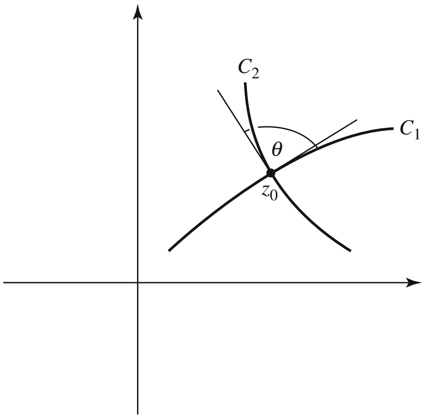
Chapter 13
Introduction to Conformal Mapping

In this chapter, we take a closer look at the mapping properties of an analytic function. Throughout the chapter, all curves $z(t)$ are assumed to be such that $\dot{z}(t) \neq 0$ for all t.

13.1 Conformal Equivalence

13.1 Definition

Suppose two smooth curves C_1 and C_2 intersect at z_0. Then the *angle from* C_1 *to* C_2 *at* z_0, $\angle C_1, C_2$, is defined as the angle measured counterclockwise from the tangent line of C_1 at z_0 to the tangent line of C_2 at z_0.



13.2 Definition

Suppose f is defined in a neighborhood of z_0. f is said to be *conformal at* z_0 if f preserves angles there. That is, for each pair of smooth curves C_1 and C_2 intersecting

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at z_0, $\angle C_1,\ C_2 = \angle\Gamma_1,\ \Gamma_2$ where $\Gamma_1 = f(C_1),\ \Gamma_2 = f(C_2)$. Similarly, we say f is conformal in a region D if f is conformal at all points $z \in D$.

Note that $f(z) = z^2$ is *not* conformal at $z = 0$. For example, the positive real axis and the positive imaginary axis are mapped onto the positive real axis and negative real axis, respectively. However, as we shall see below, it is conformal at all other points of the complex plane.

13.3 Definition

a. f is *locally* 1-1 at z_0 if for some $\delta > 0$ and any distinct $z_1,\ z_2 \in D(z_0;\delta)$, $f(z_1) \neq f(z_2)$.
b. f is locally 1-1 throughout a region D if f is locally 1-1 at every $z \in D$.
c. f is a 1-1 function in a region D, if for *every* distinct $z_1,\ z_2 \in D,\ f(z_1) \neq f(z_2)$.

Again, note that $f(z) = z^2$ is not locally 1-1 at $z = 0$ since $f(z) = f(-z)$ for all z. However, f is locally 1-1 at all points $z \neq 0$ (see Exercise 1).

13.4 Theorem

Suppose f is analytic at z_0 and $f'(z_0) \neq 0$. Then f is conformal and locally 1-1 *at z_0.*

Proof (Conformality)

Let $C : z(t) = x(t) + iy(t)$ be a smooth curve with $z(t_0) = z_0$. Then the tangent line to C at z_0 has the direction of $\dot{z}(t_0) = x'(t_0) + iy'(t_0)$ so that its angle of inclination with the positive real axis is Arg $\dot{z}(t_0)$. If we set $\Gamma = f(C)$, then Γ is given by $\omega(t) = f(z(t))$ and the angle of inclination of its tangent line at $f(z_0)$ is equal to

$$\text{Arg}\ \dot{\omega}(t_0) = \text{Arg}\ \left[f'(z_0)\dot{z}(t_0)\right] = \text{Arg}\ f'(z_0) + \text{Arg}\ \dot{z}(t_0).$$

Hence the function f maps all curves at z_0 in such a way that the angles of inclination are increased by the constant Arg $f'(z_0)$. Thus, if C_1 and C_2 meet at z_0 and Γ_1, Γ_2 are their respectively images under f, it follows that $\angle\Gamma_1, \Gamma_2 = \angle C_1, C_2$.

To show f is 1-1 in a neighborhood of z_0, let $f(z_0) = \alpha$ and take $\delta' > 0$ small enough so that $f(z) - \alpha$ has no other zeroes in $D(z_0;\ \delta')$. Such a δ' can always be found for otherwise we would have $f'(z_0) = 0$ (Theorem 6.10).

If we let $C = C(z_0; \delta')$ and $\Gamma = f(C)$, it follows by the Argument Principle (10.9) that

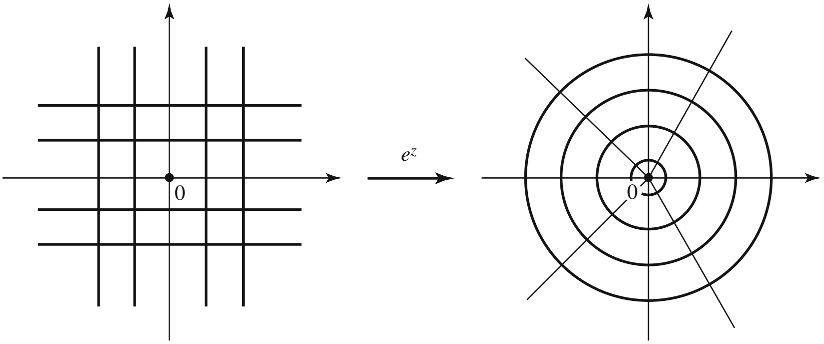
$$\begin{aligned} 1 &= \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-\alpha}\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{d\omega}{\omega-\alpha} = \frac{1}{2\pi i}\int_\Gamma \frac{d\omega}{\omega-\beta} \end{aligned}$$

for all β in some sufficiently small disc $D(\alpha;\epsilon)$, since the winding number is locally constant (see following 10.3). If we then take $\delta \le \delta'$ so that $D(z_0;\delta) \subset f^{-1}(D(\alpha;\epsilon))$ it follows that for any $z_1, z_2 \in D(z_0;\delta)$

$$\begin{aligned} 1 &= \frac{1}{2\pi i}\int_\Gamma \frac{d\omega}{\omega - f(z_1)} = \frac{1}{2\pi i}\int_\Gamma \frac{d\omega}{\omega - f(z_2)} \\ &= \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-f(z_1)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-f(z_2)}\,dz; \end{aligned}$$

i.e., the values $f(z_1)$ and $f(z_2)$ are both assumed once inside C so that $f(z_1) \ne f(z_2)$ if $z_1 \ne z_2$. □

EXAMPLE 1
$f(z) = e^z$ has a nonzero derivative at all points, hence it is everywhere conformal and locally 1-1. (Note that is it not globally 1-1 since $f(z+2\pi i) = f(z)$.) By the conformality of f, the images of the orthogonal lines $x =$ constant and $y =$ constant under the mapping f are themselves orthogonal. We leave it as an exercise to verify this by showing that f maps the vertical lines $x =$ constant onto circles centered at the origin and maps the horizontal lines $y =$ constant onto rays from the origin.



