - tz} \quad \text{for } |z| < 1.$$

The integral above is analytic throughout the complex plane minus $[1, \infty)$. According to Proposition 17.10 this extension of $f$ has a discontinuity at every point of the interval $[1, \infty)$.

ii. Since
$$\int_0^\infty e^{-nt^2} \, dt = \frac{1}{\sqrt{n}} \int_0^\infty \frac{e^{-u}}{2\sqrt{u}} \, du, \quad \frac{1}{\sqrt{n}} = c \int_0^\infty e^{-nt^2} \, dt,$$

where $c$ is a positive constant. (We will show in the next section that the value of $c$ is $2/\sqrt{\pi}$.) Hence

$$\begin{aligned}
\sum_{n=1}^\infty \frac{z^n}{\sqrt{n}} &= c \sum_{n=1}^\infty \left[ \int_0^\infty (ze^{-t^2})^n \, dt \right] \\
&= c \int_0^\infty \left[ \sum_{n=1}^\infty (ze^{-t^2})^n \right] dt, \quad \text{for } |z| < 1 \\
&= c \int_0^\infty \frac{z}{e^{t^2} - z} \, dt. \qquad \Diamond
\end{aligned}$$

Again, while the interchange of summation and integration is valid only in the original domain $|z| < 1$, the integral defines an analytic extension to the larger region: $\mathbb{C} \backslash [1, \infty)$. Again, by 17.10, the integral has a discontinuity at every point of $[1, \infty)$.

Many problems of this type can be solved by expressing the coefficients $c_n$ in the form

$$c_n = \int_0^\infty e^{-nt} g(t) \, dt.$$

(In this case, $c_n$ is obtained as the "Laplace Transform" of $g$ at the integer $n$.) Some well-known formulae are listed below:

$$\begin{aligned}
\frac{1}{n+a} &= \int_0^\infty e^{-nt} \, e^{-at} \, dt \\
\frac{a}{n^2 + a^2} &= \int_0^\infty e^{-nt} \sin at \, dt \\
\frac{n}{n^2 + a^2} &= \int_0^\infty e^{-nt} \cos at \, dt \\
\frac{1}{n^p} &= c_p \int_0^\infty e^{-nt} \, t^{p-1} \, dt, \quad p > 0.
\end{aligned}$$

(The constants $c_p$ are determined in terms of the $\Gamma$ function which we will study in the next section. See Exercise 5.)

EXAMPLE
Let

$$f(z) = \sum_{n=0}^{\infty} \frac{n^2}{n^2+1} z^n.$$

Then

$$f(z) = z \frac{d}{dz} \left( \sum_{n=0}^{\infty} \frac{n z^n}{n^2+1} \right).$$

Using one of the above formulae

$$\sum_{n=0}^{\infty} \frac{n}{n^2+1} z^n = \sum_{n=0}^{\infty} \left[ \int_0^{\infty} (e^{-nt} \cos t) z^n \, dt \right]$$
$$= \int_0^{\infty} \frac{e^t \cos t}{e^t - z} \, dt \quad \text{for } |z| < 1.$$

Thus

$$f(z) = z \int_0^{\infty} \frac{e^t \cos t}{(e^t - z)^2} \, dt.$$

[Alternatively, we could write

$$f(z) = \sum_{n=0}^{\infty} \left(1 - \frac{1}{n^2+1}\right) z^n = \frac{1}{1-z} - \sum_{n=0}^{\infty} \frac{1}{n^2+1} z^n, \quad \text{etc.\,.\,.\,.\,.}]$$ ◊

## 18.2 Analytic Continuation of Dirichlet Series

Dirichlet series, unlike power series, do not necessarily have a singularity on their boundary of convergence. For example, we will see in the next section that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$$

can actually be continued to the full complex plane. However, if all the coefficients $a_n$ are positive, we have the following analogue of Theorem 18.3.

