Chapter 13
Beyond Einstein's Gravity

To conclude our introductory course on general relativity, in this final chapter we will briefly present the main theoretical problems that plague Einstein's gravity as well as some attempts to solve them. The discussion will be necessarily at a very qualitative level, without details and far from being complete. A more accurate and complete study of these topics would be beyond the purposes of the present textbook.

13.1 Spacetime Singularities

Generally speaking, a spacetime singularity is a "region" of the spacetime with some pathological properties. For example, there are several physically relevant solutions of Einstein's equations in which the spacetime is *geodesically incomplete*; that is, there are geodesics that cannot be extended beyond a certain point. At the end of the 1960s, Roger Penrose and Stephen Hawking discussed in a number of theorems the conditions that make the formation of this kind of spacetime singularities unavoidable [8, 9]. The singularities at $r = 0$ in the Schwarzschild and in the Reissner–Nordström solutions are singularities in the sense that the spacetime is geodetically incomplete there. In the case of the Kerr metric, $r = 0$ is geodetically incomplete only for the geodesics in the equatorial plane, while the spacetime can be extended beyond $r = 0$ for off-equatorial trajectories (see Sect. 10.6). The cosmological models discussed in Sects. 11.3.2 and 11.3.3 are singular at $t = 0$ as geodesics cannot be extended into the past.

Note that the fact that a spacetime is geodetically incomplete has profound physical implications. If we cannot extend a geodesic beyond a certain point, predictability is lost. We simply do not know what is going on at the singularity, where standard calculation methods clearly break down. With the terminology of Appendix C, at the

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singularity we do not have a differentiable manifold, and therefore we cannot do any calculation. This is not a minor issue. For instance, in the case of the Schwarzschild spacetime, every particle crossing the event horizon reaches the central singularity in a finite time. So we do not know what happens to the matter swallowed by the black hole!

Some spacetime singularities are curvature singularities, i.e. some curvature invariants, like the scalar curvature R and/or the Kretschmann scalar $\mathscr{K}$, diverge there. Physical quantities that diverge are often a symptom of the breakdown of a theory.

In Einstein's gravity, there is only one dimensional coupling constant, Newton's constant of gravitation $G_{\rm N}$. If we combine $G_{\rm N}$ with the speed of light c and Dirac's constant $\hbar$, we obtain the *Planck length* $L_{\rm Pl}$, the *Planck time* $T_{\rm Pl}$, the *Planck mass* $M_{\rm Pl}$, and the *Planck energy* $E_{\rm Pl}$

$$
\begin{aligned}
L_{\rm Pl} &= \sqrt{\frac{G_{\rm N}\hbar}{c^3}} = 1.616 \cdot 10^{-33}\ {\rm cm}\,, \\
T_{\rm Pl} &= \sqrt{\frac{G_{\rm N}\hbar}{c^5}} = 5.391 \cdot 10^{-44}\ {\rm s}\,, \\
M_{\rm Pl} &= \sqrt{\frac{\hbar c}{G_{\rm N}}} = 2.176 \cdot 10^{-5}\ {\rm g}\,, \\
E_{\rm Pl} &= \sqrt{\frac{\hbar c^5}{G_{\rm N}}} = 1.221 \cdot 10^{19}\ {\rm GeV}\,.
\end{aligned} \tag{13.1}
$$

In units in which $c = \hbar = 1$, $L_{\rm Pl} = T_{\rm Pl} = 1/M_{\rm Pl} = 1/E_{\rm Pl}$, and therefore we can use just one of them. We can generically talk about *Planck scale* and often we use the Planck mass $M_{\rm Pl}$ in energy units: $M_{\rm Pl} = 10^{19}\,{\rm GeV}$.

In Einstein's gravity, the Planck scale looks like the natural UV cut-off and therefore we should expect new physics beyond it (see the next section for more details). New physics may thus show up when curvature invariants approach the Planck scale, e.g. $R \to M_{\rm Pl}^2$ and $\mathscr{K} \to M_{\rm Pl}^4$, before they diverge to infinity.

It is often claimed that the problem of spacetime singularities should be fixed by a yet unknown theory of quantum gravity. As we will briefly discuss in the next section, Einstein's gravity can be quantized, but the result is an effective theory valid at scales much smaller than the Planck one, so it is not the theoretical framework to address the problem of spacetime singularities. In extensions of Einstein's gravity, at least some spacetime singularities can be avoided, but this is not an easy job in general. Moreover, there are many attempts to extend Einstein's gravity. Every model has its own predictions, which are difficult or impossible to test, and therefore it is extremely challenging to make progress in the field.