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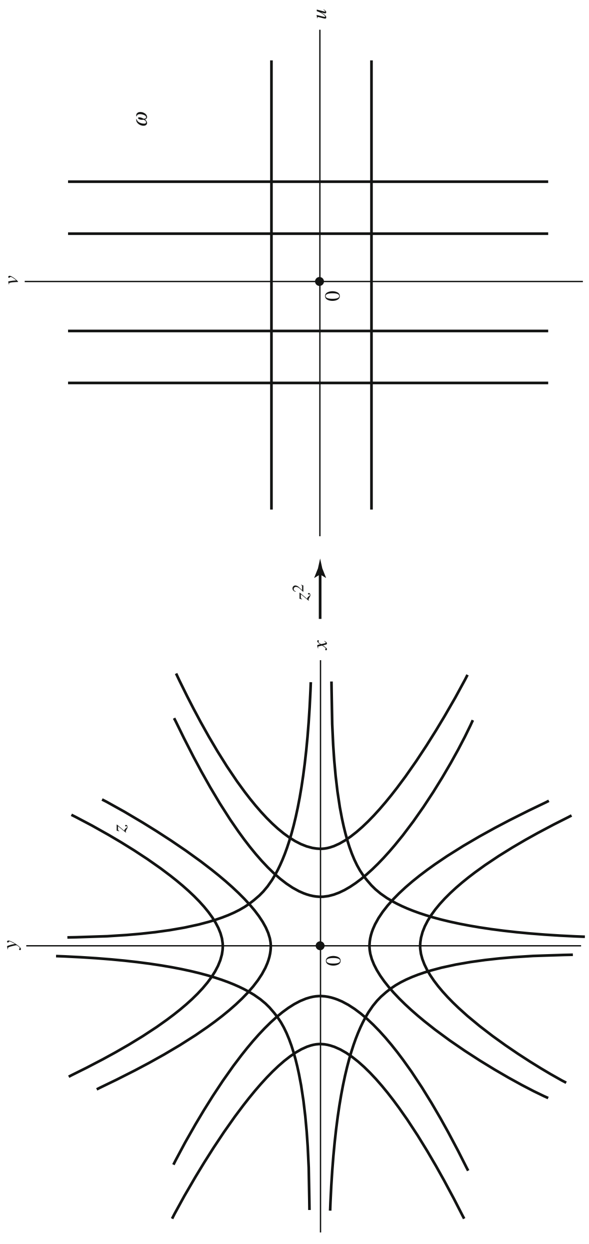
EXAMPLE 2
Let $f(z) = z^2$. Since $f'(z) = 2z \ne 0$ except at $z = 0$, f is conformal throughout $z \ne 0$. Thus, if we set $f = u + iv$, it follows that the preimages of the curves $u = c_1$, $v = c_2$ for $c_1, c_2 \ne 0$ must be orthogonal. Indeed, since $u(z) = x^2 - y^2$, $v(z) = 2xy$

these preimages are the orthogonal systems of hyperbolas given by

$$x^2 - y^2 = c_1, \quad 2xy = c_2 \quad \text{(see the figure below).}$$



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To analyze the mapping properties of a function f at a point z where $f'(z) = 0$, we first consider the following special case.

13.5 Definition

Let k be a positive integer. f is a k*-to-*1 *mapping* of D_1 onto D_2 if for every $\alpha \in D_2$, the equation $f(z) = \alpha$ has k roots (counting multiplicity) in D_1.

13.6 Lemma

Let $f(z) = z^k$, k *a positive integer. Then* f *magnifies angles at* 0 *by a factor of* k *and maps the disc* $D(0; \delta)$, $\delta > 0$, *onto the disc* $D(0; \delta^k)$ *in a* k*-to-*1 *manner.*

Proof

Since $f(re^{i\theta}) = r^k e^{ik\theta}$, f maps the ray from 0 with argument θ onto the ray from 0 with argument $k\theta$. Hence the angle at 0 between any two rays is magnified by a factor of k. To see that $f(z) = \alpha$, $\alpha \in D(0; \delta^k)$ has k roots in $D(0; \delta)$ recall that if $\alpha \neq 0$ there are k distinct roots all lying on the circle $|z| = |\alpha|^{1/k}$. If $\alpha = 0$, the equation $z^k = \alpha$ has a k-fold root at the origin. □

We can now "complete" Theorem 13.4.

13.7 Theorem

Suppose f *is analytic at* z_0 *with* $f'(z_0) = 0$. *Then, unless* f *is constant, in some sufficiently small open set containing* z_0, f *is a k-to-1 mapping and* f *magnifies angles at* z_0 *by a factor of* k, *where* k *is the least positive integer for which* $f^{(k)}(z_0) \neq 0$.

Proof

We may assume, without loss of generality, that $f(z_0) = 0$. [Otherwise, we could first consider $f(z) - f(z_0)$.] Then, by hypothesis, the power series expansion of f about z_0 is of the form

$$\begin{aligned} f(z) &= a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + a_{k+2}(z-z_0)^{k+2} + \cdots \\ &= (z-z_0)^k \left[a_k + a_{k+1}(z-z_0) + a_{k+2}(z-z_0)^2 + \cdots\right] \end{aligned}$$

with $a_k = f^{(k)}(z_0)/k! \neq 0$.

If we let $g(z)$ represent the bracketed power series, we note that $g(z_0) \neq 0$ so that g has an analytic k-th root in some disc $D(z_0; \delta)$ (see the comments following Theorem 8.8). Thus, in that disc,

$$f(z) = [h(z)]^k$$

where h is an analytic function defined by

$$h(z) = (z - z_0) g^{1/k}(z)$$

and

$$h(z_0) = 0, \quad h'(z_0) = g^{1/k}(z_0) \neq 0.$$

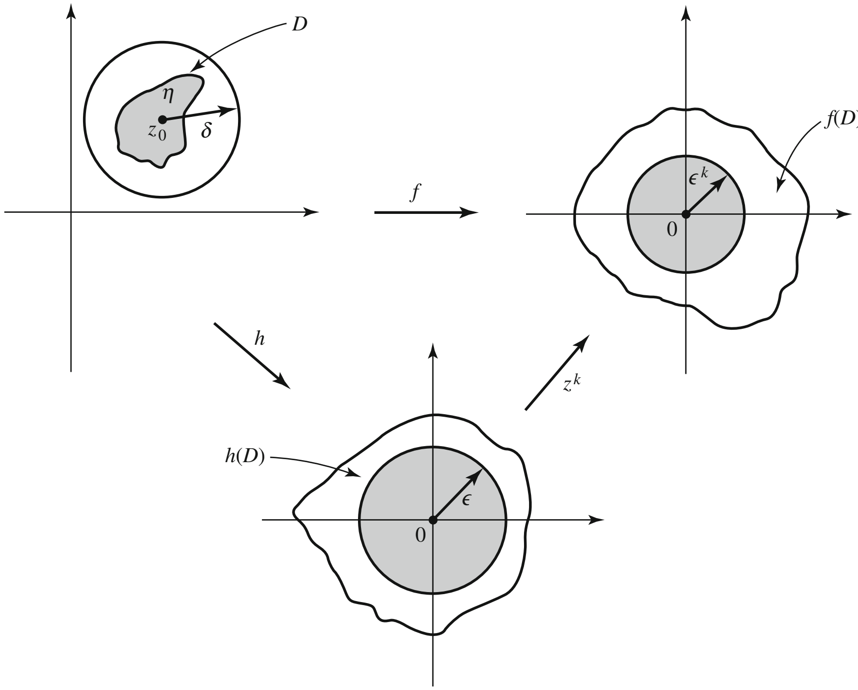
Hence, in a sufficiently small neighborhood D of z_0, f is the composition of the 1-1 and conformal mapping h followed by the mapping z^k. Since z^k magnifies angles at 0 by a factor of k, it follows that f magnifies angles at z_0 by k. Also, since z^k is k-to-1 on discs about 0, it follows that if

$$D(0;\epsilon) \subset h(D)$$

and

$$\eta = h^{-1}(D(0;\epsilon)),$$

then f is k-to-1 on η. □



The previous results combine to yield the following properties of 1-1 analytic functions.

13.8 Theorem

Suppose f is a 1-1 *analytic function in a region D. Then*

a. f^{-1} *exists and is analytic in* $f(D)$,
b. f *and* f^{-1} *are conformal in D and* $f(D)$, *respectively.*

Proof

Since f is 1-1, $f' \neq 0$. Hence f^{-1} is also analytic (Proposition 3.5). Furthermore, $(f^{-1})' = 1/f'$ so that f^{-1} also has a nonzero derivative. Thus f and f^{-1} are both conformal. □

Theorem 13.8 motivates the following definitions:

13.9 Definitions

a. A 1-1 analytic mapping is called a *conformal mapping*.
b. Two regions D_1 and D_2 are said to be *conformally equivalent* if there exists a conformal mapping of D_1 onto D_2.

We leave it as an exercise to verify that "conformal equivalence" satisfies the usual axioms of an equivalence relation. In particular, we note that the transitive property follows from the fact that the composition of two conformal mappings is also a conformal mapping, and we will use this fact throughout the remainder of the chapter.

