Chapter 11
Cosmological Models

In this chapter, we will show how we can construct some simple models for the description of the Universe.[1] More details on cosmological models and on the evolution of the Universe can be found in standard textbooks on cosmology, like [1].

11.1 Friedmann–Robertson–Walker Metric

Our starting point to construct some simple cosmological models is the so-called *Cosmological Principle*:

Cosmological Principle. The Universe is homogeneous and isotropic.

As a matter of fact, the Universe is far from being homogeneous and isotropic. We observe a lot of structures around us (stars, galaxies, clusters of galaxies). However, we can expect that the Universe can be well approximated as homogeneous and isotropic if we average over large volumes. The Cosmological Principle can be seen as a sort of Copernican Principle: there are no preferred points or preferred directions in the Universe. Our current model for the description of the Universe is called the Standard Model of Cosmology and is based on the Cosmological Principle. Nevertheless, today there is debate on the applicability of this principle. Since there are structures, the assumptions of homogeneity and isotropy inevitably introduce systematic effects in the measurements of the properties of the Universe, and the impact of these systematic effects on current measurements of cosmological parameters, which are more and more precise, is not clear.

[1]Note that it is common to use the initial capital letter in Universe only when we refer to our Universe. If we mean a generic universe/cosmological model, we write universe.

C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4_11

The Cosmological Principle requires that there are no preferred points (homogeneity, i.e. invariance under spatial translations) and no preferred directions (isotropy, i.e. invariance under spatial rotations) in the 3-dimensional spacetime. The spacetime geometry is still allowed to depend on time.[2] These assumptions strongly constrain the metric of the spacetime. The only background compatible with the Cosmological Principle is the *Friedmann–Robertson–Walker metric*. Its derivation requires some calculations, which are outlined in Appendix H. The line element reads

$$ds^2 = -c^2dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) , \tag{11.1}$$

where $a = a(t)$ is the *scale factor*, which depends on the temporal coordinate t and is independent of the spatial coordinates (r, θ, ϕ), and k is a constant. In general, k can be positive, zero, or negative. Nevertheless, it is always possible to rescale the radial coordinate r and have $k = 1, 0$, or -1. If $k = 1$, we have a *closed universe*. If $k = 0$, we have a *flat universe*. If $k = -1$, we have an *open universe*. Note that, in general, a flat universe is not a flat spacetime, namely the Minkowski spacetime of special relativity. The spacetime is flat only when a is independent of t and $k = 0$. In such a case, we can redefine the radial coordinate to absorb the scale factor and the line element (11.1) becomes that of the Minkowski spacetime in spherical coordinates

$$ds^2 = -c^2dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 . \tag{11.2}$$

It is straightforward to compute some invariants of the Friedmann–Robertson–Walker spacetime with specific Mathematica packages (see Appendix E). For instance, the scalar curvature is

$$R = 6 \, \frac{kc^2 + \dot{a}^2 + \ddot{a}a}{a^2c^2} . \tag{11.3}$$

When $k = 0$ and a is constant, R vanishes, which is indeed the case of the Minkowski spacetime (but R may vanish even if the spacetime is curved; in Einstein's gravity $R = 0$ if $T^\mu_\mu = 0$). Note also that R diverges for $a \to 0$. Typical cosmological models start from a singularity, namely $a = 0$ at the initial time (*Big Bang*). The Kretschmann scalar is

$$\mathscr{K} = 12 \, \frac{k^2c^4 + 2kc^2\dot{a}^2 + \dot{a}^4 + \ddot{a}^2a^2}{a^4c^4} , \tag{11.4}$$

which also diverges for $a \to 0$.

Note that the Cosmological Principle is quite a strong assumption only based on the requirement that the Universe is homogeneous and isotropic. It is completely independent of the Einstein equations. If we impose the Cosmological Principle in a

[2] If we impose that the spacetime geometry is also independent of time ("Perfect" Cosmological Principle), we find cosmological models in disagreement with observations.

4-dimensional spacetime, the geometry is described by the Friedmann–Robertson–Walker metric in (11.1). If we specify that we are in Einstein's gravity and we know the matter content of the Universe, we can find $a(t)$ and k. In another gravity theory and/or for a different matter content we would find, in general, different solutions for $a(t)$ and k.

