Chapter 4
Line Integrals and Entire Functions

Introduction

Recall that, according to Theorem 2.9, an everywhere convergent power series represents an entire function. Our main goal in the next two chapters is the somewhat surprising converse of that result: namely, that *every* entire function can be expanded as an everywhere convergent power series. As an immediate corollary, we will be able to prove that every entire function is infinitely differentiable. To arrive at these results, however, we must begin by discussing integrals rather than derivatives.

4.1 Properties of the Line Integral

4.1 Definition

Let $f(t) = u(t) + iv(t)$ be any continuous complex-valued function of the real variable t, $a \le t \le b$.

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

4.2 Definition

a. Let $z(t) = x(t) + iy(t)$, $a \le t \le b$. The curve determined by $z(t)$ is called *piecewise differentiable* and we set

$$\dot{z}(t) = x'(t) + iy'(t)$$

if x and y are continuous on $[a, b]$ and continuously differentiable on each subinterval $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$ of some partition of $[a, b]$.

b. The curve is said to be *smooth* if, in addition, $\dot{z}(t) \neq 0$ (i.e., $x'(t)$ and $y'(t)$ do not both vanish) except at a finite number of points.

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Throughout the remainder of the text, all curves will be assumed to be smooth unless otherwise stated.

Finally, we define the important concept of a line integral.

4.3 Definition

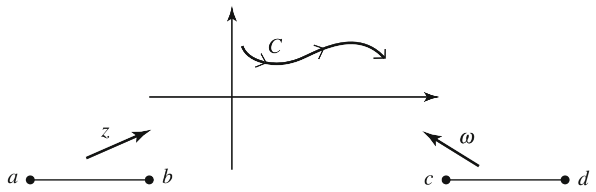
Let C be a smooth curve given by $z(t),\ a \le t \le b$, and suppose f is continuous at all the points $z(t)$. Then, the *integral of f along C* is

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt.$$

Note that the integral along the curve C depends not only on the points of C but on the direction as well. However, we will show that it is independent of the particular parametrization. Intuitively, if $z(t)$, $a \le t \le b$, and $\omega(t)$, $c \le t \le d$, trace the same curve in the same direction, then $\lambda = z^{-1} \circ \omega$ will be a 1-1 mapping of $[c, d]$ onto $[a, b]$ such that

$$\omega(t) = z(\lambda(t)). \tag{1}$$

However, if z is not 1-1, it is difficult to define z^{-1}. Instead, we take the existence of some λ that satisfies (1) as the definition for equivalent curves.



4.4 Definition

The two curves

$$C_1 : z(t), \quad a \le t \le b$$

and

$$C_2 : \omega(t), \quad c \le t \le d$$

are *smoothly equivalent* if there exists a 1-1 C^1 mapping $\lambda(t) : [c, d] \to [a, b]$ such that $\lambda(c) = a$, $\lambda(d) = b$, $\lambda'(t) > 0$ for all t, and

$$\omega(t) = z(\lambda(t)).$$

(It is easy to verify that the above is an equivalence relation. See Exercise 1.)

4.5 Proposition

If C_1 and C_2 are smoothly equivalent, then

$$\int_{C_1} f = \int_{C_2} f.$$

Proof

Suppose $f(z) = u(z) + iv(z)$, C_1 and C_2 as above. Then, by definition

$$\int_{C_1} f = \int_a^b u(z(t))x'(t)dt - \int_a^b v(z(t))y'(t)dt$$
$$+ i\int_a^b u(z(t))y'(t)dt + i\int_a^b v(z(t))x'(t)dt \qquad (1)$$

while

$$\int_{C_2} f = \int_c^d [u(z(\lambda(t))) + iv(z(\lambda(t)))][x'(\lambda(t)) + iy'(\lambda(t))]\lambda'(t)dt. \qquad (2)$$

Expanding the integrand in (2) and analyzing the four terms separately, we find that they are exactly equal to the four corresponding terms in (1).