### 18.6 Landau's Theorem

*Suppose that $a_n \geq 0$ for all $n$, and that $b$ is the real boundary point of the region of convergence of*

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}.$$

*Then $f$ has a singularity at $b$.*

**Proof**

We will show that if $f$ is regular at $b$; that is, if it can be analytically extended to a region containing the point $b$, then the Dirichlet series will converge at some real number less than $b$, which contradicts the definition of $b$. Toward that end, choose a real number $a > b$, and consider the power series representation of $f$, centered at $z = a$. Since

$$f^{(k)}(z) = \sum_{n=1}^{\infty} \frac{a_n(-\log n)^k}{n^z},$$

the power series representation for $f$ in a disc centered at $z = a$ is

$$f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k \text{ with } c_k = \sum_{n=1}^{\infty} \frac{a_n(-\log n)^k}{n^a k!} \tag{1}$$

If $f$ is regular at $z = b$, the radius of convergence of the series in (1) is greater than $a - b$ so that the series converges at a point of the form $b - \varepsilon$, with $\varepsilon > 0$. That is,

$$\sum_{k=0}^{\infty}\Big(\sum_{n=1}^{\infty} \frac{a_n(\log n)^k}{n^a k!}\Big)(a-b+\varepsilon)^k \tag{2}$$

converges. Since $a_n \geq 0$ for all $n$, all the terms in (2) are nonnegative. Hence it is an absolutely convergent series and, as such, its terms can be rearranged in any form. Suppose then that we first sum over $k$. Then

$$\sum_{k=0}^{\infty} \frac{(\log n)^k}{k!}(a-b+\varepsilon)^k = e^{(a-b+\varepsilon)\log n} = n^{a-b+\varepsilon}$$

and the convergent series in (2) becomes

$$\sum_{n=1}^{\infty} \frac{a_n n^{a-b+\varepsilon}}{n^a}$$

which is exactly the Dirichlet series with $z = b - \varepsilon$. □

## 18.7 Corollary

*If a Dirichlet series has nonnegative coefficients and can be analytically continued to the entire complex plane, then it converges throughout the complex plane.*

**Proof**

If the series did not converge for all $z$, according to Theorem 18.6, the function represented by the Dirichlet series would have a singularity at the real boundary point of its region of convergence. □

## 18.3 The Gamma and Zeta Functions

*The Gamma Function.* Consider the integral

$$I_n = \int_0^\infty e^{-t} t^n dt \quad n = 0, 1, 2 \ldots .$$

Integration by parts shows that

$$I_n = n \int_0^\infty e^{-t} t^{n-1} dt = n I_{n-1}.$$

Since $I_0 = 1$, the above recurrence relation implies

$$I_n = n!$$

for all positive integers $n$. Moreover, the above integral allows us to extend this "factorial" function to the complex plane. Note that

$$|t^z| = |e^{z \log t}| = e^{(\mathrm{Re}\, z) \log t} = t^{\mathrm{Re}\, z} \quad \text{for } t \geq 0$$

so that if we replace $n$ by the complex variable $z$, the resulting function $f(z) = \int_0^\infty e^{-t} t^z dt$ is uniformly convergent for $\mathrm{Re}\, z > -1$. A translate of this function,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1}$$

is the classical Gamma Function. Thus $\Gamma$ is analytic in the right half-plane $\mathrm{Re}\, z > 0$ and $\Gamma(n) = (n-1)!$ for all positive integers $n$.

It is clear that $\Gamma$ has a singularity at $z = 0$ since

$$\Gamma(\epsilon) = \int_0^\infty \frac{e^{-t}}{t^{1-\epsilon}} dt \to \infty \quad \text{as } \epsilon \to 0^+.$$

On the other hand, although (1) defines $\Gamma$ only in the right half-plane, the function can be extended to the whole plane with the exception of isolated poles. We may carry out this extension in several ways.