The Riemann Mapping Theorem, which we will prove in the next chapter, asserts that any two simply connected domains (besides the whole plane) are conformally equivalent. In the next section, we will consider certain special transformations that will enable us to explicitly display conformal mappings between many familiar simply connected regions.

13.2 Special Mappings

I *Elementary Transformations*

(i) $\omega = az + b$.

The linear map $\omega = az+b$ is a 1-1 analytic map of the entire plane onto itself. The effect of the mapping on a given domain can be seen by viewing it as a composition $\omega = \omega_3 \circ \omega_2 \circ \omega_1$ of the mappings

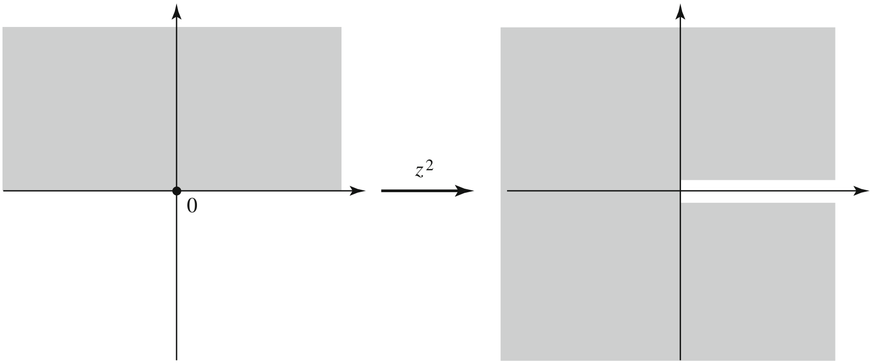
1. $\omega_1 = kz, \quad k = |a|$
2. $\omega_2 = e^{i\theta} z, \quad \theta = \text{Arg}\, a$
3. $\omega_3 = z + b$.

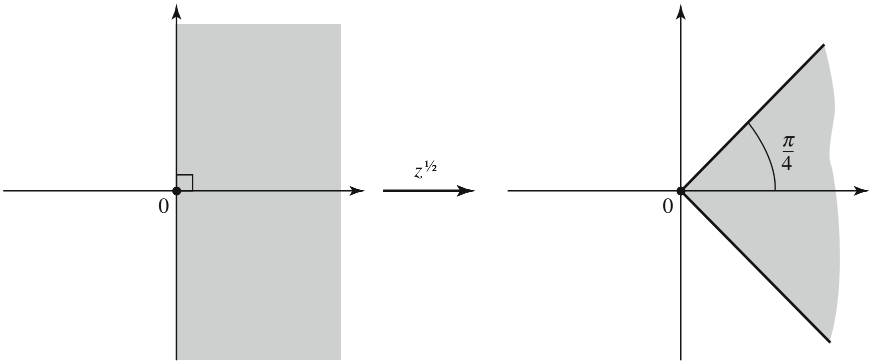
A mapping of the form $\omega = kz,\ k > 0$ is called a *magnification*. It sends each point onto another point along the same ray from the origin, multiplying its magnitude by a factor of k. The mapping $\omega = e^{i\theta} z$ is a counterclockwise *rotation* through an angle θ. Finally, $\omega = z + b$ is called a *translation* since it translates each point by the complex number, or vector, b.

(ii) $\omega = z^\alpha,\ \alpha > 0$.

As we noted in Chapter 8, the function $\omega = z^\alpha$ given by $z^\alpha = e^{\alpha \log z}$ is analytic in every simply connected domain that does not contain 0. If we take the branch of $\log z$ which is real on the positive axis, then z^α will also map the positive axis onto itself. The point $z = re^{i\theta}$ is mapped onto $r^\alpha e^{i\alpha\theta}$ and hence $\omega = z^\alpha$ maps the wedge $S = \{z : \theta_1 < \text{Arg}\, z < \theta_2\}$ onto the wedge $T = \{\omega : \alpha\theta_1 < \text{Arg}\, \omega < \alpha\theta_2\}$. If, moreover, $\alpha\theta_2 - \alpha\theta_1 \leq 2\pi$; i.e., if $\theta_2 - \theta_1 \leq 2\pi/\alpha$, the mapping is a conformal mapping of S onto T.

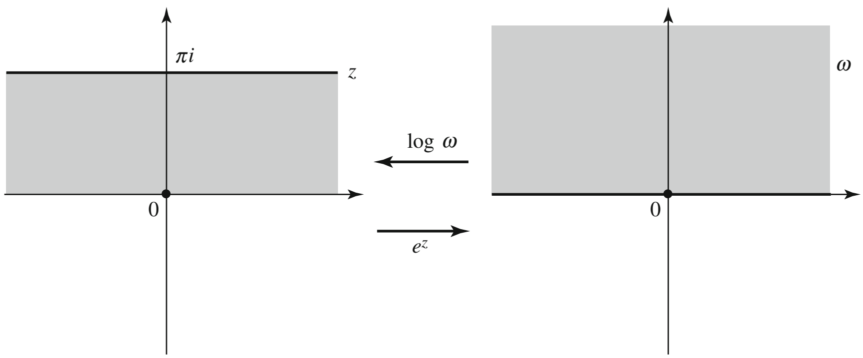
Some examples are sketched below:





(iii) $\omega = e^z$.

Since $e^z = e^x e^{iy}$, the function $\omega = e^z$ maps the strip: $y_1 < y < y_2$ onto the wedge: $y_1 < \text{Arg}\,\omega < y_2$. If $y_2 - y_1 \leq 2\pi$, the mapping is 1-1. For example, the strip $0 < y < \pi$ is mapped conformally onto the upper half-plane.



II *The Bilinear Transformation* $\boldsymbol{\omega = (az + b)/(cz + d)}$

The mapping given by

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \tag{1}$$

is called a bilinear transformation. The condition $ad - bc \neq 0$ insures that f is neither identically constant nor meaningless. Since

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0,$$

f is locally 1-1 and conformal. In fact, a bilinear transformation is globally 1-1 since

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$$

implies that

$$(ad-bc)(z_1 - z_2) = 0$$

and hence that

$$z_1 = z_2.$$

The bilinear transformation (1) maps the full plane, minus the point $-d/c$, onto the full plane minus the point a/c, since the equation

$$\frac{az+b}{cz+d} = \omega$$

has the explicit solution

$$z = \frac{d\omega - b}{-c\omega + a} \quad \text{for every } \omega \neq \frac{a}{c}.$$

In fact, if we consider the limiting values $f(\infty) = (a/c)$ and $f(-d/c) = \infty$ we can say that f is a 1-1 mapping of the Riemann sphere onto itself. (See Section 1.5.)

The set of bilinear transformations forms a group under composition. It is easily seen that the inverse of a bilinear transformation is also bilinear since, as above,

$$\omega = \frac{az+b}{cz+d}$$

admits the solution

$$z = \frac{d\omega - b}{-c\omega + a}, \text{ and } da - (-b)(-c) = ad - bc \neq 0.$$

We leave the verification of the other group properties as an exercise.

A very useful property of bilinear transformations is that they map circles and lines onto other circles and lines. We prove this first for the special case $f(z) = 1/z$.

13.10 Lemma

If S is a circle or line, and $f(z) = 1/z$, then $f(S)$ is also a circle or line.

Proof

(A proof involving the Riemann sphere is outlined in Exercises 27 and 28 of Chapter 1. The following proof is more direct.)

