Chapter 9
Classical Tests of General Relativity

In 1916, Albert Einstein proposed three tests to verify his theory [2]:

1. The gravitational redshift of light.
2. The perihelion precession of Mercury.
3. The deflection of light by the Sun.

These three tests are today referred to as the *classical tests of general relativity*, even if, strictly speaking, the gravitational redshift of light is a test of the Einstein Equivalence Principle [6], while the other two are tests of the Schwarzschild solution in the weak field limit.[1] In 1964, Irwin Shapiro proposed another test [4], which is often called the *fourth classical test of general relativity*. Like the perihelion precession and the light deflection, even the test put forward by Shapiro is actually a test of the Schwarzschild solution in the weak field limit.

In the case of the perihelion precession of Mercury and of the deflection of light by the Sun, we can test, respectively, the trajectories of massive and massless test-particles in the Schwarzschild background. As discussed in Sect. 8.4, we have two differential equations, Eqs. (8.64) and (8.65), which we rewrite here for convenience

$$u' = 0 \,, \tag{9.1}$$

$$u'' - k\frac{G_{\mathrm{N}}M}{L_z^2} + u - \frac{3G_{\mathrm{N}}M}{c^2}u^2 = 0 \,, \tag{9.2}$$

where $k = 1$ for massive particles and $k = 0$ for massless particles. These two equations are the counterpart of Eqs. (1.100) and (1.101) of Newton's gravity. Equations (1.100) and (9.1) are identical and are simply the equation for a circle, so there

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[1]Note, for example, that the Schwarzschild metric is a solution even in some alternative theories of gravity. Since the Mercury perihelion precession and the light deflection are only sensitive to the trajectories of particles, these tests can only verify the Schwarzschild metric (assuming geodesic motion), they cannot distinguish Einstein's gravity from those alternative theories of gravity in which the Schwarzschild metric is a solution of their field equations.

are no interesting implications. Equation (1.101) has the solution (1.103), and the orbits are ellipses, parabolas, or hyperbolas according to the value of the constant A. Equation (9.2) has the last term proportional to u^2, which is absent in Newton's gravity and introduces relativistic corrections.

9.1 Gravitational Redshift of Light

Let us consider a static and spherically symmetric gravitational field. As already seen in Sect. 8.1, the line element can be written as

$$ds^2 = -f(r)c^2dt^2 + g(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{9.3}$$

Let us also consider two observers at rest in this coordinate system. Observer A has spatial coordinates (r_A, θ, ϕ) and observer B has spatial coordinates (r_B, θ, ϕ); that is, the coordinates of the two observers differ only in the radial coordinate. At the location of observer A there is an emission of monochromatic electromagnetic radiation. Observer A measures the frequency ν_A for a time interval $\Delta\tau_A$. The number of wavefronts is

$$n = \nu_A \Delta\tau_A . \tag{9.4}$$

The radiation then reaches observer B, who measures the frequency ν_B for a time interval $\Delta\tau_B$, and therefore the number of wavefronts $n = \nu_B \Delta\tau_B$. Since the observers A and B must measure the same number of wavefronts n, we have

$$\frac{\nu_A}{\nu_B} = \frac{\Delta\tau_B}{\Delta\tau_A} . \tag{9.5}$$

For the electromagnetic signal propagating from the location of observer A to the location of observer B, $ds^2 = 0$. Moreover, since A and B have the same value for the coordinates θ and ϕ, along the trajectory of the signal we have

$$f(r)c^2dt^2 = g(r)dr^2 . \tag{9.6}$$

If we integrate over dt and dr, we find the time interval measured in the coordinate system (ct, r, θ, ϕ) that the first wavefront takes to go from the location of observer A to that of observer B

$$t_B^1 - t_A^1 = \int dt' = \frac{1}{c}\int_{r_A}^{r_B} dr'\sqrt{\frac{g(r')}{f(r')}} . \tag{9.7}$$

In the same way, we can compute the time interval that the last wavefront takes to go from observer A to observer B. Since the right hand side of Eq. (9.7) is independent

of time, we have

$$t_B^n - t_A^n = t_B^1 - t_A^1 \,, \tag{9.8}$$

which we rewrite as

$$t_A^n - t_A^1 = t_B^n - t_B^1 \,. \tag{9.9}$$

Equation (9.9) shows that, as measured in the coordinate system (ct, r, θ, ϕ), the time intervals of the electromagnetic signal at the location of observer A and at the location of observer B are the same.

