Chapter 14
The Riemann Mapping Theorem

14.1 Conformal Mapping and Hydrodynamics

Before proving the Riemann Mapping Theorem, we examine the relation between conformal mapping and the theory of fluid flow. Our main goal is to motivate some of the results of the next section and the treatment here will be less formal than that of the remainder of the book.

Consider a fluid flow which is independent of time and parallel to a given plane, which we take to be the complex plane. The flow (or velocity) function g is then a two-dimensional or complex variable of two variables and we can write it in the form $g(z) = u(z) + iv(z)$ where u and v are real-valued. If we let σ and τ denote, respectively, the circulation around and the flux across a closed curve C, it can be shown that

$$\int_C \overline{g(z)} dz = \sigma + i\tau \qquad \text{(see Appendix II)} . \qquad (1)$$

We will confine our attention to incompressible fluids and flows which are locally irrotational and source-free. That is, for any point z in our domain D, we assume there exists a $\delta > 0$ such that the circulation around and flux across any closed curve C in $D(z; \delta)$ is zero. Thus, for all such curves, if we define $f(z) = \overline{g(z)}$ it follows by (1) that $\int_C f(z)dz = 0$. We will assume moreover that g (and hence f) is continuous so that, by Morera's Theorem (7.4), f is analytic. Conversely, given an analytic $f = u - iv$ in a domain D, its conjugate $g = u + iv$ can be viewed as a locally irrotational and source-free flow in D.

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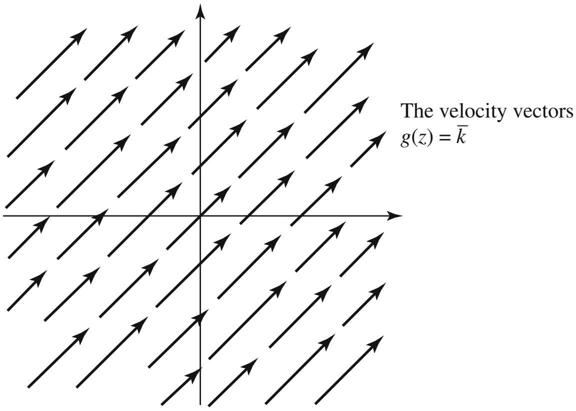
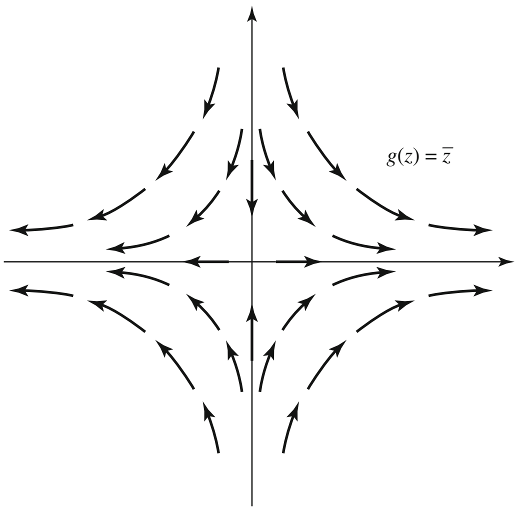
EXAMPLES

i. Suppose $f(z) = k$. Then $g(z) = \bar{k}$ represents a constant flow throughout $\mathbb{C}$.



ii Let $f(z) = z$. Then $g(z) = \bar{z}$ represents a flow which is tangent to the real and imaginary axes.



iii. Let $f(z) = 1/z, z \neq 0$.

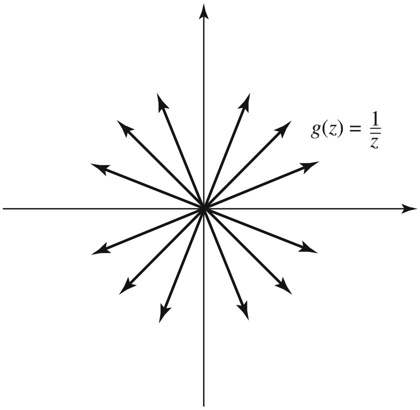
Then

$$g(z) = \overline{f(z)} = \frac{x + iy}{x^2 + y^2}$$

denotes a flow whose direction at z is the same as that of the vector (from 0) to z. In this case

$$\int_{|z|=\delta} f(z) dz = 2\pi i,$$

so that there is a nonzero flux across a circle centered at 0. Nevertheless the flow is locally irrotational and source-free in $z \neq 0$. (The flow is said to have a "source" at the origin.) $\Diamond$



As the examples above suggest, the possible fluid flows of the type considered are as abundant as the analytic functions in a given region. To focus on the particular flow related to a canonical conformal mapping of a region D, we make the following further assumptions.

A1. $\tilde{D}$ is the closure of a bounded simply-connected region. (We will refer to $\tilde{D}$ as a "barrier.")
A2. $g(z) = 1$ at ∞. That is, $\lim_{z\to\infty} g(z) = 1$.
A3. g has the direction of the tangent at the boundary of D (except for isolated points at which it may be zero or infinite).
A4. The flow is *totally* irrotational and source-free; i.e., $\int_C f(z)dz = 0$ for *every* closed curve C contained in D.

Under the above assumptions, suppose $z_0 \in D$ and set $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$. By assumption (A4), F is well-defined and hence analytic in D. Moreover, according to (A2), $F(z) \sim z$ at ∞. Finally, suppose the boundary of D is given by $z(t),\ a \le t \le b$. Then

$$\frac{d}{dt}F(z(t)) = F'(z(t))\dot{z}(t) = \overline{g(z(t))}\dot{z}(t),$$

which is real-valued according to (A3). Hence F maps ∂D onto a horizontal segment.

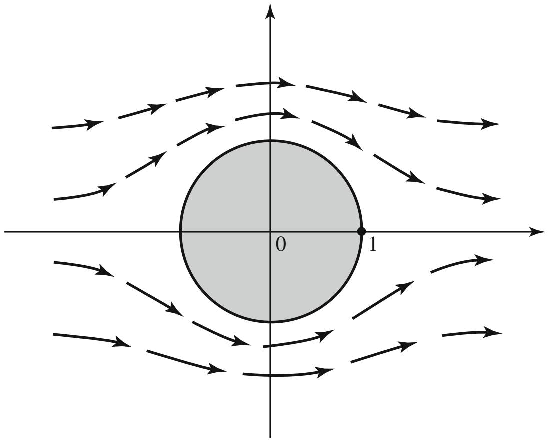
The converse is equally valid. If F maps D conformally onto the exterior of a horizontal interval and $F(z) \sim z$ at ∞, then $g(z) = \overline{F'(z)}$ will represent a fluid flow in D, satisfying assumptions (A1)–(A4).

