Chapter 9
Isolated Singularities of an Analytic Function

9.1 Classification of Isolated Singularities; Riemann's Principle and the Casorati-Weierstrass Theorem

Introduction While we have concentrated until now on the general properties of analytic functions, we now focus on the special behavior of an analytic function in the neighborhood of an "isolated singularity."

We will use the term *deleted neighborhood of* z_0 to denote a set of the form $\{z : 0 < |z - z_0| < d\}$.

9.1 Definition

f is said to have an *isolated singularity at* z_0 if f is analytic in a deleted neighborhood D of z_0 but is *not* analytic at z_0.

Note that, by Theorem 7.7, f must be *discontinuous* at an isolated singularity.

EXAMPLES

i. The function defined by $f(z) = \begin{cases} \sin z & z \neq 2 \\ 0 & z = 2 \end{cases}$
has an isolated singularity at $z = 2$.

ii. $g(z) = 1/(z - 3)$ has an isolated singularity at $z = 3$.

iii. $\exp(1/z)$ has an isolated singularity at $z = 0$. ◊

As we shall soon see, the above examples represent the different types of isolated singularities. These may be classified as follows.

9.2 Definition

Suppose f has an isolated singularity at z_0.

i. If there exists a function g, analytic at z_0 and such that $f(z) = g(z)$ for all z in some deleted neighborhood of z_0, we say f has a *removable singularity* at z_0 (i.e., if the value of f is "corrected" at the point z_0, it becomes analytic there).

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ii. If, for $z \neq z_0$, f can be written in the form $f(z) = A(z)/B(z)$ where A and B are analytic at z_0, $A(z_0) \neq 0$, and $B(z_0) = 0$, we say f has a *pole* at z_0. (If B has a zero of order k at z_0, we say that f has a pole of order k.)

iii. If f has neither a removable singularity nor a pole at z_0, we say f has an *essential singularity* at z_0.

The following theorems show how the nature of the singularity possessed by a function may be determined by its behavior in a deleted neighborhood of the singularity.

9.3 Riemann's Principle of Removable Singularities

If f has an isolated singularity at z_0 and if $\lim_{z \to z_0}(z - z_0)f(z) = 0$, then the singularity is removable.

Proof

Consider

$$h(z) = \begin{cases} (z - z_0)f(z) & z \neq z_0 \\ 0 & z = z_0. \end{cases}$$

By hypothesis, h is continuous at z_0. Since h, like f, is analytic in a deleted neighborhood of z_0, it follows that h is analytic at z_0 (Theorem 7.7). Since $h(z_0) = 0$, $g(z) = h(z)/(z - z_0)$ is likewise analytic at z_0 and equals f for $z \neq z_0$. □

9.4 Corollary

If f is bounded in a deleted neighborhood of an isolated singularity, the singularity is removable.

9.5 Theorem

If f is analytic in a deleted neighborhood of z_0 and if there exists a positive integer k such that

$$\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0 \quad \textit{but} \quad \lim_{z \to z_0} (z - z_0)^{k+1} f(z) = 0,$$

then f has a pole of order k at z_0.

Proof

If we set

$$g(z) = \begin{cases} (z - z_0)^{k+1} f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

then g is continuous and hence analytic at z_0. Furthermore, since $g(z_0) = 0$,

$$A(z) = \frac{g(z)}{z - z_0} = (z - z_0)^k f(z)$$

is likewise analytic at z_0, and by hypothesis $A(z_0) \neq 0$. Since

$$f(z) = \frac{A(z)}{(z - z_0)^k} \quad \text{for } z \neq z_0$$

the proof is complete. □

Note that according to the previous two theorems, there is no analytic function which approaches ∞ like a fractional power of $1/(z - z_0)$ in the neighborhood of an isolated singularity z_0. For example, if f were analytic in a deleted neighborhood of 0 and satisfied $|f(z)| \leq 1/\sqrt{|z|}$, then by 9.3, f would be bounded since the singularity would be removable. Similarly, given that

$$|f(z)| \leq \frac{1}{|z|^{5/2}},$$

we conclude that $z^2 f(z)$ has a removable singularity at 0. Hence f has a pole of order at most 2 at the origin and, in fact, $f(z) \leq A/|z|^2$.

