Chapter 12
Further Contour Integral Techniques

12.1 Shifting the Contour of Integration

We have already seen how the Residue Theorem can be used to evaluate real line integrals. The techniques involved, however, are in no way limited to real integrals. To evaluate an integral along any contour, we can always switch to a more "convenient" contour as long as we account for the pertinent residues of the integrand.

EXAMPLE 1
Consider

$$\int_I \frac{e^z dz}{(z+2)^3}$$

where I is the line $z(t) = 1 + it$, $-\infty < t < \infty$.

Let C_R be the left semicircle of radius $R > 3$ centered at $z = 1$. Then

$$\int_{1-iR}^{1+iR} \frac{e^z dz}{(z+2)^3} + \int_{C_R} \frac{e^z dz}{(z+2)^3} = 2\pi i \operatorname{Res}\left(\frac{e^z}{(z+2)^3}; -2\right).$$

Since e^z is bounded by e in the lefl half-plane $x \le 1$, as $R \to \infty$

$$\int_{C_R} \frac{e^z dz}{(z+2)^3} \to 0$$

and

$$\int_I \frac{e^z dz}{(z+2)^3} = 2\pi i \operatorname{Res}\left(\frac{e^z}{(z+2)^3}; -2\right).$$

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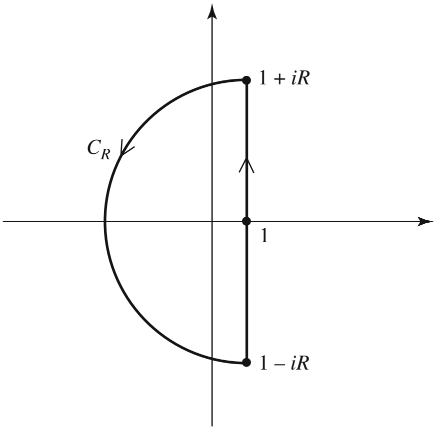

To evaluate the residue, we write

$$e^z = e^{-2}e^{z+2} = e^{-2}\left(1 + (z+2) + \frac{(z+2)^2}{2} + \cdots\right)$$

so that

$$\operatorname{Res}\left(\frac{e^z}{(z+2)^3}; -2\right) = \frac{1}{2e^2}$$

and

$$\int_I \frac{e^z dz}{(z+2)^3} = \frac{\pi i}{e^2}.$$

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EXAMPLE 2
Evaluate

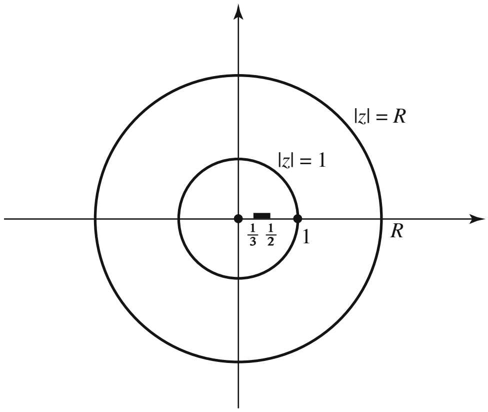
$$\int_{|z|=1} \frac{dz}{\sqrt{6z^2 - 5z + 1}}$$

where the square root is the positive $\sqrt{2}$ at the point $z = 1$.

Recall (see Exercise 10–16) that since $6z^2 - 5z + 1$ has its zeroes at $z = \frac{1}{2}$ and $z = \frac{1}{3}$, $\sqrt{6z^2 - 5z + 1}$ is analytic in the plane minus the interval $\frac{1}{3} \le z \le \frac{1}{2}$.