Note that the Cosmological Principle cannot determine some global properties of the Universe. For example, the Friedmann–Robertson–Walker metric can describe topologically different universes. If we assume that the Universe has a trivial topology, the 3-volume V is finite if $k = 1$ and infinite for $k = 0$ and -1:

$$V = \int_V \sqrt{^3g}\, d^3x = a^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{r_k} \frac{r^2 dr}{\sqrt{1 - kr^2}}\,, \tag{11.5}$$

where ^3g is the determinant of the spatial 3-metric,[3] $r_k = 1$ for $k = 1$, and $r_k = \infty$ for $k = 0$ and -1. If we integrate Eq. (11.5), we find

$$V = \begin{cases} \pi^2 a^3 & \text{for } k = 1 \\ \infty & \text{for } k = 0,\ -1 \end{cases}. \tag{11.7}$$

In the case of a universe with non-trivial topology, the picture is more complicated, and depends on the specific configuration. Closed universes always have a finite volume, but in the case of flat and open universes the volume can be either finite or infinite. Current astrophysical data suggest that the Universe is almost flat, and we cannot say if $k = 1, 0,$ or -1. In such a situation, even assuming that the Universe has a trivial topology we cannot say whether its volume is finite or infinite. In the case of a universe with non-trivial topology, the measurement of k is not enough to assert if it is finite or infinite. In principle, in the case of a non-trivial topology, the volume of the Universe may be evaluated by looking for "ghost images" of astronomical sources,[4] namely images of the same source coming from different directions. Since we currently have no evidence of the presence of ghost images due to the possible non-trivial topology of the Universe, we only have lower bounds of the possible finite size of the Universe in the case of non-trivial topologies.

[3] The line element of the spatial 3-metric of the Friedmann–Robertson–Walker spacetime reads

$$dl^2 = a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right). \tag{11.6}$$

[4] In general, ghost images can be potentially seen in any universe (with either trivial or non-trivial topology) with at least one space dimension of finite size. However, it is when the topology is non-trivial that the detection of ghost images seems to be the simplest way to infer if the Universe has at least one space dimension of finite size.

11.2 Friedmann Equations

If we assume the Cosmological Principle, the metric of the spacetime must be described by the Friedmann–Robertson–Walker solution. The only undetermined quantities are the scale factor $a(t)$ and the constant k, which can be obtained once we specify the gravity theory (e.g. Einstein's gravity) and the matter content, and we solve the corresponding field equations.

The simplest cosmological models are constructed assuming that the matter in the Universe can be described by the energy-momentum tensor of a perfect fluid

$$T^{\mu\nu} = (\rho + P)\,\frac{u^\mu u^\nu}{c^2} + P g^{\mu\nu}\,, \tag{11.8}$$

where ρ and P are, respectively, the energy density and the pressure of the fluid, and u^μ is the fluid 4-velocity. In the coordinate system of the Friedmann–Robertson–Walker metric, the Universe is manifestly homogeneous and isotropic. It corresponds to the rest-frame of the fluid, where the fluid 4-velocity becomes $u^\mu = (c, \mathbf{0})$. Let us note that the expression in Eq. (11.8) can be employed even when there are several matter components. In such a case, ρ and P are the total energy density and the total pressure, namely

$$\rho = \sum_i \rho_i\,, \quad P = \sum_i P_i\,, \tag{11.9}$$

where ρ_i and P_i are, respectively, the energy density and the pressure of the component i.

If we plug the Friedmann–Robertson–Walker metric (11.1) and the energy-momentum tensor of a perfect fluid (11.8) with $u^\mu = (c, \mathbf{0})$ into the Einstein equations, we find the field equations to solve. The tt component of the Einstein equations gives the *first Friedmann equation*:

$$H^2 = \frac{8\pi G_{\rm N}}{3c^2}\rho - \frac{kc^2}{a^2}\,, \tag{11.10}$$

where $H = \dot{a}/a$ is the *Hubble parameter*. The rr, $\theta\theta$, and $\phi\phi$ components of the Einstein equations provide the same equation, which is called the *second Friedmann equation* and reads

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_{\rm N}}{3c^2}\left(\rho + 3P\right)\,. \tag{11.11}$$

Instead of the Einstein equations, we may use the covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, which is a consequence of the Einstein equations. In such a case, we find

$$\dot{\rho} = -3H\,(\rho + P)\ . \tag{11.12}$$

Note that Eqs. (11.10), (11.11), and (11.12) are not three independent equations. Only two equations are independent, and it is possible to obtain the third equation from the other two. For instance, if we derive the first Friedmann equation with respect to t, we have

$$\begin{aligned} \frac{2\dot{a}\ddot{a}a^2 - 2a\dot{a}^3}{a^4} &= \frac{8\pi G_\mathrm{N}}{3c^2}\dot{\rho} + \frac{2\dot{a}kc^2}{a^3}\,, \\ H\frac{\ddot{a}}{a} - H\frac{\dot{a}^2}{a^2} &= \frac{4\pi G_\mathrm{N}}{3c^2}\dot{\rho} + \frac{kc^2}{a^2}H\,. \end{aligned} \tag{11.13}$$

We replace the term $\dot{a}^2/a^2$ with the expression on the right hand side of the first Friedmann equation and $\dot{\rho}$ with the expression in Eq. (11.12)

$$H\frac{\ddot{a}}{a} - H\frac{8\pi G_\mathrm{N}}{3c^2}\rho + H\frac{kc^2}{a^2} = \frac{4\pi G_\mathrm{N}}{3c^2}\left[-3H\,(\rho + P)\right] + \frac{kc^2}{a^2}H\,, \tag{11.14}$$

and eventually we recover the second Friedmann equation.