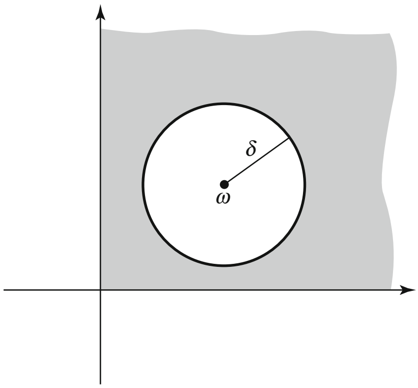
It also follows that in the neighborhood of an essential singularity, a function f must be not only unbounded but such that, for each integer N, $(z - z_0)^N f(z) \not\to 0$ as $z \to z_0$. It does not follow, however, that $f(z) \to \infty$ as $z \to z_0$. In fact, the following theorem shows that the set of values assumed by a function in the neighborhood of an essential singularity is "dense" in the whole complex plane. That is, the range of f intersects every disc in $\mathbb{C}$.

9.6 Casorati-Weierstrass Theorem

If f has an essential singularity at z_0 and if D is a deleted neighborhood of z_0, then the range $R = \{f(z): z \in D\}$ is dense in the complex plane.

Proof

Assume there exists some disc with center ω and radius δ which does not intersect R.



Then $|f(z) - \omega| > \delta$ and

$$\left|\frac{1}{f(z) - \omega}\right| < \frac{1}{\delta} \text{ throughout } D.$$

By Riemann's Principle (9.3), it follows that $1/(f(z) - \omega)$ has (at most) a removable singularity at z_0. Hence

$$\frac{1}{f(z) - \omega} = g(z)$$

where g is analytic at z_0. But then

$$f(z) = \omega + \frac{1}{g(z)}$$

so that f has either a pole (if $g(z_0) = 0$) or a removable singularity (if $g(z_0) \neq 0$) at z_0. □

There is, in fact, a much stronger form of the Casorati-Weierstrass Theorem–known as Picard's Theorem–which asserts that an analytic function takes every value with at most a single exception in the neighborhood of an essential singularity.

9.2 Laurent Expansions

In Chapter 6, we saw that functions analytic in a disc could be represented there by power series. A somewhat similar representation–by "two-sided power series" of the form $\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$–can be derived for functions analytic in an annulus $R_1 < |z - z_0| < R_2$. These two-sided power series, known as Laurent expansions, are valuable tools in the study of isolated singularities.

9.7 Definition

We say $\sum_{k=-\infty}^{\infty} \mu_k = L$ if both $\sum_{k=0}^{\infty} \mu_k$ and $\sum_{k=1}^{\infty} \mu_{-k}$ converge and if the sum of their sums is L.

9.8 Theorem

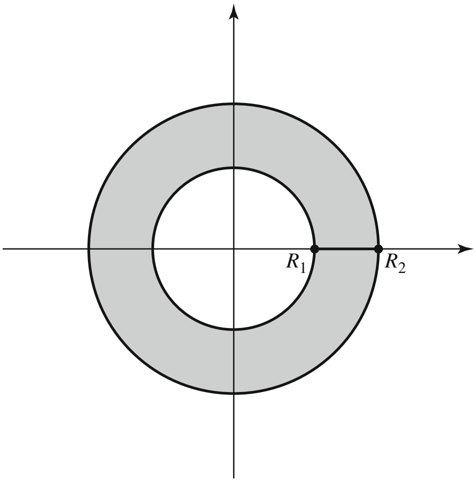
$f(z) = \sum_{-\infty}^{\infty} a_k z^k$ *is convergent in the domain*

$$D = \{z : R_1 < |z| \text{ and } |z| < R_2\}$$

where

$$R_2 = 1/ \lim_{k\to\infty} \sup |a_k|^{1/k}$$
$$R_1 = \lim_{k\to\infty} \sup |a_{-k}|^{1/k}.$$

If $R_1 < R_2$, D is an annulus and f is analytic in D.