For example

$$\int_c^d u(z(\lambda(t)))x'(\lambda(t))\lambda'(t)dt = \int_a^b u(z(t))x'(t)dt,$$

by the change-of-variable theorem for ordinary real integrals, and the proof is complete. □

The following proposition points out the dependence of the line integral on the direction of the curve.

4.6 Definition

Suppose C is given by $z(t)$, $a \le t \le b$. Then $-C$ is defined by $z(b + a - t)$, $a \le t \le b$. (Intuitively, $-C$ is the point set of C traced in the opposite direction.)

4.7 Proposition

$$\int_{-C} f = -\int_C f.$$

Proof

$$\int_{-C} f = -\int_a^b f(z(b + a - t))\dot{z}(b + a - t)dt.$$

Again, expanding the integral into real and imaginary parts and applying the change-of-variable theorem to each (real) integral, we find

$$\int_{-C} f = \int_b^a f(z(t))\dot{z}(t)dt = -\int_C f.$$

□

EXAMPLE 1

Suppose $f(z) = x^2 + iy^2$ (where x and y denote the real and imaginary parts of z, respectively), and consider

$$C : z(t) = t + it, \quad 0 \leq t \leq 1.$$

Then

$$\dot{z}(t) = 1 + i$$

and

$$\int_C f(z)dz = \int_0^1 (t^2 + it^2)(1+i)dt = (1+i)^2 \int_0^1 t^2 dt = 2i/3.$$

◊

EXAMPLE 2

Let

$$f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2},$$

and set

$$C : z(t) = R\cos t + iR\sin t, \quad 0 \leq t \leq 2\pi, \quad R \neq 0.$$

Then

$$\int_C f(z)dz = \int_0^{2\pi} \left(\frac{\cos t}{R} - i\frac{\sin t}{R}\right)(-R\sin t + iR\cos t)dt$$
$$= \int_0^{2\pi} i\,dt = 2\pi i \quad \text{(See Exercise 8.)}$$

That is, the integral of $1/z$ around any circle centered at the origin (traversed counterclockwise) is $2\pi i$. ◊

EXAMPLE 3

Suppose $f(z) \equiv 1$, and let C be any smooth curve. Then

$$\int_C f(z)dz = \int_a^b \dot{z}(t)dt = z(b) - z(a).$$

◊

The integrals defined above are natural generalizations of the definite integral and, not too surprisingly, they share many of the same properties.

4.8 Proposition

Let C be a smooth curve; let f and g be continuous functions on C; and let α be any complex number. Then

I. $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$

II. $\int_C \alpha f(z)dz = \alpha \int_C f(z)dz.$

Proof

Exercise 4. □

Notation: If α and β are complex numbers, the symbol $\alpha \ll \beta$ will be used to denote the inequality $|\alpha| \le |\beta|$.

4.9 Lemma

Suppose $G(t)$ is a continuous complex-valued function of t. Then

$$\int_a^b G(t)dt \ll \int_a^b |G(t)|dt.$$

Proof

Suppose

$$\int_a^b G(t)dt = Re^{i\theta}, R \ge 0. \tag{1}$$

By Proposition 4.8, then

$$\int_a^b e^{-i\theta} G(t)dt = R. \tag{2}$$

Suppose further that $e^{-i\theta}G(t) = A(t) + iB(t)$, with A and B real-valued. Then, according to (2),

$$R = \int_a^b A(t)dt = \int_a^b \operatorname{Re}(e^{-i\theta}G(t))dt.$$

But $\operatorname{Re} z \le |\operatorname{Re} z| \le |z|$, hence

$$R \le \int_a^b |G(t)|dt. \tag{3}$$

A comparison of (1) and (3) then gives the desired result. □

4.10 *M-L* Formula

Suppose that C is a (smooth) curve of length L, that f is continuous on C, and that $f \ll M$ throughout C. Then

$$\int_C f(z)dz \ll ML.$$

Proof

Suppose C is given by $z(t) = x(t)+iy(t),\ a \le t \le b$. Then, by the previous lemma,

