Chapter 7
Einstein's Gravity

In Chap. 6, we discussed non-gravitational phenomena in curved spacetimes for a generic metric theory of gravity. General covariance is the basic principle: once we have the metric of the spacetime, we can describe non-gravitational phenomena. A different issue is the calculation of the metric of the spacetime. The Principle of General Covariance is not enough to determine the metric. We need the field equations of a specific gravity theory. With Einstein's gravity, we refer to the gravity theory described by the Einstein–Hilbert action or, equivalently, by the Einstein equations.

7.1 Einstein Equations

As in the case of the construction of the Lagrangian of a physical system, we do not have any direct way to infer the field equations of the gravity theory we are looking for. Thus, we have to start by listing some "reasonable" requirements that our theory and its field equations should satisfy and then test their predictions with observations.

1. The gravitational field should be completely described by the metric tensor of the spacetime. As we saw in Chaps. 5 and 6, the spacetime metric is potentially capable of describing non-gravitational phenomena in a gravitational field if the Einstein Equivalence Principle holds. While additional degrees of freedom cannot be excluded, the requirement that the gravitational field is only described by the metric tensor is the *minimal* scenario and thus the first one to explore.
2. The field equations must be tensor equations; that is, they should be written in manifestly general covariant form in order to be explicitly independent of the choice of the coordinate system.
3. The field equations should be partial differential equations at most of second order, in analogy with the field equations of the known physical systems. As in point 1, this is the minimal scenario.

C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4_7

4. The field equations must have the correct Newtonian limit and therefore we must recover the Poisson equation $\Delta\Phi = 4\pi G_{\mathrm{N}}\rho$, where ρ is the mass density.
5. Since in Newton's gravity the source of the gravitational field is the mass density, now the source must be somehow related to the energy density. Since we want a tensor equation, the best candidate seems to be the matter energy-momentum tensor $T^{\mu\nu}$.
6. In the absence of matter, we must recover the Minkowski spacetime.

From conditions 2 and 5, the field equations can be written as

$$G^{\mu\nu} = \kappa T^{\mu\nu} , \tag{7.1}$$

where $G^{\mu\nu}$ is the tensor to find and κ is a proportionality constant somehow related to G_{N}. Since the matter energy-momentum tensor is covariantly conserved and symmetric

$$\nabla_\mu T^{\mu\nu} = 0 , \quad T^{\mu\nu} = T^{\nu\mu} , \tag{7.2}$$

we need that

$$\nabla_\mu G^{\mu\nu} = 0 , \quad G^{\mu\nu} = G^{\nu\mu} . \tag{7.3}$$

Conditions 1, 3, and 6 are compatible with the following choice

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R . \tag{7.4}$$

In the next section, we will show that this choice also meets condition 4 of the correct Newtonian limit.

If we relax conditions 4 and 6, we can also write

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} , \tag{7.5}$$

where Λ is called the *cosmological constant*. If its value is sufficiently small, the choice (7.5) can also be consistent with observations. For the moment, we assume that $\Lambda = 0$, but a non-vanishing value of the cosmological constant can be relevant in cosmological models (see Chap. 11).

The field equations in Einstein's gravity are the *Einstein equations* and read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} , \tag{7.6}$$

where κ is Einstein's constant of gravitation (we will find its relation to Newton's constant of gravitation G_{N} in the next section). If we want to consider the possibility of a non-vanishing cosmological constant, we have[1]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \,. \tag{7.8}$$

In this section, we have arrived at these equations by imposing conditions 1–6. However, there is not a recipe to obtain the right field equations. Every theory has its own field equations. In Einstein's gravity, the field equations are the Einstein equations in (7.6) or in (7.8). Its predictions agree well with current observational data. However, there are also significant efforts to find alternative gravity theories with different field equations. The latter should be able to explain experimental data in order to be considered viable candidates as alternative theories. Once we find an observation that cannot be explained by one of these theories (Einstein's gravity included), the theory is ruled out.