13.2 Quantization of Einstein's Gravity

Contrary to what is sometimes claimed, Einstein's gravity can be quantized, and the result is a self-consistent theory [5, 6, 10] (see Ref. [4] for a pedagogical introduction). However, it is an *effective field theory*[1] valid for energies $E \ll M_{\rm Pl}$. At low energies, quantum corrections are extremely small, and experimental tests seem to be very unlikely, even in the future. If we consider processes approaching the Planck scale, the theory breaks down, and therefore it cannot address the most interesting questions concerning spacetime singularities at the center of black holes and in cosmology.

The procedure to quantize Einstein's gravity can be summarized as follows. We write the spacetime metric $g_{\mu\nu}$ as a background field $\bar{g}_{\mu\nu}$ plus a perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{\kappa} h_{\mu\nu} \, . \tag{13.2}$$

In general, $\bar{g}_{\mu\nu}$ is a solution of Einstein's equations, not necessarily the Minkowski metric $\eta_{\mu\nu}$. $h_{\mu\nu}$ is the field to quantize. $\tilde{\kappa}^2 = 32\pi/M^2_{\rm Pl}$ is introduced to provide the right dimensions to $h_{\mu\nu}$. The inverse metric reads

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \tilde{\kappa} h^{\mu\nu} + \tilde{\kappa}^2 h^{\mu}_{\rho} h^{\rho\nu} + \cdots \, . \tag{13.3}$$

We expand the action around the background metric $\bar{g}_{\mu\nu}$

$$S = \int d^4x \sqrt{-\bar{g}} \left(\mathscr{L}^{(0)} + \mathscr{L}^{(1)} + \mathscr{L}^{(2)} + \cdots \right) \, , \tag{13.4}$$

where

$$\begin{aligned}
\mathscr{L}^{(0)} &= \frac{2}{\tilde{\kappa}^2} \bar{R} \, , \\
\mathscr{L}^{(1)} &= \frac{1}{\tilde{\kappa}} h_{\mu\nu} \left(\bar{g}^{\mu\nu} \bar{R} - 2\bar{R}^{\mu\nu} \right) \, , \\
\mathscr{L}^{(2)} &= \frac{1}{2} \left(\bar{\nabla}_{\rho} h_{\mu\nu} \right) \left(\bar{\nabla}^{\rho} h^{\mu\nu} \right) - \frac{1}{2} \left(\bar{\nabla}_{\mu} h \right) \left(\bar{\nabla}^{\mu} h \right) \\
&\quad + \left(\bar{\nabla}_{\mu} h \right) \left(\bar{\nabla}_{\nu} h^{\mu\nu} \right) - \left(\bar{\nabla}_{\rho} h_{\mu\nu} \right) \left(\bar{\nabla}^{\nu} h^{\mu\rho} \right) \\
&\quad + \frac{1}{2} \bar{R} \left(\frac{1}{2} h^2 - h_{\mu\nu} h^{\mu\nu} \right) + \left(2h^{\rho}_{\mu} h_{\nu\rho} - h h_{\mu\nu} \right) \bar{R}^{\mu\nu} \, .
\end{aligned} \tag{13.5}$$

[1] An effective field theory is a field theory that provides reliable predictions in its realm of validity but it breaks down beyond it. The crucial point of the known effective field theories is the possibility of separating the physics at low energies from that at much higher energies. This permits us to make predictions at low energies without making unwarranted assumptions about what is going on at high energies.

The quantities with a bar are evaluated with the background metric $\bar{g}_{\mu\nu}$. h is the trace of $h_{\mu\nu}$, i.e. $h = h^{\mu}_{\mu}$.

At this point, there are some technical problems to solve. It is necessary to fix the gauge, but this causes the appearance of some non-physical degrees of freedom. The latter can be removed with some tricks. After that, we can infer the Feynman rules of the theory in the same way as when we quantize other interactions. The result is a non-renormalizable theory: all amplitudes are divergent at a sufficiently high order in the perturbation, and therefore it is not possible to absorb the divergent terms into a finite number of observables. The theory is predictable at low energies ($E \ll M_{\rm Pl}$), when low order terms are dominant and higher order terms can be neglected, and breaks down when we approach the Planck scale. We have a situation similar to the Fermi theory of the weak interactions, which is a viable effective theory for energies $E \ll M_W$, where $M_W = 80$ GeV is the mass of the W-boson, while it breaks down when $E \to M_W$.