Examples

i. $F(z) = z + 1/z$ maps the exterior of the unit disc conformally onto the exterior of the interval $[-2, 2]$, and clearly $F(z) \sim z$ at ∞. Thus

$$g(z) = \overline{F'(z)} = 1 - \frac{1}{\bar{z}^2}$$

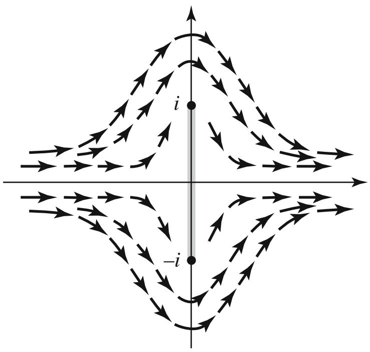
represents a fluid flow in the given region, satisfying (A1)–(A4). Note that $g(-1) = g(1) = 0$ and that the maximum speed is assumed at $\pm i$ where $|g(z)| = 2$.



ii. Suppose D is the complement of the interval I from $-i$ to i. Then an analytic $\sqrt{1+z^2}$ can be defined there (see Chapter 10, Exercise 16). Note that if $\sqrt{1+z^2}$ is taken to be positive on the positive axis, it is negative on the negative axis and maps D onto the exterior of $[-1, 1]$. Also $\sqrt{1+z^2} \sim z$ at ∞ so that $F(z) = \sqrt{1+z^2}$ is the desired conformal mapping and the flow is given by

$$g(z) = \overline{\left(\frac{z}{\sqrt{1+z^2}}\right)}.$$

In this (idealized) case, $g(0) = 0$ and $g(\pm i) = \infty$. ◊

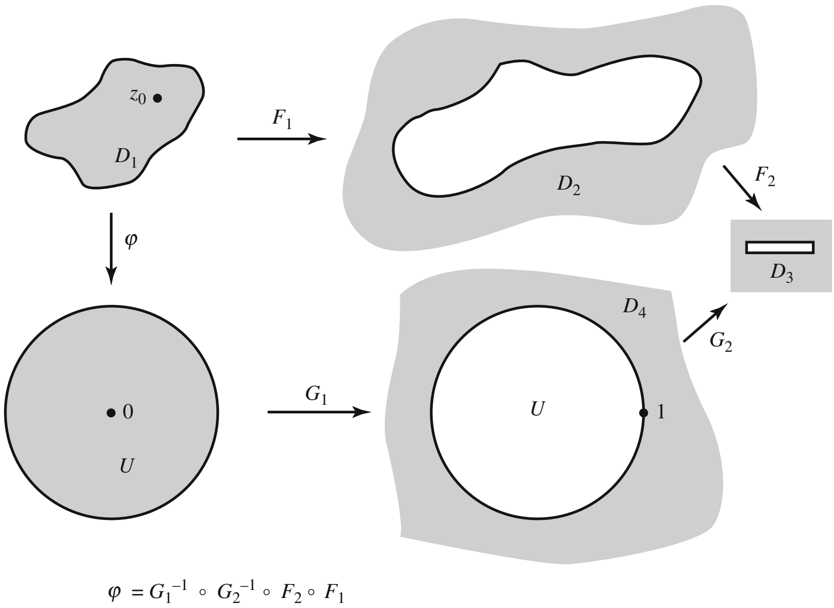


The examples above show how the appropriate mapping function enables us to obtain the fluid flow in a region. On the other hand, certain physical properties of the flow yield the following insights into conformal mapping.

I *Existence and Uniqueness of Conformal Mappings* As we have seen, the existence of a conformal mapping of a "single-barrier" domain onto the exterior of a horizontal interval is equivalent to the existence of a flow function satisfying (A1)–(A4). However, it is known that a flow satisfying these assumptions exists and is unique. According to Kelvin's Theorem [Milne-Thomson, p.95], the totally irrotational flow of a fluid occupying a region of the type considered (with the conditions at ∞ and along the boundary) is the unique flow with the least possible kinetic energy. The proof that such a unique flow exists can thus be given in terms of the partial differential equations governing the flow. We will not proceed that far with the physics, but we will be guided by the notion that a conformal mapping of the type sought is given by the solution to an extremal problem. We cast the problem in mathematical terms and complete the details in the next section.

II *Conformal Mapping of Other Types of Domains* By considering fluid flow throughout other types of domains, we can identify canonical domains to which they can be conformally mapped. In fact, reasoning like the above suggests that all domains with n "barriers" can be conformally mapped onto the plane slit along n horizontal line segments.

In the case of a simply connected domain D_1, the standard canonical domain is the unit disc. For, if we fix $z_0 \in D_1$, the mapping given by $F_1(z) = 1/(z - z_0)$ maps D_1 onto a single barrier domain D_2, sending z_0 into ∞. We then can map D_2 conformally onto D_3, the exterior of a horizontal interval, by a mapping F_2. Similarly, the unit disc U is mapped



by $G_1(z) = 1/z$ onto the exterior of the unit circle, D_4. Because D_4 is a single barrier domain, we have a conformal mapping G_2 of D_4 onto D_3. Finally, the mapping

$$\varphi = G_1^{-1} \circ G_2^{-1} \circ F_2 \circ F_1$$

maps the simply-connected domain D_1 conformally onto the unit disc U. Note that in changing our canonical region from the exterior of an interval to the unit disc the condition $F(\infty) = \infty$ is replaced by $\varphi(z_0) = 0$.

14.2 The Riemann Mapping Theorem

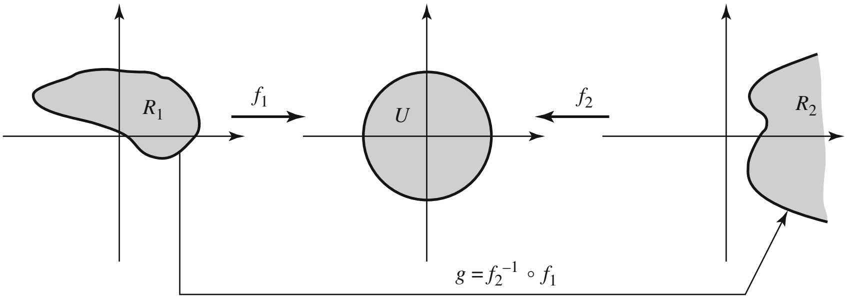
The Riemann Mapping Theorem, in its most common form, asserts that any two simply connected, proper subdomains of the plane are conformally equivalent. That is, if $R_1,\ R_2(\neq \mathbb{C})$ are simply connected regions, there exists a 1-1 analytic map of R_1 onto R_2.

Note that the condition that $R_1,\ R_2 \neq \mathbb{C}$ is necessary, as a consequence of Liouville's Theorem. See Exercise 10.

To prove the theorem, it suffices to show that for any simply connected region $R(\neq \mathbb{C})$ there exists a conformal mapping of R onto U. For then, if $R_1,\ R_2$ are two simply connected, proper subdomains of $\mathbb{C}$, we have conformal mappings

$$f_1 : R_1 \to U$$
$$f_2 : R_2 \to U$$

and $g = f_2^{-1} \circ f_1$ is a conformal mapping of R_1 onto R_2.