To evaluate the integral, we switch to the contour $|z| = R$. Then, since $\sqrt{6z^2 - 5z + 1} \sim \sqrt{6}\, z$ for large z, it follows that

$$\int_{|z|=R} \frac{dz}{\sqrt{6z^2 - 5z + 1}} \to \frac{1}{\sqrt{6}} \int_{|z|=R} \frac{dz}{z} = \frac{2\pi i}{\sqrt{6}}.$$



To formally justify the last step, suppose (in general) that $f(z) = z + \epsilon(z)$, where $\epsilon(z)/z \to 0$ as $z \to \infty$. Then

$$\int_{|z|=R} \frac{1}{f(z)} dz - \int_{|z|=R} \frac{dz}{z} = -\int_{|z|=R} \frac{\epsilon(z)}{z(z+\epsilon(z))} dz$$
$$\ll 2\pi \max_{|z|=R} \left| \frac{\epsilon(z)}{z+\epsilon(z)} \right| \to 0 \text{ as } R \to \infty. \qquad \lozenge$$

EXAMPLE 3
Based on numerical evidence, it was conjectured that

$$\sum_{k=0}^{n} (-1)^k \sqrt{\binom{n}{k}} \to 0 \text{ as } n \to \infty.$$

A proof of the conjecture can be given as follows:
Note that

$$f(z) = \frac{\sin \pi z}{\pi z(1-z)(1-z/2)\dots(1-z/n)}$$

satisfies

$$f(k) = \binom{n}{k} \qquad \text{for any nonnegative integer } k.$$

Because $f(z)$ is zero-free in $-1 < \operatorname{Re} z < n+1$, $\sqrt{f(z)}$ (taken as positive at the origin) is analytic there. By the Residue Theorem, then

$$\sum_{k=0}^{n} (-1)^k \sqrt{\binom{n}{k}} = \frac{1}{2\pi i} \int_C \sqrt{f(z)} \frac{\pi}{\sin \pi z} dz \tag{1}$$

where C is any contour in $-1 < \operatorname{Re} z < n+1$ which winds *once* about each integer $0, 1, \dots, n$ and *never* about any other integer.

Suppose we let $C = C_M$ be the rectangle formed by the lines $\operatorname{Re} z = -1/2$, $\operatorname{Re} z = n + 1/2$ and $\operatorname{Im} z = \pm M$. Then

$$\int_{C_M} \sqrt{f(z)} \frac{\pi}{\sin \pi z} dz = \int_{C_M} \frac{\sqrt{\pi}\, dz}{\sqrt{z(1-z)(1-z/2)\dots(1-z/n)\sin \pi z}}$$

and letting $M \to \infty$, we conclude

$$\sum_{k=0}^{n} (-1)^k \sqrt{\binom{n}{k}} = \frac{1}{2\pi i}\left[\int_{-1/2+i\infty}^{-1/2-i\infty} + \int_{n+1/2-i\infty}^{n+1/2+i\infty} \sqrt{f(z)} \frac{\pi}{\sin \pi z} dz\right].$$

Since the integrand is invariant (aside from a $\pm$ sign) under the substitution $z \to n-z$, we need only estimate the first integral. Now, when $\operatorname{Re} z = -\frac{1}{2}$,

$$\begin{aligned}
|z(1-z)(1-z/2)\dots(1-z/n)| &\geq \frac{1}{2}\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\dots\left(1+\frac{1}{2n}\right) \\
&\geq \frac{1}{2}\sqrt{1+1}\sqrt{1+\frac{1}{2}}\dots\sqrt{1+\frac{1}{n}} \\
&= \frac{\sqrt{n+1}}{2}
\end{aligned}$$

and so the first integral is bounded by

$$\frac{1}{\sqrt{2\pi}\,\sqrt[4]{n+1}} \int_{\operatorname{Re} z=-1/2} \left|\frac{dz}{\sqrt{\sin \pi z}}\right| \leq \frac{A}{\sqrt[4]{n}}.$$

Hence

$$\sum_{k=0}^{n} (-1)^k \sqrt{\binom{n}{k}} \to 0 \text{ as } n \to \infty.$$

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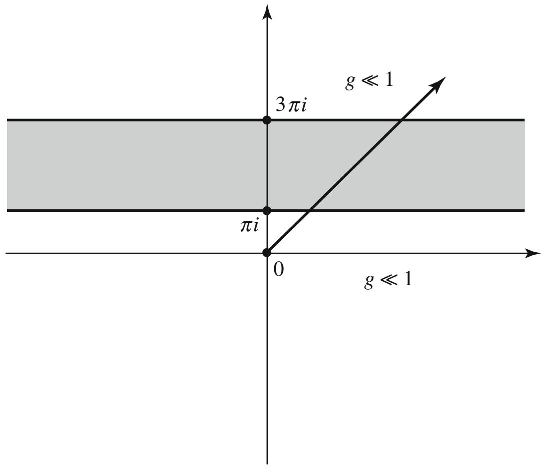
12.2 An Entire Function Bounded in Every Direction

