Chapter 6
Properties of Analytic Functions

Introduction

In the last two chapters, we studied the connection between everywhere convergent power series and entire functions. We now turn our attention to the more general relationship between power series and analytic functions. According to Theorem 2.9 every power series represents an analytic function inside its circle of convergence. Our first goal is the converse of this theorem: we will show that a function analytic in a disc can be represented there by a power series. We then turn to the question of analytic functions in arbitrary open sets and the local behavior of such functions.

6.1 The Power Series Representation for Functions Analytic in a Disc

6.1 Theorem

Suppose f is analytic in $D = D(\alpha; r)$. If the closed rectangle R and the point a are both contained in D and Γ represents the boundary of R,

$$\int_\Gamma f(z)dz = \int_\Gamma \frac{f(z) - f(a)}{z - a} dz = 0.$$

Proof

The proof is exactly the same as those of Theorems 4.14 and 5.1. The only requirement there was that f be analytic throughout R, and this is satisfied since $R \subset D$. □

To simplify notation, we adopt the following convention. If $f(z)$ is analytic in a region D, including the point a, the function

$$g(z) = \frac{f(z) - f(a)}{z - a}$$

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will denote the function given by

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & z \in D, \quad z \neq a \\ f'(a) & z = a. \end{cases}$$

The fact that g is analytic at a is proven in Proposition 6.7. (Compare with Proposition 5.8.)

6.2 Theorem

If f is analytic in $D(\alpha; r)$, and $a \in D(\alpha; r)$, there exist functions F and G, analytic in D and such that

$$F'(z) = f(z), \qquad G'(z) = \frac{f(z) - f(a)}{z - a}.$$

Proof

We define

$$F(z) = \int_{\alpha}^{z} f(\zeta)d\zeta$$

and

$$G(z) = \int_{\alpha}^{z} \frac{f(\zeta) - f(a)}{\zeta - a}\, d\zeta$$

where the path of integration consists of the horizontal and then vertical segments from α to z. Note that for any $z \in D(\alpha; r)$ and h small enough, $z + h \in D(\alpha; r)$ so that, as in 4.15, we may apply the Rectangle Theorem to the respective difference quotients to conclude

$$F'(z) = f(z)$$

and

$$G'(z) = \frac{f(z) - f(a)}{z - a}. \qquad \square$$

6.3 Theorem

If f and a are as above and C is any (smooth) closed curve contained in $D(\alpha; r)$,

$$\int_C f(z)dz = \int_C \frac{f(z) - f(a)}{z - a}\, dz = 0.$$

Proof

According to Theorem 6.2, there exists G, analytic in $D(\alpha; r)$ and such that

$$G'(z) = \frac{f(z) - f(a)}{z - a}.$$

Hence,

$$\int_C \frac{f(z) - f(a)}{z - a} dz = \int_C G'(z)dz = G(z(b)) - G(z(a)) = 0$$

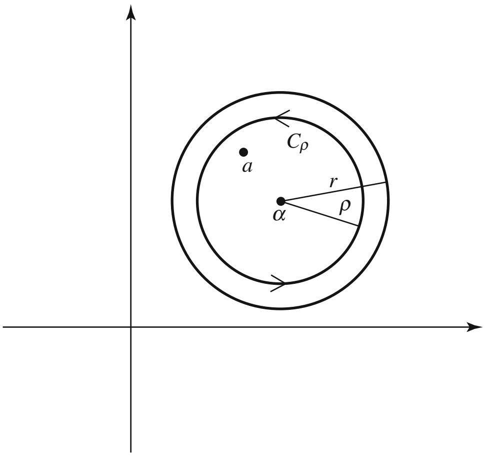
since the initial and terminal points $z(a)$ and $z(b)$ coincide. Similarly, $\int_C f(z)dz = 0$. □

6.4 Cauchy Integral Formula

Suppose f is analytic in $D(\alpha; r)$, $0 < \rho < r$, and $|a - \alpha| < \rho$. Then

$$f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - a} dz$$

where C_ρ represents the circle $\alpha + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$.



Proof

$$\int_{C_\rho} \frac{f(z) - f(a)}{z - a} dz = 0$$

so that

$$f(a) \int_{C_\rho} \frac{dz}{z - a} = \int_{C_\rho} \frac{f(z)}{z - a} dz.$$

Moreover, according to Lemma 5.4,

$$\int_{C_\rho} \frac{dz}{z - a} = 2\pi i$$

and the proof is complete. □

6.5 Power Series Representation for Functions Analytic in a Disc

If f is analytic in $D(\alpha; r)$ there exist constants C_k such that

$$f(z) = \sum_{k=0}^{\infty} C_k (z-\alpha)^k$$

for all $z \in D(\alpha; r)$.