Assume first that $S = C(\alpha; r)$ and let

$$f(S) = \left\{\omega = \frac{1}{z} : z \in S\right\}.$$

Writing the equation for S in the form

$$(z - \alpha)(\bar{z} - \bar{\alpha}) = r^2$$

we have

$$z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = r^2 - |\alpha|^2$$

or, in terms of ω,

$$\frac{1}{\omega\bar{\omega}} - \frac{\alpha}{\bar{\omega}} - \frac{\bar{\alpha}}{\omega} = r^2 - |\alpha|^2. \tag{1}$$

Note then that if $r = |\alpha|$; i.e., if S passes through the origin, (1) is equivalent to

$$1 - \alpha\omega - \bar{\alpha}\bar{\omega} = 0$$

or

$$\text{Re}\,\alpha\omega = \frac{1}{2}.$$

In that case, if $\alpha = x_0 + iy_0$ and $\omega = u + iv$, the equation for ω becomes

$$ux_0 - vy_0 = \frac{1}{2}\,;$$

i.e., $f(S)$ is a line in the ω-plane.

If, on the other hand, $r \neq |\alpha|$, then (1) is equivalent to

$$\omega\bar{\omega} - \left(\frac{\bar{\alpha}}{|\alpha|^2 - r^2}\right)\bar{\omega} - \left(\frac{\alpha}{|\alpha|^2 - r^2}\right)\omega = \frac{-1}{|\alpha|^2 - r^2},$$

and setting

$$\beta = \frac{\bar{\alpha}}{|\alpha|^2 - r^2}$$

we obtain

$$\omega\bar{\omega} - \beta\bar{\omega} - \bar{\beta}\omega + |\beta|^2 = \frac{r^2}{(|\alpha|^2 - r^2)^2}.$$

Thus

$$|\omega - \beta|^2 = \left(\frac{r}{|\alpha|^2 - r^2}\right)^2,$$

so that $f(S)$ is a circle with center β and radius $|r/(|\alpha|^2 - r^2)|$.

Finally, if S is a straight line, then there exist real-valued $a,\ b,\ c$ such that if $z = x + iy \in S$,

$$ax + by = c. \tag{2}$$

Letting $\alpha = a - bi$, (2) is equivalent to

$$\text{Re } \alpha z = c$$

or

$$\alpha z + \bar{\alpha}\,\bar{z} = 2c.$$

It follows then, as above, that $f(S)$ is either a circle or a line. □

13.11 Theorem

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

maps circles and lines onto circles and lines.

Proof

If $c = 0,\ f$ is a linear map and the result is immediate. Otherwise, we can write

$$\frac{az + b}{cz + d} = \frac{1}{c}\left[\frac{acz + ad - ad + bc}{cz + d}\right] = \frac{1}{c}\left[a - \left(\frac{ad - bc}{cz + d}\right)\right].$$

Thus f is the composition $f = f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = cz + d,$$

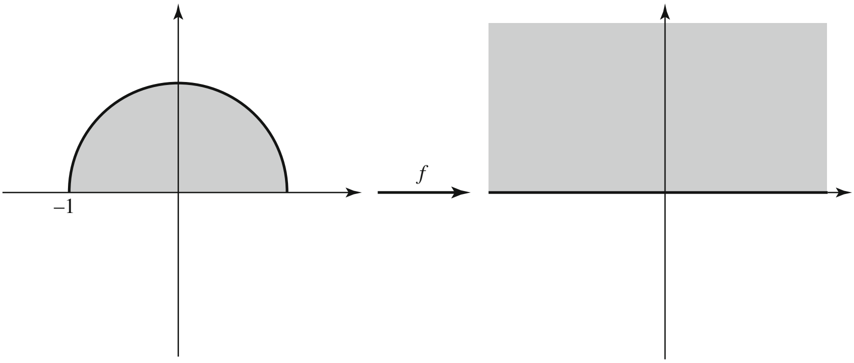
$$f_2(z) = \frac{1}{z},$$

$$f_3(z) = \frac{a}{c} - \left(\frac{ad - bc}{c}\right)z.$$

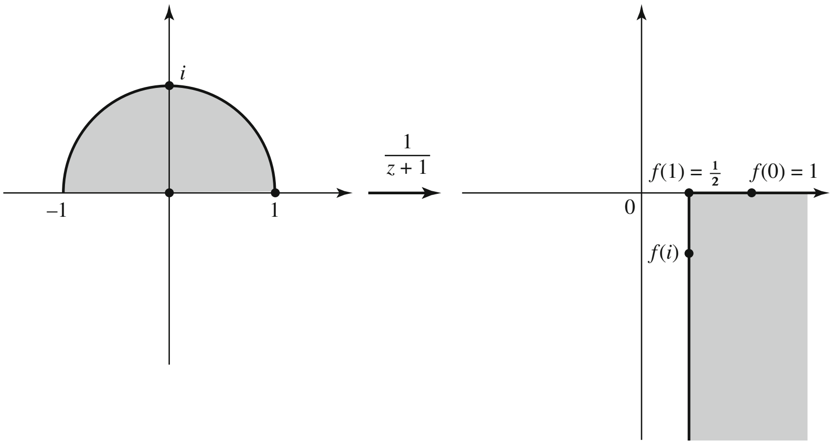
f_1 and f_3 are linear; hence they map circles and lines into circles and lines. By Lemma 13.10, the same is true for f_2, and thus it follows that f has the desired property. □

The above properties of bilinear transformations make them a very handy tool in solving many conformal mapping and miscellaneous geometric problems.

EXAMPLE 1
Find a conformal mapping f of the semi-disc $S = \{z : |z| < 1,\ \text{Im}\, z > 0\}$ onto the upper half-plane.



Note that because $g(z) = 1/(z+1)$ has a pole at -1, it maps the line segment [**-1, 1**] and the upper semi-circle onto infinite rays. Furthermore, the two rays must intersect at $g(1) = \frac{1}{2}$, and by the conformality of g, they intersect orthogonally. By considering several points, it can then be seen that g maps the segments onto the lines indicated below and maps the semi-disc onto the quadrant bounded by the orthogonal rays.



Thus, the desired mapping f is given by

$$f(z) = \left[i\left(g(z) - \frac{1}{2}\right)\right]^2 = \frac{-(z-1)^2}{4(z+1)^2}.$$

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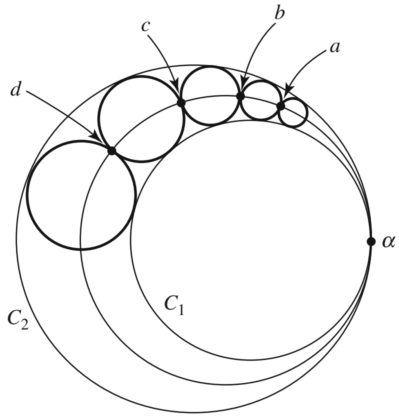
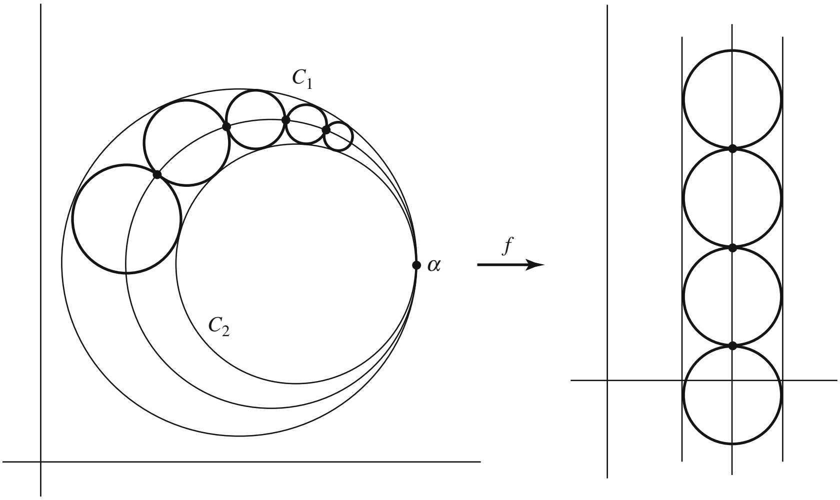
EXAMPLE 2

Suppose two circles C_1 and C_2 are tangent at a point α and a chain of circles tangent to C_1 and C_2 and to each other is constructed as indicated. Prove that the points of tangency $a,\ b,\ c,\ \ldots$ thus created all lie on a circle.



