Chapter 16
Harmonic Functions

16.1 Poisson Formulae and the Dirichlet Problem

In this chapter, we focus on the real parts of analytic functions and their connection with real harmonic functions.

16.1 Definition

A real-valued function $u(x, y)$ which is twice continuously differentiable and satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

throughout a domain D is said to be *harmonic* in D.

Although one may talk of complex-valued harmonic functions, the term "harmonic" throughout this chapter will always refer to a real-valued function.

16.2 Theorem

If $f = u + iv$ is analytic in D, u and v are harmonic there.

Proof

u and v both have continuous partial derivatives of all orders since f is analytic. By the Cauchy-Riemann equations

$$u_x = v_y; \qquad u_y = -v_x$$

so that

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy},$$

hence u is harmonic. By the same argument, v is harmonic since it is the real part of the analytic function $-if$. □

The converse of the above is not true. For example,

$$u(x, y) = \log(x^2 + y^2)$$

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is harmonic in the punctured plane but is not the real part of an analytic function there. (See Exercise 4.) We do have the following partial converse:

16.3 Theorem

If u is harmonic in D, then

a. *u_x is the real part of an analytic function in D;*
b. *if D is simply connected, u is the real part of an analytic function in D.*

Proof

a. Let $f = u_x - iu_y$. Since $u \in C^2$, f has continuous first-order partial derivatives. Moreover, by the harmonicity of u

$$f_y = u_{xy} - iu_{yy} = u_{yx} + iu_{xx} = if_x$$

so that f satisfies the Cauchy-Riemann equations. Hence f is analytic in D.
b. If D is simply connected, by the Integral Theorem (8.5), $f = u_x - iu_y$ is the derivative of an analytic function F. But then if $F = A + iB$

$$F'(z) = A_x + iB_x = A_x - iA_y = u_x - iu_y$$

so that

$$A(x, y) = u(x, y) + C.$$

Hence $u(x, y)$ is the real part of the analytic function $F(z) - C$. □

EXAMPLE

$u(x, y) = x - e^x \sin y$ is harmonic in the whole plane. Hence $f(z) = u_x(z) - iu_y(z) = 1 - e^x \sin y + ie^x \cos y$ is entire. In fact, $f(z) = 1 + ie^z$ and if we set

$$F(z) = \int_0^z f(\zeta)d\zeta = z + ie^z - i,$$

then

$$u(z) = \text{Re}F(z).$$

◊

The fact that a harmonic function is, at least locally, the real part of an analytic function allows us to apply some of the theory of analytic functions to harmonic functions.

16.4 Mean-Value Theorem for Harmonic Functions

If u is harmonic in $D(z_0; R)$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

for all positive $r < R$.

Proof

Let $u = \operatorname{Re} f$. By 6.12

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta,$$

and the result follows by taking the real parts of the above. □

16.5 Maximum-Modulus Theorem for Harmonic Functions

If u is a nonconstant harmonic function in a region D, u has no maximum or minimum points in D.

Proof

The theorem may be derived as a corollary of the above Mean-Value Theorem. It follows even more immediately, however, from the Open Mapping Theorem (7.1). For in a disc $D(z_0; \delta) \subset D$ about any point $z_0 \in D$, u is the real part of an analytic function f. Since f maps $D(z_0; \delta)$ onto an open set, u takes both larger and smaller values than $u(z_0)$ in the open disc. □

Note that the Maximum-Modulus Theorem for analytic functions (6.13) asserts only that $|f|$ has no interior maximum point; $|f|$ can have a local minimum if it is equal to zero. By contrast, Theorem 16.5 shows that a non-constant harmonic function has neither a maximum nor a minimum point in the interior of a domain.

We let the term C-*harmonic* refer to a function which is harmonic in the interior of a domain and continuous on the closure. The previous theorem implies then that a C-harmonic function in a compact domain must assume its maximum and minimum values on the boundary of that domain.

16.6 Corollary

If two C-harmonic functions u_1 and u_2 agree on the boundary of a compact domain D, then $u_1 = u_2$ throughout D.