The time interval of the electromagnetic signal measured by observer A is [remember Eq. (8.43)]

$$\Delta \tau_A = \int_{t_A^1}^{t_A^n} dt \sqrt{f(r_A)} = \sqrt{f(r_A)} \left(t_A^n - t_A^1 \right) \,. \tag{9.10}$$

Observer B measures the time interval

$$\Delta \tau_B = \sqrt{f(r_B)} \left(t_B^n - t_B^1 \right) \,. \tag{9.11}$$

Employing Eq. (9.9), we find

$$\frac{\nu_A}{\nu_B} = \sqrt{\frac{f(r_B)}{f(r_A)}} \,. \tag{9.12}$$

In the Newtonian limit, $f = 1 + 2\Phi/c^2$, as we found in Eq. (6.14). We can thus rewrite Eq. (9.12) as

$$\frac{\nu_A}{\nu_B} = \sqrt{\frac{1 + 2\Phi_B/c^2}{1 + 2\Phi_A/c^2}} \approx 1 + \frac{\Phi_B}{c^2} - \frac{\Phi_A}{c^2} \,. \tag{9.13}$$

The relative variation of the frequency is

$$\frac{\Delta \nu}{\nu} = \frac{\nu_B - \nu_A}{\nu_B} = \frac{\Phi_A - \Phi_B}{c^2} = -\frac{\Delta \Phi}{c^2} \,. \tag{9.14}$$

This phenomenon is called the *gravitational redshift* of light. It was measured for the first time by Robert Pound and Glen Rebka in 1960 [3]. They used a moving atomic source such that the Doppler blueshift could exactly compensate the gravitational redshift that the photons experienced to reach the detector located at a height of 22.5 m.

For small distances, the Earth's gravitational field can be approximated as constant. If we send an electromagnetic signal from the ground to a detector at the height

h, the gravitational redshift is

$$\frac{\Delta \nu}{\nu} = -\frac{gh}{c^2} \,, \tag{9.15}$$

where g is the gravitational acceleration on Earth. For $g = 9.81\,\text{m/s}^2$ and $h = 22.5\,\text{m}$, we find

$$\frac{\Delta \nu}{\nu} = -2 \cdot 10^{-15} \,. \tag{9.16}$$

9.2   Perihelion Precession of Mercury

Let us study Eq. (9.2) for massive particles ($k = 1$). We introduce the dimensionless variable $y = Ru$, where R is the characteristic value of the radial coordinate of the orbit of the particle and therefore we expect a solution $y = O(1)$. Equation (9.2) becomes

$$y'' - \alpha + y - \varepsilon y^2 = 0 \,, \tag{9.17}$$

where α and ε are

$$\alpha = \frac{G_\text{N} M R}{L_z^2} \,, \quad \varepsilon = \frac{3 G_\text{N} M}{c^2 R} = \frac{3}{2} \frac{r_\text{S}}{R} \,. \tag{9.18}$$

Note that $\varepsilon \ll 1$. For the Sun, the Schwarzschild radius is $r_\text{S} = 3\,\text{km}$. The characteristic orbital radius of Mercury is $R \sim 5 \cdot 10^7\,\text{km}$, and therefore $\varepsilon \sim 10^{-7}$. We can use ε as an expansion parameter and write y as

$$y = y_0 + \varepsilon y_1 + O(\varepsilon^2) \,. \tag{9.19}$$

Equation (9.17) becomes

$$y_0'' + \varepsilon y_1'' + y_0 + \varepsilon y_1 = \alpha + \varepsilon y_0^2 + O\left(\varepsilon^2\right) \,, \tag{9.20}$$

and we have to solve two differential equations

$$y_0'' + y_0 = \alpha \,, \tag{9.21}$$
$$y_1'' + y_1 = y_0^2 \,. \tag{9.22}$$

Equation (9.21) is the equation of Newton's gravity (see Sect. 1.9) and the solution is

$$y_0 = \alpha + A \cos \phi \,, \tag{9.23}$$

We plug the solution (9.23) into Eq. (9.22) and we find

$$y_1'' + y_1 = \alpha^2 + A^2 \cos^2 \phi + 2\alpha A \cos \phi$$
$$= \alpha^2 + \frac{A^2}{2} + \frac{A^2}{2} \cos(2\phi) + 2\alpha A \cos \phi \,. \tag{9.24}$$