I. Integration by parts shows that

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for Re z} > 0,$$

or equivalently,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \quad \text{for } \mathrm{Re}\, z > 0. \tag{2}$$

Identity (2) allows us to define an extension of $\Gamma$ to the half-plane $\mathrm{Re}\, z > -1, z \neq 0$. This extension is analytic for $-1 < \mathrm{Re}\, z < 0$ and is continuous along the nonzero $y$-axis since the "original" $\Gamma$ is continuous on the line $\mathrm{Re}\, z = 1$. That is,

$$\lim_{z \to iy} \Gamma(z) = \lim_{z \to iy} \frac{\Gamma(z+1)}{z} = \frac{\Gamma(iy+1)}{iy} = \Gamma(iy), \quad y \neq 0.$$

Hence by Morera's Theorem the extended function is analytic throughout $\text{Re}\, z > -1$, $z \neq 0$. Identity (2) also reveals the nature of the singularity at $z = 0$, since as $z \to 0$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \sim \frac{\Gamma(1)}{z} = \frac{1}{z}.$$

Hence $\Gamma$ has a simple pole with residue 1 at $z = 0$.

Continuing in the same manner, we can define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} \quad \text{for Re}\, z > -2,$$
$$\Gamma(z) = \frac{\Gamma(z+3)}{z(z+1)(z+2)} \quad \text{for Re}\, z > -3, \ldots,$$
$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k)} \quad \text{for Re}\, z > -k-1. \tag{3}$$

Note then that the only singularities are the isolated (simple) poles at the non-positive integers, and as $z \to -k$

$$\Gamma(z) \sim \frac{\Gamma(1)}{(-k)(-k+1)\cdots(-1)(z+k)} = \frac{(-1)^k}{k!(z+k)}.$$

Hence

$$\text{Res}(\Gamma(z); -k) = \frac{(-1)^k}{k!}.$$

II. Set $\Gamma(z) = \Gamma_1(z) + \Gamma_2(z)$, where

$$\Gamma_1(z) = \int_0^1 e^{-t} t^{z-1} dt$$
$$\Gamma_2(z) = \int_1^\infty e^{-t} t^{z-1} dt, \quad \text{Re}\, z > 0.$$

Since $|t^{z-1}| = t^{\text{Re}\, z-1}$, $\Gamma_2$ is uniformly convergent in any compact subset and represents an entire function. Thus, to extend $\Gamma$, we need only to extend $\Gamma_1$. But for $\text{Re}\, z > 0$

$$\Gamma_1(z) = \int_0^1 \left(1 - t + \frac{t^2}{2!} - + \cdots \right) t^{z-1} dt$$
$$= \int_0^1 t^{z-1} dt - \int_0^1 t^z dt + \int_0^1 \frac{t^{z+1}}{2!} - + \cdots$$
$$= \frac{1}{z} - \frac{1}{(z+1)} + \frac{1}{2!(z+2)} - + \cdots .$$

The above series defines an analytic extension of $\Gamma_1$ to the whole plane except for isolated poles at $0, -1, -2, \ldots$. Note again that

$$\text{Res}(\Gamma;\ -k) = \text{Res}(\Gamma_1;\ -k) = \frac{(-1)^k}{k!}.$$

III. Using the fact that $(1 - t/n)^n$ converges to $e^{-t}$ as $n \to \infty$, one can show that

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty} \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n dt \\ &= \lim_{n\to\infty} \frac{1}{n^n} \int_0^n t^{z-1}(n-t)^n dt, \quad \operatorname{Re} z > 0.\end{aligned}$$

(See Exercise 7.)

Integrating by parts, we have

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty} \frac{1}{n^n} \cdot \frac{n}{z} \int_0^n t^z (n-t)^{n-1} dt \\ &= \lim_{n\to\infty} \frac{1}{n^n} \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} \int_0^n t^{z+n-1} dt \\ &= \lim_{n\to\infty} \frac{n^z}{z} \left(\frac{1}{z+1}\right)\left(\frac{2}{z+2}\right)\cdots\left(\frac{n}{z+n}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \lim_{n\to\infty} z n^{-z} (1+z)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right) \\ &= \lim_{n\to\infty} z n^{-z} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right).\end{aligned}$$

To examine the above limit, we insert "convergence factors" $e^{-z/k}$ and obtain