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}dt \ll \int_a^b |f(z(t))\dot{z}(t)|dt.$$

According to the Mean-Value Theorem for Integrals applied to the positive functions $|f(z(t))|$ and $|\dot{z}(t)|$

$$\int_C f(z)dz \ll \max_{z\in C} |f(z)| \int_a^b |\dot{z}(t)|dt. \tag{4}$$

Finally, recall that for any curve given parametrically by $(x(t), y(t)),\ a \le t \le b$, the arc length L is given by

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 dt} = \int_a^b |\dot{z}(t)|dt,$$

so that according to (4) $\int_C f(z)dz \ll ML$. □

EXAMPLE

Let C be the unit circle and suppose $f \ll 1$ on C. Then $M = 1$, $L = 2\pi$, and

$$\int_C f(z)dz \ll 2\pi.$$

To see that the upper bound of 2π can actually be achieved, consider Example 2 above. ♢

4.11 Proposition

Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ uniformly on the smooth curve C. Then

$$\int_C f(z)dz = \lim_{n\to\infty} \int_C f_n(z)dz.$$

Proof

$$\int_C f(z)dz - \int_C f_n(z)dz = \int_C [f(z) - f_n(z)]dz$$

by Proposition 4.8. Taking n large enough so that $|f(z) - f_n(z)| < \epsilon$ for all $z \in C$, and applying Proposition 4.10, shows that

$$\int_C f(z) - \int_C f_n(z)dz \ll \epsilon \cdot (\text{length of } C)$$

for any pre-assigned $\epsilon > 0$, and hence that

$$\lim_{n\to\infty}\int_C f_n(z)dz = \int_C f(z)dz. \qquad \square$$

The following generalization of the Fundamental Theorem of Calculus will be crucial in the development of this chapter.

4.12 Proposition

Suppose f is the derivative of an analytic function F–that is, $f(z) = F'(z)$, where F is analytic on the smooth curve C. Then

$$\int_C f(z)dz = F(z(b)) - F(z(a)).$$

Proof

The proof depends on a complex analogue of the chain-rule for differentiation. Letting

$$\gamma(t) = F(z(t)), \quad a \le t \le b,$$

we wish to show that

$$\dot{\gamma}(t) = f(z(t))\dot{z}(t)$$

at the all-but-finite number of points where $\dot{z}(t)$ exists and is nonzero.

Note first that for any smooth curve $\lambda(t)$, by considering the real and imaginary parts of λ separately, it is easily seen that

$$\dot{\lambda}(t) = \lim_{\substack{h\to 0\\ h \text{ real}}} \frac{\lambda(t+h) - \lambda(t)}{h}.$$

Hence

$$\begin{aligned}\dot{\gamma}(t) &= \lim_{h\to 0}\frac{F(z(t+h)) - F(z(t))}{h}\\ &= \lim_{h\to 0}\frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h}.\end{aligned}$$

[Since $\dot{z}(t) \neq 0$, we can find $\delta > 0$ so that $|h| < \delta$ implies $z(t+h) - z(t) \neq 0$.] Thus

$$\dot{\gamma}(t) = f(z(t))\dot{z}(t).$$

Proposition 4.12 follows then by noting that

$$\begin{aligned}\int_C f(z)dz &= \int_a^b f(z(t))\dot{z}(t)dt = \int_a^b \dot{\gamma}(t)dt\\ &= \gamma(b) - \gamma(a) = F(z(b)) - F(z(a)).\end{aligned} \qquad \square$$

4.2 The Closed Curve Theorem for Entire Functions

4.13 Definition

A curve C is *closed* if its initial and terminal points coincide–i.e., if C is given by $z(t), a \leq t \leq b$, with $z(a) = z(b)$. C is a *simple closed curve* if no other points coincide; i.e., if $z(t_1) = z(t_2)$ with $t_1 < t_2$ implies $t_1 = a$ and $t_2 = b$.