It is an easy exercise to show that given a conformal mapping f of R onto U and $z_0 \in R$, one can compose f with an appropriate automorphism of U so that the composite function φ maps R conformally onto U with the added properties that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$. [See Exercise 6.] In fact, if we insist that φ be a conformal mapping of R onto U with these two additional properties, then the mapping is unique. We first prove the uniqueness and then we will prove the Riemann Mapping Theorem by showing that such a unique mapping φ exists.

Riemann Mapping Theorem

For any simply connected domain $R(\neq \mathbb{C})$ and $z_0 \in R$, there exists a unique conformal mapping φ of R onto U such that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$.

Proof (Uniqueness)

Suppose φ_1 and φ_2 were two mappings with the above properties. Then $\Phi = \varphi_1 \circ \varphi_2^{-1}$ would be an automorphism of the unit disc with $\Phi(0) = 0$ and $\Phi'(0) > 0$. By 13.14, then, $\Phi(z) = e^{i\theta} z$ and since $\Phi'(0) = e^{i\theta} > 0$, it follows that Φ is the identity mapping. Hence, $\varphi_1 \equiv \varphi_2$.

(Existence) As we mentioned in the last section, we will find φ as the solution to an extremal problem. We recall some of the solutions to extremal problems for analytic mappings of U onto U that we obtained in Chapter 7. We found that, for fixed $\alpha \in U$, the 1-1 analytic mappings φ which maximize $|\varphi'(\alpha)|$ are precisely those of the form

$$\varphi(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z};$$

that is, those φ which

i. map α onto 0 and
ii. map U *onto* U.

(See Example 2 after 7.2 and Exercises 10 and 11 of Chapter 7.) This suggests a strategy for proving the existence of the conformal mapping φ of an arbitrary simply connected domain $R(\neq \mathbb{C})$ onto U. Namely, given R and $z_0 \in R$, we will consider the collection $\mathcal{F}$ of all 1-1 analytic functions $f: R \to U$ satisfying $f'(z_0) > 0$ and take φ to be such that $\varphi'(z_0) = \sup_{f \in \mathcal{F}} f'(z_0)$. The details which we must show are the following.

A. $\mathcal{F}$ is nonempty.
B. $\text{Sup}_{f \in \mathcal{F}} f'(z_0) = M < \infty$ and there exists a function $\varphi \in \mathcal{F}$ such that $\varphi'(z_0) = M$.
C. With φ as in (B), φ is a conformal mapping of R onto U such that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$. [The facts that $\varphi(z_0) = 0$ and that φ is an onto mapping are not guaranteed in (B).] □

Proof of (A)*:* Since $R \neq \mathbb{C}$, there exists a point $\rho_0 \in \tilde{R}$. [If $\tilde{R}$ contains a disc $D(\rho_0; \delta)$, we can simply set $f(z) = \delta/(z - \rho_0)$ and it would clearly follow that $|f| < 1$ throughout R. It is possible, however, that $\tilde{R}$ contains no discs at all so we must use a different approach.] Since R is simply connected, there exists an analytic function

$$g(z) = \sqrt{\frac{z - \rho_0}{z_0 - \rho_0}}$$

with $g(z_0) = 1$. It follows then that g must remain bounded away from -1. For if

$$g(\xi_n) = \sqrt{\frac{\xi_n - \rho_0}{z_0 - \rho_0}} \to -1$$

then

$$\frac{\xi_n - \rho_0}{z_0 - \rho_0} \to 1$$

so that $\xi_n \to z_0$. But then, by the continuity of g at z_0, it would follow that $g(\xi_n) \to +1$ and the contradiction is apparent. Hence, for some $\eta > 0$, $|g(z)+1| > \eta$ throughout R. Thus, if we set $f(z) = \eta/(g(z)+1)$, we will have $|f| < 1$. Since f is the composition of 1-1 functions, it too is 1-1 in R. Finally, since all the above properties are invariant under multiplication by $e^{i\theta}$ we can assume that $f'(z_0) > 0$ and hence $f \in \mathcal{F}$.

Proof of (B): Note, first, that since R is open, there exists some disc $D(z_0; 2\delta) \subset R$ and hence, for any $f \in \mathcal{F}$,

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\int_{C(z_0;\delta)} \frac{f(z)}{(z-z_0)^2}\,dz\right| \le \frac{1}{\delta},$$

by the usual $M - L$ estimate.

Suppose then that $M = \sup_{f\in F} f'(z_0)$ and let $f_1, f_2, \ldots$, be chosen so that $f_n'(z_0) \to M$ as $n \to \infty$. To show that there exists a function $\varphi \in \mathcal{F}$, such that $\varphi'(z_0) = M$, we will find a subsequence of $\{f_n\}_{n=1}^{\infty}$ which converges uniformly on compacta of R. To that end, let $\xi_1, \xi_2, \ldots$ be a countable dense subset of R. (For example, the $\xi's$ may be chosen as the points of R with rational coordinates.) Since $\{f_n(\xi_1)\}_{n=1}^{\infty}$ is a bounded sequence, there exists a subsequence $\{f_{1n}\}_{n=1}^{\infty}$ such that $\{f_{1n}(\xi_1)\}_{n=1}^{\infty}$ converges to some limit which we will denote $\varphi(\xi_1)$. Similarly $\{f_{1n}\}$ has a subsequence $\{f_{2n}\}$ such that $\{f_{2n}(\xi_2)\}$ converges and we denote its limit by $\varphi(\xi_2)$. Continuing in this manner, we obtain a nested sequence of subsequences $\{\{f_{kn}\}_{n=1}^{\infty}\}_{k=1}^{\infty}$ such that $\{f_{kn}\}_{n=1}^{\infty}$ converges at $\xi_1, \xi_2 \ldots, \xi_k$. If we then take the "diagonal" subsequence $\{\varphi_n(z)\}_{n=1}^{\infty}$ with $\varphi_n = f_{nn}$, it follows that $\varphi_n(z)$ converges to the function denoted by φ for $z = \xi_1, \xi_2, \ldots$.