Proof

Pick $a \in D(\alpha; r)$ and $\rho > 0$ such that $|a-\alpha| < \rho < r$.

By the previous integral formula, if $|z-\alpha| < |a-\alpha|$

$$f(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\omega)}{\omega - z}\, d\omega$$

and using the fact that

$$\frac{1}{\omega-\alpha} + \frac{z-\alpha}{(\omega-\alpha)^2} + \frac{(z-\alpha)^2}{(\omega-\alpha)^3} + \cdots$$

converges uniformly to $1/(\omega - z)$ throughout C_ρ (see Lemma 5.4)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_\rho} f(\omega) \left[\frac{1}{\omega-\alpha} + \frac{z-\alpha}{(\omega-\alpha)^2} + \frac{(z-\alpha)^2}{(\omega-\alpha)^3} + \cdots \right] d\omega \\ &= C_0(\rho) + C_1(\rho)(z-\alpha) + C_2(\rho)(z-\alpha)^2 + \cdots \qquad (1) \end{aligned}$$

where

$$C_k(\rho) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\omega)}{(\omega-\alpha)^{k+1}}\, d\omega.$$

Note, then, that the coefficients $C_k(\rho)$ are actually independent of ρ. For once again, as in 5.5, we can apply (1) to conclude that f is infinitely differentiable at α and

$$C_k(\rho) = \frac{f^{(k)}(\alpha)}{k!} \text{ for each } \rho,\ 0 < \rho < r, \text{ and all } k.$$

Hence, for *all* $z \in D(\alpha; r)$

$$f(z) = \sum_{k=0}^{\infty} C_k (z-\alpha)^k$$

with

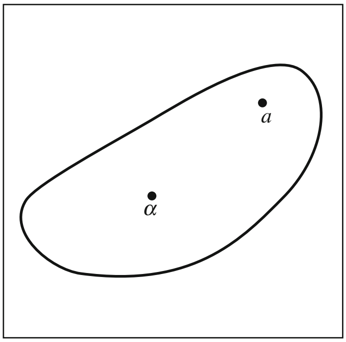
$$C_k = \frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z-\alpha)^{k+1}}\, dz. \qquad \square$$

6.2 Analytic in an Arbitrary Open Set

The methods used above cannot be generalized to find a single power series equal to a given analytic function in an arbitrary open set. In fact no such generalization is possible even to the most elementary of domains–e.g., a square. The breakdown in the previous strategy arises when, given a point a in the square, we try to find a contour C surrounding a and the center α of the square such that

$$\left|\frac{a-\alpha}{\omega-\alpha}\right| < 1 \quad \text{for all } \omega \in C$$

(see the diagram). As we shall soon see, this is not simply a technical difficulty but a reflection of the fact that in general, no such power series exists! However, we can apply our previous results to obtain the following general theorem.



6.6 Theorem

If f is analytic in an arbitrary open domain D, then for each $\alpha \in D$, there exist constants C_k such that

$$f(z) = \sum_{k=0}^{\infty} C_k (z-\alpha)^k$$

for all points z inside the largest disc centered at α and contained in D.

Proof

This is a simple reformulation of Theorem 6.5. □

EXAMPLES

i. $f(z) = 1/(z-1)$ is analytic at $z = 2$ and in a disc of radius 1 centered at $z = 2$. To find a power series representation for f in that disc, we write

$$\frac{1}{z-1} = \frac{1}{1+(z-2)} = 1-(z-2)+(z-2)^3-(z-2)^3+-\cdots \qquad (1)$$

which converges as long as $|z-2| < 1$.

Note that the power series diverges throughout $|z-2|>1$ despite the fact that $f(z)=1/(z-1)$ is analytic everywhere except at the single point $z=1$.