The quantum theory obtained from this procedure is predictable at low energies. As an example, we can consider the Schwarzschild spacetime. The classical Schwarzschild metric in the harmonic gauge reads (see e.g. Ref. [11])

$$\begin{aligned}
g_{00} &= -\frac{1 - \frac{G_{\rm N}M}{c^2 r}}{1 + \frac{G_{\rm N}M}{c^2 r}} = -\left(1 - \frac{2G_{\rm N}M}{c^2 r} + \frac{2G_{\rm N}^2 M^2}{c^4 r^2} + \cdots\right) \\
g_{0i} &= 0 \\
g_{ij} &= \left(1 + \frac{G_{\rm N}M}{c^2 r}\right)^2 \delta_{ij} + \frac{G_{\rm N}^2 M^2}{c^4 r^2}\left(\frac{1 + \frac{G_{\rm N}M}{c^2 r}}{1 - \frac{G_{\rm N}M}{c^2 r}}\right)\frac{x_i x_j}{r^2} \\
&= \left(1 + \frac{2G_{\rm N}M}{c^2 r} + \frac{G_{\rm N}^2 M^2}{c^4 r^2}\right)\delta_{ij} + \frac{G_{\rm N}^2 M^2}{c^4 r^2}\frac{x_i x_j}{r^2} + \cdots . \qquad (13.6)
\end{aligned}$$

In Ref. [3], the authors employed a particular set of Feynman diagrams and found long distance quantum corrections to the solution in (13.6). The quantum-corrected metric reads

$$\begin{aligned}
g_{00} &= -\left(1 - \frac{2G_{\rm N}M}{c^2 r} + \frac{2G_{\rm N}^2 M^2}{c^4 r^2} + \frac{62 G_{\rm N}^2 M \hbar}{15\pi c^5 r^3} + \cdots\right) \\
g_{0i} &= 0 \\
g_{ij} &= \left(1 + \frac{2G_{\rm N}M}{c^2 r} + \frac{G_{\rm N}^2 M^2}{c^4 r^2} + \frac{14 G_{\rm N}^2 M \hbar}{15\pi c^5 r^3}\right)\delta_{ij} \\
&\quad + \left(\frac{G_{\rm N}^2 M^2}{c^4 r^2} + \frac{76 G_{\rm N}^2 M \hbar}{15\pi c^5 r^3}\right)\frac{x_i x_j}{r^2} + \cdots . \qquad (13.7)
\end{aligned}$$

If we compare the metric in Eq. (13.6) with that in Eq. (13.7), we see that the leading order quantum corrections are proportional to $\hbar$, as it was to be expected. However, the contribution of these corrections is very small and therefore any observational test turns out to be extremely challenging.

The "UV completion", namely how to extend the theory in order to have a good model even at high energies, is a completely open problem. There are many attempts in the literature, but none is satisfactory at the moment. The lack of experimental tests because of the huge value of the Planck energy with respect to particle physics energies is an additional limitation of this line of research.

13.3 Black Hole Thermodynamics and Information Paradox

As we have pointed out in Sect. 10.4, the black holes of Einstein's gravity are very simple objects, in the sense that they are characterized by a small number of parameters. In the case of a Kerr–Newman black hole, there are three parameters: the mass M, the spin angular momentum J, and the electric charge Q. If the black hole swallows a number of particles, the black hole only changes the values of M, J, and Q, and the particles disappear. At first, one may naively argue that such a process could permit the reduction of the entropy of the whole system, violating the Second Law of Thermodynamics. Entropy is a measurement of the number of microscopic configurations that a thermodynamical system can have for certain macroscopic variables. When the system is made of a black hole and many particles, we have a large number of possible configurations, because every particle has its own position and velocity. After the black hole has swallowed all particles, we have just a black hole characterized by a few parameters and nothing more.

Such a conclusion is not correct. It turns out that the entropy of a black hole is proportional to the area of its event horizon [1]

$$S_{\rm BH} = \frac{k_{\rm B} A_{\rm H}}{4 L_{\rm Pl}^2} \, , \tag{13.8}$$

where $k_{\rm B}$ is Boltzmann's constant and $A_{\rm H}$ is the area of the event horizon of the black hole. Equation (13.8) is usually referred to as the *Bekenstein-Hawking formula*. However, the exact origin of the black hole entropy is not yet known.