Consider the image of the above diagram under the mapping $f(z) = 1/(z - \alpha)$. Since the mapping is 1-1 and has a pole at $\alpha,\ C_1$ and C_2 are mapped into a pair of parallel lines. Furthermore, all the other circles are mapped into circles and, again since f is 1-1, the circles will be tangent to the parallel lines and to each other.



It is clear that the points $f(a),\ f(b),\ f(c),\ \ldots$ lie on a straight line (between $f(C_1)$ and $f(C_2)$). Finally, then, $a,\ b,\ c\ldots$ all lie on the image of this line under the inverse transformation f^{-1}. Since f^{-1} is also bilinear, this image is a circle and the result is established. ◊

In light of Theorem 13.11, it will come as no surprise that bilinear functions can be used to map half-planes and discs conformally onto other half-planes and discs. In fact, as we shall see below, all such mappings are given by bilinear transformations.

13.12 Definition

A conformal mapping of a region onto itself is called an *automorphism* of that region.

13.13 Lemma

Suppose $f: D_1 \to D_2$ *is a conformal mapping. Then*

a. *any other conformal mapping* $h: D_1 \to D_2$ *is of the form* $g \circ f$;
b. *any automorphism* h *of* D_1 *is of the form* $f^{-1} \circ g \circ f$, *where* g *is an automorphism of* D_2.

Proof

a. If f and h are both conformal mappings of D_1 onto D_2, then $h \circ f^{-1}$ is an automorphism of D_2; i.e., $h \circ f^{-1} = g$ and $h = g \circ f$.
b. If h is an automorphism of D_1, $f \circ h \circ f^{-1}$ is an automorphism of D_2; thus $f \circ h \circ f^{-1} = g$ and $h = f^{-1} \circ g \circ f$. □

We now consider the problem of determining all the automorphisms of the unit disc.

13.14 Lemma

The only automorphisms of the unit disc with $f(0) = 0$ *are given by* $f(z) = e^{i\theta} z$.

Proof

If f maps the unit disc 1-1 onto itself and $f(0) = 0$, then by Schwarz' Lemma (7.2)

$$|f(z)| \le |z| \quad \text{for } |z| < 1. \tag{3}$$

Moreover, since f^{-1} also maps the disc onto itself and $f^{-1}(0) = 0$, by the same argument,

$$|f^{-1}(z)| \le |z| \quad \text{for } |z| < 1. \tag{4}$$

However, (3) and (4) can both be valid only if $|f(z)| = |z|$ and, by Schwarz' Lemma once again, it follows that

$$f(z) = e^{i\theta} z.$$

□

Suppose now that we wish to find an automorphism f of the unit disc with $f(\alpha) = 0$, for a fixed α, $0 < |\alpha| < 1$. If we assume that f is bilinear, then since f is globally 1-1, it must map the unit circle onto itself and we can thus apply the Schwarz Reflection Principle (7.8) (see also Exercise 19, Chapter 7) to conclude that

$f(1/\bar{\alpha}) = \infty$. Hence f must be of the form

$$f(z) = c\left(\frac{z-\alpha}{z-1/\bar{\alpha}}\right).$$

Setting

$$|f(1)| = |c\alpha| = 1$$

we have $|c| = (1/|\alpha|)$, and f may be written in the form

$$f(z) = e^{i\theta}\left(\frac{z-\alpha}{1-\overline{\alpha}z}\right).$$

This suggests the following theorem.

13.15 Theorem

The automorphisms of the unit disc are of the form

$$g(z) = e^{i\theta}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right), \quad |\alpha| < 1.$$

Proof

Let $g(z) = (z-\alpha)/(1-\bar{\alpha}z)$. Then, as we noted previously (following 7.2), $|g(z)| = 1$ for $|z| = 1$. Since $g(\alpha) = 0$, it follows that g is indeed an automorphism of the unit disc. Now assume that f is an automorphism of the unit disc with $f(\alpha) = 0$. Then $h = f \circ g^{-1}$ is an automorphism with $h(0) = 0$, so that according to the previous lemma

$$h(z) = e^{i\theta}z$$

or

$$f(z) = e^{i\theta}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right).$$

□

Suppose next that we wish to determine a conformal mapping h of the upper half-plane onto the unit disc. Again, let us first assume that h is bilinear and $h(\alpha) = 0$, for fixed α with Im $\alpha > 0$. Then, since the real axis is mapped into the unit circle, it follows by the Schwarz Reflection Principle that $h(\bar{\alpha}) = \infty$ so that

$$h(z) = c\left(\frac{z-\alpha}{z-\bar{\alpha}}\right).$$

13.16 Theorem

The conformal mappings h of the upper half-plane onto the unit disc are of the form

$$h(z) = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right), \quad \text{Im}\,\alpha > 0.$$

Proof

Let $f(z) = (z-\alpha)/(z-\bar{\alpha})$. Since $|z-\alpha| = |z-\bar{\alpha}|$ for real z, f maps the real axis onto the unit circle. Also, since $f(\alpha) = 0$ and $\operatorname{Im} \alpha > 0$, it follows that f maps the upper half-plane onto the unit disc. Suppose then that h is any conformal mapping of the upper half-plane onto the unit disc and $h(\alpha) = 0$. By Lemma 13.13, h is of the form

$$h = g \circ f$$

where g is an automorphism of the disc. However, since $h(\alpha) = g(0) = 0$, it follows that $g(z) = e^{i\theta} z$ (13.14) and

$$h(z) = e^{i\theta} \left(\frac{z-\alpha}{z-\bar{\alpha}} \right).$$

□

13.17 Theorem

The automorphisms of the upper half-plane are of the form

$$h(z) = \frac{az+b}{cz+d}$$

with a, b, c, d *real and* $ad - bc > 0$.

Proof

Let h be as above. Then clearly h maps the real axis onto itself. Also,

$$\operatorname{Im} f(i) = \frac{ad-bc}{c^2+d^2} > 0,$$

so that i is mapped into the upper half-plane and hence f is an automorphism of the upper half-plane. To show that there are no other automorphisms, we can apply Lemma 13.13 and Theorem 13.15 to show that any such automorphism h must be of the form $h = f^{-1} \circ g \circ f$ with

$$f(z) = \frac{z-i}{z+i} \quad \text{and} \quad g(z) = e^{i\theta} \left(\frac{z-\alpha}{1-\bar{\alpha}z} \right), \quad |\alpha| < 1.$$

We leave it as an exercise to verify that such a mapping can be written in the form

$$h(z) = \frac{az+b}{cz+d}; \quad a, b, c, d, \text{ real}; \quad ad - bc > 0.$$

(See Exercise 16.) □

In the results that follow, we will see that there is a unique bilinear mapping sending any three distinct points z_1, z_2, z_3 onto any three distinct points $\omega_1, \omega_2, \omega_3$, respectively.

13.18 Definition

z_0 is called a *fixed point* of the function f if $f(z_0) = z_0$.

13.19 Proposition

A bilinear transfomation (other than the identity mapping $f(z) = z$) has at most two fixed points.

Proof

Let $f(z) = (az + b)/(cz + d)$. If $c \neq 0$, the equation $f(z) = z$ is equivalent to the quadratic equation $az + b = cz^2 + dz$ and hence has at most two solutions. (Note also that in this case $f(\infty) = a/c \neq \infty$.) If $c = 0$, f is linear and, unless $a/d = 1$, $f(z) = z$ has one solution in the finite complex plane. In this case, since $f(\infty) = \infty$, the point at infinity may be considered a second fixed point. Finally, if $f(z) = z + b$, there are no fixed points in $\mathbb{C}$. □

13.20 Lemma

The unique bilinear mapping sending z_1, z_2, z_3 *into* $\infty, 0, 1$, *respectively, is given by*

$$T(z) = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}.$$

Proof

Certainly T has the desired properties. If S is another bilinear transformation which maps $z_1,\ z_2,\ z_3$ into $\infty,\ 0,\ 1$, then $T \circ S^{-1}$ is a bilinear map with three fixed points, so that $T \circ S^{-1}$ is the identity map and $T \equiv S$. □

Note that the lemma, with the appropriate modifications, is valid also if some $z_i = \infty$. If $z_1 = \infty$, the map is given by

$$T(z) = \frac{z - z_2}{z_3 - z_2}.$$

If $z_2 = \infty$,

$$T(z) = \frac{z_3 - z_1}{z - z_1};$$

and if $z_3 = \infty$,

$$T(z) = \frac{z - z_2}{z - z_1}.$$

13.21 Definition

The *cross-ratio* of the four complex numbers z_1, z_2, z_3, z_4—denoted (z_1, z_2, z_3, z_4)—is the image of z_4 under the bilinear map which maps $z_1,\ z_2,\ z_3$ into $\infty,\ 0,\ 1$, respectively.