Proof

$u = u_1 - u_2$ is C-harmonic in D; hence it takes its maximum and minimum on the boundary. Since $u \equiv 0$ on the boundary, it follows that $u \equiv 0$ throughout D and that $u_1 \equiv u_2$. □

Corollary 16.6 shows that a C-harmonic function is determined by its values on the boundary of a compact domain. But this result is of a purely theoretical nature. How to determine the value at an interior point from a knowledge of u on the boundary is the subject of the next theorem. We begin by considering C-harmonic functions in the unit disc.

16.7 Theorem

Suppose u is C-harmonic in $D(0;1)$. Then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\mathcal{K}(\theta,\ z)d\theta$$

where $\mathcal{K}(\theta, z)$ is the "Poisson Kernel,"

$$\mathcal{K}(\theta, z) = \mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right].$$

In polar form,

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(e^{i\theta})(1-r^2)}{1-2r\cos(\theta-\varphi)+r^2}d\theta.$$

Proof

[To simplify the notation, we will assume $u = \mathrm{Re} f$ where f is analytic on the *closed* unit disc. To justify the assumption, we could first prove the theorem for $u^*(z) = u(rz)$ where $r < 1$ and then take the limit as $r \to 1$ since u is uniformly continuous on $\overline{D(0;1)}$.]

By the Cauchy Integral Formula (6.4)

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}d\zeta$$

or

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left[\frac{e^{i\theta}}{e^{i\theta}-z}\right]d\theta \tag{1}$$

If we replace z by the symmetric point $1/\bar{z}$ which lies outside the unit disc, then by the Closed Curve Theorem (8.6)

$$0 = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - \frac{1}{\bar{z}}}d\zeta$$

or

$$0 = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left[\frac{e^{i\theta}}{e^{i\theta}-1/\bar{z}}\right]d\theta. \tag{2}$$

Note that

$$\begin{aligned}\frac{e^{i\theta}}{e^{i\theta}-1/\bar{z}} &= \frac{\bar{z}e^{i\theta}}{\bar{z}e^{i\theta}-1} = \frac{-\bar{z}}{e^{-i\theta}-\bar{z}}\\ &= 1 - \frac{e^{-i\theta}}{e^{-i\theta}-\bar{z}} = \overline{\left[1-\frac{e^{i\theta}}{e^{i\theta}-z}\right]},\end{aligned}$$

so that subtracting (2) from (1) yields

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left[\frac{e^{i\theta}}{e^{i\theta}-z} + \overline{\left(\frac{e^{i\theta}}{e^{i\theta}-z}\right)} - 1\right]d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left[2\,\mathrm{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-z}\right) - 1\right]d\theta,$$

or

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]d\theta. \tag{3}$$

Finally, taking the real parts of the above, we obtain

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]d\theta. \tag{4}$$

□

By mapping the unit disc onto other domains, we can obtain similar results for any simply connected domain. For example, if u is harmonic in $D(0; R)$, $u = \mathrm{Re} f$, we can apply the above results to $g(\zeta) = f(R\zeta)$. Thus

$$f(R\zeta) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\mathrm{Re}\left[\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\right]d\theta,$$

and if we let $R\zeta = z$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\mathrm{Re}\left[\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right]d\theta, \tag{5}$$

and

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\mathrm{Re}\left[\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right]d\theta. \tag{6}$$

The above is known as the Poisson Integral Formula for a disc. The Poisson Formula for a bounded harmonic function in a half-plane

$$u(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yu(t)}{(t-x)^2+y^2}dt \tag{7}$$

is derived in Exercise 6.

The Dirichlet Problem The Dirichlet Problem is the problem of proving the existence of a function u which is C-harmonic in a domain and assumes prescribed boundary values. This differs from the attitude in the last section where a function u was assumed to be C-harmonic in a domain and we sought a formula for u in terms of its boundary values. Nevertheless, the previous theorems offer a starting point. Suppose, for example, that D is the unit disc. Then if there is a harmonic function u

in D with limit values $u(e^{i\theta})$ on the boundary, u must be of the form

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\text{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]d\theta,$$

or

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(e^{i\theta})(1-r^2)}{1-2r\cos(\theta-\varphi)+r^2}d\theta.$$

The fact that this Poisson Integral does indeed provide the solution to the Dirichlet Problem is proven below.