Proof

By Theorem 2.8,

$$f_1(z) = \sum_{k=0}^{\infty} a_k z^k \text{ converges for } |z| < R_2$$

and similarly

$$f_2(z) = \sum_{-\infty}^{-1} a_k z^k = \sum_{1}^{\infty} a_{-k} (\frac{1}{z})^k$$

converges for

$$\left|\frac{1}{z}\right| < \frac{1}{R_1}, \text{ or } |z| > R_1.$$

Hence $\sum_{-\infty}^{\infty} a_k z^k$ converges for all z in the intersection. Also, since f_1 is a power series and $f_2(z) = g(1/z)$ where g is a power series, f_1 and f_2 are both analytic in their respective domains of convergence. Hence f is analytic in the intersection of these domains. □

9.9 Theorem

If f is analytic in the annulus A: $R_1 < |z| < R_2$, then f has a Laurent expansion, $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$, in A.

Proof

Let C_1 and C_2 represent circles centered at 0 of radii r_1 and r_2, respectively, with $R_1 < r_1 < r_2 < R_2$. Fix z with $r_1 < |z| < r_2$. Then

$$g(w) = \frac{f(w) - f(z)}{w - z}$$

is analytic in A, and by Cauchy's Theorem

$$\int_{C_2-C_1} g(w)dw = 0.$$

(See Example 2 following Theorem 8.6.) Thus

$$\int_{C_2-C_1} \frac{f(z)}{w-z}dw = \int_{C_2-C_1} \frac{f(w)}{w-z}dw. \tag{1}$$

Note then that $\int_{C_2} dw/(w-z) = 2\pi i$, according to Lemma 5.4, while $\int_{C_1} dw/(w-z) = 0$ by Cauchy's Theorem so that

$$\int_{C_2-C_1} \frac{f(z)}{w-z}dw = 2\pi i f(z). \tag{2}$$

Combining (1) and (2), we have

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}dw. \tag{3}$$

Now, on C_2, $|w| > |z|$ so that

$$\frac{1}{w-z} = \frac{1}{w\left(1-\frac{z}{w}\right)} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \cdots$$

while on C_1, since $|w| < |z|$,

$$\frac{1}{w-z} = \frac{-1}{z-w} = -\frac{1}{z} - \frac{w}{z^2} - \frac{w^2}{z^3} - \cdots,$$

the convergence being uniform in both cases. Substitution into (3), then, yields

$$f(z) = \frac{1}{2\pi i}\int_{C_2}\left(\sum_{k=0}^{\infty} \frac{f(w)z^k}{w^{k+1}}\right)dw + \frac{1}{2\pi i}\int_{C_1}\left(\sum_{k=-1}^{-\infty} \frac{f(w)z^k}{w^{k+1}}\right)dw$$

and switching the order of summation and integration,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \quad a_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}dw$$

where C is any circle in A centered at 0, for *all* $z \in A$. For although, in the course of the proof, we have

$$C = \begin{cases} C_2 & \text{for } k \geq 0 \\ C_1 & \text{for } k < 0, \end{cases}$$

in fact C can be taken as any circle in A centered at 0. This follows again from the fact that

$$g(w) = \frac{f(w)}{w^{k+1}}$$

is analytic in A and from the Cauchy Closed Curve Theorem. □

Note that the Laurent expansion is unique. That is, if

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n$$

in an annulus, then

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}} dz \tag{4}$$

where C is as above. For if $\sum_{-\infty}^{\infty} a_n z^n$ converges in A, it converges uniformly along C, and thus

$$\int_C \frac{f(z)}{z^{k+1}} dz = \sum_{n=-\infty}^{\infty} \int_C a_n z^{n-k-1} dz. \tag{5}$$

Since

$$\int_C z^p dz = \begin{cases} 2\pi i & p = -1 \\ 0 & \text{any integer } p \neq -1, \end{cases}$$

it follows that

$$\int_C \frac{f(z)}{z^{k+1}} dz = 2\pi i a_k,$$

proving (4).

9.10 Corollary

If f is analytic in the annulus $R_1 < |z - z_0| < R_2$, then f has a unique representation

$$f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz$$

and $C = C(z_0; R)$ with $R_1 < R < R_2$.

Proof

Simply apply the previous results to $g(z) = f(z + z_0)$, which is analytic in an annulus centered at 0. □

If we set $R_1 = 0$, we obtain:

9.11 Corollary

If f has an isolated singularity at z_0, then for some $\delta > 0$, and $0 < |z - z_0| < \delta$

$$f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k,$$

where the a_k are defined as in Corollary 9.10.