At this point, we have two independent equations and three unknown functions of time (a, ρ, and P). In order to close the system and find a, ρ, and P, we need another equation. We can introduce the *equation of state* of the matter in the Universe. The simplest form is

$$P = w\rho\,, \tag{11.15}$$

where w is a constant. While this is a very simple equation of state, it includes the main physically relevant cases: dust ($w = 0$), radiation ($w = 1/3$), and vacuum energy ($w = -1$).

The first Friedmann equation for $k = 0$ reads

$$H^2 = \frac{8\pi G_\mathrm{N}}{3c^2}\rho_\mathrm{c}\,, \tag{11.16}$$

and defines the *critical density* ρ_c as the energy density of a flat universe. The value of the critical energy density today is

$$\begin{aligned} \rho_\mathrm{c}^0 = \frac{3H_0^2c^2}{8\pi G_\mathrm{N}} &= 1.88 \cdot 10^{-29}\ \ h_0^2\ c^2\ \ \mathrm{g \cdot cm^{-3}} \\ &= 11\, h_0^2\ \ \mathrm{protons \cdot m^{-3}}\,, \end{aligned} \tag{11.17}$$

where H_0 is the *Hubble constant*, namely the value of the Hubble parameter today, and can be written as[5]

$$H_0 = 100\ h_0\ \frac{\text{km}}{\text{s} \cdot \text{Mpc}}\,. \tag{11.18}$$

h_0 is a dimensionless parameter of order 1. Since the value of the Hubble constant was not known with good precision in the past, it was common to use the expression in (11.18) and keep the parameter h_0 in all the equations. Today we know that $h_0 \approx 0.7$.

11.3 Cosmological Models

If we plug the equation of state (11.15) into Eq. (11.12), we have

$$\frac{\dot{\rho}}{\rho} = -3\,(1+w)\,\frac{\dot{a}}{a}\,. \tag{11.19}$$

The solution of this equation is

$$\rho \propto a^{-3(1+w)}\,. \tag{11.20}$$

In particular, we have the following relevant cases

$$\begin{array}{cccc} w = 0 & \rightarrow & \rho \propto 1/a^3 & \text{(dust)}, \\ w = 1/3 & \rightarrow & \rho \propto 1/a^4 & \text{(radiation)}, \\ w = -1 & \rightarrow & \rho = \text{constant} & \text{(vacuum energy)}, \end{array} \tag{11.21}$$

If we plug (11.20) into the first Friedmann equation, we neglect the term kc^2/a^2 (whose contribution can always be ignored at sufficiently early times in a universe made of dust or radiation because the term with ρ is dominant with respect to kc^2/a^2 for $a \rightarrow 0$), and we write $a \propto t^\alpha$, the first Friedmann equation reads

$$t^{-2} \propto t^{-3\alpha(1+w)}\,, \tag{11.22}$$

and we find

$$\alpha = \frac{2}{3\,(1+w)}\,. \tag{11.23}$$

[5] The parsec (pc) is a common unit of length in astronomy and cosmology. 1 pc = $3.086 \cdot 10^{16}$ m. 1 Mpc = 10^6 pc.

11.3.1 Einstein Universe

For ordinary matter $\rho + 3P > 0$, and therefore the second Friedmann equation implies that $\ddot{a} < 0$; that is, the Universe cannot be static. However, this was against the common belief at the beginning of the 20th century. This apparent problem led Einstein to introduce the cosmological constant Λ into the theory, replacing the field equations (7.6) with (7.8). In the presence of Λ, the first and the second Friedmann equations read

$$H^2 = \frac{8\pi G_N}{3c^2}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}\,, \tag{11.24}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}(\rho + 3P) + \frac{\Lambda c^2}{3}\,. \tag{11.25}$$

The so-called Einstein universe is a cosmological model in which matter is described by dust ($P = 0$) and there is a non-vanishing cosmological constant to make the universe static. If we require that $\dot{a} = \ddot{a} = 0$, from Eqs. (11.24) and (11.25) we find

$$\rho = \frac{\Lambda c^4}{4\pi G_N}\,, \quad a = \frac{1}{\sqrt{\Lambda}}\,, \quad k = 1\,. \tag{11.26}$$

The Einstein universe is unstable, namely small perturbations make it either collapse or expand. After the discovery of the expansion of the Universe by Hubble in 1929, the cosmological constant was (temporarily) removed from the Einstein equations.