Furthermore, according to Theorem 2.14 any other power series $\sum a_k(z-2)^k$ which equals $1/(z-1)$ in *any* disc around $z=2$ would have to be identical with the power series in (1). Hence, there is *no* power series $\sum a_k(z-2)^k$ equal to $1/(z-1)$ throughout its domain of analyticity.

ii. To find a power series representation for $1/z^2$ near $z=3$, we set

$$\frac{1}{z^2}=\left[\frac{1}{3+(z-3)}\right]^2=\frac{1}{9}\left[\frac{1}{1+(z-3)/3}\right]^2$$

$$=\frac{1}{9}\left[1-\frac{(z-3)}{3}+\frac{(z-3)^2}{9}-\frac{(z-3)^3}{27}+-\cdots\right]^2$$

$$=\frac{1}{9}\left[1-\frac{2(z-3)}{3}+\frac{3(z-3)^2}{9}-\frac{4(z-3)^3}{27}+-\cdots\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{9}\frac{(k+1)}{3^k}(z-3)^k.$$

Note again that the radius of convergence

$$1/\limsup|C_k|^{1/k}=\lim\left(\frac{9\cdot 3^k}{k+1}\right)^{1/k}=3$$

represents the radius of the largest disc centered at $z=3$ in which $1/z^2$ is analytic.

iii. To find the first three terms of the power series for $f(z)=\sin(1/z)$ around $z=1$, because no immediate formula suggests itself, we evaluate the coefficients directly using the formula

$$C_k=\frac{f^{(k)}(1)}{k!}.$$

Thus we find

$$f(z)=\sin\frac{1}{z}=\sin 1-\cos 1(z-1)+\frac{(2\cos 1-\sin 1)}{2}(z-1)^2+\cdots \qquad \Diamond$$

6.3 The Uniqueness, Mean-Value, and Maximum-Modulus Theorems; Critical Points and Saddle Points

We now consider some of the implications of the power series representations discussed in Theorem 6.6. We begin with a local version of Proposition 5.8.

6.7 Proposition

If f is analytic at α, so is

$$g(z) = \begin{cases} \dfrac{f(z) - f(\alpha)}{z - \alpha} & z \neq \alpha \\ f'(\alpha) & z = \alpha. \end{cases}$$

Proof

By Theorem 6.6, in some neighborhood of α,

$$f(z) = f(\alpha) + f'(\alpha)(z - \alpha) + \frac{f''(\alpha)}{2!}(z - \alpha)^2 + \cdots .$$

Thus g has the power series representation

$$g(z) = f'(\alpha) + \frac{f''(\alpha)}{2!}(z - \alpha) + \frac{f^{(3)}(\alpha)}{3!}(z - \alpha)^2 + \cdots$$

in the same neighborhood, and by 2.9, g is analytic at α. □

6.8 Theorem

If f is analytic at z, then f is infinitely differentiable at z.

Proof

We need only recall that, by definition, f is analytic at a point z if it is analytic in an open set containing z. By 6.6, then, in some disc containing z, f may be expressed as a power series. This completes the proof, since power series are infinitely differentiable (Corollary 2.10). □

6.9 Uniqueness Theorem

Suppose that f is analytic in a region D and that $f(z_n) = 0$ where $\{z_n\}$ is a sequence of distinct points and $z_n \to z_0 \in D$. Then $f \equiv 0$ in D.

Proof

Since f has a power series representation around z_0, by the Uniqueness Theorem for Power Series, $f = 0$ throughout some disc containing z_0. To show that $f \equiv 0$ in the whole domain D, we split D into two sets:

$$A = \{z \in D\colon z \text{ is a limit of zeroes of } f\},$$
$$B = \{z \in D\colon z \notin A\}.$$

By definition, $A \cap B = \emptyset$. A is open by the Uniqueness Theorem for power series: if z is a limit of zeroes of f, $f \equiv 0$ in an entire disc around z and that disc is contained in A. B is open since for each $z \in B$, there must be some $\delta > 0$ such that $f(\omega) \neq 0$ for $0 < |z - \omega| < \delta$. The disc $D(z; \delta)$ would then be contained in B. By the connectedness of D, then, either A or B must be empty. But, by hypothesis, $z_0 \in A$. Thus B is empty and every $z \in D$ is a limit of zeroes of f. By the continuity of f, then, $f \equiv 0$ in D. □

6.10 Corollary

If two functions f and g, analytic in a region D, agree at a set of points with an accumulation point in D, then $f \equiv g$ through D.