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \lim_{n\to\infty} z n^{-z} e^{z(1+1/2+\cdots+1/n)} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) e^{-z/k} \\ &= \lim_{n\to\infty} e^{z(1+1/2+\cdots+1/n-\log n)} \left[z \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) e^{-z/k}\right].\end{aligned}$$

By the lemma below, $1 + \frac{1}{2} + \cdots + 1/n - \log n$ approaches a positive limit $\gamma$ (known as the Euler constant) so that

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-z/k}.$$

Using the above identity to define an extension of $\Gamma$ to the left half-plane, we obtain

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) = -z \frac{\sin \pi z}{\pi}.$$

Thus

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z \sin \pi z},$$

and since $\Gamma(1-z) = -z\Gamma(-z)$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{4}$$

Two immediate consequences of identity (4) are

i. $\Gamma$ is zero-free,
ii. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Applying the identity $\Gamma(z+1) = z\Gamma(z)$, we have also $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(5/2) = 3\sqrt{\pi}/4$, etc.

## 18.8 Lemma

*If* $s_n = 1 + \frac{1}{2} + \cdots + 1/n - \log n$, *then* $\lim_{n\to\infty} s_n$ *exists. This limit is called the Euler constant,* $\gamma$.

**Proof**

$t_n = 1 + \frac{1}{2} + \cdots + 1/(n-1) - \log n$ increases with $n$. Geometrically this is obvious since $t_n$ represents the area of the $n-1$ regions between the upper Riemann sum and the exact value for $\int_1^n (1/x)\,dx$. We can write

$$t_n = \sum_{k=1}^{n-1}\left[\frac{1}{k} - \log\left(\frac{k+1}{k}\right)\right]$$

and

$$\lim_{n\to\infty} t_n = \sum_{k=1}^{\infty}\left[\frac{1}{k} - \log\left(1+\frac{1}{k}\right)\right].$$

The series above converges to a positive constant since

$$0 < \frac{1}{k} - \log\left(1+\frac{1}{k}\right) = \frac{1}{2k^2} - \frac{1}{3k^3} + \frac{1}{4k^4} - + \cdots \le \frac{1}{2k^2}.$$

This proves the lemma, because $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$. □

*The Zeta Function.* Recall that the Zeta Function $\zeta(z)$ is defined by the Dirichlet series

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots, \quad \text{Re}\, z > 1.$$

This function is of special interest in number theory because it provides a link between the prime numbers and analytic function theory. To see this connection, note that

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \cdots$$

so that

$$\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \cdots.$$

Similarly,

$$\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\zeta(z) = 1+\frac{1}{5^z}+\frac{1}{7^z}+\frac{1}{11^z}+\cdots,$$

and because of the unique prime factorization of the integers, we can continue indefinitely to obtain (in the limit)

$$\prod_{p\text{ prime}}\left(1-\frac{1}{p^z}\right)\zeta(z) = 1.$$

That is

$$\zeta(z) = 1 \Bigg/ \prod_{p\text{ prime}}\left(1-\frac{1}{p^z}\right), \quad \operatorname{Re} z > 1. \tag{5}$$

To best exploit identity (5), we need to extend $\zeta$ beyond the domain $\operatorname{Re} z > 1$. Note that $\zeta$ does have a singularity at $z = 1$ since $\zeta(1+\epsilon) \to \infty$ as $\epsilon \to 0^+$. We shall see below that this is the only singularity of the $\zeta$ function.