From the Einstein equations, we can write the scalar curvature in terms of the matter content as follows

$$\begin{aligned} & g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \kappa g^{\mu\nu} T_{\mu\nu} \,, \\ & R - 2R = \kappa T \,, \\ & R = -\kappa T \,, \end{aligned} \tag{7.9}$$

where $T = T^{\mu}_{\mu}$ is the trace of the matter energy-momentum tensor. In Einstein's gravity, the scalar curvature thus vanishes either in vacuum or for $T = 0$ (e.g. the energy-momentum tensor of an electromagnetic field, see Sect. 4.5). In other theories of gravity, this may not be true. The Einstein equations can be rewritten as

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \,. \tag{7.10}$$

Note that $R_{\mu\nu} = 0$ does not imply no gravitational field, but just no matter at that point of the spacetime. For example, if we consider a distribution of matter of finite extension, $R_{\mu\nu} \neq 0$ in the region with matter and $R_{\mu\nu} = 0$ in the exterior region.

In four dimensions, the Einstein equations are a system of 10 differential equations to determine the 10 components of the metric tensor $g_{\mu\nu}$ ($G_{\mu\nu}$ and $g_{\mu\nu}$ have 16 components, but the tensors are symmetric). Even if we fix the initial conditions, there is not a unique solution because it is always possible to perform a coordinate transformation: even if the solution looks different after a coordinate transformation,

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[1] If we use the convention of a metric with signature $(+---)$, Eq. (7.8) reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \,, \tag{7.7}$$

i.e. the sign in front of Λ is $-$ instead of $+$ (employing the common convention of positive/negative Λ).

it is physically equivalent. This, in particular, means that we do not have 10 physical degrees of freedom.

Lastly, note that the Einstein equations relate the geometry of the spacetime (on the left hand side) to the matter content (on the right hand side). If we know the matter content, we can determine the spacetime metric. In principle, we can obtain any kind of spacetime for a proper choice of the matter energy-momentum tensor. For example, even unphysical spacetimes with closed time-like curves (i.e. trajectories in which massive particles can go backwards in time, as in a time machine) are possible for an unphysical matter energy-momentum tensor. In other words, the Einstein equations can make clear predictions only when we clearly specify the matter content. If this is not the case, every metric is allowed.

7.2 Newtonian Limit

In order to be consistent with observations, the Einstein equations must be able to recover the correct Newtonian limit. This should also provide the relation between Einstein's constant of gravitation κ appearing in (7.6) and Newton's constant of gravitation $G_{\rm N}$ appearing in Newton's Law of Universal Gravitation.

As in Sect. 6.3, we impose that the gravitational field is weak and stationary; that is,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1\,, \tag{7.11}$$

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0\,. \tag{7.12}$$

Let us also choose a coordinate system in which all the components of the matter energy-momentum tensor vanish with the exception of the tt component, which describes the energy density and in the Newtonian limit reduces to the mass density ρ multiplied by the square of the speed of light c^2,

$$T_{tt} = \rho c^2\,. \tag{7.13}$$

This assumption is justified by the fact that in Newton's gravity the source of the gravitational field is only the mass. With our choice, the trace of the matter energy-momentum tensor is $T = -\rho c^2$ and the tt component of Eq. (7.10) turns out to be

$$R_{tt} = \kappa\left(\rho c^2 + \frac{1}{2}\eta_{tt}\rho c^2\right) = \frac{\kappa c^2}{2}\rho\,. \tag{7.14}$$

Neglecting terms of second order in $h_{\mu\nu}$ and employing Eq. (6.8), we have

$$R_{tt} = \frac{\partial \Gamma^i_{tt}}{\partial x^i} + O(h^2) = -\frac{1}{2}\frac{\partial}{\partial x^i}\left(\eta^{ij}\frac{\partial h_{tt}}{\partial x^j}\right) + O(h^2)$$
$$= -\frac{1}{2}\Delta h_{tt} + O(h^2)\,, \tag{7.15}$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplacian. As seen in Sect. 6.3, $h_{tt} = -2\Phi/c^2$, and therefore

$$R_{tt} = \frac{\Delta\Phi}{c^2}\,. \tag{7.16}$$

After replacing R_{tt} with $\Delta\Phi/c^2$ in Eq. (7.14), we have

$$\Delta\Phi = \frac{\kappa c^4}{2}\rho\,. \tag{7.17}$$

The formal solution is

$$\Phi(\mathbf{x}) = -\frac{\kappa c^4}{8\pi}\int \frac{\rho(\tilde{\mathbf{x}})}{|\mathbf{x} - \tilde{\mathbf{x}}|}d^3\tilde{\mathbf{x}}\,. \tag{7.18}$$