11.3.2 Matter Dominated Universe

Let us now consider a universe filled with dust. The equation of state is $P = 0$, namely $w = 0$. The energy density scales as $1/a^3$, so we can write

$$\rho a^3 = \text{constant} \equiv C_1\,. \tag{11.27}$$

The first Friedmann equation becomes

$$\dot{a}^2 = \frac{8\pi G_N}{3c^2}\frac{C_1}{a} - kc^2\,. \tag{11.28}$$

In order to solve Eq. (11.28), we introduce the variable η

$$\frac{d\eta}{dt} = \frac{1}{a}\,. \tag{11.29}$$

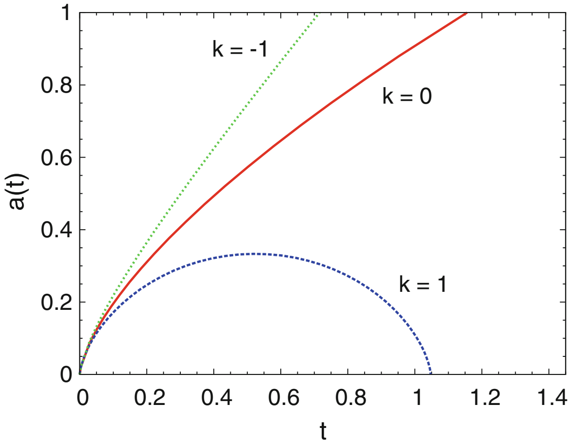

Fig. 11.1 Scale factor a as a function of the cosmological time t for matter dominated universes. The scale factor is expressed in units in which $8\pi G_N C_1/c^2 = A = 1$

We replace t with η in Eq. (11.28) and the first Friedmann equation reads

$$a'^2 = \frac{8\pi G_N}{3c^2} C_1 a - kc^2 a^2 \,, \tag{11.30}$$

where the prime $'$ indicates the derivative with respect to η, i.e. $' = d/d\eta$. With the initial condition $a = 0$ at $t = 0$, we get the following parametric solutions for a and t. In the case of closed universes ($k = 1$), we have

$$a = \frac{4\pi G_N}{3c^2} C_1 A \left(1 - \cos\frac{\eta}{\sqrt{A}}\right) , \quad t = \frac{4\pi G_N}{3c^2} C_1 A \left(\eta - \sqrt{A}\sin\frac{\eta}{\sqrt{A}}\right) . \tag{11.31}$$

For flat universes ($k = 0$), we have

$$a = \frac{2\pi G_N}{3c^2} C_1 \eta^2 \,, \quad t = \frac{2\pi G_N}{9c^2} C_1 \eta^3 \,. \tag{11.32}$$

Lastly, for open universes ($k = -1$), we have

$$a = \frac{4\pi G_N}{3c^2} C_1 A \left(\cosh\frac{\eta}{\sqrt{A}} - 1\right) , \quad t = \frac{4\pi G_N}{3c^2} C_1 A \left(\sqrt{A}\sinh\frac{\eta}{\sqrt{A}} - \eta\right) . \tag{11.33}$$

$A = 1/(|k|c^2)$ has dimensions of time squared.

The scale factor a as a function of the cosmological time t for matter dominated universes is shown in Fig. 11.1. At $t = 0$, $a = 0$ and the matter density diverges. This is the Big Bang and the universe starts expanding. A closed universe expands up to a critical point and then recollapses. An open universe expands forever. The flat universe expands forever and represents the critical case separating closed and open universes.

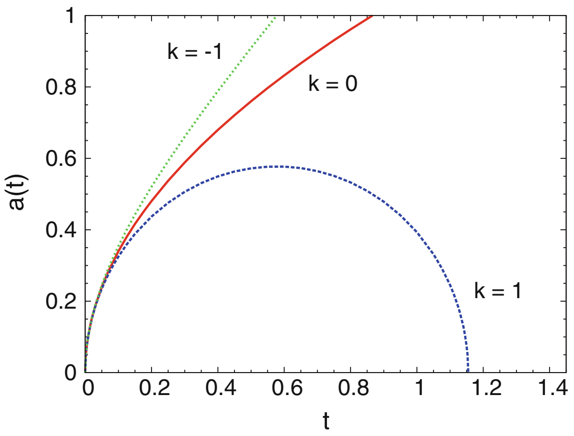

Fig. 11.2 Scale factor a as a function of the cosmological time t for radiation dominated universes. The scale factor is expressed in units in which $8\pi G_N C_2/c^2 = c = 1$

11.3.3 Radiation Dominated Universe

If the universe is filled with radiation, the energy density scales as $1/a^4$, and we can write

$$\rho a^4 = \text{constant} \equiv C_2\,. \tag{11.34}$$

The first Friedmann equation becomes

$$\dot{a}^2 = \frac{8\pi G_N}{3c^2}\frac{C_2}{a^2} - kc^2\,. \tag{11.35}$$

With the initial condition $a = 0$ at $t = 0$, we find the following solution

$$a = \left[\sqrt{\frac{32\pi G_N C_2}{3c^2}}\,t - kc^2t^2\right]^{1/2}\,. \tag{11.36}$$

where $k = 1$, 0, or -1 for closed, flat, or open universes, respectively.