Let us now rewrite y_1 as the sum of three functions

$$y_1 = y_{11} + y_{12} + y_{13} \,, \tag{9.25}$$

and we split Eq. (9.24) into three parts as follows

$$y_{11}'' + y_{11} = \alpha^2 + \frac{A^2}{2} \,, \tag{9.26}$$

$$y_{12}'' + y_{12} = \frac{A^2}{2} \cos(2\phi) \,, \tag{9.27}$$

$$y_{13}'' + y_{13} = 2\alpha A \cos \phi \,. \tag{9.28}$$

The homogeneous solutions have the form $B \cos \phi$. The inhomogeneous solutions are

$$y_{11} = \alpha^2 + \frac{A^2}{2} \,, \tag{9.29}$$

$$y_{12} = -\frac{A^2}{6} \cos(2\phi) \,, \tag{9.30}$$

$$y_{13} = \alpha A \phi \sin \phi \,. \tag{9.31}$$

We can thus write the solution of Eq. (9.17) up to $O\left(\varepsilon^2\right)$:

$$y = \alpha + A \cos \phi + \varepsilon \left[\alpha^2 + \frac{A^2}{2} - \frac{A^2}{6} \cos(2\phi) + \alpha A \phi \sin \phi + B \cos \phi \right] + O\left(\varepsilon^2\right) \,. \tag{9.32}$$

Note that the corrections to the Newtonian orbit are proportional to ε, which is roughly the ratio between the Schwarzschild radius of the Sun and the characteristic radius of the orbit, and that the only term that grows with the number of orbital revolutions is $\alpha A \phi \sin \phi$; that is, this term becomes more and more important when we consider longer and longer time intervals. We rewrite the solution in (9.32) neglecting the terms that do not grow with the number of orbital revolutions and employing the relation

$$\cos\left(\phi - \varepsilon\alpha\phi\right) = \cos\phi\cos\left(\varepsilon\alpha\phi\right) + \sin\phi\sin\left(\varepsilon\alpha\phi\right)$$
$$= \cos\phi + \varepsilon\alpha\phi\sin\phi + O\left(\varepsilon^2\right) . \qquad (9.33)$$

The result is

$$y = \alpha + A\cos\left(\phi - \varepsilon\alpha\phi\right) , \qquad (9.34)$$

and, in terms of the coordinate r,

$$\frac{1}{r} = \frac{\alpha}{R} + \frac{A}{R}\cos\left(\phi - \varepsilon\alpha\phi\right) . \qquad (9.35)$$

The function in (9.35) is periodic with period 2π with respect to the argument $\phi - \varepsilon\alpha\phi$. We can thus write

$$\phi\left(1 - \varepsilon\alpha\right) = 2\pi n , \qquad (9.36)$$

and

$$\phi = 2\pi n\left(1 + \varepsilon\alpha\right) + O\left(\varepsilon^2\right) , \qquad (9.37)$$

After n orbital revolutions, there is a shift $\delta\phi$ with respect to the Newtonian prediction

$$\delta\phi = \phi - 2\pi n = 2\pi\varepsilon\alpha n . \qquad (9.38)$$

Comparing Eq. (9.35) with Eqs. (D.6) and (D.7) in Appenix D, we see that

$$\alpha = \frac{R}{a\left(1 - e^2\right)} , \qquad (9.39)$$

and Eq. (9.38) can be rewritten as

$$\delta\phi = \frac{6\pi\,G_N M}{c^2}\frac{n}{a\left(1 - e^2\right)} . \qquad (9.40)$$

Note that the perihelion precession of planets in the Solar System is already expected in Newtonian mechanics and the latter is larger than the relativistic contribution. Table 9.1 shows the case of the perihelion precession of Mercury. The dominant contribution comes from the equinox precession due to the fact that we observe Mercury from Earth. A minor but still large contribution comes from the perturbation of Mercury's orbit by other planets, in particular Venus, Earth, and Jupiter. In the end, the perihelion precession due to relativistic effects is a small contribution. Urbain Le Verrier was the first, in 1859, to point out an anomaly in the perihelion precession of Mercury within Newton's theory. The origin of this anomaly was under debate for a long time. It was the first test passed by Einstein's gravity, but the

Table 9.1 Contributions to the Mercury perihelion precession. $\delta\phi$ is in seconds of arc per century. Table readapted from [1]

Cause	$\delta\phi$ (as/100 years)
Mercury	0.03 ± 0.00
Venus	277.86 ± 0.68
Earth	90.04 ± 0.08
Mars	2.54 ± 0.00
Jupiter	153.58 ± 0.00
Saturn	7.30 ± 0.01
Uranus	0.14 ± 0.00
Neptune	0.04 ± 0.00
Solar oblateness	0.01 ± 0.02
Equinox precession	5025.65 ± 0.50
Sum	5557.18 ± 0.85
Observed	5599.74 ± 0.41
Difference	42.56 ± 0.94
Relativistic effect	43.03 ± 0.03

explanation was not immediately accepted, because of a certain skepticism towards this theory by a large fraction of the scientific community and the difficulties in the measurements of the Newtonian effects of the perihelion precession of Mercury.