By the preceding lemma

$$(z_1,\ z_2,\ z_3,\ z_4) = \left(\frac{z_4 - z_2}{z_4 - z_1}\right)\left(\frac{z_3 - z_1}{z_3 - z_2}\right).$$

13.22 Proposition

The cross-ratio of four points is invariant under bilinear transformations: i.e., if S is bilinear, $(Sz_1, Sz_2, Sz_3, Sz_4) = (z_1, z_2, z_3, z_4)$.

Proof [Ahlfors]

Let T be the bilinear map which sends $z_1,\ z_2,\ z_3$ into $\infty,\ 0,\ 1$. Then $T \circ S^{-1}$ maps $Sz_1,\ Sz_2,\ Sz_3$ into $\infty,\ 0,\ 1$ and by definition $(Sz_1,\ Sz_2,\ Sz_3,\ Sz_4) = T \circ S^{-1}(Sz_4) = Tz_4 = (z_1,\ z_2,\ z_3,\ z_4)$. □

13.23 Theorem

The unique bilinear transformation $\omega = f(z)$ *mapping* z_1, z_2, z_3 *into* $\omega_1, \omega_2, \omega_3$*, respectively, is given by*

$$\frac{(\omega - \omega_2)(\omega_3 - \omega_1)}{(\omega - \omega_1)(\omega_3 - \omega_2)} = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}. \tag{5}$$

Proof

The existence of the mapping f is easily established. If we let $T_1,\ T_2$ be the bilinear maps with

$$T_1 : z_1,\ z_2,\ z_3 \to \infty,\ 0,\ 1$$

$$T_2 : \omega_1,\ \omega_2,\ \omega_3 \to \infty,\ 0,\ 1$$

then $f = T_2^{-1} \circ T_1$. To show that $\omega = f(z)$ must satisfy (5), we need only invoke Proposition 13.22 that the cross-ratio of any four points is preserved under a bilinear map and hence

$$(\omega_1,\ \omega_2,\ \omega_3,\ \omega) = (z_1,\ z_2,\ z_3,\ z).$$

(The appropriate modification of (5) if some z_i or some $\omega_i = \infty$ is left as an exercise.) □

Note that (5) affords a direct method to find the desired mapping by simply solving for ω.

EXAMPLE
To map $z_1 = 1,\ z_2 = 2,\ z_3 = 7$ onto $\omega_1 = 1,\ \omega_2 = 2,\ \omega_3 = 3$ we set

$$\frac{(\omega - 2)(3 - 1)}{(\omega - 1)(3 - 2)} = \frac{(z - 2)(7 - 1)}{(z - 1)(7 - 2)}$$

or, solving for ω,

$$\omega = \frac{7z - 4}{2z + 1}.$$

◊

13.3 Schwarz-Christoffel Transformations

I *Mapping a Semi-Infinite Strip Onto a Half-Plane*

We will show that $f(z) = \sin z$ maps the semi-infinite strip:

$$\frac{-\pi}{2} < \text{Re } z < \frac{\pi}{2}; \ \text{Im } z > 0$$

conformally onto the upper half-plane by considering its behavior on the rectangle R:

$$\frac{-\pi}{2} \le \text{Re } z \le \frac{\pi}{2}; 0 \le \text{Im } z \le N$$

for large N.

Of course, the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is mapped onto $[-1, 1]$. For complex z, we use the identity

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \tag{1}$$

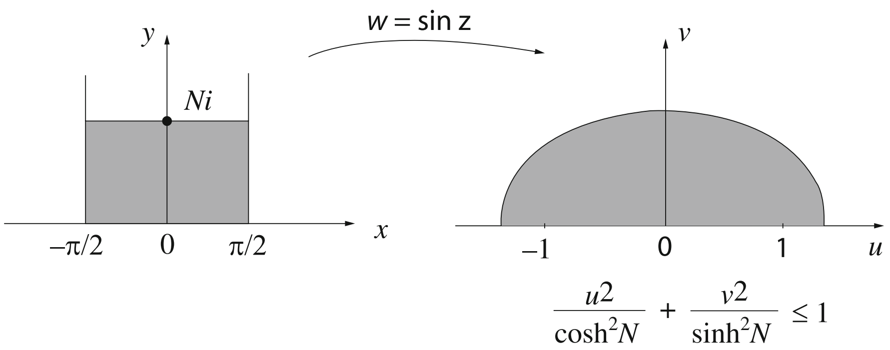
to show that $\sin(\frac{\pi}{2} + iy) = \cosh y$, which is real-valued and increases from 1 to $\cosh N$ as y increases from 0 to N. Note that the mapping $f(z) = \sin z$ doubles the vertex angle of R at $z = \frac{\pi}{2}$, as we could have anticipated since $f'(z)$ has a simple zero at that point.

Along the line $\text{Im } z = N$,

$$\sin z = \sin x \cosh N + i \cos x \sinh N \tag{2}$$

For large N, $\sinh N$ is just slightly smaller than $\cosh N$ since both are very close to $\frac{1}{2}e^N$. Hence, according to (2), as x varies from $\frac{\pi}{2}$ to $\frac{-\pi}{2}$, $\sin z$ traces an "almost circular" elliptical path from $\cosh N$ counterclockwise to $-\cosh N$.

Finally, $\sin z$ maps the interval connecting $\frac{-\pi}{2} + iN$ to $\frac{-\pi}{2}$ onto the interval $[-\cosh N, -1]$. So $\sin z$ maps the boundary of R onto the boundary of a region S whose base is the real interval $[-\cosh N, \cosh N]$ and which is very close to a semicircular region in the upper half-plane. By the Argument Principle, then, f maps the interior of R onto the interior of S (see Remark 2 following Corollary 10.9).



It follows, letting $N \to \infty$, that $\sin z$ maps the semi-infinite strip:

$$\frac{-\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}; \operatorname{Im} z > 0$$

conformally onto the upper half-plane. □

A mapping of any other semi-infinite strip onto a half-plane or onto any of the domains previously considered can be found by composing $\sin z$ with the appropriate conformal mappings.

It is interesting to examine the inverse function, $\sin^{-1} z$, which can be defined by the familiar integral formula:

$$\sin^{-1} z = \int_0^z \frac{1}{\sqrt{1-\zeta^2}} d\zeta$$

Unlike $\sin z$, which is an entire function, $\sin^{-1} z$ is not analytic at the points $z = \pm 1$. This follows immediately from the fact that its derivative, $\frac{1}{\sqrt{1-z^2}}$, approaches ∞ as $z \to \pm 1$. It is also evident geometrically since $\sin^{-1} z$ maps the straight angles at $z = \pm 1$ onto right angles. The function $\sin^{-1} z$ is analytic, however, as is $\frac{1}{\sqrt{1-z^2}}$, in the plane slit along the two rays $z = \pm 1 - iy$, $0 \leq y < \infty$. In particular, it is analytic in the upper half-plane and continuous to the boundary, including the points ± 1, since the improper definite integral $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ converges to $\pi/2$. As was the case with $\sin z$, it is easy to determine the behavior of $\sin^{-1} z$ along the boundary; i.e., along the $x - axis$.