We next wish to show that $\{\varphi_n\}$ converges throughout R and uniformly on any compact subset $\mathcal{K} \subset R$. We leave it as an exercise to show that any compact $\mathcal{K} \subset R$ is contained in a finite union of closed discs contained in R. Hence, we may assume, without loss of generality, that $\mathcal{K}$ is itself a fixed compact disc in R. Note that $d(\mathcal{K}, \tilde{R})$, the distance from $\mathcal{K}$ to the closed set $\tilde{R}$, is positive and we can set $d(\mathcal{K}, \tilde{R}) = 2d > 0$. Hence, since $|\varphi_n| \le 1$

$$|\varphi_n'(z)| = \left|\frac{1}{2\pi i}\int_{C(z;d)} \frac{\varphi_n(\xi)}{(\xi - z)^2}d\xi\right| \le \frac{1}{2\pi}\cdot\frac{2\pi d}{d^2} = \frac{1}{d},\ z \in \mathcal{K}$$

and

$$|\varphi_n(z_1) - \varphi_n(z_2)| = \left|\int_{z_1}^{z_2} \varphi_n'(z)dz\right| \le \frac{|z_1 - z_2|}{d}.$$

Thus, $\{\varphi_n\}$ is an "equicontinuous" sequence of functions on $\mathcal{K}$. That is, for each $\epsilon > 0$ and *all n*,

$$|\varphi_n(z_1) - \varphi_n(z_2)| \le \epsilon$$

as long as $|z_1 - z_2| \le \epsilon d$. If we then take $z \in \mathcal{K}$ and $\epsilon > 0$, we can write $|\varphi_n(z) - \varphi_m(z)| \le |\varphi_n(z) - \varphi_n(\xi_k)| + |\varphi_n(\xi_k) - \varphi_m(\xi_k)| + |\varphi_m(\xi_k) - \varphi_m(z)|$ and choosing ξ_k such that $|\xi_k - z| < \epsilon d/3$, it follows that

$$|\varphi_n(z) - \varphi_m(z)| < \epsilon$$

once n and m are chosen large enough so that

$$|\varphi_n(\xi_k) - \varphi_m(\xi_k)| < \frac{\epsilon}{3}.$$

Thus $\{\varphi_n(z)\}_{n=1}^{\infty}$ satisfies the Cauchy Criterion and converges for any $z \in \mathcal{K}$. Moreover, the limit function φ is continuous since

$$|\varphi(z_1) - \varphi(z_2)| = \lim_{n\to\infty} |\varphi_n(z_1) - \varphi_n(z_2)| < \epsilon, \qquad z_1, z_2 \in \mathcal{K},$$

as long as $|z_1 - z_2| < \epsilon d$.

Finally, to show that $\varphi_n \to \varphi$ uniformly on compacta, we apply the following standard argument. Suppose $\epsilon > 0$ is given and set

$$S_j = \{z \in \mathcal{K}: |\varphi_n(z) - \varphi(z)| < \epsilon \qquad \text{for } n > j\}.$$

Clearly $\mathcal{K} \subset \bigcup_{j=1}^{\infty} S_j$. Hence, since the sets S_j are open (by the equicontinuity of the functions φ_n) and $\mathcal{K}$ is compact, we can choose N such that $\mathcal{K} \subset \bigcup_{j=1}^{N} S_j$. Thus, for all $z \in \mathcal{K}$, $|\varphi_n(z) - \varphi(z)| < \epsilon$ when $n > N$ and the convergence is uniform.

Since $\varphi_n \to \varphi$ uniformly on compacta, φ is analytic (Theorem 7.6). Also, according to Theorem 10.12

$$\varphi'(z_0) = \lim_{n\to\infty} \varphi_n'(z_0) = M > 0$$

so that φ is nonconstant. Since it is the uniform limit of 1-1 functions, φ is 1-1 in R (Theorem 10.15).

Proof of (C): It remains only to show that $\varphi(z_0) = 0$ and that φ maps R *onto* U. To see the former, assume that $\varphi(z_0) = \alpha$, $0 < |\alpha| < 1$. Then

$$f(z) = \frac{\varphi(z) - \alpha}{1 - \overline{\alpha}\varphi(z)}$$

is also a 1-1 analytic map of R into U with

$$f'(z_0) = \frac{\varphi'(z_0)}{1 - |\alpha|^2}.$$

Thus $f'(z_0) > \varphi'(z_0)$, which is impossible.

Assume next that $\varphi(z) \neq \omega$, $\omega = -t^2 e^{i\theta}$, $0 < t < 1$. If we set $g(z) = e^{-i\theta}\varphi(z)$, g too will map R into U, $g(z_0) = 0$ and $|g'(z_0)| = \varphi'(z_0)$. Moreover, $g(z) \neq -t^2$ for all $z \in R$. If we then set

$$f_1(z) = \frac{g(z) + t^2}{1 + t^2 g(z)}$$

it follows that f_1 maps R into U with $f_1(z_0) = t^2$. Since $g(z) \neq -t^2$, $f_1(z) \neq 0$ and there exists an analytic square root

$$f_2(z) = \sqrt{f_1(z)}$$

with $f_2(z_0) = t$. Next, let

$$f_3(z) = \frac{f_2(z) - t}{1 - tf_2(z)}.$$

Clearly f_3 is 1-1 and direct calculation shows that

$$f_1'(z_0) = g'(z_0)(1 - t^4)$$
$$f_2'(z_0) = \frac{f_1'(z_0)}{2\sqrt{f_1(z_0)}} = \frac{f_1'(z_0)}{2t}$$
$$f_3'(z_0) = \frac{f_2'(z_0)}{1 - t^2}$$

so that, combining the above equations,

$$f_3'(z_0) = \frac{g'(z_0)(1 + t^2)}{2t} \gg g'(z_0)$$

since $1 + t^2 > 2t$ for $0 < t < 1$. If we set $f(z) = e^{i\theta} f_3(z)$ we will have $f \in \mathcal{F}$ and such that $f'(z_0) > \varphi'(z_0)$ which is impossible. Hence, φ must be onto and the proof is complete. □

Note: Consider the original sequence $f_1, f_2, \ldots$ such that $f_n'(z_0) \to M$ as $n \to \infty$. While the Riemann mapping function φ was obtained as the limit of a subsequence of $\{f_n\}$, it turns out that the original (full) sequence $\{f_n\}$ converges to φ. For suppose there existed some subsequence $f_{n_1}, f_{n_2}, \cdots$ such that

$$|f_{n_i}(z) - \varphi(z)| > \epsilon \tag{1}$$

for a fixed $z \in R$ and $\epsilon > 0$.

Then since $f_{n_k}'(z_0) \to M$ as $k \to \infty$, we could apply the previous proof to show that it has a subsequence which converges to the unique mapping function φ. But then (1) is impossible.

14.3 Mapping Properties of Analytic Functions on Closed Domains

Introduction

While the Riemann Mapping Theorem showed that any two open simply connected sets, other than $\mathbb{C}$, are conformally equivalent, there is no such theorem for closed or even for compact connected sets. In fact, there is often no *analytic* mapping of one closed domain onto another. As an example, there is no analytic mapping of the closed upper half-plane onto the closed first quadrant. If f were such an analytic function, it would map some real number x_0 onto the origin. Suppose $f'(x_0) \neq 0$. Then, according to Theorem 13.4, f would map the rays $I_1 = \{x_0 + t, 0 \le t < \infty\}$ and $I_2 = \{x_0 - t, 0 \le t < \infty\}$ onto two curves whose tangent lines form a straight angle.