The perihelion precession of other planets in the Solar System is smaller, essentially because their orbital radius is larger and their eccentricity is lower, but it can still be measured for Venus and Earth. In the case of binary pulsars, the relativistic contribution to the orbital precession can be much larger. For instance, in the binary PSR1913+16, the relativistic contribution to the perihelion precession is about 4° per year [5].

9.3 Deflection of Light

Let us now consider the case of massless particles ($k = 0$). As before, we introduce the variable $y = Ru$, where R is still the characteristic value of the radial coordinate of the particle. Equation (9.2) is now

$$y'' + y - \varepsilon y^2 = 0 \,. \tag{9.41}$$

If R is the radius of the Sun, $R = 7 \cdot 10^5$ km, and $\varepsilon \sim 10^{-5}$. We proceed as in the previous section and we write y as an expansion in ε

$$y = y_0 + \varepsilon y_1 + O(\varepsilon^2) \,. \tag{9.42}$$

Equation (9.41) becomes

$$y_0'' + \varepsilon y_1'' + y_0 + \varepsilon y_1 = \varepsilon y_0^2 + O\left(\varepsilon^2\right), \tag{9.43}$$

and we have to solve the following equations

$$y_0'' + y_0 = 0, \tag{9.44}$$
$$y_1'' + y_1 = y_0^2. \tag{9.45}$$

Equation (9.44) provides the Newtonian solution

$$y_0 = A \cos \phi, \tag{9.46}$$

which can be rewritten in terms of the radial coordinate r

$$\frac{1}{r} = \frac{A}{R} \cos \phi. \tag{9.47}$$

This is the equation of a straight line in polar coordinates. For $\phi = 0$, we find the impact parameter $b = R/A$ (see Fig. 9.1).

We plug the solution (9.46) into Eq. (9.45) and we get

$$y_1'' + y_1 = \frac{R^2}{b^2} \cos^2 \phi = \frac{R^2}{2b^2} + \frac{R^2}{2b^2} \cos(2\phi). \tag{9.48}$$

We write y_1 as the sum of two contributions

$$y_1 = y_{11} + y_{12}, \tag{9.49}$$

and Eq. (9.48) can be split into two parts

$$y_{11}'' + y_{11} = \frac{R^2}{2b^2}, \tag{9.50}$$

$$y_{12}'' + y_{12} = \frac{R^2}{2b^2} \cos(2\phi), \tag{9.51}$$

The homogeneous solutions have the form $B \cos \phi$. The inhomogeneous solutions are

$$y_{11} = \frac{R^2}{2b^2}, \tag{9.52}$$

$$y_{12} = -\frac{R^2}{6b^2} \cos(2\phi). \tag{9.53}$$

y_1 is thus given by

$$y_1 = \frac{R^2}{2b^2} - \frac{R^2}{6b^2} \cos(2\phi) + B \cos\phi$$

$$= \frac{2R^2}{3b^2} - \frac{R^2}{3b^2} \cos^2\phi + B \cos\phi \, , \tag{9.54}$$

and the solution of Eq. (9.41) up to $O\left(\varepsilon^2\right)$ is

$$\frac{1}{r} = \frac{1}{b} \cos\phi + \varepsilon \left(\frac{2R}{3b^2} - \frac{R}{3b^2} \cos^2\phi + \frac{B}{R} \cos\phi \right) + O\left(\varepsilon^2\right) . \tag{9.55}$$

For $\phi = 0$, we find the minimum value of the radial coordinate of the orbit of the particle

$$\frac{1}{r_{\min}} = \frac{1}{b} + \varepsilon \left(\frac{R}{3b^2} + \frac{B}{R} \right) . \tag{9.56}$$

We define

$$\tilde{\varepsilon} = \frac{\varepsilon R}{b} \, , \quad \tilde{B} = \frac{B}{R} \, , \tag{9.57}$$

and we consider the limit $r \to \infty$. Equation (9.55) becomes

$$-\frac{\tilde{\varepsilon}}{3} \cos^2\phi + \left(1 + \varepsilon \tilde{B} b \right) \cos\phi + \frac{2}{3} \tilde{\varepsilon} = 0 \, ,$$

$$\cos^2\phi - \frac{3\left(1 + \varepsilon \tilde{B} b\right)}{\tilde{\varepsilon}} \cos\phi - 2 = 0 \, , \tag{9.58}$$

which is a second order equation in $\cos\phi$. The solution is

$$\cos\phi = \frac{3\left(1 + \varepsilon \tilde{B} b\right)}{2\tilde{\varepsilon}} \left[1 \pm \sqrt{1 + \frac{8}{9} \frac{\tilde{\varepsilon}^2}{\left(1 + \varepsilon \tilde{B} b\right)^2}} \right] . \tag{9.59}$$