To that end, note that its derivative:

$$\frac{d}{dz} \sin^{-1} z = \frac{1}{\sqrt{1-z^2}}$$

is positive on the interval $-1 < z < 1$. So $\sin^{-1} z$ maps the closed interval $[-1, 1]$ monotonically onto the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. To see how the analytic $\frac{1}{\sqrt{1-z^2}}$ behaves on the remainder of the line, consider $\sqrt{1+z}$ as z varies along a small semicircular arc from $-1 + r$ to $-1 - r$. That is, let $z = -1 + re^{i\theta}$, $0 \leq \theta \leq \pi$. Then

$$\sqrt{1+z} = \sqrt{re^{i\theta}} = \sqrt{r} e^{i\theta/2}$$

and at $z = -1 - r$, i.e. when $\theta = \pi$, $\sqrt{1+z}$ is a multiple of i. It follows that $\frac{1}{\sqrt{1+z}}$ and $\frac{1}{\sqrt{1-z^2}}$ are negative multiples of i throughout the interval $(-\infty, -1)$. This property of its derivative, and the fact that $\int_{-\infty}^{-1} \frac{1}{\sqrt{1-\zeta^2}} d\zeta$ diverges, show that $\sin^{-1} z$ maps $(-\infty, -1)$ onto the ray from $-\pi/2 + i\infty$ to $-\pi/2$. By analyzing $\sqrt{1-z}$ in a semicircular arc around $z = 1$, we can see that $\sin^{-1} z$ maps the interval $(1, \infty)$ onto the ray from $\pi/2$ to $\pi/2 + i\infty$. (This also follows from the Schwarz Reflection

Principle (7.9) since $\frac{1}{\sqrt{1-z^2}}$ is real-valued on the imaginary axis, so that $\sin^{-1} z$ maps the imaginary axis into itself.)

While our insights into the mapping properties of $\sin z$ were very dependent on formula (1), our analysis of the behavior of $\sin^{-1} z$ along the real line can easily be adapted to a wide range of problems. This will ultimately lead to the general Schwarz-Christoffel formula, but first we consider the following special case:

II *Mapping the Upper Half-Plane Onto a Rectangle.*

As we saw in the last section, if the argument of $f'(z)$ is constant along a straight line, $f(z)$ will map that line into another line. To be specific, recall that

$$f(z) - f(z_0) = \int_{z_0}^{z} f'(\zeta)d\zeta$$

Hence if γ represents the ray: $z = z_0 + re^{i\alpha}$, $r > 0$, and if $Arg(f') = \beta$, then $\Gamma = f(\gamma)$ will travel along the ray

$$\omega = f(z_0) + se^{i(\alpha+\beta)}, s > 0$$

More specifically, if z travels to the right along the real axis from a real point z_0, and if f' has a constant argument of θ, then $f(z)$ will travel along the ray from $f(z_0)$ with argument θ.

If we want to find a function f which maps the upper half-plane onto a rectangle, we would like f to map the real line onto the four sides of the rectangle. This suggests that f' should have exactly four different arguments on the segments of the real line and that its argument should increase by $\frac{\pi}{2}$ as we move (to the right) from one segment to the next. To create such a function f', note that if z_0 represents any real number, $z - z_0$ has a constant argument of π for real $z < z_0$, and a constant argument of 0 for real $z > z_0$. If we define the analytic function $(z - z_0)^{-\alpha}$ so that it is positive for real $z > z_0$, its argument will increase from $-\alpha\pi$ to 0 as z crosses the point z_0. In particular, with $\alpha = 1/2$, the argument of $(z - z_0)^{-\alpha}$ increases by $\pi/2$ as z crosses over the point z_0. So, to define f', we can pick four arbitrary real numbers $a < b < c < d$ and let

$$f'(z) = 1/\sqrt{(z-a)(z-b)(z-c)(z-d)}.$$

The square root in the denominator is defined as the product of the square roots of each of its linear factors, and each of these is defined to be positive for large positive values of z. We can then define

$$f(z) = \int_0^z f'(\zeta)d\zeta = \int_0^z \frac{1}{\sqrt{(\zeta-a)(\zeta-b)(\zeta-c)(\zeta-d)}}d\zeta \qquad (3)$$

Note that although $f'(z)$ is undefined at the (real) points a, b, c, d, it is analytic in the entire complex plane slit along the four rays from t to $t - i\infty$, for

$t = a, b, c, d$. So it is analytic on the closed upper half-plane minus the points a, b, c, d. Technically, the path of integration in (3) should be indented slightly to avoid these points. However, since the associated improper integrals are all convergent, the path of integration can be along the real line for all real z, including $z = a, b, c, d$, and f is continuous at all points of the real line. Moreover, since the real integral

$$\int_d^\infty \frac{1}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}dx$$

converges, $\lim_{z\to\infty} f(z)$ exists, as does $\lim_{z\to-\infty} f(z)$.

According to our earlier remarks, f maps the interval $(-\infty, a]$ onto a finite interval parallel to the real line, and it maps the four successive intervals: $[a, b]$,$[b, c]$, $[c, d]$, $[d, \infty)$ onto intervals, each of which represents a counterclockwise rotation of $\pi/2$ from its predecessor. These facts alone do not guarantee that the image of the real line is the boundary of a rectangle. It does follow, however, once we can show that $f(-\infty) = f(\infty)$; i.e., that

$$\int_{-\infty}^\infty \frac{1}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}dx = 0 \qquad (4)$$

This follows by the type of argument used in Chapter 11. Let C_R be the closed contour consisting of the real interval $[-R, R]$ followed by Γ_R, the upper semicircle of radius R, traversed counterclockwise from R to $-R$, and let $C_{R,\varepsilon}$ be the contour formed by replacing each interval in C_R of the form $[t-\varepsilon, t+\varepsilon]$, $t = a, b, c, d$ with a semicircle in the upper half-plane centered at t, with radius ε. By the Cauchy Closed Curve Theorem,

$$\int_{C_{R,\varepsilon}} f'(\zeta)d\zeta = 0$$

and, letting $\varepsilon \to 0$, we see that

$$\int_{C_R} f'(\zeta)d\zeta = 0$$

That is,

$$\int_{-R}^{R} \frac{1}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}dx + \int_{\Gamma_R} f'(\zeta)d\zeta = 0$$

Since $|f'(z)|$ is asymptotic to $1/R^2$ throughout Γ_R, the usual $M - L$ estimate shows that the second integral above approaches 0 as $R \to \infty$, thus proving equation (4).

An analogous argument can be used to show that f maps the upper half-plane conformally onto the inside of the rectangle. We need only note that for any point ω inside the rectangle, if R is sufficiently large, f maps C_R onto a contour which is just a slight perturbation of the boundary of the rectangle and hence winds around the point ω exactly once. By the Argument Principle, f takes the value ω exactly once inside C_R. The general result follows by letting $R \to \infty$. □

It is interesting that we never directly established that the definite integrals which yield the lengths of opposite sides of the rectangle have the same magnitude. In fact, if it weren't an obvious corollary of our other arguments, it would be hard to verify directly. For example, if we let a, b, c, d equal 1, 2, 5, 9, respectively, it follows that

$$\int_1^2 \frac{1}{\sqrt{(x-1)(2-x)(5-x)(9-x)}}dx = \int_5^9 \frac{1}{\sqrt{(x-1)(x-2)(x-5)(9-x)}}dx$$

One particularly nice choice for a, b, c, d is the set of values $-1/k, -1, 1, 1/k$ with $k < 1$. The resulting formula for $f(z)$ given by (3), and with a suitable additional constant factor, is

$$f(z) = \int_0^z \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}d\zeta$$

This is known as an elliptic integral of the first kind. In this form it is easy to verify directly that opposite sides of the rectangle obtained have equal length. It is also easy to show that by choosing an appropriate value of k, the rectangle obtained can have adjacent sides of any desired ratio.