But two such curves, which meet at the origin, cannot both lie in the first quadrant. Similarly, if f has a zero of order $k \geq 2$ at x_0, we can complete the argument by considering the effect of f on I_1 and on the ray $\{x_0 + te^{\pi i/k}, 0 \leq t < \infty\}$.

The above argument might leave the impression that if we somehow "rounded the corner" of the first quadrant, the resulting region might possibly be the image of the closed upper half-plane under some analytic mapping. That, in fact, is not the case (Corollary 14.6), but it requires a separate argument.

On the positive side, while an analytic mapping of one closed domain onto another is not always possible, the theorem below shows that for certain types of regions, any conformal mapping between the interiors extends to a 1-1 *continuous* map between the closures.

Recall that a Jordan curve is a simple closed curve. As we noted in Chapter 10, the very intuitive (but difficult to prove) Jordan Curve Theorem asserts that any Jordan curve disconnects its complement in the complex plane into two disjoint regions: a bounded component known as its interior and an unbounded component known as its exterior.

14.2 Definition

A region R will be called a Jordan region if it is the interior of a Jordan curve.

14.3 Theorem (Carathéodory-Osgood):

Any conformal mapping between two Jordan regions can be extended to a homeomorphism between the closures of the two regions.

Like the Jordan Curve Theorem, the Carathéodory-Osgood Theorem is not easy to prove. Although we will use the result throughout this section, we refer the interested reader to the proofs in the classic texts of Ahlfors and Carathéodory.

A Jordan curve γ is *positively-oriented* if its has winding number one around its interior points. For circles, as we have seen, this is the counterclockwise direction. As with circles, the positive orientation is the one which has the interior points on the left as the curve is traversed. We can also define a triple $\{a_1, a_2, a_3\} \subset \partial R$ to be positively oriented with respect to ∂R if the parametrization of ∂R which passes through a_1, a_2, a_3 in that order is positively-oriented. Suppose f is a conformal mapping between two Jordan regions R_1 and R_2. Then, according to the Argument Principle (see the comments following Corollary 10.9), the induced mapping between the boundaries must be orientation-preserving.

Based on the Carathéodory-Osgood Theorem, there are two additional ways to characterize a unique conformal mapping between any two Jordan regions.

14.4 Proposition:

Let R be any Jordan region, and let $\mathbb{D} = D(0; 1), \mathbb{S} = \partial\mathbb{D} = C(0; 1)$. Then

(i) *given a positively-oriented triple, $\{a_1, a_2, a_3\} \subset \partial R$ and a positively-oriented triple, $\{b_1, b_2, b_3\} \subset \mathbb{S}$, there exists a unique conformal mapping $f\colon R \to \mathbb{D}$ such that $f(a_k) = b_k$ for $k = 1, 2, 3$;*

(ii) *given $z_0 \in R$, $a \in \partial R$ and $b \in \mathbb{S}$, there exists a unique conformal mapping from R to $\mathbb{D}$ with $f(z_0) = 0$ and $f(a) = b$.*

Proof

To prove (i), take any conformal mapping of R onto $\mathbb{D}$ and follow it with an automorphism of $\mathbb{D}$ which maps the images of the three boundary points of R onto the three points of $\mathbb{S}$. [Such a mapping exists since Theorem 13.23 showed that there is a bilinear mapping T, sending the images of a_1, a_2, a_3 onto b_1, b_2, b_3, respectively. Moreover, since both triples are positively oriented, T will also map $\mathbb{D}$ onto itself.] To show that the mapping is unique, use the fact that if there were two such mappings f_1 and f_2, $f_2 \circ f_1^{-1}$ would be an automorphism of the unit disc with three fixed points on the unit circle, and hence would be the identity (Proposition 13.19). To prove (ii), follow the usual Riemann mapping with the appropriate rotation so that the image of the boundary point a is mapped onto b. The uniqueness follows from Schwarz' Lemma. □

Next we would like to consider the possibility of finding analytic (not necessarily conformal or even locally conformal) mappings from one closed region onto another. It is worth recalling that in this general context the boundary of a region is not necessarily mapped entirely onto the boundary of its image. For a simple example, consider the image of the rectangle $0 \le x \le 1$, $0 \le y \le 2\pi$ under the mapping $f(z) = e^z$. (Also, see exercise 3 of chapter 7.) In many cases, however, we can determine the image of the boundary by using the following theorem.

Rigidity of Analytic Arcs

An arc $\gamma : I \to \mathbb{C}$ (I being a real interval) is said to be *analytic* if γ is the restriction to I of a function $\tilde{\gamma}$ which is analytic on an open set of $\mathbb{C}$ containing I.

14.5 Theorem

Let $\gamma : I \to \mathbb{C}$ be an analytic arc where I is a compact real interval; let $\ell \subset \mathbb{C}$ be a circle or a line. If $\gamma\,[I] \cap \ell$ is infinite, then $\gamma\,[I] \subset \ell$.

Note that a very straightforward proof can be given for the case where ℓ is a line. By making a simple change of variables, we can assume that ℓ is a subset of the real line. In that case, $\operatorname{Im} \gamma(t)$ is a "real" analytic function for $t \in I$. Since $\operatorname{Im} \gamma(t)$ has infinitely many zeroes in I, it follows from the uniqueness theorem for real analytic functions that $\operatorname{Im} \gamma(t)$ is identically zero and $\gamma(t) \in \mathbb{R}$ for all $t \in I$. A modified form of the proof can also be given for the case where ℓ is a circle. The following argument, however, is applicable to both lines and circles and highlights the fact that the theorem is really about analytic arcs.

Proof of Theorem 14.5. Let $\tilde{\gamma}$ be analytic on some domain Ω containing I with $\tilde{\gamma}|_I = \gamma$; we may arrange that Ω be symmetric about $\mathbb{R}$. Let $(\ \ )^\dagger$ denote reflection across ℓ and consider $\omega(z) = (\tilde{\gamma}(\overline{z}))^\dagger$ for $z \in \Omega$. By the Schwarz reflection principle and by the symmetry of Ω, ω (like $\tilde{\gamma}$) is analytic in Ω. For $t \in I$ with $\gamma(t) \in \ell$, $\omega(t) = \gamma(t)$. But $\gamma(t) \in \ell$ for infinitely many values of t, so that according to the uniqueness theorem (6.10), $\omega \equiv \tilde{\gamma}$ throughout Ω. In particular, for all $t \in I$,

$$\gamma(t) = \tilde{\gamma}(t) = \omega(t) = (\gamma(t))^\dagger ,$$

which implies that $\gamma\,[I] \subset \ell$. □

14.6 Corollary

If f is an analytic mapping of the closed upper half-plane, and if the boundary of its image contains a line segment J, then the boundary of the image is a subset of the line containing J.