If we compare Eq. (7.17) with the Poisson equation $\Delta\Phi = 4\pi G_{\rm N}\rho$ that holds in the Newtonian theory, we find the relation between κ and $G_{\rm N}$

$$\kappa = \frac{8\pi G_{\rm N}}{c^4}\,. \tag{7.19}$$

Note that in the presence of a non-vanishing cosmological constant Eq. (7.17) would be

$$\frac{\Delta\Phi}{c^2} = \frac{\kappa c^2}{2}\rho - \Lambda\,, \tag{7.20}$$

and we cannot recover the Poisson equation.

7.3 Einstein–Hilbert Action

It is sometimes convenient to have the action of a certain theory and be able to derive the equations governing the dynamics of the system by employing the Least Action Principle. While it is not guaranteed that such an action exits, for the known physical systems we have one. Einstein's gravity is not an exception. In this section we want thus to discuss the action that, when we impose the Least Action Principle, provides the Einstein equations.

The total action will be the sum of the action of the gravitational sector, say S_g, and of the action of the matter sector, say S_m,

$$S = S_g + S_m \,. \tag{7.21}$$

The natural candidates for the Lagrangian coordinates of the action of the gravitational field are the metric coefficients and their first derivatives, namely $g_{\mu\nu}$ and $\partial_\rho g_{\mu\nu}$. The matter sector will have its own Lagrangian coordinates. A coupling constant will connect the gravity and the matter sectors and establish the strength of the interaction. The Einstein equations should be obtained by considering the following variation

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \,, \tag{7.22}$$

with $\delta g_{\mu\nu} = 0$ at the boundary of the integration region:

$$\delta S_g + \delta S_m = 0 \quad \Rightarrow \quad G^{\mu\nu} - \kappa T^{\mu\nu} = 0 \,. \tag{7.23}$$

The field equations of the matter sector in the gravitational field should instead be obtained by considering variations with respect to the Lagrangian coordinates of the matter sector

The Einstein equations can be obtained by applying the Least Action Principle to the *Einstein–Hilbert action*, which reads

$$S_{EH} = \frac{1}{2\kappa c} \int R\sqrt{-g}\, d^4x \,, \tag{7.24}$$

or, with Newton's constant of gravitation G_N instead of κ,

$$S_{EH} = \frac{c^3}{16\pi G_N} \int R\sqrt{-g}\, d^4x \,. \tag{7.25}$$

S_{EH} describes the action of the gravitational sector, S_g in Eq. (7.21). The Einstein equations require also the action for the matter sector, S_m. In what follows, we will check that, through the Least Action Principle, the Einstein–Hilbert action provides the left hand side part in the Einstein equations.

Let us calculate the effect of a variation of the metric coefficients of the form in (7.22) on $g_{\rho\sigma} g^{\sigma\nu}$. We find

$$0 = \delta g^\nu_\rho = \delta \left(g_{\rho\sigma} g^{\sigma\nu} \right) = \left(\delta g_{\rho\sigma} \right) g^{\sigma\nu} + g_{\rho\sigma} \left(\delta g^{\sigma\nu} \right) \,, \tag{7.26}$$

and therefore

$$g_{\rho\sigma} \delta g^{\sigma\nu} = -g^{\sigma\nu} \delta g_{\rho\sigma} \,. \tag{7.27}$$

$\delta g^{\mu\nu}$ is thus

$$\delta g^{\mu\nu} = g^{\mu\rho} g_{\rho\sigma} \delta g^{\sigma\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} \, . \tag{7.28}$$

Let us now consider the effect of a variation of the metric coefficients on $\sqrt{-g}$. Employing Eq. (5.56), we have

$$\begin{aligned} \delta\sqrt{-g} &= \frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = -\frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = -\frac{1}{2} \frac{1}{\sqrt{-g}} g g^{\mu\nu} \delta g_{\mu\nu} \\ &= \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \, , \end{aligned} \tag{7.29}$$