Recall that, according to Liouville's Theorem, every nonconstant entire function is unbounded. Nevertheless, one may wonder whether there is a nonconstant entire function which is bounded along every ray from the origin. The answer to this question is yes! However, there seems to be no way of describing such a function in closed form. Instead, the function will be given in integral form and the crucial estimate will then be obtained by switching the contour of integration.

The strategy is as follows: We will find a nonconstant entire function f which is bounded by 1 outside of the strip $|\operatorname{Im} z| \leq \pi$. If we consider

$$g(z) = f(z - 2\pi i),$$



then it follows that $g \ll 1$ outside the strip $\pi \le \operatorname{Im} z \le 3\pi$ and hence g will be bounded on every ray. As a final touch, we might then consider

$$h(z) = \frac{g(z) - g(0)}{z}$$

which is an entire function that approaches zero along every ray!
Construction of f: Define

$$f(z) = \int_0^\infty \frac{e^{zt}}{t^t} dt.$$

The integral converges absolutely since for any $z = x + iy$,

$$\int_0^\infty \left| \frac{e^{zt}}{t^t} \right| dt = \int_0^\infty \frac{e^{xt}}{t^t} dt < \infty.$$

Furthermore, f is continuous and for any rectangle R

$$\int_{\partial R} f(z) dz = \int_{\partial R} \left(\int_0^\infty \frac{e^{zt}}{t^t} dt \right) dz = \int_0^\infty \left(\int_{\partial R} \frac{e^{zt}}{t^t} dz \right) dt = \int_0^\infty 0 \, dt = 0.$$

The absolute convergence of the integral justifies the change in the order of integration. Hence, by Morera's Theorem, f is entire.

We see from our definition of f that f is real-valued along the real axis. Thus, by the Schwarz Reflection Principle, $f(\bar{z}) = \overline{f(z)}$ and we need only show that f is bounded for $z = x + iy$, $y > \pi$. In fact, we will show that for $z = x + iy$, $y = \pi/2 + c$, $|f(z)| \le 1/c$.

To derive the stated upper bound for

$$f(z) = \int_0^\infty \frac{e^{zt}}{t^t} dt,$$

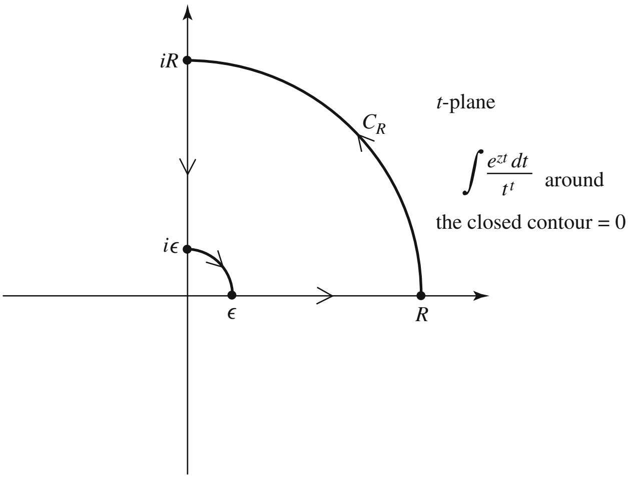
note that the integrand is an analytic function of t in the right half-plane and hence we can replace the integral

$$\int_{\epsilon}^{R} \frac{e^{zt}}{t^t} dt$$

along the positive axis by the integral along the quarter-circle from ϵ to $i\epsilon$ plus the integral along the imaginary axis from $i\epsilon$ to iR minus the integral along the quarter-circle C_R of radius R. (See below.) Since the integrand approaches 1 at $t = 0$, the integral along the quarter-circle of radius ϵ is negligible. As $\epsilon \to 0$ and $R \to \infty$,

$$f(z) = \int_0^{\infty} \frac{e^{zt}}{t^t} dt = -\lim_{R \to \infty} \int_{C_R} \frac{e^{zt}}{t^t} dt + \int_I \frac{e^{zt}}{t^t} dt$$

where I is the positive imaginary axis. (See the diagram below.)