16.8 Theorem

Suppose $u(e^{i\theta})$ is continuous on $C(0;1)$. Then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\mathcal{K}(\theta,\ z)d\theta$$

is the restriction to $D(0;1)$ of a C-harmonic function in the closed unit disc with boundary values $u(e^{i\theta})$.

Proof

Let

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]d\theta, \quad |z|<1.$$

Since $(e^{i\theta}+z)/(e^{i\theta}-z)$ is an analytic function of z for each θ and since g is continuous, it follows by Morera's Theorem that g is analytic in $D(0;1)$. Moreover, $u(z) = \text{Re}\, g(z)$ so that u is harmonic. To show that u has the limit $u(e^{i\theta})$ as $z \to e^{i\theta}$, we note the following properties of the Poisson Kernel

$$\mathcal{K}(\theta,z) = \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}, \quad z = re^{i\varphi}. \tag{8}$$

i. $\mathcal{K}(\theta,\ z) > 0$.
   The numerator is obviously positive and the denominator is bigger than $(1-r)^2$.
ii. $(1/2\pi)\int_0^{2\pi}\mathcal{K}(\theta,z)d\theta = 1$.
   This follows on applying the Poisson Formula (16.7) with $u \equiv 1$.
iii. For every $\delta > 0$

$$\left[\int_0^{\varphi-\delta}\mathcal{K}(\theta,z)d\theta + \int_{\varphi+\delta}^{2\pi}\mathcal{K}(\theta,z)d\theta\right] \to 0 \quad \text{as } z \to e^{i\varphi}.$$

Note that the denominator in (8) is bounded away from zero for $|\theta-\varphi| > \delta$ while the numerator approaches 0 as z approaches the boundary.

According to (ii) we can write

$$u(re^{i\varphi}) - u(e^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} [u(e^{i\theta}) - u(e^{i\varphi})]\mathcal{K}(\theta, z)d\theta.$$

Let $M = \text{Max}_\theta |u(e^{i\theta})|$. By the continuity of u, given $\epsilon > 0$ we can find $\delta > 0$ so that $|u(e^{i\theta}) - u(e^{i\varphi})| < \epsilon$ when $|\theta - \varphi| < \delta$. Then by (ii) and (iii)

$$\begin{aligned} |u(re^{i\varphi}) - u(e^{i\varphi})| &\le \frac{M}{\pi}\left[\int_0^{\varphi-\delta} \mathcal{K}(\theta, z)d\theta + \int_{\varphi+\delta}^{2\pi} \mathcal{K}(\theta, z)d\theta\right] \\ &\quad + \frac{1}{2\pi}\int_{\varphi-\delta}^{\varphi+\delta} \epsilon\mathcal{K}(\theta, z)d\theta \\ &\le \frac{M}{\pi}\left[\int_0^{\varphi-\delta} \mathcal{K}(\theta, z)d\theta + \int_{\varphi+\delta}^{2\pi} \mathcal{K}(\theta, z)d\theta\right] + \epsilon, \end{aligned}$$

and as $r \to 1$

$$\overline{\lim}_{r\to 1} |u(re^{i\varphi}) - u(e^{i\varphi})| \le \epsilon.$$

Hence

$$\lim_{r\to 1} u(re^{i\varphi}) = u(e^{i\varphi}). \tag{9}$$

Since u was assumed to be continuous on the unit circle and u is harmonic (and hence continuous) in the disc, it follows from (9) that u is continuous in $\bar{D}$ with the prescribed values on the boundary. □

Remarks

1. According to Corollary 16.6, the above solution to the Dirichlet Problem is unique.
2. The arguments above show that for any integrable function u on the unit circle, there is a harmonic function in $D(0; 1)$ with limit $u(e^{i\varphi})$ at any point of continuity of u along the boundary.
3. By considering the appropriate conformal mapping f of D onto U, we can solve the Dirichlet Problem for any bounded simply connected domain. To find a harmonic function u_1 in D with given boundary values, we first determine a harmonic function u_2 in U with the values $u_1(f^{-1}(z))$ along the boundary. Since u_2 is the real part of an analytic function g,

$$u_1(z) = u_2(f(z)) = \text{Re } g(f(z))$$

is the desired harmonic function in D.
4. In many simple cases, an explicit solution to the Dirichlet Problem can be obtained (without recourse to the Poisson Integral) by determining an analytic function with the appropriate real part.