We extend $\zeta$ by the method of moments. Note that

$$\int_0^\infty e^{-nt}t^{z-1}dt = \frac{1}{n^z}\int_0^\infty e^{-t}t^{z-1}dt = \frac{\Gamma(z)}{n^z}$$

so that

$$\begin{aligned}\Gamma(z)\sum_{n=1}^\infty \frac{1}{n^z} &= \int_0^\infty t^{z-1}\left(\sum_{n=1}^\infty e^{-nt}\right)dt\\ &= \int_0^\infty \frac{t^{z-1}}{e^t-1}dt.\end{aligned}$$

That is,

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}dt$$

or

$$\zeta(z) = \frac{1}{\Gamma(z)}\left[\int_0^1 \frac{t^{z-1}}{e^t-1}dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}dt\right]. \tag{6}$$

Recall that $1/\Gamma(z)$ (with the appropriate limiting value of zero at the poles of $\Gamma$) is entire, as is $\int_1^\infty (t^{z-1}/(e^t-1))dt$. Furthermore, the Laurent Expansion for $1/(e^t-1)$ around $t = 0$,

$$\frac{1}{e^t-1} = \frac{1}{t} + A_0 + A_1 t + A_2 t^2 + \cdots,$$

converges absolutely for $t = 1$ so that

$$\int_0^1 \frac{t^{z-1}}{e^t - 1} dt = \int_0^1 (t^{z-2} + A_0 t^{z-1} + A_1 t^z + \cdots) dt$$
$$= \frac{1}{z-1} + \frac{A_0}{z} + \frac{A_1}{z+1} + \cdots \tag{7}$$

provides an analytic extension of $\int_0^1 (t^{z-1}/(e^t - 1))\, dt$ except for isolated poles. According to (6), then

$$\zeta(z) = \frac{1}{\Gamma(z)} \left[ \left( \frac{1}{z-1} + \frac{A_0}{z} + \frac{A_1}{z+1} \cdots \right) + g(z) \right] \tag{8}$$

where $g(z)$ is entire. Note that while the bracketed expression above has a simple pole at $z = 1$ as well as at every non-positive integer, all these poles are cancelled by the zeros of $1/\Gamma(z)$ except $z = 1$. Hence $\zeta$ has a single (simple) pole at $z = 1$ with residue 1.
For future reference, then, we record

### 18.9 Theorem

*The only singularity of the Zeta function $\zeta(z)$ is a simple pole with residue 1 at $z = 1$.*

According to (5), $\zeta$ is zero-free for $\operatorname{Re} z > 1$. The celebrated Riemann hypothesis asserts that all the complex zeroes of the Zeta function lie on the line $\operatorname{Re} z = \frac{1}{2}$. While this hypothesis has neither been proved nor disproved, the following theorem offers an important extension of the zero-free region of $\zeta$.

### 18.10 Theorem

*$\zeta$ is zero-free throughout $\operatorname{Re} z \geq 1$.*

**Proof**

A key element in the proof is the observation that if $\zeta(1 + ia) = 0$, then the function $f(z) = \zeta(z)\zeta(z + ia)$ is entire. At $z = 1$, the pole of $\zeta(z)$ is cancelled by the zero of $\zeta(z + ia)$. Also, since $\zeta(z)$ is real-valued for real $z$, according to the Schwarz Reflection Principle, $\zeta(\overline{z}) = \overline{\zeta(z)}$. Hence $\zeta(1 - ia) = 0$, and the pole of $\zeta(z + ia)$ at $z = 1 - ia$ is cancelled by the zero of $\zeta(z)$ at that point. Note that $f(z - ia) = \zeta(z - ia)\zeta(z)$ will also be entire as will the product $g(z) = f(z) f(z - ia) = \zeta^2(z)\zeta(z + ia)\zeta(z - ia)$. The desired contradiction will be based, in part, on the fact that the Dirichlet series for $g(z)$ has all nonnegative coefficients. To see that, we first consider $\log(g(z))$ which, according to Euler's formula for $\zeta(z)$, is given by

$$\log(g(z)) = \sum_p [-2\log(1 - p^{-z}) - \log(1 - p^{-z+ia}) - \log(1 - p^{-z-ia})]$$
$$= \sum_{p,n} \frac{1}{np^{nz}} (2 + p^{-ina} + p^{ina})$$