Proof

Consider $f - g$. □

Note that a non-trivial analytic function may have infinitely many zeroes. For example, $\sin z$, which is entire, is equal to 0 at all the points $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. In fact, $\sin(1/z) = 0$ on the set

$$\left\{ \frac{1}{n\pi} : n = \pm 1, \pm 2, \ldots \right\}$$

which has an accumulation point at 0! Because this limit point is not in the domain of analyticity of $\sin(1/z)$, however, $\sin(1/z)$ does not satisfy the hypothesis of Theorem 6.9.

6.11 Theorem

If f is entire and if $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial.

Proof

By hypothesis, there is some $M > 0$ such that $|z| > M$ implies that $|f(z)| > 1$. We conclude that f has at most a finite number of zeroes $\alpha_1, \alpha_2, \ldots, \alpha_N$. Otherwise, the set of zeroes would have an accumulation point in $D(0; M)$, and by the Uniqueness Theorem f would be identically zero, contradicting the original hypothesis. If we divide out the zeroes of f,

$$g(z) = \frac{f(z)}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_N)}$$

is likewise entire (Corollary 5.9), and never equal to zero; hence

$$h(z) = \frac{1}{g(z)} = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_N)/f(z)$$

is also entire. Since $f \to \infty$ as $z \to \infty$, $|h(z)| \leq A + |z|^N$; therefore, by Theorem 5.11, h is a polynomial. But $h = 1/g \neq 0$, hence according to the

Fundamental Theorem of Algebra, h is a constant k. Thus

$$f(z) = \frac{1}{k}(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_N). \qquad \square$$

The Uniqueness Theorem is often used to demonstrate the validity in the complex plane of functional equations known to be true on the real line. For example, to prove the identity

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{2}$$

we first take z_2 to be a fixed real number. Then $e^{z_1+z_2}$ and $e^{z_1} \cdot e^{z_2}$ represent two entire functions of z_1 which agree at all real points and hence by the Uniqueness Theorem, they agree for all complex z_1 as well. Finally, for any fixed z_1, we consider the two sides of (2) as analytic functions in z_2 which agree for real z_2, and again applying the Uniqueness Theorem, we conclude that they agree for all complex z_2 as well. Hence (2) is valid for all complex z_1 and z_2. Similarly, equations such as

$$\tan^2 z = \sec^2 z - 1,$$

which are known to be true for real z, are valid throughout their domains of analyticity.

In general, if there is an "analytic" relationship among analytic functions: that is, a functional equation of the form

$$F(f, g, h, \ldots) = 0$$

which is satisfied by the analytic function $F(f, g, h, \ldots)$ on a set with an accumulation point in its region of analyticity, then the equation holds throughout the region.

We now examine the local behavior of analytic functions.

6.12 Mean Value Theorem

If f is analytic in D and $\alpha \in D$, then $f(\alpha)$ is equal to the mean value of f taken around the boundary of any disc centered at α and contained in D. That is,

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})d\theta$$

when $D(\alpha; r) \subset D$.

Proof

This is a reformulation of the Cauchy Integral Formula (6.4) with $a = \alpha$. That is,

$$f(\alpha) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z - \alpha}\, dz,$$

and introducing the parameterization $z = \alpha + re^{i\theta}$, we see that

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})d\theta. \qquad \square$$

In analogy with the real case, we will call a point z a *relative maximum* of f if $|f(z)| \geq |f(\omega)|$ for all complex ω in some neighborhood of z. A relative minimum is defined similarly.

6.13 Maximum-Modulus Theorem

A non-constant analytic function in a region D does not have any interior maximum points: For each $z \in D$ and $\delta > 0$, there exists some $\omega \in D(z;\delta) \cap D$, such that $|f(\omega)| > |f(z)|$.

Proof

The fact that

$$|f(\omega)| \geq |f(z)|$$

for some ω near z follows immediately from the Mean-Value Theorem. Since for $r > 0$ such that $D(z;r) \subset D$ we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})d\theta,$$

it follows that

$$|f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z + re^{i\theta})|d\theta \leq \max_\theta |f(z + re^{i\theta})|. \tag{3}$$

Similarly, we may deduce that $|f(\omega)| > |f(z)|$ for some $\omega \in D(z;r)$. For, to obtain equality in (3), $|f|$ would have to be constant throughout the circle $C(z;r)$ and since this holds for all sufficiently small $r > 0$, $|f|$ would be constant throughout a disc. But then by Theorem 3.7, f would be constant in that disc, and by the Uniqueness Theorem, f would be constant throughout D. $\square$

Ironically, the Maximum-Modulus Theorem actually asserts that an analytic function has no relative maximum. It is sometimes given a more positive flavor as follows.