Proof

Note that the boundary of the image is a subset of the analytic curve $f(\mathbb{R})$. If the boundary includes a line segment J, a standard set-theoretic argument shows that, for sufficiently large N, $f([-N, N])$ contains infinitely many points of J. But then, according to Theorem 14.5, the boundary is entirely contained in the line determined by J. □

It is a well-known consequence of the maximum modulus principle that any nonconstant function g which is C-analytic in $\mathbb{D}$, mapping $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$ and $\mathbb{S}$ into $\mathbb{S}$ is of the form

$$g(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z} \quad \text{for some } n \in \mathbb{N}, \alpha_k \in \mathbb{D}, \text{ and } \theta \in \mathbb{R}. \tag{1}$$

(See the solution to Chapter 7, exercise 5.) According to Theorem 14.5, if g is analytic in $\overline{\mathbb{D}}$, the condition that g maps $\mathbb{S}$ into $\mathbb{S}$ can be dispensed with. Thus we have

14.7 Corollary

An analytic function mapping $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$ is of the form (1).

Proof

Given such an analytic function f, it suffices to show that $f[\mathbb{S}] = \mathbb{S}$. Note that $\mathbb{S} = \partial(f[\overline{\mathbb{D}}]) \subset f[\mathbb{S}]$ by the open mapping theorem. Thus $f[\mathbb{S}] \cap \mathbb{S} = \mathbb{S}$. Theorem 14.5 then shows that the analytic arc $f[\mathbb{S}]$ is a subset of $\mathbb{S}$. Hence $f[\mathbb{S}] = \mathbb{S}$. □

14.8 Corollary

If an entire function f maps $\mathbb{D}$ onto $\mathbb{D}$, then $f(z) = cz^n$ for some $c \in \mathbb{S}$.

Proof

Clearly $f[\overline{\mathbb{D}}] = \overline{\mathbb{D}}$. By Corollary 14.7, $f(z) = e^{i\theta} \prod_{k=1}^{n}(z - \alpha_k)/(1 - \overline{\alpha_k} z)$ for some $\alpha_k \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Since f is entire, all the points α_k must be zero, so that $f(z) = e^{i\theta} z^n$.

□

It follows that *any entire function f mapping some disc onto a disc is of the form*

$$f(z) = \alpha(z - z_0)^n + w_0$$

for some $\alpha, z_0, w_0 \in \mathbb{C}$ and $n \in \mathbb{N}$.

Analytic Mappings Between Polygons

Does there exist any analytic function which maps a closed convex n-gon onto another closed convex n-gon? Of course, if the polygons are similar, there is an elementary linear mapping between them. But what if they are not similar? The somewhat surprising answer is a very general NO, even if all the angles are identical, as would be the case with two dissimilar rectangles. In fact, we begin our inquiry by first considering rectangles. We then answer the question for the smallest n (i.e., $n = 3$) and finally address the general problem by induction on n. We note at the end of this section that the convexity condition on the polygons is actually not necessary.

14.9 Theorem

An analytic function f that maps a closed rectangle R onto another closed rectangle S is a linear polynomial.

Theorem 14.5 will play a crucial role here, allowing us to show that $f[\partial R] = \partial S$.

Proof

Assume without loss of generality that

$$R = [0, a] \times [0, b] \quad \text{and} \quad S = [0, c] \times [0, d].$$

First, note that f will *not* turn a straight line into an angle unequal in measure to an integral multiple of π, because, where $f' \neq 0$, f is conformal, and, where $f' = 0$, f magnifies angles by an integral factor. Therefore, f will not map any nonvertex point on ∂R to a vertex of S. We thus make the following observation:

Each vertex of S has precisely one preimage, which is a vertex of R. Hence, the image of each vertex of R must be a vertex of S and f gives a one-to-one correspondence between the vertices of R and those of S.

By the open mapping theorem, no interior point of R will be mapped by f to ∂S, and therefore $\partial S \subset f[\partial R]$. It is a simple set-theoretic matter that there is a side ℓ of ∂R such that $f[\ell] \cap [0, c]$ is infinite. Theorem 14.5 then implies that $f[\ell] \subset [0, c]$. Without loss of generality, we may assume that $\ell = [0, a]$, in which case either $f(0) = 0$ and $f(a) = c$, or $f(0) = c$ and $f(a) = 0$. But the latter possibility cannot happen, because f, being analytic, must preserve the orientation of the boundary. It then follows from the intermediate value theorem that $f[\ell] = [0, c]$.

Similarly we conclude that the side $\{c + is : s \in [0, d]\} \subset S$ must be the image of some side of R, and, since $f(a) = c$, that side of R must be $\{a + ir : r \in [0, b]\} \subset R$. Continuing until we exhaust all four sides of S, we then have established:

f maps each side of R onto a side of S.

Finally, reflection across the sides of R allows us to extend f (by the Schwarz reflection principle) to rectangles adjacent and congruent to R. Continuing this reflection process, we obtain an *entire* function, which by construction has at most linear growth in modulus. Hence, by the Extended Liouville Theorem (5.11), f must be a linear polynomial. □

We note those elements of the preceding argument that are applicable in general.

14.10 Lemma

Suppose an analytic function f maps some closed n-gon R onto a closed n-gon S. Then:

(a) *Each vertex of S has precisely one preimage, which is a vertex of R, and f maps each side of R monotonically onto a side of S.*
(b) *If R and S are both convex, each interior angle of R has the same measure as the corresponding interior angle of S.*

Proof

For part (a), given the argument for Theorem 14.9, we only need to establish monotonicity of f on each side of R. Let ℓ be a side of R. If $f(z)$ were to reverse direction as z traverses ℓ, there would be some $\mu \in \ell$ with $f'(\mu) = 0$. At the critical point μ, f magnifies angle by an integral factor. Hence the image of any μ-centered semidisc contained in R would lie partly outside of S, contradicting our hypothesis!

To show part (b), denote by $m(V)$ the measure of an interior angle with vertex V. For any vertex A of R, let $A' = f(A)$. Then, by analyticity of f, $m(A') = k_A m(A)$ for some $k_A \in \mathbb{N}$ (and, by convexity of S, $k_A m(A) < \pi$). Since R and S are both convex n-gons,

$$\sum_A m(A) = \sum_A m(A') = \sum_A k_A m(A).$$

Hence $k_A = 1$ for every vertex A. □

We turn now to the general case. The strategy will be to first solve the problem for triangles and then to use induction on the number of sides of the polygons in question to prove the general case.

14.11 Lemma

If an analytic function f maps a closed n-gon R onto a closed n-gon S, then f gives a conformal equivalence between their interiors: $\mathring{R}$ and $\mathring{S}$.

It is interesting that it is the isolated *singularities* (i.e., vertices) on ∂R and ∂S that force f to be conformal. Were the boundaries analytic Jordan curves, no conclusion about the valence of f on R could be drawn (as can be seen by considering z^n on the unit disc).