The solution with the sign $+$ has no physical meaning, because $\cos\phi$ cannot exceed 1. The physical solution is that with the sign $-$ and we have

$$\cos\phi \approx \frac{3\left(1 + \varepsilon \tilde{B} b\right)}{2\tilde{\varepsilon}} \left[1 - 1 - \frac{4}{9} \frac{\tilde{\varepsilon}^2}{\left(1 + \varepsilon \tilde{B} b\right)^2} \right] = -\frac{2}{3} \frac{\tilde{\varepsilon}}{\left(1 + \varepsilon \tilde{B} b\right)}$$

$$= -\frac{2}{3} \tilde{\varepsilon} + O\left(\varepsilon^2\right) = -\frac{r_S}{b} + O\left(\varepsilon^2\right) . \tag{9.60}$$

For $\delta \equiv r_S/b \ll 1$, we have

$$\cos\left(\pm\frac{\pi}{2} \pm \delta\right) = -\delta + O\left(\delta^2\right) , \tag{9.61}$$

and we see that the solution of ϕ in Eq. (9.60) is

$$\phi = \pm\frac{\pi}{2} \pm \delta . \tag{9.62}$$

As we can see from Fig. 9.1, the total deflection of the light ray is Δ

$$\Delta = 2\delta = \frac{2r_S}{b} = \frac{4G_N M}{c^2 b} . \tag{9.63}$$

If b is the radius of the Sun, we find $\Delta = 1.75$ as.

In reality, it is not possible to observe light rays with impact parameter b equal to the radius of the Sun $R_\odot$ because of the presence of the Solar corona. We have to consider photons with $b > R_\odot$ and in this case the deflection angle is

$$\Delta' = \Delta\frac{R_\odot}{b} . \tag{9.64}$$

The observation of the deflection of light rays by the Sun in 1919 by a group led by Arthur Eddington was the first test specifically done to verify Einstein's gravity. The observation was done during a Solar eclipse, when the light from the Sun was blocked by the Moon and it was thus possible to observe stars close to the Sun. From the comparison of photographs of the same region of the sky during the eclipse and when the Sun was not there, it was possible to measure the deflection angle Δ'
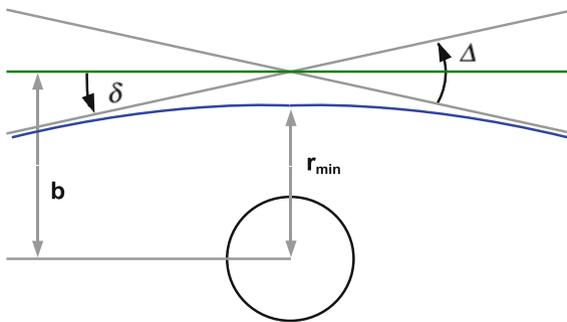
Fig. 9.1 Trajectory of a massless particle. The green line is the trajectory in Newton's gravity and is described by $r = b/\cos\phi$, where r is the radial coordinate of the polar coordinate system centered at the center of the massive body. The blue line is the trajectory in the Schwarzschild metric, Eq. (9.55). At $r \to \infty$, the polar angle is $\phi = \pm\pi/2 \pm \delta$. The total deflection angle is thus $\Delta = 2\delta$

and then infer Δ. This kind of optical observations is quite challenging and the final measurement is affected by systematic effects. Today, radio observations can provide more reliable and precise measurements.

9.4 Shapiro's Effect

In the perihelion precession of Mercury and in the deflection of light, we see how relativistic effects change the trajectories predicted in Newton's gravity. In 1964, Irwin Shapiro proposed a new test, often called the fourth classical test of general relativity, which is based on the measurement of the time delay of an electromagnetic signal to move from one point to another in the Solar System with respect to the time that the same signal would take in a flat spacetime [4].

Figure 9.2 is a sketch to illustrate Shapiro's effect. The Sun is at the point O. We want to calculate the time that an electromagnetic signal takes to go from point A with spatial coordinates $(r_A, \pi/2, \phi_A)$ to point B with spatial coordinates $(r_B, \pi/2, \phi_B)$. As before, without loss of generality, we consider the equatorial plane $\theta = \pi/2$. C is the point of the trajectory of the electromagnetic signal with the smallest value of the radial coordinate, which we call r_C.