Note also that if we omitted the fourth point d in formula (3), the resulting function

$$f(z) = \int_0^z f'(\zeta)d\zeta = \int_0^z \frac{1}{\sqrt{(\zeta-a)(\zeta-b)(\zeta-c)}}d\zeta$$

would still map the (closed) upper half-plane, with the point at ∞, onto a closed rectangle. This follows from the behavior of $f(z)$ along the real axis and from the fact that in this case, as in formula (3), $f(-\infty) = f(\infty)$. Here, the point at infinity takes the place of the "missing" point d, and is mapped by f onto one of the vertices of the rectangle.

III *Mapping the Upper Half Plane Onto any Convex Polygon*

The ideas of the previous section are easily generalized to find a conformal mapping f of the upper half-plane onto a convex polygon with any number of sides and any interior angles. To assure that f maps the real line (with the point at infinity) onto the boundary of such a polygon, we choose n real points $a_1 < a_2 < \cdots < a_n$, and define

$$f'(z) = (z-a_1)^{-\alpha_1}(z-a_2)^{-\alpha_2}\cdots(z-a_n)^{-\alpha_n}$$

where the n exterior angles of the polygon are equal in order to $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$. As in the previous section, each of the analytic functions $(z-a_i)^{-\alpha_i}$ is defined so that its argument is 0 for real $z > a_i$ and $-\alpha_i\pi$ for real $z < a_i$. Since the desired image polygon is convex, $0 < \alpha_i < 1$ for each i, and $\sum \alpha_i = 2$. So if we once again define $f(z) = \int_0^z f'(\zeta)d\zeta$, it follows that the argument of f' increases by $\alpha_i \pi$, and f has a corresponding change in direction, as z crosses the real point a_i. Moreover, by the same reasoning used in the previous section, $\lim_{z\to\infty} f(z)$ and $\lim_{z\to-\infty} f(z)$ both exist and are equal. Taken together, these properties assure that f is a conformal mapping of the upper half-plane onto a polygon of the desired type.

Here, too, if we omitted the final factor: $(z - a_n)^{\alpha_n}$ from the formula for f, it would still map the upper half-plane onto a polygon of the desired type. As in the case of the rectangle, the point at ∞ would map into one of the n vertices of the polygon.

In this general setting, it is more difficult to show that, with the proper choices of $a_1, a_2, \ldots a_n$, the function f can be made to map the upper half-plane onto an arbitrary polygon; that is, onto a polygon with any desired shape. On the other hand, it is not hard to define the mappings onto certain special polygons. This is especially easy for triangles since their shape is entirely determined by their angles. Thus, it follows e.g. that

$$f(z) = \int_0^z (\zeta^2 - 1)^{-2/3} d\zeta$$

maps the upper half-plane onto an equilateral triangle. □

Note: The techniques used above can be applied equally well to finding mappings from the upper half-plane onto a wide variety of polygonal regions. Many such examples, and much more information regarding Schwarz-Christoffel mappings, can be found in the classic book of Nehari.

Exercises

1. Verify directly that $f(z) = z^k$ is locally 1-1 for $z \neq 0$, k any nonzero integer.
2. Find the image under $\omega = e^z$ of the lines $x =$ constant and $y =$ constant.
3. Find a conformal mapping f between the regions S and T, where

   i. $S = \{z = x + iy : -2 < x < 1\}$; $T = D(0; 1)$
   ii. $S = T =$ the upper half-plane; $f(-2) = -1$, $f(0) = 0$ and $f(2) = 2$
   iii. $S = \{re^{i\theta} : r > 0$ and $0 < \theta < \pi/4\}$; $T = \{x + iy : 0 < y < 1\}$
   iv. $S = D(0; 1)\backslash[0, 1]$; $T = D(0; 1)$.

   [*Hint*: For (iv) use the mapping of the upper semi-disc onto a quadrant.]

4.* Find a conformal mapping of the region "between" the circles: $|z| = 2$ and $|z - 1| = 1$ onto the unit disc.

5.* Find a conformal mapping of the semi-infinte strip: $x > 0$, $0 < y < 1$ onto the unit disc.

6.* Find a conformal mapping of the semi-disc $S = \{z : |z| < 1, \text{Im } z > 0\}$ onto the unit disc.

7.* Verify that "conformal equivalence" satisfies the reflexive, symmetric and transitive properties of an equivalence relation.

8. a. Prove that a linear function maps polygons onto polygons.
   b. Suppose f is entire and, for some rectangle R, $f(R)$ is a rectangle. Prove f is linear.

9. Prove that bilinear mappings form a group under composition.

10. Find the image of the circle $|z| = 1$ under the mappings

    a. $\omega = \dfrac{1}{z}$,
    b. $\omega = \dfrac{1}{z-1}$,
    c. $\omega = \dfrac{1}{z-2}$.

11. Show that the only automorphism of the unit disc with $f(0) = 0$, $f'(0) > 0$ is the identity map $f(z) \equiv z$.

12. Suppose f_1 and f_2 are both conformal mappings of a region D onto the unit disc and for some $z_0 \in D$,

$$f_1(z_0) = f_2(z_0) = 0; \quad f_1'(z_0),\ f_2'(z_0) > 0.$$

    Prove $f_1 \equiv f_2$.

13. Show that all conformal mappings of a half-plane or disc onto a half-plane or disc are given by bilinear transformations.

14. What is the image of the upper half-plane under a mapping of the form

$$f(z) = \frac{az+b}{cz+d} \quad a,\ b,\ c,\ d \text{ real}; \quad ad - bc < 0?$$

15. Find a formula for all the automorphisms of the first quadrant.

16. Complete Theorem 13.17 by showing h is of the form

$$h(z) = \frac{az+b}{cz+d} \quad a,\ b,\ c,\ d \text{ real}; \quad ad - bc > 0.$$

    [*Hint*: Write $h = h_1 \circ h_2$ where

$$h_1(z) = \left(\frac{z-i}{z+i}\right)^{-1} \circ e^{i\theta} z \circ \left(\frac{z-i}{z+i}\right)$$
$$h_2(z) = \left(\frac{z-i}{z+i}\right)^{-1} \circ \left(\frac{z-\alpha}{1-\bar{\alpha}z}\right) \circ \left(\frac{z-i}{z+i}\right).$$

    Show, then that

$$h_1(z) = \frac{(1+\cos\theta)z + \sin\theta}{(-\sin\theta)z + (1+\cos\theta)}$$
$$h_2(z) = \frac{(1-\operatorname{Re}\alpha)z + \operatorname{Im}\alpha}{(\operatorname{Im}\alpha)z + (1+\operatorname{Re}\alpha)}.]$$

17. Find the fixed points of the mappings
    a. $\omega = \dfrac{z-1}{z+1}$, b. $\omega = \dfrac{z}{z+1}$.

18. Prove that $(z_1,\ z_2,\ z_3,\ z_4)$ is real-valued if and only if the four points $z_1,\ z_2,\ z_3,\ z_4$ lie on a circle or line.

19. Find the bilinear mappings which send

    a. $1, i, -1$ onto $-1, i, 1$, respectively
    b. $-i, 0, i$ onto $0, i, 2i$, respectively
    c. $-i, i, 2i$ onto $\infty, 0, \frac{1}{3}$ respectively.

20. Find a conformal map f of the region between the two circles $|z| = 1$ and $|z - \frac{1}{4}| = \frac{1}{4}$ onto an annulus $a < |z| < 1$. [*Hint*: Find a bilinear map which simultaneously maps $|z| < 1$ onto $|z| < 1$ and $|z - \frac{1}{4}| < \frac{1}{4}$ onto a disc of the form $|z| < a$.]

21.* Find the image of the upper half-plane under the mapping $f(z) = \int_0^z 1/\sqrt{\zeta^2 - 1} d\zeta$. How is this function related to $\sin^{-1} z$?

22.* Find a mapping of the upper half-plane onto an isosceles right triangle.

23.* Find a mapping of the upper half-plane onto a square. [Hint: Let the point at infinity map onto one of the vertices of the square.]