This is not the end of the story. A black hole is not completely "black", in the sense that it can only swallow matter. It has instead a finite temperature and, therefore, it emits radiation (*Hawking radiation*) [7]. In the case of a Schwarzschild black hole, the temperature is

$$T_{\rm BH} = \frac{\hbar c^3}{8\pi G_{\rm N} M k_{\rm B}} \tag{13.9}$$

The black hole temperature is proportional to $\hbar$, which means that classically ($\hbar \to 0$) the temperature vanishes. Since $\hbar \neq 0$, the temperature is finite, but in the case of astrophysical black holes it is extremely low and can be ignored. For a Solar mass Schwarzschild black hole, we have

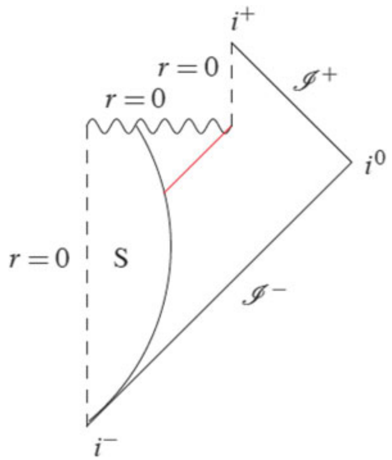

Fig. 13.1 Penrose diagram for the formation and the evaporation of a black hole. The letter S indicates the interior region of the collapsing star. The black arc extending from i^- to the singularity (the horizontal line with wiggles) is the surface of the star. See the text for the details

$$T_{\rm BH} = 6 \cdot 10^{-8} \left(\frac{M_\odot}{M} \right) \, {\rm K} \, . \tag{13.10}$$

A black hole emits radiation (almost) as a black body and its luminosity is given by

$$L_{\rm BH} \sim \sigma_{\rm BH} A_{\rm H} T_{\rm BH}^4 \sim 10^{-21} \left(\frac{M_\odot}{M} \right)^2 \, {\rm erg/s} \, . \tag{13.11}$$

where $\sigma_{\rm BH}$ is a Stefan–Bolzmann-like constant whose numerical value depends on the particles that can be emitted, so it depends on the black hole mass and the particle content of the theory. It is clear that a luminosity like that predicted by Eq. (13.11) will not likely be detected, even in the future.

Since a black hole emits radiation, it loses mass and eventually can "evaporate". It turns out that the Penrose diagram of the gravitational collapse of a massive body that creates a black hole, illustrated in Fig. 10.8, changes when the evaporation process is taken into account. The (possible) new Penrose diagram is shown in Fig. 13.1. As an artifact of the coordinates of the Penrose diagram, it seems like the evaporation is an instantaneous process, but this is not true. The evaporation process is actually very slow and can be estimated from the luminosity in (13.11), since $L_{\rm BH} = dM/dt$. At the end we have a Minkowski spacetime.

The evaporation of a black hole presents the following open problem. Let us assume that the initial collapsing body is a well-defined quantum state (a "pure" state described by a single ket vector). In quantum mechanics, the evolution operator is unitary and a pure state evolves into a pure state. However, in the formation and evaporation of a black hole, it seems that a pure state could be transformed into thermal radiation, which is a mixed state. If so, the laws of quantum mechanics are not consistent with the black hole evaporation process. This is the *black hole information paradox*. There are several proposals to solve the problem, but we do not know which one, if any, is correct. For example, it is possible that there are small but non-negligible deviations from the standard semiclassical predictions at the event horizon and that there is no information loss once these corrections are properly taken into account.

13.4 Cosmological Constant Problem

When we construct the action of a theory, we have to add to the Lagrangian any term that is not forbidden by the symmetries of the theory. Even if we did not do so, such terms would show up after quantum corrections are taken into account, as nothing can prevent their presence. This is well established in particle physics, where it is possible to test quantum field theory.

From this point of view, when we write the action of the gravitational sector, we should have

$$S = \int d^4x \sqrt{-g} \left(-\frac{\Lambda}{\kappa c} + \frac{R}{2\kappa c} + a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + \cdots \right) \tag{13.12}$$

instead of the simple Einstein-Hilbert action of Eq. (7.24). The terms proportional to R^2 and $R_{\mu\nu}R^{\mu\nu}$, as well as those of higher order that are omitted in Eq. (13.12), become important only when we approach the Planck scale, which is not the case in astrophysical or cosmological environments that can be observed (at least for the moment). So they can be safely neglected for most purposes. On the contrary, the cosmological constant Λ cannot be ignored and a number of theoretical arguments would suggest a value much higher than the one compatible with that in the Universe. This is the *cosmological constant problem*.

If we write the action of the gravitational sector with a cosmological constant term and we do not want to introduce a new scale, we should expect that the effective energy density associated to the cosmological constant is

$$\rho_\Lambda = \frac{\Lambda}{\kappa} \sim M_{\rm Pl}^4 \sim 10^{76}\ {\rm GeV}^4\,, \tag{13.13}$$

because $M_{\rm Pl}$ is the only scale in the gravitational sector. However, if Λ introduces a new scale in the system, we cannot make any prediction about its value, that should thus be obtained from observations.