The sum is taken over all primes $p$ and all positive integers $n$. Since $2 + p^{-ina} + p^{ina} = 2 + 2\cos(na \log p) \geq 0$, all of the coefficients in the above Dirichlet series for $\log(g(z))$ are nonnegative. But if a Dirichlet series $S(z) = \sum a_n n^{-z}$ has all nonnegative coefficients, so does

$$e^{S(z)} = \prod_n \sum_k \frac{a_n^k}{n^{kz} k!}$$

Hence $g(z)$ represents an entire function whose Dirichlet series has all nonnegative coefficients. According to Corollary 17.12, then, its Dirichlet series must converge for all $z$! But this is clearly impossible. Since all of the coefficients of

$$g(z) = \left(\sum_n n^{-z}\right)^2 \sum_n n^{-z-ia} \sum_n n^{-z+ia}$$

are nonnegative, the sum is clearly positive for all real $z$. Moreover, the sum must be larger than the sum over any subset of the positive integers. So if we consider nonnegative real values of $z$ and limit ourselves to the subseries corresponding to integers $n$ of the form $2^k$, we have

$$|g(z)| > \frac{1}{(1-2^{-z})^2} \cdot \frac{1}{1-2^{-z-ia}} \cdot \frac{1}{1-2^{-z+ia}}$$

Finally, since $z$ is nonnegative, $|(1-2^{-z-ia})(1-2^{-z+ia})| \leq 4$, and

$$|g(z)| > \frac{1}{4(1-2^{-z})^2}$$

Letting $z \to 0$ through positive real values, then, shows that the Dirichlet series for $g$ diverges at 0. □

## Exercises

1. Let $f(z) = \log z$, $\text{Re}\, z > 0$, $\text{Im}\ z > 0$. Let $g_1$ be the continuation of $f$ to the plane minus the negative axis (and 0) and let $g_2$ be the continuation of $f$ to the plane minus the negative imaginary axis (and 0). Show that $g_1 \neq g_2$ throughout the third quadrant.

2.* a. Suppose $f(z) = \sum a_n z^n$ has radius of convergence 1 and assume that an analytic continuation of $f$ has a pole at $z = 1$. Show that $\sum a_n z^n$ diverges at every point on the unit circle. (Hint: Show that if $\{a_n\} \to 0$, then $(1-z)f(z) \to 0$ as $z \to 1$ from below, along the $x$−axis.)
   b. Generalize the result; i.e. show that if $f(z) = \sum a_n z^n$ has a positive radius of convergence and an analytic continuation of $\sum a_n z^n$ has a pole at any point on its circle of convergence, then $\sum a_n z^n$ diverges at *all* points on the circle of convergence.

3. Prove: If $\sum_{n=0}^{\infty} (-1)^n a_n z^n$, $a_n \geq 0$ has a finite radius of convergence, then it has a singularity on the negative axis.

4. Define an analytic continuation of

$$\text{a.} \quad \sum_{n=1}^{\infty} \frac{z^n}{\sqrt[3]{n}}, \qquad \text{b.} \quad \sum_{n=0}^{\infty} \frac{z^n}{n^2+1}.$$

5. Show that

$$\int_0^\infty e^{-nt} t^{p-1} dt = \frac{\Gamma(p)}{n^p} \quad \text{for } p > 0.$$

6. Use the Gamma Function to show

$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$

7. Prove

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n t^{z-1} \left(1 - \frac{t}{n}\right)^n dt, \quad \operatorname{Re} z > 0.$$

[*Hint*: First show, for $t \le n$,

$$0 \le e^{-t/n} - \left(1 - \frac{t}{n}\right) \le \frac{t^2}{2n^2}$$

and then use the identity

$$a^n - b^n \le n a^{n-1}(a-b) \quad \text{for } a > b$$

to show

$$\left| e^{-t} - \left(1 - \frac{t}{n}\right)^n \right| \le e^{-t} \left(\frac{e t^2}{2n}\right). ]$$

8.* Use the product formula for $1/\Gamma(z)$ to prove that $\Gamma'(1) = -\gamma$.

9. Show that

$$1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + - \cdots$$

can be continued analytically to the full plane. That is, show that it represents an entire function.

10. Use identity (5) to prove $\sum_{p \text{ prime}} (1/p)$ diverges.