Finally we will show that the latter integral is bounded by $1/c$ and that the limit on the right is 0.

Using the obvious parametrization $t = iv,\ 0 \le v < \infty$

$$\int_I \frac{e^{zt}}{t^t} dt = i \int_0^{\infty} \frac{e^{ivz}}{(iv)^{iv}} dv \ll \int_0^{\infty} \left| \frac{e^{ivz}}{(iv)^{iv}} \right| dv$$

but for $\operatorname{Im} z = \pi/2 + c$,

$$\left| \frac{e^{ivz}}{(iv)^{iv}} \right| = \frac{e^{-v(\pi/2+c)}}{|e^{iv \log iv}|} = \frac{e^{-v/(\pi/2+c)}}{e^{-v\pi/2}} = e^{-cv}.$$

Hence

$$\int_I \frac{e^{zt}}{t^t} dt \ll \int_0^{\infty} e^{-cv} dv = \frac{1}{c}.$$

To estimate

$$\int_{C_R} \frac{e^{zt}}{t^t} dt, \quad \text{let } t = Re^{i\theta}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

Then $\log t = \log R + i\theta$ and

$$\left|\frac{e^{zt}}{t^t}\right| = \left|\frac{\exp[(x+iy)(R\cos\theta + iR\sin\theta)]}{\exp[(\log R + i\theta)(R\cos\theta + iR\sin\theta)]}\right|$$
$$= \exp -[(\log R - x)R\cos\theta + (y-\theta)R\sin\theta].$$

Taking R large enough so that $\log R - x > y > y - \theta$,

$$\left|\frac{e^{zt}}{t^t}\right| \le \exp -[(y-\theta)R] \le e^{-cR}$$

and

$$\int_{C_R} \frac{e^{zt}}{t^t} dt \ll \frac{\pi}{2} R \cdot e^{-cR},$$

which approaches 0 as $R \to \infty$.

We note that for every nonconstant entire function, there is always *some* polygonal line along which the function is not only unbounded but actually approaches infinity. We prove this result in Chapter 15. (See Exercise 6.)

Exercises

1.* a. Evaluate

$$\int_I \frac{e^z}{(z+1)^4} dz,$$

where I is the imaginary axis (from $-i\infty$ to $+i\infty$).

b. Evaluate

$$\int_{1-i\infty}^{1+i\infty} \frac{a^z}{z^2} dz, \quad 0 < a < \infty.$$

2.* Evaluate

$$\int_{|z|=2} \frac{dz}{\sqrt{4z^2 - 8z + 3}}.$$

3. Evaluate $\int_\gamma e^z \log z dz$ where $\log z$ is that branch for which $\log 1 = 0$ and γ is the parabola: $\gamma(t) = 1 - t^2 + it, \ -\infty < t < \infty$.

4. Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^{1/3} \to 0 \qquad \text{as } n \to \infty.$$

5. a. Obtain an improved estimate for

$$\sum_{k=0}^{n}(-1)^k\sqrt{\binom{n}{k}}$$

by integrating along the lines $\operatorname{Re} z = -3/4$ and $\operatorname{Re} z = n + 3/4$.

b. Estimate

$$\sum_{k=0}^{n}(-1)^k\sqrt{\binom{n}{k}}$$

by integrating along $\operatorname{Re} z = -1 + \delta$ and $\operatorname{Re} z = n + 1 - \delta$. Find an optimal δ. [*Note*: Numerical evidence suggests

$$\sum_{k=0}^{n}(-1)^k\sqrt{\binom{n}{k}} \sim \frac{1}{\sqrt{n}} \text{ for even } n].$$

6. * Suppose g is the entire function (bounded on every ray) described in the last section. Show that $g(x + 2\pi i) \to \infty$ as $x \to \infty$.