The cosmological constant problem arises because the effective cosmological constant, namely the one appearing in the Friedmann equations and contributing to the expansion of the Universe, should be the sum of a number of different contributions. While we cannot make exact predictions of the final value resulting from this sum, we can estimate the order of magnitude of the single contributions. An almost perfect cancellation from so many different terms of different magnitude would be very unnatural.

For example, if we have the scalar field ϕ with the action (in this discussion, we use natural units in which $\hbar = c = 1$ for convenience)

$$S = -\int \left[\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) + V(\phi) \right] \sqrt{-g}\, d^4x \,, \tag{13.14}$$

its energy-momentum tensor is

$$T^{\mu\nu} = \left(\partial^\mu \phi\right) \left(\partial^\nu \phi\right) - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \left(\partial_\rho \phi\right) \left(\partial_\sigma \phi\right) - V(\phi) g^{\mu\nu} \,. \tag{13.15}$$

In the state of minimum energy, the kinetic energy of the scalar field vanishes, and ϕ sits at the minimum of the potential. Equation (13.15) becomes

$$T^{\mu\nu} = -V(\phi_{\rm min})\, g^{\mu\nu} \,. \tag{13.16}$$

This generates the vacuum energy $\rho_\Lambda = V(\phi_{\rm min})$.

From particle physics, we would expect a number of terms similar to this example. In the Standard Model of Particle Physics, we have the Higgs boson. The absolute value of its potential cannot be measured in particle colliders, because with the exception of gravity only energy differences matter. However, the Higgs sector is at the electroweak scale $M_{\rm EW} \sim 100$ GeV, and therefore it is natural to expect $V(\phi_{\rm min}) \sim (100 \text{ GeV})^4$. If we believe in the Grand Unification Theories, we have similar situations but involving the GUT scale $M_{\rm GUT} \sim 10^{16}$ GeV, and therefore $V(\phi_{\rm min}) \sim (10^{16} \text{ GeV})^4$. From quantum chromodynamics (QCD), we expect that the world around us is a condensate of quarks and gluons. The QCD scale is of order 100 MeV, and the corresponding contribution to the effective cosmological constant should be roughly $(100 \text{ MeV})^4$. Note that the value of the quark and gluon condensates can be measured by observations, even if we cannot measure the absolute value contributing to the cosmological constant.

Lastly, a non-vanishing cosmological constant should be expected even from the quantum fluctuation of any field of the theory. Within a semiclassical approach, we treat gravity as a classical field and we quantize matter. The result is that the Einstein equations read

$$G_{\mu\nu} = \frac{8\pi G_{\rm N}}{c^4} \langle T_{\mu\nu} \rangle \,, \tag{13.17}$$

where $\langle T_{\mu\nu} \rangle$ is the expectation value of the matter energy-momentum tensor $T_{\mu\nu}$. An effective cosmological constant arises from the fluctuations of the matter field. After standard renormalization procedures, we obtain a renormalized cosmological constant [2], whose value should be measured in experiments. While it is not possible to make theoretical predictions on it, it would still be natural to expect $\rho_\Lambda \sim M_{\rm Pl}^4$, because there are no other scales in the gravity sector.

In the end, from theoretical arguments we cannot make any clear prediction about the value of the effective cosmological constant in the Universe, but it would be quite natural to have $\rho_\Lambda \sim M_{\rm Pl}^4$, or at least $\rho_\Lambda \sim M_{\rm EW}^4$. However, this is not what we see in the Universe. Such a high value of ρ_Λ would lead to a very fast expansion of the Universe, making impossible the formation of any structure, including galaxies and galaxy clusters. Current astronomical data are consistent with a non-vanishing vacuum energy density of order

$$\rho_\Lambda \sim 10^{-47}\ {\rm GeV}^4\,. \tag{13.18}$$

This is $\sim$120 orders of magnitude smaller than the expectation $\rho_\Lambda \sim M_{\rm Pl}^4$!

There are several proposals on how to solve the cosmological constant problem, but none seems to be satisfactory. Note that there is an implicit but strong assumption on this issue: we assume that we can discuss this problem within an effective field theory, in which low energy physics is decoupled from that at much higher energies and therefore we can get reliable predictions at low energies without knowing what happens at much higher energies. Since we know a number of expected contributions to an effective cosmological constant from the known physics below the electroweak scale, we argue that at least those contributions should be there, and other contributions coming from the physics at higher energies may be added. Such a conclusion is not guaranteed. It is instead possible that the explanation to this puzzle comes from unknown physics at higher energies.

Problem

13.1 Get a rough estimate of the evaporation time of a Schwarzschild black hole of one Solar mass.

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