Figure 11.2 shows the scale factor a as a function of the cosmological time t for radiation dominated universes. As in the matter dominated universes, closed universes expand up to a critical point and then recollapse, while flat and open universes expand forever.

11.3.4 Vacuum Dominated Universe

In the case of vacuum energy, the equation of state is $P = -\rho$, and we find that ρ is constant. Equivalently, we can write the Friedmann equations without matter ($\rho = P = 0$) and with a non-vanishing cosmological constant

$$H^2 = \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}\,, \quad \frac{\ddot{a}}{a} = \frac{\Lambda c^2}{3}\,. \tag{11.37}$$

For $\Lambda > 0$, the solutions for closed, flat, and open universes are

$$k = 1 \to a = \sqrt{\frac{3k}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}\,ct\right)\,. \tag{11.38}$$

$$k = 0 \to a = a(t = 0)\, \exp\left(\sqrt{\frac{\Lambda}{3}}\,ct\right)\,. \tag{11.39}$$

$$k = -1 \to a = \sqrt{\frac{-3k}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}}\,ct\right)\,. \tag{11.40}$$

All universes expand forever. For $k = 1$ and 0, the scale factor never vanishes, so there is no Big Bang.

If $\Lambda < 0$, we can only find a solution for $k = -1$

$$k = -1 \to a = \sqrt{\frac{3k}{\Lambda}} \cos\left(\sqrt{-\frac{\Lambda}{3}}\,ct\right)\,. \tag{11.41}$$

If $\Lambda = 0$, we have only the trivial case $k = 0$ and a constant corresponding to the Minkowski spacetime.

11.4 Properties of the Friedmann–Robertson–Walker Metric

11.4.1 Cosmological Redshift

Now we want to study the geodesic motion of test-particles in the Friedmann–Robertson–Walker metric. The Lagrangian is

$$L = \frac{1}{2} g_{\mu\nu} x'^{\mu} x'^{\nu}\,, \tag{11.42}$$

where $g_{\mu\nu}$s are the metric coefficients of the Friedmann–Robertson–Walker solution and the prime $'$ is used to indicate the derivative with respect to the proper time/affine parameter of the trajectory (in this chapter the dot $\dot{}$ is used for the derivative of the coordinate t; e.g. $da/dt = \dot{a}$). Since in the Friedmann–Robertson–Walker metric the scale factor a depends on the time t, it is clear that the energy of the particle is not a constant of motion.

In the case of a photon, $g_{\mu\nu}x'^{\mu}x'^{\nu} = 0$. If we employ a coordinate system in which the motion is only along the radial direction, we have

$$c^2 t'^2 - a^2 \frac{r'^2}{1 - kr^2} = 0\,. \tag{11.43}$$

The Euler-Lagrange equation for the t coordinate reads

$$t'' = -\frac{a\dot{a}}{c^2}\frac{r'^2}{1 - kr^2}\,. \tag{11.44}$$

If we use Eq. (11.43) into Eq. (11.44), we find

$$t'' = -\frac{\dot{a}}{a}t'^2 = -\frac{a'}{a}t'\,. \tag{11.45}$$

Since t' is proportional to the energy of the photon, say E, $t''/t' = E'/E$. Equation (11.45) tells us that the the energy of a photon propagating in a Friedmann–Robertson–Walker spacetime scales as the inverse of the scale factor

$$E \propto 1/a\,. \tag{11.46}$$

This is the *cosmological redshift* and it is due to the expansion of the Universe. It is different from the Doppler redshift, which is due to the relative motion between the source and the observer and is present already in special relativity. It is also different from the gravitational redshift, which is due to climbing in a gravitational potential. It is instead connected to the behavior of the energy density of radiation, which scales as $1/a^4$: the photon number density scales as the inverse of the volume, $1/a^3$, while the photon energy scales as $1/a$, so the result is that the energy density scales as $1/a^4$.

11.4.2 Particle Horizon

Within a cosmological model with a Big Bang, the *particle horizon* at the time t is defined as the distance covered by a photon from the time of the Big Bang to the time t. It is an important concept because it defines the causally connected regions at any time: two points at a distance larger than the particle horizon have never exchanged any information. Let us consider a flat universe ($k = 0$), which can be seen as a valid approximation for any matter or radiation dominated universe for sufficiently early times. For simplicity, we choose a coordinate system in which the photon is at the origin $r = 0$ at the time $t = 0$ (Big Bang). From $ds^2 = 0$, we have

$$r = \int_0^r d\tilde{r} = c \int_0^t \frac{d\tilde{t}}{a} \, . \tag{11.47}$$

If we write the scale factor as $a \propto t^\alpha$ (see Sect. 11.3) and we integrate in $d\tilde{t}$, we find

$$r = \frac{ct}{a\,(1-\alpha)} \, . \tag{11.48}$$

The proper distance at the time t between the origin and a point with the radial coordinate r is $d = a\,r$. The particle horizon is thus

$$d = \frac{ct}{1-\alpha} \, . \tag{11.49}$$

If we have a universe filled with dust, $\alpha = 2/3$ and $d = 3ct$. If the universe is filled with radiation, $\alpha = 1/2$ and $d = 2ct$. For ordinary matter $w \geq 0$, the particle horizon grows linearly with time, and the scale factor grows slower, because $a \propto t^\alpha$ and $\alpha < 1$. In such a case, more and more regions in the universe get causally connected at later times.