For massless particles, Eq. (8.55) is

$$g \left(\frac{dr}{d\lambda} \right)^2 + \frac{L_z^2}{r^2} - \frac{E^2}{c^2 f} = 0 \,. \tag{9.65}$$

Employing Eq. (8.51), we can write

$$\frac{dr}{d\lambda} = \frac{dr}{dt} \frac{dt}{d\lambda} = \frac{dr}{dt} \frac{E}{c^2 f} \,, \tag{9.66}$$

and we remove the affine parameter λ in Eq. (9.65)

$$\frac{E^2}{c^4 f^3} \left(\frac{dr}{dt} \right)^2 + \frac{L_z^2}{r^2} - \frac{E^2}{c^2 f} = 0 \,, \tag{9.67}$$

where we have also exploited the fact that $g = 1/f$ in the Schwarzschild metric. At $r = r_C$, we have $dr/dt = 0$, and therefore

$$L_z^2 = \frac{E^2 r_C^2}{c^2 f(r_C)} \,. \tag{9.68}$$

With this expression we can rewrite Eq. (9.67) without the constants that depend on the parametrization of the orbit

Fig. 9.2 Shapiro's effect.
The Sun is at point O and we
want to measure the time that
an electromagnetic signal
takes to go from point A to B
and return to A
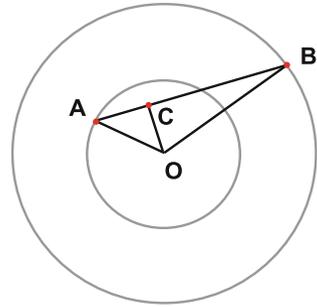
$$\frac{1}{f^3}\left(\frac{dr}{dt}\right)^2 + \frac{c^2}{f(r_C)}\frac{r_C^2}{r^2} - \frac{c^2}{f} = 0. \tag{9.69}$$

We find

$$dt = \pm\frac{1}{c}\frac{dr}{\sqrt{\left[1 - \frac{f(r)}{f(r_C)}\frac{r_C^2}{r^2}\right]f^2(r)}}. \tag{9.70}$$

The time, as measured by the coordinate system (ct, r, θ, ϕ), that an electromagnetic signal takes to go from point A to point C is

$$t_{AC} = \frac{1}{c}\int_{r_C}^{r_A}\frac{dr}{\sqrt{\left[1 - \frac{f(r)}{f(r_C)}\frac{r_C^2}{r^2}\right]f^2(r)}}. \tag{9.71}$$

We only consider the first order terms in the expansion in r_S/r and r_S/r_C

$$\left[1 - \frac{f(r)}{f(r_C)}\frac{r_C^2}{r^2}\right]f^2(r) = \left[1 - \frac{\left(1 - \frac{r_S}{r}\right)}{\left(1 - \frac{r_S}{r_C}\right)}\frac{r_C^2}{r^2}\right]\left(1 - \frac{r_S}{r}\right)^2$$

$$= \left[1 - \left(1 - \frac{r_S}{r} + \frac{r_S}{r_C}\right)\frac{r_C^2}{r^2}\right]\left(1 - \frac{2r_S}{r}\right)$$

$$= 1 - \frac{r_C^2}{r^2} - \frac{2r_S}{r} + \frac{3r_S}{r}\frac{r_C^2}{r^2} - \frac{r_S}{r_C}\frac{r_C^2}{r^2}, \tag{9.72}$$

and therefore

$$\frac{1}{\sqrt{\left[1 - \frac{f(r)}{f(r_C)} \frac{r_C^2}{r^2}\right] f^2(r)}} = \frac{1}{\sqrt{1 - \frac{r_C^2}{r^2}}} \left[1 - \frac{1}{2} \frac{1}{1 - \frac{r_C^2}{r^2}} \left(\frac{3r_S}{r} \frac{r_C^2}{r^2} - \frac{2r_S}{r} - \frac{r_S}{r_C} \frac{r_C^2}{r^2}\right)\right]$$

$$= \sqrt{\frac{r^2}{r^2 - r_C^2} - \frac{1}{2} \left(\frac{r^2}{r^2 - r_C^2}\right)^{3/2} \left(\frac{3r_S}{r} \frac{r_C^2}{r^2} - \frac{2r_S}{r} - \frac{r_S}{r_C} \frac{r_C^2}{r^2}\right)}$$

$$= \sqrt{\frac{r^2}{r^2 - r_C^2} + \frac{1}{2} \frac{r_S}{\left(r^2 - r_C^2\right)^{3/2}} \left(2r^2 + r_C r - 3r_C^2\right)} . \quad (9.73)$$

The leading order term of the integral (9.71) is

$$\frac{1}{c} \int_{r_C}^{r_A} \frac{r \, dr}{\sqrt{r^2 - r_C^2}} = \frac{1}{c} \sqrt{r_A^2 - r_C^2} . \tag{9.74}$$