EXAMPLES

i. To determine the C-harmonic function u in $D(0; 1)$ with boundary values $u(x, y) = x^2$, note that

$$\text{Re } z^2 = x^2 - y^2$$

is everywhere harmonic and equals $2x^2 - 1$ on the boundary. Hence

$$u(x, y) = \frac{1}{2}(x^2 - y^2) + \frac{1}{2}.$$

By taking linear combinations of the above with the harmonic polynomials $1, x, y$ and xy, we can find a C-harmonic function in $D(0; 1)$ with boundary values equal to any given quadratic polynomial on $C(0; 1)$.

ii. $\log r = \text{Re} \log z$ is a harmonic function in the punctured plane $z \neq 0$ which depends only on the modulus. [Although $\log z$ is only analytic in a slit plane, $\text{Re} \log z = \log |z|$ is continuous and hence harmonic in the entire punctured plane.]

Thus if A is an annulus: $r_1 \leq |z| \leq r_2$, we can find a harmonic function in A with arbitrary constant values on the inner and outer circles by setting $u(re^{i\varphi}) = a \log r + b$ for appropriate a and b. ◊

An Application to Heat Problems. Suppose we consider a solid whose temperature u is constant in one direction. (This is a reasonable model for a cylindrical solid with insulated faces or for a "very long" cylindrical solid.) If we assume that the temperature is independent of time, then, thinking of the solid as resting in a region of the z plane, the temperature depends only on the x, y position and can be shown to be a harmonic function. [See Appendix III.] For that reason, Laplace's equation: $u_{xx} + u_{yy} = 0$, is sometimes called the heat equation and Dirichlet problems can be thought of as boundary-value heat problems. Such problems can thus be solved by the methods discussed. It is often helpful to first map the given region onto a simpler one where a solution to the corresponding problem is known.

EXAMPLES

i. Suppose the annulus $1 \leq |z| \leq 2$ represents the cross-section of an "infinite" cylindrical solid with temperature $100°$ maintained on the outer rim and temperature $0°$ on the inner rim. Then, as in the previous example, the temperature is given by

$$u(re^{i\phi}) = \left(\frac{\log r}{\log 2}\right) 100°.$$

In particular, the isothermal line with temperature $50°$ is the circle of radius $\sqrt{2}$.

ii. Next we find the "steady-state" temperature function in the unit disc with boundary values 1 on the upper semi-circle and 0 on the lower semi-circle. Note that $w = (z - 1)/(z + 1)$ maps the disc onto the left half-plane with the upper and lower semi-circles mapping onto the positive and negative imaginary axes, respectively. In the left half-plane, $\text{Arg } z = \text{Im} \log z$ is harmonic with boundary values $\pi/2$

and $3\pi/2$, so that

$$\frac{3}{2} - \frac{\text{Arg } z}{\pi}$$

has the desired boundary values. The solution to the given problem, then, is

$$u(z) = \frac{3}{2} - \frac{1}{\pi} \text{Arg}\left(\frac{z-1}{z+1}\right).$$

◊

16.2 Liouville Theorems for Re f; Zeroes of Entire Functions of Finite Order

The following theorem offers a formula, much like the Poisson Integral Formula, for the value of a C-analytic function in $D(0; R)$ in terms of its real part. This, in turn, will allow us to obtain estimates on the magnitude of an entire function from given bounds on its real part alone.