The following theorem is the first of several which show that, under rather general conditions, the integral of an analytic function along a closed curve is zero. Of course, Example 2 showed that this is not always the case. We begin cautiously, by considering entire functions.

Note: Throughout the text, *the boundary of a rectangle* will mean a simple closed curve parametrized so that the rectangle it bounds lies on the left as the curve is traced out for increasing t.

4.14 Rectangle Theorem

Suppose f is entire and Γ is the boundary of a rectangle R. Then

$$\int_\Gamma f(z)dz = 0.$$

Lemma

If f is a linear function and if Γ is as above, then

$$\int_\Gamma f(z)dz = 0.$$

Proof of Lemma

Let $f(z) = \alpha + \beta z$ and let Γ be given by

$$\Gamma : z(t), \quad a \leq t \leq b.$$

Since $f(z)$ is everywhere the derivative of the analytic function $F(z) = \alpha z + \beta z^2/2$,

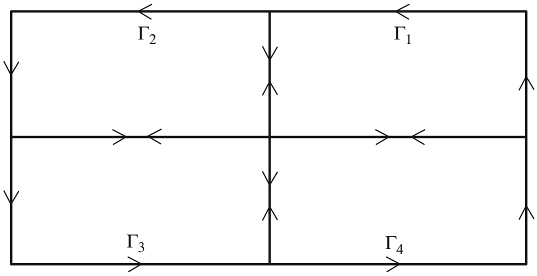
$$\int_\Gamma f(z)dz = \int_\Gamma F'(z)dz = F(z(b)) - F(z(a)) = 0$$

by Proposition 4.12 and the observation that Γ is a closed curve. (An alternate, more direct proof is outlined in Exercise 7.) □

Proof of Theorem 4.14

Let $\int_\Gamma f(z)dz = I$. To show that $I = 0$, we use the method of continued bisection. That is, we split the rectangle R into four congruent subrectangles, by bisecting each of the sides. If we let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ denote the boundaries of the four subrectangles

$$\int_{\Gamma} f = \sum_{i=1}^{4} \int_{\Gamma_i} f$$



since the integrals along the interior lines appear in opposite directions and thus cancel, by Proposition 4.7. Hence for some Γ_k, $1 \leq k \leq 4$, which we will denote $\Gamma^{(1)}$,

$$\int_{\Gamma^{(1)}} f(z)dz \gg \frac{I}{4}.$$

Let $R^{(1)}$ be the rectangle bounded by $\Gamma^{(1)}$. Continuing in this manner, dividing $R^{(k)}$ into four congruent rectangles, we obtain a sequence of rectangles

$$R^{(1)} \supset R^{(2)} \supset R^{(3)} \supset \cdots$$

and their boundaries

$$\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \ldots$$

such that diam $R^{(k+1)} = \frac{1}{2}\text{diam} R^{(k)}$ and such that

$$\int_{\Gamma^{(k)}} f(z)dz \gg \frac{I}{4^k}. \tag{1}$$

Let $z_0 \in \cap_{k=1}^{\infty} R^{(k)}$. The proof will follow by considering the analyticity of f at z_0. That is, since

$$\frac{f(z) - f(z_0)}{z - z_0} \to f'(z_0)$$

we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon_z(z - z_0)$$

where $\epsilon_z \to 0$ as $z \to z_0$.