Proof

By Lemma 14.10(a), $f[\partial R] = \partial S$ and $f|_{\partial R}: \partial R \to \partial S$ is univalent. Thus the winding number of $f|_{\partial R}$ around any point in $\mathring{S}$ is exactly one. By the argument principle, f is a conformal equivalence between $\mathring{R}$ and $\mathring{S}$. □

We now apply Lemma 14.11 to resolve the triangle case.

14.12 Lemma

An analytic function f that maps some closed triangle R onto another closed triangle S is a linear polynomial.

Proof

By Lemma 14.10(b), the two triangles R and S have equal corresponding angles and are therefore similar. We then can easily construct an affine map g mapping R onto S, which is *a fortiori* conformal. If R is not equilateral, g is unique; if R is equilateral, let g be such that it agrees with f on the vertices of R. By Lemma 14.11, f gives a conformal equivalence between $\mathring{R}$ and $\mathring{S}$. Since f and g agree on the three vertices of R, by Proposition 14.4, $f = g$! □

Finally, we are able to completely answer the question raised at the beginning of this section.

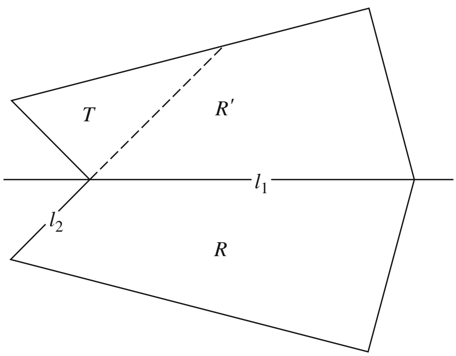
14.13 Theorem

An analytic function mapping some closed convex n-gon R onto another closed convex n-gon S is a linear polynomial.

Proof

If R is a rectangle, then, according to Lemma 14.10(b), S is also a rectangle. By Theorem 14.9, f is a linear polynomial.

Suppose then that R is a nonrectangular quadrilateral. Note that there is at least one interior angle of R that is obtuse. Denote by ℓ_1 and ℓ_2 the two sides of R that form this angle. Reflect R across ℓ_1 and denote by R' the reflected image of R. Extend ℓ_2 to a full line L_2. Then L_2 divides R' into two regions, one of which is a triangle. Call this triangle T. See the diagram below.



Observe that, by the Schwarz reflection principle, f can be analytically continued to R' and this extended map (which we also call f) maps R' onto the similarly-constructed quadrilateral S'. Since $f[\ell_2]$ is a side of S, by Theorem 14.5, $f[L_2 \cap R']$ is also a line segment. Therefore $f[\partial T]$ is the boundary of a triangle Δ; by considering the winding number of $f|_{\partial T}$, we deduce that $f[\mathring{T}] = \mathring{\Delta}$. Hence $f[T] = \Delta$, and, according to Lemma 14.12, f must be a linear polynomial.

To complete the proof for $n > 4$, we proceed as in the quadrilateral case and use induction on n. Given a convex n-gon R with $n > 4$, there must be an interior angle that is obtuse. Otherwise, the formula for the sum of interior angles would be violated. Reflect R across one of the two sides forming this obtuse angle and extend the other side to divide R' (the reflected image of R) into two regions, one of which is a convex polygon T with fewer than n sides. As in the quadrilateral case, f extends analytically to R' and maps T onto another convex polygon; it is immediate that the polygon $f[T]$ has the same number of sides as T. The induction hypothesis then guarantees that f is a linear polynomial. □

In the rectangle case, we first showed that the analytic function in question can be extended to an entire function, and then used its order of growth to prove that it is linear. In the case of other convex polygons, we showed that the analytic function in question can be analytically continued to a mapping between two triangles, which allowed us to conclude that it is a linear polynomial. In both cases, the Schwarz reflection principle and Theorem 14.5 played pivotal roles.

Note also that the convexity condition on the polygons in Theorem 14.13 can be dispensed with. This slightly more general result will follow easily. The idea is that, by extending the sides forming an interior angle greater than a straight angle, we can find a convex polygon that is mapped analytically to a convex polygon. We leave the details to the interested reader.

Conformal Mappings Between Dissimilar Rectangles

Among entire functions, linear polynomials are the only ones that are conformal on every domain. Since a linear polynomial is a composition of rotation, real multiplication, and translation, the two rectangles in Theorem 14.9 and the two n-gons in Theorem 14.13 are actually similar.

Let R and S be two closed rectangles that are *not similar* to each other and let f be a *conformal* map from $\mathring{R}$ onto $\mathring{S}$ (whose existence is guaranteed by the Riemann mapping theorem). By the Carathéodory-Osgood theorem, f extends to a homeomorphism $\tilde{f}: R \to S$. The argument in the proof of Theorem 14.9 shows that

- the extension $\tilde{f}$ of f to ∂R fails to be analytic at some point on ∂R;
- at least one vertex of R fails to be mapped by $\tilde{f}$ to a vertex of S.

We can attempt to make $\tilde{f}$ "as analytic as possible" by requiring that three of the four vertices of R be mapped to vertices of S; then by the Schwarz reflection principle, $\tilde{f}$ will be analytic on the two sides of R bounded by the three vertices. Note that, according to Proposition 14.4, the requirement that three chosen vertices of R be mapped to three chosen vertices of S uniquely determines the conformal equivalence f between $\mathring{R}$ and $\mathring{S}$. One can express such a map explicitly with the aid of the Schwarz-Christoffel formulae, and this will offer an illustration of the fact that the interiors of two dissimilar rectangles cannot be put into a conformal equivalence that extends to a vertex-preserving homeomorphism.

To focus on a simple concrete example, let $R = [-1, 1] \times [0, r]$ and $S = [-1, 1] \times [0, s]$. Let $\mathbb{H}$ denote the upper half-plane and let $F_R : \mathbb{H} \to R$ be the Riemann map with

$$\tilde{F}_R(-1) = -1, \quad \tilde{F}_R(0) = 0, \quad \tilde{F}_R(1) = 1. \tag{1}$$

Then F_R is given by the elliptic integral

$$F_R(z) = C_r \int_0^z \frac{1}{\sqrt{(1-\xi^2)(1-k_r^2\xi^2)}} d\xi$$

where $k_r = 1/a_r$ and $C_r = 1/\int_0^1 [(1-\xi^2)(1-k_r^2\xi^2)]^{-1/2} d\xi$. It follows from the definition of F_R that $a_r = \tilde{F}_R^{-1}(1+ir)$ increases monotonocally with r and that