16.9 Theorem

If $f = u + iv$ *is* C*-analytic in* $D(0; R)$, *then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left[\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right] d\theta + iv(0).$$

Proof

We have already proven (following Theorem 16.7) that if $f = u + iv$ is C-analytic in $D(0; R)$ then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) \text{ Re}\left[\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right] d\theta. \tag{1}$$

Moreover, as we noted in the proof of 16.8 (with $R = 1$)

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left[\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right] d\theta \tag{2}$$

is also analytic in $D(0; R)$. A comparison of (1) and (2) shows, however, that f and g have the same real parts

$$\text{Re } f(z) = \text{Re } g(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \text{ Re}\left[\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right] d\theta.$$

Hence

$$f(z) = g(z) + i\lambda$$

or

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left[\frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right] d\theta + i\lambda.$$

To determine λ, set $z = 0$. Then, by the Mean-Value Theorem (16.4), the integral on the right equals $u(0)$, so that

$$f(0) = u(0) + i\lambda$$

and

$$\lambda = v(0). \qquad \square$$

Analogues of Liouville's Theorems for Re f. The original Liouville Theorem (5.10) states that a bounded entire function is constant. Note that the condition $|f| \leq M$ implies the four inequalities

$$-M \leq \text{Re } f \leq M$$
$$-M \leq \text{Im } f \leq M.$$

However, according to the Weierstrass Theorem (9.6), any one of the four inequalities would suffice to prove that f is constant. For if any one of the inequalities is satisfied, the set of values assumed by f is not dense in the whole plane and f must be constant. The next theorem shows that the same reduction in hypothesis is possible for the Extended Liouville Theorem (5.11).

16.10 Theorem

If f is entire and any one of the four inequalities

$$-A|z|^n \leq \text{Re } f(z) \leq A|z|^n$$
$$-A|z|^n \leq \text{Im } f(z) \leq A|z|^n$$

holds for sufficiently large z, then f is a polynomial of degree $\leq n$.

Proof

Without loss of generality, we may assume Re $f(z) \leq A|z|^n$ for large z. (In the other cases, we could consider $-f$ or *if*.) Then applying 16.9 with $R = 2|z|$

$$\left| \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right| \leq 3$$

and

$$|f(z)| \leq \frac{3}{2\pi} \int_0^{2\pi} |u(Re^{i\theta})| d\theta + |f(0)|, \quad \text{where } u = \text{Re } f.$$

To estimate the integral above, we set

$$u^+(\zeta) = \begin{cases} u(\zeta) & \text{if } u(\zeta) > 0 \\ 0 & \text{if } u(\zeta) \leq 0. \end{cases}$$

Then according to the hypothesis, if $|z|$ is large enough,

$$\frac{1}{2\pi}\int_0^{2\pi} u^+(Re^{i\theta})\,d\theta \le AR^n = A2^n|z|^n$$

and by the Mean-Value Theorem (16.4)

$$\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\,d\theta = u(0).$$

By the lemma below

$$\frac{1}{2\pi}\int_0^{2\pi} |u(Re^{i\theta})|\,d\theta \le A2^{n+1}|z|^n + |u(0)|$$

so that

$$|f(z)| \le A_1|z|^n + A_2$$

and by the Extended Liouville Theorem, f is a polynomial of degree at most n. □

16.11 Lemma

Let g be real-valued and continuous on $[a,b]$. If $\int_a^b g(x)dx = \alpha$ and if

$$\int_a^b g^+(x)dx \le \beta,$$

then

$$\int_a^b |g(x)|\,dx \le 2\beta + |\alpha|.$$

Proof

Recall that

$$g^+(x) = \begin{cases} g(x) & \text{if } g(x) > 0 \\ 0 & \text{if } g(x) \le 0. \end{cases}$$

If we set

$$g^-(x) = \begin{cases} -g(x) & \text{if } g(x) < 0 \\ 0 & \text{if } g(x) \ge 0, \end{cases}$$

then

$$g = g^+ - g^-$$

and

$$|g| = g^+ + g^-.$$

By hypothesis

$$\int_a^b g^+(x) \le \beta$$

and

$$\int_a^b g^-(x)\,dx = \int_a^b g^+(x)\,dx - \alpha \le \beta - \alpha,$$

so that

$$\int_a^b |g(x)|\,dx \le 2\beta - \alpha \le 2\beta + |\alpha|. \qquad \square$$

16.12 Definition

An entire function f is said to be of *finite order* if for some k and some $R > 0$, $|f(z)| \le \exp(|z|^k)$ for all z with $|z| \ge R$.