Suppose a function f is analytic in a bounded region D and continuous on $\bar{D}$. (We will, henceforth, use the expression "f is C-analytic in D" to denote this hypothesis.) Somewhere in the compact domain $\bar{D}$, the continuous function $|f|$ must assume its maximum value. The Maximum-Modulus Theorem may then be invoked to assert that this maximum is always assumed on the boundary of the domain.

6.14 Minimum Modulus Theorem

If f is a non-constant analytic function in a region D, then no point $z \in D$ can be a relative minimum of f unless $f(z) = 0$.

Proof

Suppose that $f(z) \neq 0$ and consider $g = 1/f$. If z were a minimum point for f, it would be a maximum point for g. Hence g would be constant in D, contrary to our hypothesis on f. □

Remark

We can also prove the Maximum-Modulus Theorem by analyzing the local power series representation for an analytic function. That is, for any point α, consider the power series

$$f(z) = C_0 + C_1(z - \alpha) + C_2(z - \alpha)^2 + \cdots,$$

which is convergent in some disc around α. To find z near α and such that $|f(z)| > |f(\alpha)|$, we first assume $C_1 \neq 0$ and set $z = \alpha + \delta e^{i\theta}$, with $\delta > 0$ "small", and θ chosen so that C_0 and $C_1 \delta e^{i\theta}$ have the same argument. Then

$$|f(\alpha)| = |C_0|$$

$$|f(z)| \geq |C_0 + C_1(z - \alpha)| - |C_2(z - \alpha)^2 + C_3(z - \alpha)^3 + \cdots|$$

$$\geq |C_0| + |C_1 \delta| - \delta^2 |C_2 + C_3(z - \alpha) + \cdots|.$$

Since the last expression represents a convergent series,

$$|f(z)| \geq |C_0| + |C_1 \delta| - A\delta^2 \geq |C_0| + \frac{1}{2}|C_1 \delta| > |f(\alpha)|$$

as long as $\delta < |C_1|/2A$. Hence α cannot be a maximum point. Note that if $C_1 = 0$, the same argument can be applied by focusing on the first non-zero coefficient C_k.

This technique of studying the local behavior of an analytic function by considering the first terms of its power series expansion can be used to derive the following result.

Recall that in calculus, relative maximum points were found among the critical points (those points at which $f' = 0$) of a differentiable function f. The proposition below shows a somewhat surprising contrast in the behavior of an analytic function at a point where it assumes its maximum modulus.

6.15 Theorem

Suppose f is nonconstant and analytic on the closed disc D, and assumes its maximum modulus at the boundary point z_0. Then $f'(z_0) \neq 0$.

Proof (G. Pólya and G. Szegő)

Assume that $f'(z_0) = 0$. For any complex number ξ of sufficiently small modulus we have

$$f(z_0 + \xi) = f(z_0) + \frac{f^{(k)}(z_0)}{k!}\xi^k + \cdots,$$

where k is the least integer with $f^{(k)}(z_0) \neq 0$ and the omitted terms are all of higher order in ξ than ξ^k. Multiplying the above expression by its conjugate shows

$$|f(z_0 + \xi)|^2 = f(z_0 + \xi)\overline{f(z_0 + \xi)}$$

$$= |f(z_0)|^2 + \frac{2}{k!}\operatorname{Re}\left(\overline{f(z_0)}f^{(k)}(z_0)\xi^k\right) + \cdots .$$

Since $|f(z_0)| = \max_{z\in D}|f(z)|$, $f(z_0) \neq 0$. Write $\overline{f(z_0)}f^{(k)}(z_0) = Ae^{i\alpha}$ with $A > 0$, and let $e^{i\theta} = \xi/|\xi|$. Then

$$|f(z_0 + \xi)|^2 = |f(z_0)|^2 + \frac{2A}{k!}|\xi|^k\cos(k\theta + \alpha) + \cdots,$$

and, for ξ of sufficiently small modulus, $|f(z_0 + \xi)| - |f(z_0)|$ has the same sign as $\cos(k\theta + \alpha)$. It follows that

$|f(z)| > |f(z_0)|$ *if z is in any of the k wedges of the form*

$$\left\{z_0 + r_\theta e^{i\theta} : \theta \in \left(\frac{-\pi + 4\pi j - 2\alpha}{2k}, \frac{\pi + 4\pi j - 2\alpha}{2k}\right) \text{ and } r_\theta \in (0, \varepsilon_\theta)\right\} \quad (4)$$

for some positive ε_θ and $j = 0, 1, \ldots, k-1$ (and $|f(z)| < |f(z_0)|$ if z is in any of the alternate wedges).