EXAMPLES

(i)

$$\frac{(z+1)^2}{z} = \frac{1}{z} + 2 + z \text{ for all } z \neq 0.$$

(ii)

$$\begin{aligned} \frac{1}{z^2(1-z)} &= \frac{1}{z^2}(1 + z + z^2 + \cdots) \\ &= \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \text{ for } 0 < |z| < 1. \end{aligned}$$

(iii)

$$\begin{aligned} \frac{1}{z^2(1-z)} = \frac{-1}{z^2(z-1)} &= \frac{-1}{[1 + (z-1)]^2\,(z-1)} \\ &= \frac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - + \cdots \\ &\text{for } 0 < |z-1| < 1. \end{aligned}$$

(iv)

$$\exp(1/z) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots \text{ for } z \neq 0.$$

◊

9.12 Definition

If $f(z) = \sum a_k (z - z_0)^k$ is the Laurent expansion of f about an isolated singularity z_0, $\sum_{-\infty}^{-1} a_k (z - z_0)^k$ is called the *principal part* of f at z_0; $\sum_0^{\infty} a_k (z - z_0)^k$ is called the *analytic part*.

Because of the uniqueness of the Laurent expansion, we can derive the following characterizations of the principal parts around the different types of singularities.

(i) If f has a removable singularity at z_0, all the coefficients C_{-k} of its Laurent expansion about z_0, for $k > 0$, are 0.

Proof

Since $f(z) = g(z)$ for $z \neq z_0$, the Laurent expansion for f must agree with the *Taylor* expansion for g around z_0. □

EXAMPLE

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \cdots .$$

◊

(ii) If f has a pole of order k at z_0, $C_{-k} \neq 0$ but $C_{-N} = 0$ for all $N > k$.

Proof

Since $f(z) = A(z)/B(z)$ where $A(z_0) \neq 0$ and B has a zero of order k at z_0,

$$f(z) = \frac{Q(z)}{(z - z_0)^k},$$

where Q is analytic and nonzero at z_0. Hence if $Q(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, then

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{(z - z_0)^n}{(z - z_0)^k} = \sum_{j=-k}^{\infty} C_j (z - z_0)^j$$

where $C_j = a_{j+k}$. Thus, $C_{-k} = a_0 = Q(z_0) \neq 0$. □

(iii) If f has an essential singularity at z_0, it must have infinitely many nonzero terms in its principal part.

Proof

Otherwise $(z - z_0)^N f(z)$ would be analytic at z_0 for large enough N and f would have a pole at z_0. □

The so-called partial fraction decomposition of proper rational functions can be derived as a corollary of the theory of Laurent expansions.

9.13 Partial Fraction Decomposition of Rational Functions

Any proper rational function

$$\mathcal{R}(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z - z_1)^{k_1} (z - z_2)^{k_2} \cdots (z - z_n)^{k_n}},$$

where P *and* Q *are polynomials with* $\deg P < \deg Q$, *can be expanded as a sum of polynomials in* $1/(z - z_k)$, $k = 1, 2, \ldots, n$.

Proof

Since $\mathcal{R}$ has a pole of order at most k_1 at z_1,

$$\mathcal{R}(z) = P_1\left(\frac{1}{z - z_1}\right) + A_1(z)$$

where $P_1(1/(z - z_1))$ is the principal part of $\mathcal{R}$ around z_1 and A_1 is the analytic part. Furthermore

$$A_1(z) = \mathcal{R}(z) - P_1\left(\frac{1}{z - z_1}\right)$$

has a removable singularity at z_1 and the same principal parts as $\mathcal{R}$ at $z_2, \ \ldots, z_n$. Thus, if we take $P_2(1/(z - z_2))$ to be the principal part of $\mathcal{R}$ around z_2 and proceed inductively, we find

$$A_n(z) = \mathcal{R}(z) - \left[P_1\left(\frac{1}{z - z_1}\right) + P_2\left(\frac{1}{z - z_2}\right) + \cdots + P_n\left(\frac{1}{z - z_n}\right)\right]$$

is an entire function (once it is defined "correctly" at $z_1, \ z_2, \ldots z_n$). Furthermore, A_n is bounded since $\mathcal{R}$ and all its principal parts approach 0 as $z \to \infty$. Thus, by Liouville's Theorem (5.10), A_n is constant; indeed $A_n \equiv 0$. Hence.