Theorem 16.10 can be used to prove the existence of zeroes for many entire functions of finite order. To show, for example, that $e^z - z$ must have a zero, we first assume that $e^z - z \ne 0$. Then $g(z) = \log(e^z - z)$ would be entire with

$$\mathrm{Re}(g(z)) = \log|e^z - z| \le |z| + 1 \quad \text{for } |z| \ge e.$$

But then, according to Theorem 16.10, g would be a linear polynomial; that is,

$$\log(e^z - z) = az + b$$

or

$$e^z - z = e^{az+b}.$$

Expanding both sides in power series would lead us to conclude

$$1 + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^b\left(1 + az + a^2\frac{z^2}{2!} + \cdots\right),$$

which is impossible.

Similarly, we can show that $e^z - z$ must have *infinitely* many zeroes. For if $e^z - z$ had only finitely many zeroes $\alpha_1, \alpha_2, \ldots, \alpha_N$, we could apply the above argument to

$$(e^z - z)/(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_N)$$

to conclude that

$$e^z - z = (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_N)e^{az+b}. \tag{3}$$

By considering the growth of both sides as $z \to \infty$, however, it is easily seen that (3) cannot hold.

16.13 Theorem

Suppose f is an entire function of finite order. Then either f has infinitely many zeroes or

$$f(z) = Q(z)e^{P(z)}$$

where Q and P are polynomials.

Proof

Suppose f has a finite number of zeroes, $\alpha_1, \ldots, \alpha_k$. Then we may write

$$f(z) = \mathrm{Q}(z)g(z)$$

where

$$Q(z) = (z - \alpha_1) \cdot \cdots \cdot (z - \alpha_k)$$

and g is an entire function that is never zero. Thus we can define an entire function

$$P(z) = \log g(z),$$

which by our hypothesis must satisfy

$$|\mathrm{Re}\, P(z)| \leq |z|^k \quad \text{for } |z| \geq R$$

for some k and R. Hence P is a polynomial and $f(z) = Q(z)e^{P(z)}$, as desired. □

An entire function does not have to assume every value in the complex plane. However, according to Theorem 16.13, *if f is of finite order,* and if $f(z) \neq a$ for all z, then

$$f(z) - a = e^{P(z)}$$

It follows that f assumes every *other* complex value b infinitely often since P assumes each of the infinitely many values of $\log(b - a)$.

The Little Picard Theorem asserts that the above is true for *all* entire functions. While a proof of this theorem would take us too far afield, we can prove that it is true for a very broad class of functions. Let $E_1(z) = \exp(z^k)$ for any fixed positive integer k, and let $E_{n+1}(z) = \exp(E_n(z))$, so that E_j is the j-fold exponential of z^k. We will show that Picard's Theorem is applicable to any entire function which grows no faster than E_j for some j. To be precise, we will say that f is of $j-$fold exponential order if, for some $R > 0$, the j-fold logarithm: $\log(\log(\log \cdots (|f(z)|))) < |z|^k$, for some fixed k and $|z| > R$. Note that if f is of $j-$fold exponential order, then $\log f$ is of $(j - 1)-$fold exponential order.

16.14 Theorem

Suppose f is an entire function of $j-$fold exponential order, for some j. Then, if $f(z) \neq a$ for all z, f assumes every other complex value b infinitely often.