Note, then, that

$$\begin{aligned} \int_{\Gamma^{(n)}} f(z)dz &= \int_{\Gamma^{(n)}} [f(z_0) + f'(z_0)(z - z_0) + \epsilon_z \cdot (z - z_0)]dz \\ &= \int_{\Gamma^{(n)}} \epsilon_z \cdot (z - z_0)dz \quad \text{by the lemma.} \end{aligned}$$

To estimate the integral, let us assume that the largest side of the original boundary Γ was of length s. Then, by elementary geometric considerations,

$$\int_{\Gamma^{(n)}} |dz| = \text{length of } \Gamma^{(n)} \leq \frac{4s}{2^n}$$

and

$$|z - z_0| \leq \frac{\sqrt{2} \cdot s}{2^n} \quad \text{for all } z \in \Gamma^{(n)}.$$

Given $\epsilon > 0$, we choose N so that

$$|z - z_0| \leq \frac{\sqrt{2} \cdot s}{2^N} \quad \text{implies that } \epsilon_z \ll \epsilon.$$

Then for $n \geq N$, we have by the *M-L* formula (Proposition 4.10)

$$\int_{\Gamma^{(n)}} f(z)dz \ll \epsilon \cdot \frac{4\sqrt{2}s^2}{4^n}. \tag{2}$$

A combination of (1) and (2) shows that for $n \geq N$

$$\frac{I}{4^n} \ll \epsilon \frac{4\sqrt{2}s^2}{4^n}$$

or

$$I \ll \epsilon \cdot 4\sqrt{2}s^2.$$

Since this holds for all $\epsilon > 0$, we may conclude that $I = 0$. □

Note: Although the orientation of Γ was chosen to be counterclockwise, the same result would hold with the opposite orientation. This follows from Proposition 4.7. The counterclockwise orientation was chosen primarily to fix a direction. In later chapters, we will see that the counterclockwise direction along the boundary is also the more "natural" one in a sense for functions analytic inside a region. Hence, unless otherwise specified, the integral around any convex curve will always be taken in the counterclockwise direction.

4.15 Integral Theorem

If f is entire, then f is everywhere the derivative of an analytic function. That is, there exists an entire F such that $F'(z) = f(z)$ for all z.

Proof

We define $F(z)$ as

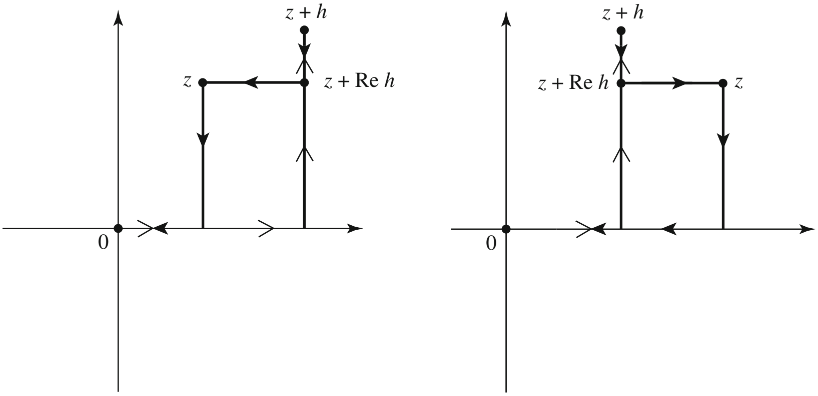
$$\int_0^z f(\zeta)d\zeta$$

where $\int_0^z$ denotes the integral along the straight lines from 0 to $\operatorname{Re} z$ and from $\operatorname{Re} z$ to z.

Note that

$$F(z+h) = F(z) + \int_z^{z+h} f(\zeta)d\zeta$$

where the integral denotes the integral along the line segments from z to $z + \text{Re}\, h$ and from $z + \text{Re}\, h$ to $z + h$. This follows since the difference between the two approaches is equal to the integral of f around a closed rectangle and is thus equal to zero. (See diagrams below.)