$$\mathcal{R}(z) = P_1\left(\frac{1}{z - z_1}\right) + P_2\left(\frac{1}{z - z_2}\right) + \cdots + P_n\left(\frac{1}{z - z_n}\right).$$

□

Exercises

1. Suppose $f(z) \to \infty$ as $z \to z_0$, an isolated singularity. Show that f has a pole at z_0.
2. Does there exist a function f with an isolated singularity at 0 and such that $|f(z)| \sim \exp(1/|z|)$ near $z = 0$?
3.* Suppose that f is an entire $1-1$ function. Show that $f(z) = az + b$.
4. Suppose f is analytic in the punctured plane $z \neq 0$ and satisfies $|f(z)| \leq \sqrt{|z|} + 1/\sqrt{|z|}$. Prove f is constant.
5.* Suppose f and g are entire functions with $|f(z)| \leq |g(z)|$ for all z. Prove that $f(z) = cg(z)$, for some constant c.
6. Verify directly that $e^{1/z}$ takes every value (with a single exception) in the annulus: $0 < |z| < 1$. What is the missing value?
7. Suppose f and g have poles of order m and n, respectively, at z_0. What can be said about the singularity of $f + g, \ f \cdot g, \ f/g$ at z_0?
8.* Suppose f has an isolated singularity at z_0. Show that z_0 is an essential singularity if and only if there exist sequences $\{a_n\}$ and $\{\beta_n\}$ with $\{a_n\} \to z_0$, $\{\beta_n\} \to z_0$, $\{f(a_n)\} \to 0$, and $\{f(\beta_n)\} \to \infty$.

9. Classify the singularities of

   a. $\dfrac{1}{z^4 + z^2}$
   b. $\cot z$
   c. $\csc z$
   d. $\dfrac{\exp(1/z^2)}{z - 1}$.

10.* Find the principal part of the Laurent expansion of

$$f(z) = \frac{1}{(z^2 + 1)^2}$$

about the point $z = i$.

11. Find the Laurent expansion for

   a. $\dfrac{1}{z^4 + z^2}$ about $z = 0$
   b. $\dfrac{\exp(1/z^2)}{z - 1}$ about $z = 0$
   c. $\dfrac{1}{z^2 - 4}$ about $z = 2$.

12.* Find the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ (in powers of z) for

   a. $0 < |z| < 1$
   b. $1 < |z| < 2$
   c. $|z| > 2$.

13.* Let $\{a_1, a_2, ..., a_k\}$ be a set of positive integers and

$$R(z) = \frac{1}{(z^{a_1} - 1)(z^{a_2} - 1) \cdots (z^{a_k} - 1)}.$$

Find the coefficient c_{-k} in the Laurent expansion for $R(z)$ about the point $z = 1$.

14. Show that if f is analytic in $z \neq 0$ and "odd" (i.e., $f(-z) = -f(z)$) then all the even terms in its Laurent expansion about 0 are 0.

15. Find partial fraction decompositions for

   a. $\dfrac{1}{z^4 + z^2}$ b. $\dfrac{1}{z^2 + 1}$.

16. Suppose f is analytic in a deleted neighborhood D of z_0 except for poles at all points of a sequence $\{z_n\} \to z_0$. (Note that z_0 is *not* an isolated singularity.) Show that $f(D)$ is dense in the complex plane. [*Hint*: Assume, as in the proof of the Casorati-Weierstrass Theorem, that $|f(z) - w| > \delta$ and consider $g(z) = 1/(f(z) - w)$.]

17. Show that the image of the unit disc minus the origin under

$$f(z) = \csc 1/z$$

is dense in the complex plane

a. by noting that $\sin(1/z)$ has an essential singularity at $z = 0$,
b. by applying Exercise 16 to $f(z)$.

18. Prove that the image of the plane under a nonconstant entire mapping is dense in the plane. [*Hint*: If f is not a polynomial, consider $f(1/z)$.]