Proof

If $j = 1$, f is of finite order and the result follows, as indicated above. To complete the proof, assume that f is of $(j+1)$-fold exponential order. Then $g(z) = \log(f(z) - a)$ would be of j-fold exponential order, and by induction we can assume that g assumes every value in the complex plane with at most one exception. In particular, g assumes all of the infinitely many values of $\log(b-a)$ with at most one exception. But then, since $f(z) = a + e^{g(z)}$, f assumes every complex value $b \neq a$ infinitely often. □

We leave it as an exercise to show that Theorem 16.14 is not equivalent to Picard's Little Theorem. That is, there are entire functions which are not of j−fold exponential order, for any j.

Exercises

1. Suppose $f = u + iv$ is analytic. Show then that $u + v$ and uv are harmonic.
2. Show that every partial derivative of a harmonic function is itself harmonic.
3. Show that u^2 *cannot* be harmonic for any nonconstant harmonic function u.
4. Show that $\log(x^2 + y^2)$ is harmonic in $z \neq 0$ but is not equal to the real part of a function that is analytic in $z \neq 0$.
5. a. Show that if $u(r, \theta)$ is dependent on r alone, Laplace's equation becomes

$$u_{rr} + \frac{1}{r}u_r = 0.$$

   b. Use the above to show that a harmonic function which depends on r alone must have the form $u(r, \theta) = a \log r + b$.

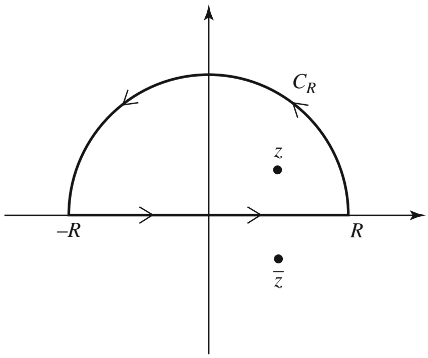
6. Derive the Poisson Formula

$$u(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y \cdot u(t)dt}{(t-x)^2 + y^2} \qquad \text{(I)}$$

for a bounded C-harmonic function in the upper half-plane. [*Hint*: Let C_R denote the indicated contour and set

$$2\pi i f(z) = \int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{C_R} \frac{f(\zeta)}{\zeta - \overline{z}}\, d\zeta,$$

where Re $f = u$. Then simplify and obtain (I) for $f(x+iy)$ by letting $R \to \infty$.]



7. Find a harmonic function in $D(0; 1)$ with boundary values $u(x, y) = x^3$.
8. Let u be harmonic in $D(0; 1)$ with boundary values: 1 on the upper semi-circle and 0 on the lower semi-circle. Show that the level curves $u(x, y) = k$, $0 \leq k \leq 1$, are all circular segments.

9. Find a harmonic function u in the upper half-plane with

$$\lim_{y\to 0} u(x, y) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

10. Find the temperature function $u(x, y)$ for a solid represented by the semi-infinite strip

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad y \ge 0$$

given that $u(x, 0) = 1, -\frac{\pi}{2} < x < \frac{\pi}{2}$,

$$u\left(-\frac{\pi}{2}, y\right) = 2, \text{ and } u\left(\frac{\pi}{2}, y\right) = 0, \quad \text{for } y > 0.$$

11. Prove $e^z - P(z)$ and $\sin z - P(z)$ have infinitely many zeroes for every non-zero polynomial P.

12. We say an entire function of finite order has order j if

$$j = \inf\left\{k : \lim_{z\to\infty} \frac{f(z)}{\exp(|z|^k)} = 0\right\}.$$

Prove that the only non-vanishing entire functions of order j are of the form $f(z) = e^{P_j(z)}$, where P_j is a polynomial of degree j.

13.* Show that $\sin z - z = c$ has a solution for every complex number c by showing that if $\sin z - z \neq c$ for all z, then $\sin z - z \neq c + 2\pi$

14.* Let $f_0(z) = z,\ f_{n+1}(z) = e^{f_n(z)}$, $n = 0, 1, 2, \ldots$ and let $g_0(t) = t,\ g_{n+1}(t) = t^{g_n(t)}$. Define

$$F(z) = \sum_{k=1}^{\infty} \frac{f_k(z)}{g_k(k)}$$

Show that $F(z)$ is an entire function but $F(z)$ is not of j-fold exponential order for any positive integer j.