Since $f'(z_0) = 0$, $k \geq 2$. To complete the proof, note that at least one of the k wedges described in (4) must intersect D. Hence $|f(z_0)|$ cannot be the maximum value of $|f|$ on D. □

Remarks

1. While the theorem asserts that $|f|$ cannot achieve an absolute maximum value at a critical point, it is equally true that $|f|$ cannot have a minimum value other than zero at a critical point. This is obvious from the parenthetical remark after (4), above. It can also be proven by considering $1/f$ (which is analytic on an open set containing D if f is nonvanishing on D).
2. Theorem 6.15 is easily generalized to a wide range of compact sets K, including those which do not have smooth boundaries. The key is that, along with each boundary point z_0, K must also contain a wedge (or "cone") of the form

$$\left\{z_0 + re^{i\theta} : \theta \in [\alpha, \beta], r \in (0, \varepsilon)\right\}$$

with $\varepsilon > 0$ and $\beta - \alpha > \pi/2$. This is sufficient since each of the wedges in (4) has a maximum vertex angle of $\pi/2$. Thus, the theorem would be equally valid for a polygon all of whose vertex angles were obtuse. Without this "cone condition", however, the theorem is no longer valid. For example, in the unit square $\{z : \operatorname{Re} z, \operatorname{Im} z \in [0, 1]\}$, $z^2 + i$ has *both* an absolute minimum *and* a critical point at 0, and $1/(z^2 + i)$ has *both* an absolute maximum *and* a critical point at 0.

3. The ideas in the proof of Theorem 6.15 can be applied to show that the set of interior critical points of an analytic function (except for those which are also zeroes) is identical with the set of its "saddle points". The details are given below.

6.16 Definition

z_0 is a *saddle point* of an analytic function f if it is a saddle point of the real-valued function $g = |f|$; that is, if g is differentiable at z_0,with $g_x(z_0) = g_y(z_0) = 0$, but z_0 is neither a local maximum nor a local minimum of g.

6.17 Theorem

z_0 is a *saddle point* of an analytic function f if and only if $f'(z_0) = 0$ and $f(z_0) \neq 0$.

Proof

Let $f = u + iv$, where u and v are real, and let $g = |f|$.

First, suppose that z_0 is a *saddle point* of f. Then $g = |f|$ is differentiable at z_0, and obviously $g(z_0) \neq 0$. Note that

$$g_x = \frac{(uu_x + vv_x)}{g}, \qquad g_y = \frac{(uu_y + vv_y)}{g}. \tag{5}$$

Since $g_x(z_0) = g_y(z_0) = 0$,

$$u(z_0)u_x(z_0) + v(z_0)v_x(z_0) = 0,$$
$$u(z_0)u_y(z_0) + v(z_0)v_y(z_0) = 0.$$

$u(z_0)$ and $v(z_0)$ are not both 0, so the above equations imply that

$$\det\begin{pmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{pmatrix} = 0.$$

From the Cauchy-Riemann equations, it follows that $u_x^2(z_0) + v_x^2(z_0) = 0$, and hence that $f'(z_0) = 0$.

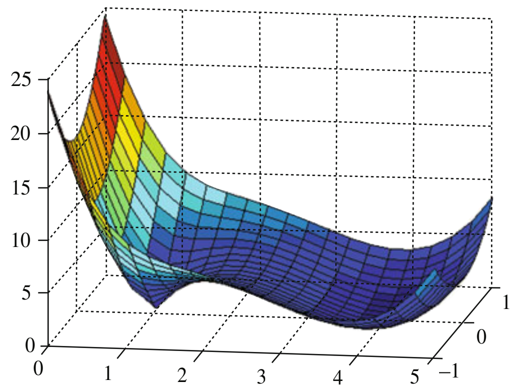
Conversely, if $f'(z_0) = 0$, then $u_x(z_0)$ and $v_x(z_0)$ are both zero, and by the Cauchy-Riemann equations, the same is true for $u_y(z_0)$ and $v_y(z_0)$. It follows from (5) that g is differentiable with $g_x(z_0) = g_y(z_0) = 0$. However, as in the proof of Theorem 6.15, the facts that $f'(z_0) = 0$ and $f(z_0) \neq 0$ guarantee that z_0 is not an extremal point of g. □