The next-to-leading order terms are

$$\frac{1}{c} \int_{r_C}^{r_A} \frac{r_S r^2 \, dr}{\left(r^2 - r_C^2\right)^{3/2}} = \frac{1}{c} \left[-\frac{r_S r}{\sqrt{r^2 - r_C^2}} + r_S \ln\left(r + \sqrt{r^2 - r_C^2}\right)\right]_{r_C}^{r_A} , \tag{9.75}$$

$$\frac{1}{2c} \int_{r_C}^{r_A} \frac{r_S r_C r \, dr}{\left(r^2 - r_C^2\right)^{3/2}} = \frac{1}{2c} \left[-\frac{r_S r_C}{\sqrt{r^2 - r_C^2}}\right]_{r_C}^{r_A} , \tag{9.76}$$

$$\frac{1}{2c} \int_{r_C}^{r_A} \frac{-3r_S r_C^2 \, dr}{\left(r^2 - r_C^2\right)^{3/2}} = \frac{1}{2c} \left[\frac{3r_S r}{\sqrt{r^2 - r_C^2}}\right]_{r_C}^{r_A} . \tag{9.77}$$

These integrals diverge at $r = r_C$, but this is because we have expanded in r_S/r and r_S/r_C and we see that the sum of those integrals with $r \to r_C$ is finite

$$(9.75) + (9.76) + (9.77) = \frac{r_S}{2c} \frac{r_A - r_C}{\sqrt{r_A^2 - r_C^2}} + \frac{r_S}{c} \ln\left(r_A + \sqrt{r_A^2 - r_C^2}\right) - \frac{r_S}{c} \ln\left(r_C\right)$$

$$= \frac{r_S}{2c} \sqrt{\frac{r_A - r_C}{r_A + r_C}} + \frac{r_S}{c} \ln\left(\frac{r_A + \sqrt{r_A^2 - r_C^2}}{r_C}\right) . \tag{9.78}$$

Eventually, the time that the electromagnetic signal takes to go from point A to point C turns out to be

$$t_{AC} = \frac{1}{c}\sqrt{r_A^2 - r_C^2} + \frac{r_S}{c} \ln\left(\frac{r_A + \sqrt{r_A^2 - r_C^2}}{r_C}\right) + \frac{r_S}{2c}\sqrt{\frac{r_A - r_C}{r_A + r_C}} . \tag{9.79}$$

In the case of a flat spacetime, the time that the electromagnetic signal would take to go from point A to point C is

$$\tilde{t}_{AC} = \frac{1}{c}\sqrt{r_A^2 - r_C^2} , \tag{9.80}$$

and corresponds to the leading order term in (9.79). The total time that the electromagnetic signal takes to go from A to B and come back to A is

$$t_{\text{tot}} = 2t_{AC} + 2t_{BC} , \tag{9.81}$$

while in flat spacetime it would be

$$\tilde{t}_{\text{tot}} = 2\tilde{t}_{AC} + 2\tilde{t}_{BC} = \frac{2}{c}\sqrt{r_A^2 - r_C^2} + \frac{2}{c}\sqrt{r_B^2 - r_C^2} . \tag{9.82}$$

The maximum time delay with respect to the flat spacetime is when $r_C = R_\odot$, where $R_\odot$ is the radius of the surface of the Sun. The result is ($R_\odot \ll r_A, r_B$)

$$\delta t_{\text{max}} = \frac{4G_N M}{c^3}\left[1 + \ln\left(\frac{4r_A r_B}{R_\odot^2}\right)\right] . \tag{9.83}$$

Note that we are using the coordinate time t, not the proper time of the observer in A. However, the correction is of the order of r_S/r_A.

The phenomenon of time delay of an electromagnetic signal passing near a massive body is commonly called *Shapiro's effect*. Current measurements are in perfect agreement with the theoretical predictions. To have an idea of the magnitude of the effect, let us estimate the time delay of an electromagnetic signal to go from Earth to Mercury and return to Earth when the two planets are at opposite sides of the Sun, so r_C is the radius of the Sun. We plug the following values into Eq. (9.83)

$$r_A = 150 \cdot 10^6 \text{ km} , \quad r_B = 58 \cdot 10^6 \text{ km} , \quad R_\odot = 0.7 \cdot 10^6 \text{ km} ,$$
$$r_S = 2.95 \text{ km} . \tag{9.84}$$