Hence

$$F(z+h) - F(z) = \int_z^{z+h} f(\zeta)d\zeta$$

and since

$$\frac{1}{h}\int_z^{z+h} 1dz = \frac{1}{h}(z+h-z) = 1,$$

(see Example 3 after Proposition 4.7)

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} [f(\zeta) - f(z)]d\zeta.$$

Finally, for each $\epsilon \geq 0$, if h is small enough, $|f(\zeta) - f(z)| \ll \epsilon$ throughout the path of integration. Applying the *M-L* formula, we obtain

$$\frac{F(z+h) - F(z)}{h} - f(z) \ll \frac{1}{h} \cdot 2h\epsilon = 2\epsilon.$$

Hence

$$F'(z) = f(z).$$

□

4.16 Closed Curve Theorem

If f is entire and if C is a (smooth) closed curve,

$$\int_C f(z)dz = 0.$$

Proof

Since f is entire, by the Integral Theorem $f(z) = F'(z)$ for some entire function F so that

$$\int_C f(z)dz = \int_C F'(z)dz = F(z(b)) - F(z(a))$$

by Proposition 4.12. Since C is closed, $z(b) = z(a)$, $F(z(b)) = F(z(a))$, and

$$\int_C f(z)dz = 0.$$

□

Remarks

While Theorem 4.16 was proven for entire functions f, the only fact we needed was that $f(z)$ is the derivative of an analytic function on C. Thus, for example,

$$\int_C \frac{1}{z^2} dz = 0$$

if C is any smooth closed curve not passing through the origin. For although $1/z^2$ is not entire, it is the derivative of $F(z) = -1/z$ which is analytic except at the origin. Similarly,

$$\int_C z^k dz = 0$$

if k is any integer except -1. Recall Example 2 which showed that $k = -1$ is an exception to the above. (See Exercise 8.)

Exercises

1. Prove that "equivalence" of smooth curves has the familiar reflexive, symmetric, and transitive properties of an equivalence relation.
2. Evaluate $\int_C f$ where $f(z) = x^2 + iy^2$ as in Example 1, but where C is given by $z(t) = t^2 + it^2$, $0 \leq t \leq 1$.
3. Evaluate $\int_C f$ where $f(z) = 1/z$ as in Example 2, and C is given by $z(t) = \sin t + i \cos t$, $0 \leq t \leq 2\pi$. Why is the result different from that of Example 2?
4. Prove Proposition 4.8. [*Hint*: Divide the integrals into real and imaginary parts.]
5. Prove the uniqueness of the integral. That is, show that $F' \equiv 0$ implies that F is a constant. [*Hint*: Use Proposition 4.12 to get an expression for $F(b) - F(a)$.]

6. Show that, if f is a continuous *real-valued* function and $f \ll 1$, then

$$\int_{|z|=1} f \ll 4.$$

[*Hint*: Show that $\int f \ll \int_0^{2\pi} |\sin t| dt$.]

7. Give a direct proof of the lemma to Theorem 4.14. That is, given any rectangle with vertices (a, c), (b, c), (b, d) and (a, d), parameterize the boundary Γ and verify directly that

$$\int_\Gamma dz = \int_\Gamma z dz = 0.$$

8. Show that $\int_C z^k dz = 0$ for any integer $k \neq -1$ and $C : z = \mathrm{Re}^{i\theta}, 0 \leq \theta \leq 2\pi$
   a. by showing that z^k is the derivative of a function analytic throughout C,
   b. directly, using the parametrization of C.

9. Evaluate $\int_C (z - i) dz$ where C is the parabolic segment:

$$z(t) = t + it^2, \quad -1 \leq t \leq 1$$

   a. by applying Proposition 4.12,
   b. by integrating along the straight line from $-1+i$ to $1+i$ and applying the Closed Curve Theorem.

10.* Evaluate
   a. $\int_0^i e^z dz$
   b. $\int_{\pi/2}^{\pi/2+i} \cos 2z \, dz$

11.* Suppose f is analytic in a convex region D and $|f'| \leq 1$ throughout D. Prove that f is a "contraction"; i.e., show that $|f(b) - f(a)| \leq |b - a|$ for all $a, b \in D$.

12.* Let a, b be two complex numbers in the left half-plane. Prove that $|e^a - e^b| < |a - b|$.