Note: Of course, if $f(z_0) = 0$, $|f|$ has an absolute minimum at z_0. If, in addition, $f'(z_0) = 0$, then it follows from the power series expansion of f about z_0 that, for z sufficiently close to z_0 and for some positive constant M,

$$||f(z)| - |f(z_0)|| \le |f(z) - f(z_0)| \le M\,|z - z_0|^2,$$

showing that $g = |f|$ is differentiable at z_0 with $g_x = g_y = 0$ there. If $f(z_0) = 0$ but $f'(z_0) \neq 0$, it can be shown that $|f|$ is not differentiable at z_0. (See Bak-Ding-Newman)

These observations can be illustrated by $f(z) = (z - 1)(z - 4)^2$, which has a simple zero at $z = 1$, a critical point but not a zero at $z = 2$, and a critical point at the double zero $z = 4$. The graph of $|f|$ is shown in Figure 1. Note that $|f|$ has a saddle point at $z = 2$ and is not differentiable at $z = 1$.



Exercises

1. Find a power series expansion for $1/z$ around $z = 1 + i$.
2.* Find a power series, centered at the origin, for the function $f(z) = \frac{1}{1-z-2z^2}$ by first using partial fractions to express $f(z)$ as a sum of two simple rational functions.
3. Using the identity $1/(1 - z) = 1 + z + z^2 + \cdots$ for $|z| < 1$, find closed forms for the sums $\sum nz^n$ and $\sum n^2 z^n$.
4. Show that if f is analytic in $|z| \le 1$, there must be some positive integer n such that $f(1/n) \neq 1/(n + 1)$.
5. Prove that $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.
6. Suppose an analytic function f agrees with $\tan x$, $0 \le x \le 1$. Show that $f(z) = i$ has no solution. Could f be entire?

7. Suppose that f is entire and that $|f(z)| \geq |z|^N$ for sufficiently large z. Show that f must be a polynomial of degree at least N.

8. Suppose f is C-analytic in $|z| \leq 1$, $f \ll 2$ for $|z| = 1, \operatorname{Im} z \geq 0$ and $f \ll 3$ for $|z| = 1, \operatorname{Im} z \leq 0$. Show then that $|f(0)| \leq \sqrt{6}$. [*Hint*: Consider $f(z) \cdot f(-z)$.]

9. Show directly that the maximum and minimum moduli of e^z are always assumed on the boundary of a compact domain.

10. Find the maximum and minimum moduli of $z^2 - z$ in the disc: $|z| \leq 1$.

11.* (A proof, due to Landau, of the maximum modulus theorem) Suppose f is analytic inside and on a circle C with $|f(z)| \leq M$ on C, and suppose z_0 is a point inside C. Use Cauchy's integral formula to show that $|f(z_0)|^n \leq KM^n$, where K is independent of n, and deduce that $|f(z_0)| \leq M$.

12. Suppose f and g are both analytic in a compact domain D. Show that $|f(z)| + |g(z)|$ takes its maximum on the boundary. [*Hint*: Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.]

13. Show that the Fundamental Theorem of Algebra may be derived as a consequence of the Minimum-Modulus Theorem.

14. Suppose $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ is bounded by 1 for $|z| \leq 1$. Show that $|P(z)| \leq |z|^n$ for all $z \gg 1$. [*Hint*: Use Exercise 6 of Chapter 5 to show $|a_n| \leq 1$ and then consider $P(z)/z^n$ in the annulus: $1 \leq |z| \leq R$ for "large" R.]

15.* Let $f(z) = (z-1)(z-4)^2$. Find the lines (through $z = 2$) on which $|f(z)|$ has a relative maximum, and the ones on which $|f(z)|$ has a relative minimum, at $z = 2$. (See the figure at the end of the chapter.)

16.* Find the saddle point of $f(z) = \frac{(z+1)^2}{z}$ and identify the lines on which it is a relative maximum or a relative minimum of $|f|$.

17.* a. Find the saddle points z_1, z_2 of

$$f(z) = \frac{(z-1)^2(z+1)}{z^2}$$

b. Show that, for $i = 1, 2$

$$|f(z_i)| = Max|f(z)| \text{ on the circle } |z| = |z_i|.$$

c. Find lines through z_i on which $|f|$ has a relative maximum or a relative minimum at z_i.