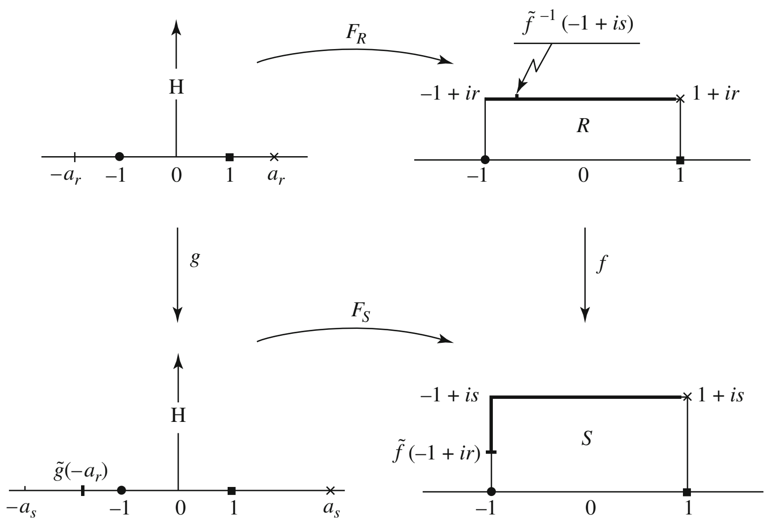
$$\tilde{F}_R(\pm a_r) = \pm 1 + ir.$$

In the same manner, let $F_S : \mathbb{H} \to S$ be the Riemann Mapping, also satisying conditions (1) and with a_s similarly defined.

Suppose $f : \mathring{R} \to \mathring{S}$ is a conformal equivalence with

$$\tilde{f}(-1) = -1, \quad \tilde{f}(1) = 1, \quad \tilde{f}(1+ir) = 1+is.$$

We will see that, if $r \neq s$, $\tilde{f}(-1+ir) \neq -1+is$ and hence $\tilde{f}^{-1}(-1+is)$ is not a vertex of R, which implies that $\tilde{f}$ cannot be analytic at $\tilde{f}^{-1}(-1+is)$. To this end, we consider $g = F_S^{-1} \circ f \circ F_R : \mathbb{H} \to \mathbb{H}$. See the diagram below.



The map g is an automorphism of the upper half-plane. Hence it is a bilinear mapping with

$$\tilde{g}(-1) = -1, \quad \tilde{g}(1) = 1, \quad \tilde{g}(a_r) = a_s, \text{ with } 1 < a_r, a_s. \tag{2}$$

If $\tilde{f}(-1+ir) = -1+is$, we would also have $\tilde{g}(-a_r) = -a_s$. Since the cross-ratio is preserved under bilinear transformations, it would follow that $(-a_r, -1, 1, a_r) = (-a_s, -1, 1, a_s)$, or equivalently:

$$\frac{(a_r+1)^2}{4a_r} = \frac{(a_s+1)^2}{4a_s} \tag{3}$$

It is easy to see, however, that (3) is possible if and only if either $a_r = a_s$ or $a_r a_s = 1$. The latter equation is impossible, according to (2). Hence $a_r = a_s$ and we have the desired conclusion: f maps vertices of R to vertices of S only if R and S are similar.

If $r < s$, then $a_r < a_s$. So, if we let $\tilde{g}(-a_r) = -b$, by the preservation of cross-ratios, $b < a_r$. This translates into the sketch of the conformal mapping f in the above diagram. Note that the top side of R is bent by $\tilde{f}$ into an L-shaped path and that $\tilde{f}$ fails to be analytic at precisely one point, i.e., at $\tilde{f}^{-1}(-1+is)$.

Finally, observe that, despite the fact that $f[R]$ is a rectangle, the image of any subrectangle $Q \subset \mathring{R}$ must *not* be a rectangle; for, otherwise, f would map $\overline{Q}$ onto a rectangle and thus have to be a linear polynomial, in which case R would necessarily be similar to S.

Exercises

1. Suppose g represents a locally irrotational and source-free flow in a simply-connected domain D and $F(z) = \int_{z_0}^{z} \bar{g}(\zeta)d\zeta$. Show that g is orthogonal to the curves given by $\text{Re}F(z) = \text{constant}$.

2. If F and g are as in (1), show that the curves $\text{Im}F(z) = \text{constant}$ are the "streamlines" of g; i.e., show that the flow is tangent to those curves.

3. Find the streamlines of the flow functions given by

   a. $g(z) = \bar{z}$
   b. $g(z) = 1/\bar{z}, z \neq 0$.

4. Verify directly that $F(z) = z + 1/z$ is the unique conformal mapping (up to an additive constant) of $|z| > 1$ onto the exterior of a horizontal interval, with $F(z) \sim z$ at ∞ [Hint: Begin with the Laurent Expansion

$$F(z) = z + A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots$$

   and use the fact that $\text{Im}F(e^{i\theta}) = \text{ constant}$ (see Markushevich, p. 189).]

5. a. Show that $w = 2z + 1/z$ maps the exterior of the unit circle conformally onto the exterior of the ellipse:

$$\frac{x^2}{9} + y^2 = 1.$$

   b. Find a conformal mapping of the exterior of the ellipse $x^2/9 + y^2 = 1$ onto the exterior of a real line segment.

6. Given a conformal mapping f of R onto U (the unit disc) and $z_0 \in R$, find a conformal mapping g of R onto U with $g(z_0) = 0$ and $g'(z_0) > 0$.

7.* Let R be a simply connected region $\neq \mathbb{C}$, which is symmetric with respect to the real axis; and suppose that f is the Riemann mapping of R onto U, with $f(z_0) = 0$, $f'(z_0) > 0$, for some real-valued $z_0 \in R$. Prove that $f(\overline{z}) = \overline{f(z)}$ for all $z \in R$.

8.* Find the unique conformal mapping of the upper half-plane onto the unit disc with

   a. $f(-1)$, $f(0)$, $f(1)$ equal to 1, i, and -1, respectively.
   b. $f(i) = 0$ and $f(1) = 1$.

9. Let R be simply-connected and assume $z_1, z_2 \in R$. Show there exists a conformal mapping of$R$ onto itself, taking z_1 into z_2. (Consider two cases: $R \neq \mathbb{C}$ and $R = \mathbb{C}$.)

10. Suppose R is any simply connected domain $\neq \mathbb{C}$. Show that there exists no conformal mapping of $\mathbb{C}$ onto R.

11. Let R be a simply connected region and $z_0 \in R$. Suppose G is defined as the set of all analytic functions $g : R \to U$ such that $g'(z_0) > 0$ (g need *not* be 1-1).

   a. Show that

$$\sup_{g \in G} g'(z_0) = M^* < \infty.$$

   b. Assuming that $\Phi'(z_0) = M^*$, show that Φ is 1-1 in R. [*Hint*: Show that Φ is the Riemann mapping function.]

12.* a. Find a conformal mapping f from the semi-disc $S = \{z : |z| < 1, \text{Im} z > 0\}$ onto the unit disc U and show that it extends to a homeomorphism between $\overline{S}$ and $\overline{U}$.
   b. Show that f is analytic on $\overline{S}$ but f^{-1} is not analytic on $\overline{U}$.

13.* Prove that there is no analytic mapping from $\overline{U}$ onto a "Norman window", which is a closed region whose boundary is a rectangle surmounted by a semicircle.