11.5 Primordial Plasma

Let us consider a gas of particles in thermal equilibrium. We assume that the space is homogeneous and isotropic. The particle number density n and the particle energy density ρ at the temperature T are, respectively,

$$n = g \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \, f(\mathbf{p}) \, , \quad \rho = g \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \, E(\mathbf{p}) \, f(\mathbf{p}) \, , \tag{11.50}$$

where g is the number of internal degrees of freedom, $\mathbf{p}$ is the particle 3-momentum, $E = \sqrt{m^2c^4 + \mathbf{p}^2c^2}$ is the particle energy, and $f(\mathbf{p})$ is the Bose-Einstein distribution (in the case of bosons, namely for particles with integer spin) or the Fermi-Dirac distribution (for fermions, namely for particles with half-integer spin)

$$f(\mathbf{p}) = \begin{cases} \frac{1}{e^{(E-\mu)/k_\mathrm{B}T}-1} & \text{Bose-Einstein distribution} \\ \frac{1}{e^{(E-\mu)/k_\mathrm{B}T}+1} & \text{Fermi-Dirac distribution} \end{cases} \, . \tag{11.51}$$

Here μ is the chemical potential and k_B is the Boltzmann constant. The reader not familiar with these concepts may refer to any textbook on statistical mechanics.

For a non-degenerate ($\mu \ll T$) relativistic ($m \ll T$) gas, the particle number density turns out to be

$$n = \frac{g}{2\pi^2 c^3 \hbar^3} \int \frac{E^2\, dE}{e^{E/k_B T} \pm 1} = \begin{cases} \frac{\zeta(3)}{\pi^2} \frac{g}{c^3\hbar^3} (k_B T)^3 & \text{for bosons} \\ \frac{3}{4} \frac{\zeta(3)}{\pi^2} \frac{g}{c^3\hbar^3} (k_B T)^3 & \text{for fermions} \end{cases} \tag{11.52}$$

where $\zeta(3) = 1.20206...$ is the Riemann Zeta function. The particle energy density is

$$\rho = \frac{g}{2\pi^2 c^3 \hbar^3} \int \frac{E^3\, dE}{e^{E/k_B T} \pm 1} = \begin{cases} \frac{\pi^2}{30} \frac{g}{c^3\hbar^3} (k_B T)^4 & \text{for bosons} \\ \frac{7}{8} \frac{\pi^2}{30} \frac{g}{c^3\hbar^3} (k_B T)^4 & \text{for fermions} \end{cases} \tag{11.53}$$

For non-relativistic particles ($m \gg T$) with arbitrary chemical potential μ, we find (in the non-relativistic limit, there is no difference between bosons and fermions)

$$n = g \left(\frac{m k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-\left(mc^2 - \mu\right)/k_B T} , \quad \rho = mc^2 n \, . \tag{11.54}$$

If the gas is made of different species of particles, the total number density and the total energy density will be given by, respectively, the sum of all number densities and the sum of all energy densities, as in Eq. (11.9). Since the energy density of non-relativistic particles in thermal equilibrium is exponentially suppressed with respect to that of relativistic particles, their contribution may be ignored and we can write the total energy density of the universe as

$$\rho = \frac{\pi^2}{30} \frac{g_{\rm eff}}{c^3\hbar^3} (k_B T)^4 \, , \tag{11.55}$$

where $g_{\rm eff}$ is the effective number of light degrees of freedom

$$g_{\rm eff} = \sum_{\rm bosons} g_b + \frac{7}{8} \sum_{\rm fermions} g_f \, . \tag{11.56}$$

In general, $g_{\rm eff}$ will depend on the plasma temperature, because some particles may be relativistic above some temperature and become non-relativistic at lower temperatures.

If we plug the expression in Eq. (11.55) into the first Friedmann equation, we get

$$H^2 = \frac{4\pi^3 G_N}{45 c^5 \hbar^3} g_{\rm eff} (k_B T)^4 - \frac{kc^2}{a^2} \, . \tag{11.57}$$

In a radiation dominated universe, we have $a \propto t^{1/2}$ at sufficiently early times, and therefore $H = 1/2t$. If we plug such an expression for the Hubble parameter into Eq. (11.57), we can get a relation between the time t of the universe and the temperature T of the plasma. Within the Standard Model of particle physics, at $T > 1$ MeV we have $g_{\rm eff} \sim 10 - 100$ [1]. The relation between time t and plasma temperature T is

$$t \sim 1 \left(\frac{1\,\text{MeV}}{T} \right)^2 \text{s}\,. \tag{11.58}$$

11.6 Age of the Universe

The cosmological models discussed in Sect. 11.3 are very simple and it is possible to obtain a compact analytic expression for their scale factor a. In more realistic cosmological models, the Universe is filled with different components. However, if we know the contribution of each component and the value of the Hubble parameter at a certain time (for example today), we can calculate the evolution in time of the scale factor. In the cosmological models starting from a vanishing scale factor, like those in Sects. 11.3.2 and 11.3.3, we can define the age of the Universe as the time interval measured with respect to the temporal coordinate of the Friedmann–Robertson–Walker metric between the Big Bang ($a = 0$) and today.