The result is

$$\delta t_{\text{max}} = 0.24 \text{ ms} . \tag{9.85}$$

9.5 Parametrized Post-Newtonian Formalism

The Parametrized Post-Newtonian (PPN) formalism is a convenient approach to test the solutions of Einstein's gravity in the weak field regime. The basic idea is to write the metric as an expansion about the Minkowski metric in terms of certain gravitational potentials. If we want to test the Schwarzschild metric in the Solar System, we write the most general static and spherically symmetric line element as an expansion in r_S/r, where r_S is the Schwarzschild radius of the Sun. The approach is traditionally formulated in isotropic coordinates and, as seen in Sect. 8.1, the most general static and spherically symmetric line element reads[2]

$$ds^2 = -\left(1 - \frac{2G_N M}{c^2 R} + \beta \frac{2G_N^2 M^2}{c^4 R^2} + \cdots\right) c^2 dt^2$$
$$+ \left(1 + \gamma \frac{2G_N M}{c^2 R} + \cdots\right) \left(dx^2 + dy^2 + dz^2\right) , \tag{9.87}$$

where β and γ are free parameters to be determined by observations.

In spherical-like coordinates, the line element (9.87) is

$$ds^2 = -\left(1 - \frac{2G_N M}{c^2 R} + \beta \frac{2G_N^2 M^2}{c^4 R^2} + \cdots\right) c^2 dt^2$$
$$+ \left(1 + \gamma \frac{2G_N M}{c^2 R} + \cdots\right) \left(dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2\right) . \tag{9.88}$$

With the transformation

$$R \to r = R\left(1 + \gamma \frac{G_N M}{c^2 R} + \cdots\right) , \tag{9.89}$$

with inverse transformation

$$R = r\left(1 - \gamma \frac{G_N M}{c^2 r} + \cdots\right) , \tag{9.90}$$

we write the line element in the more conventional Schwarzschild coordinates

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[2]In order to have the correct Newtonian limit, the line element must have the form

$$ds^2 = -\left(1 - \frac{2G_N M}{c^2 R} + \ldots\right) c^2 dt^2 + (1 + \ldots)\left(dx^2 + dy^2 + dz^2\right) . \tag{9.86}$$

For higher order terms, there are no theoretical requirements, and therefore we introduce β and γ.

$$ds^2 = -\left[1 - \frac{2G_{\text{N}}M}{c^2 r} + (\beta - \gamma)\frac{2G_{\text{N}}^2 M^2}{c^4 r^2} + \cdots \right] c^2 dt^2$$
$$+ \left(1 + \gamma \frac{2G_{\text{N}}M}{c^2 r} + \cdots \right) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) . \qquad (9.91)$$

Now we can easily see that the Schwarzschild metric is recovered when $\beta = \gamma = 1$.

In we employ the metric in (9.91) in place of the Schwarzschild one in the discussion of the Mercury precession, Eq. (9.40) becomes

$$\delta\phi = \frac{6\pi\, G_{\text{N}}M}{c^2}\frac{n}{a\left(1 - e^2\right)}\left(\frac{2 - \beta + 2\gamma}{3}\right) . \qquad (9.92)$$

If we do the same for the problem of light bending, Eq. (9.63) becomes

$$\Delta = \frac{4 G_{\text{N}}M}{c^2 b}\left(\frac{1 + \gamma}{2}\right) . \qquad (9.93)$$

For the Shapiro time delay, Eq. (9.83) becomes

$$\delta t_{\text{max}} = \frac{4 G_{\text{N}}M}{c^3}\left[1 + \left(\frac{1 + \gamma}{2}\right)\ln\left(\frac{4 r_A r_B}{R_\odot^2}\right)\right] . \qquad (9.94)$$

From the measurements of observational effects, like the perihelion precession of Mercury $\delta\phi$, the light bending Δ, and the Shapiro time delay δt, we can constrain the PPN parameters β and γ to check whether they are consistent with 1, as required by the Schwarzschild metric. The current most stringent constraints are [6]

$$|\beta - 1| < 8 \cdot 10^{-5} , \qquad (9.95)$$
$$|\gamma - 1| < 2.3 \cdot 10^{-5} . \qquad (9.96)$$

Current observations thus confirm the Schwarzschild solution within their precision.

References

1. G.M. Clemence, Rev. Mod. Phys. **19**, 361 (1947)
2. A. Einstein, Ann. Phys. **49**, 769 (1916). [Annalen Phys. **14**, 517 (2005)]
3. R.V. Pound, G.A. Rebka Jr., Phys. Rev. Lett. **4**, 337 (1960)
4. I.I. Shapiro, Phys. Rev. Lett. **13**, 789 (1964)
5. J.H. Taylor, R.A. Hulse, L.A. Fowler, G.E. Gullahorn, J.M. Rankin, Astrophys. J. **206**, L53 (1976)
6. C.M. Will, Living Rev. Relativ. **17**, 4 (2014), arXiv:1403.7377[gr-qc]