To evaluate the age of the Universe today we can proceed as follows. First, we define the effective energy density associated to a possible non-vanishing k as

$$\rho_k = -\frac{3c^4}{8\pi G_{\rm N}} \frac{k}{a^2}\,, \tag{11.59}$$

and we rewrite the first Friedmann equation as

$$H^2 = \frac{8\pi G_{\rm N}}{3c^2} \rho_{\rm c}^0 \sum_i \frac{\rho_i}{\rho_{\rm c}^0}\,, \tag{11.60}$$

where $\rho_{\rm c}^0$ is the value of the critical energy density today and the sum is over all the different components filling the Universe, including also ρ_k.

We define the *redshift factor* z as

$$1 + z \equiv \frac{a_0}{a}\,, \tag{11.61}$$

where a_0 is the scale factor today and a is the scale factor at the redshift z. Today $z = 0$. We know from observations that the Universe is expanding, namely z increases as we go backwards in time. From Sect. 11.3 and Eq. (11.59), we know how the energy densities of different components evolve with the scale factor, and therefore with the redshift factor. If we restrict attention to non-relativistic matter (dust), vacuum energy, and the effective energy associated to k (curvature), we have

$$\rho_{\rm m} = \rho_{\rm m}^0 \, (1+z)^3 \, , \quad \rho_\Lambda = \rho_\Lambda^0 \, , \quad \rho_k = \rho_k^0 \, (1+z)^2 \, . \tag{11.62}$$

where $\rho_{\rm m}$, ρ_Λ, and ρ_k are, respectively, the energy densities of non-relativistic matter, vacuum energy, and curvature at redshift z, and $\rho_{\rm m}^0$, ρ_Λ^0, and ρ_k^0 are the same energy densities today. We plug the expressions in Eq. (11.62) into Eq. (11.60) and we find

$$H^2 = H_0^2 \left[\Omega_{\rm m}^0 \, (1+z)^3 + \Omega_\Lambda^0 + \Omega_k^0 \, (1+z)^2 \right] , \tag{11.63}$$

where $\Omega_i = \rho_i / \rho_{\rm c}$ and the index 0 is to indicate their value today.

From the definition of the Hubble parameter, we find

$$H = \frac{\dot{a}}{a} = \frac{d}{dt} \ln \frac{a}{a_0} = \frac{d}{dt} \ln \frac{1}{1+z} = - \frac{1}{1+z} \frac{dz}{dt} \, , \tag{11.64}$$

and we can thus rewrite Eq. (11.63) as

$$\frac{dt}{dz} = - \frac{1}{1+z} \frac{1}{H_0 \sqrt{\Omega_{\rm m}^0 \, (1+z)^3 + \Omega_\Lambda^0 + \Omega_k^0 \, (1+z)^2}} \, . \tag{11.65}$$

Since $\Omega_{\rm m}^0 + \Omega_\Lambda^0 + \Omega_k^0 = 1$ by definition, we can write $\Omega_k^0 = 1 - \Omega_{\rm m}^0 - \Omega_\Lambda^0$ and remove Ω_k^0 in Eq. (11.65). Integrating by parts, we can find the time difference between today ($z = 0$) and the time at which the redshift of the Universe was z in terms of H_0, $\Omega_{\rm m}^0$, and Ω_Λ^0

$$\Delta t = \frac{1}{H_0} \int_0^z \frac{d\tilde{z}}{1+\tilde{z}} \frac{1}{\sqrt{\left(1 + \Omega_{\rm m}^0 \tilde{z}\right) (1+\tilde{z})^2 - \tilde{z} \, (2+\tilde{z}) \, \Omega_\Lambda^0}} \, . \tag{11.66}$$

The age of the Universe is obtained when $z \to \infty$ (corresponding to $a = 0$)

$$\tau = \frac{1}{H_0} \int_0^\infty \frac{d\tilde{z}}{1+\tilde{z}} \frac{1}{\sqrt{\left(1 + \Omega_{\rm m}^0 \tilde{z}\right) (1+\tilde{z})^2 - \tilde{z} \, (2+\tilde{z}) \, \Omega_\Lambda^0}} \, . \tag{11.67}$$

The integral in Eq. (11.67) is typically of order unity, so the age of the Universe is roughly given by $1/H_0 \approx 14$ Gyr and is not too sensitive to the exact matter content. For instance, in the simple case of a flat Universe without vacuum energy (i.e. $\Omega_{\rm m}^0 = 1$ and $\Omega_\Lambda^0 = 0$), we find

$$\tau = \frac{1}{H_0} \int_0^\infty \frac{d\tilde{z}}{(1+\tilde{z})^{5/2}} = \frac{2}{3} \frac{1}{H_0} \approx 10 \text{ Gyr} . \tag{11.68}$$

In more general cases, it is necessary to integrate Eq. (11.67) numerically. We should also take into account the contribution of relativistic matter in order to get a more accurate result, but in the case of our Universe it only introduces a small correction.

11.7 Destiny of the Universe

For a universe filled with ordinary matter, there is a simple relation between its geometry (given by the value of the constant k) and its destiny (recollapse or eternal expansion): a closed universe must recollapse, while flat or open universes expand forever. In the presence of vacuum energy, this is not true any longer. A positive cosmological constant makes a universe without matter expand for any value of k. If the universe is filled with dust and vacuum energy, its destiny is determined by their relative contribution. If there is enough dust to stop the expansion, then the universe starts recollapsing to a new singular configuration with $a = 0$. If vacuum energy starts driving the expansion before the recollapse, we are in the opposite scenario and dust becomes less and less important in the evolution of the universe.

If we consider only dust and vacuum energy, the universe is flat if

$$\Omega_{\rm m} + \Omega_\Lambda = 1 , \tag{11.69}$$

and it is closed (open) if $\Omega_{\rm m} + \Omega_\Lambda > 1$ (< 1).

The curve separating eternally expanding universes from universes initially having an expanding phase followed by a contraction is given by

$$\Omega_\Lambda = \begin{cases} 0 & \text{for } \Omega_{\rm m} \le 1 , \\ 4\Omega_{\rm m} \sin^3 \left[\frac{1}{3} \arcsin \left(\frac{\Omega_{\rm m} - 1}{\Omega_{\rm m}} \right) \right] & \text{for } \Omega_{\rm m} > 1 . \end{cases} \tag{11.70}$$

From the second Friedmann equation, we see that the universe's expansion is accelerating (decelerating) if $\rho + 3P < 0$ (> 0). If we write $\rho = \rho_{\rm m} + \rho_\Lambda$ and $P = P_\Lambda = -\rho_\Lambda$, we find

$$\ddot{a} > 0 \Rightarrow \Omega_{\rm m} < 2\Omega_\Lambda \quad (\ddot{a} < 0 \Rightarrow \Omega_{\rm m} > 2\Omega_\Lambda) . \tag{11.71}$$

Figure 11.3 shows the curves in Eqs. (11.69), (11.70), and (11.71) on the plane $(\Omega_{\rm m}, \Omega_\Lambda)$

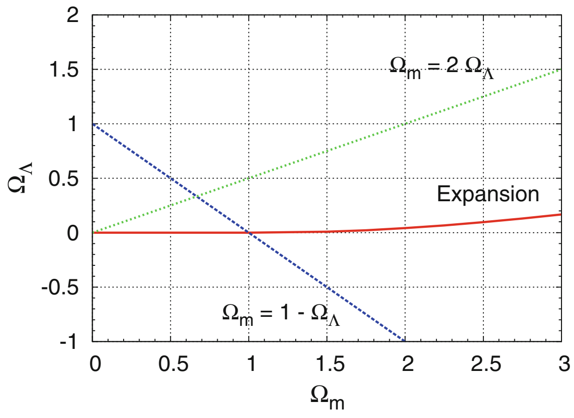

Fig. 11.3 Cosmological models with dust and vacuum energy. The line $\Omega_{\mathrm{m}} = 1 - \Omega_\Lambda$ separates closed universes ($\Omega_{\mathrm{m}} > 1 - \Omega_\Lambda$) from open universes ($\Omega_{\mathrm{m}} < 1 - \Omega_\Lambda$). The line "Expansion" (red solid line) separates universes that expand forever (above) from universes that first expand and then recollapse (below). The line $\Omega_{\mathrm{m}} = 2\Omega_\Lambda$ separates accelerating ($\ddot{a} > 0$) from decelerating ($\ddot{a} < 0$) universes

Problems

11.1 Write the tt component of the Einstein equations for the Friedmann–Robertson–Walker metric and a perfect fluid and derive the first Friedmann equation.

11.2 With the help of some Mathematica package, verify Eqs. (11.3) and (11.4).

11.3 Check that the Einstein universe is unstable.

11.4 Repeat the discussion in Sect. 11.6 about the age of the Universe in the case of a non-negligible contribution from a radiation component.

Reference

1. C. Bambi, A.D. Dolgov, *Introduction to Particle Cosmology: The Standard Model of Cosmology and its Open Problems* (Springer-Verlag, Berlin Heidelberg, 2016)