Lastly, we need to calculate the effect of a variation of the metric coefficients on the Ricci tensor

$$\begin{aligned} \delta R_{\mu\nu} &= \delta \left(\partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\sigma} \right) \\ &= \partial_\rho \left(\delta \Gamma^\rho_{\mu\nu} \right) - \partial_\nu \left(\delta \Gamma^\rho_{\mu\rho} \right) + \left(\delta \Gamma^\sigma_{\mu\nu} \right) \Gamma^\rho_{\sigma\rho} + \Gamma^\sigma_{\mu\nu} \left(\delta \Gamma^\rho_{\sigma\rho} \right) \\ &\quad - \left(\delta \Gamma^\sigma_{\mu\rho} \right) \Gamma^\rho_{\nu\sigma} - \Gamma^\sigma_{\mu\rho} \left(\delta \Gamma^\rho_{\nu\sigma} \right) \, . \end{aligned} \tag{7.30}$$

If we add and subtract the quantity $\Gamma^\sigma_{\rho\nu} \left(\delta \Gamma^\rho_{\mu\sigma} \right)$ to the previous expression, we have

$$\begin{aligned} \delta R_{\mu\nu} &= \delta R_{\mu\nu} - \Gamma^\sigma_{\nu\rho} \left(\delta \Gamma^\rho_{\mu\sigma} \right) + \Gamma^\sigma_{\rho\nu} \left(\delta \Gamma^\rho_{\mu\sigma} \right) \\ &= \left[\partial_\rho \left(\delta \Gamma^\rho_{\mu\nu} \right) + \Gamma^\rho_{\sigma\rho} \left(\delta \Gamma^\sigma_{\mu\nu} \right) - \Gamma^\sigma_{\mu\rho} \left(\delta \Gamma^\rho_{\nu\sigma} \right) - \Gamma^\sigma_{\nu\rho} \left(\delta \Gamma^\rho_{\mu\sigma} \right) \right] \\ &\quad - \left[\partial_\nu \left(\delta \Gamma^\rho_{\mu\rho} \right) + \Gamma^\rho_{\nu\sigma} \left(\delta \Gamma^\sigma_{\mu\rho} \right) - \Gamma^\sigma_{\mu\nu} \left(\delta \Gamma^\rho_{\sigma\rho} \right) - \Gamma^\sigma_{\rho\nu} \left(\delta \Gamma^\rho_{\mu\sigma} \right) \right] \, . \end{aligned} \tag{7.31}$$

Note that the quantity

$$\delta \Gamma^\rho_{\mu\nu} = \Gamma'^\rho_{\mu\nu} - \Gamma^\rho_{\mu\nu} \tag{7.32}$$

is a tensor of type (1, 2). Indeed we know that the Christoffel symbols transform with the rule in Eq. (5.21). Under a coordinate transformation $x^\mu \to \tilde{x}^\mu$, $\delta \Gamma^\rho_{\mu\nu}$ transforms as[2]

$$\delta \Gamma^\rho_{\mu\nu} \to \delta \tilde{\Gamma}^\rho_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \tilde{x}^\mu} \frac{\partial x^\gamma}{\partial \tilde{x}^\nu} \left(\Gamma'^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} \right) \, . \tag{7.33}$$

Moreover, $\delta \Gamma^\rho_{\mu\nu}$ is made of objects evaluated at the same point (objects belonging to the same tangent space with the terminology of Appendix C), and therefore it is a tensor. The covariant derivative of a tensor of type (1, 2) is

[2]Note that $\Gamma^\rho_{\mu\nu}$s and $\Gamma'^\rho_{\mu\nu}$s are the Christoffel symbols associated, respectively, to the metric tensors $g_{\mu\nu}$ and $g_{\mu\nu} + \delta g_{\mu\nu}$, both in the coordinates x^μ.

$$\nabla_\mu A^\nu{}_{\rho\sigma} = \partial_\mu A^\nu{}_{\rho\sigma} + \Gamma^\nu_{\lambda\mu} A^\lambda{}_{\rho\sigma} - \Gamma^\lambda_{\rho\mu} A^\nu{}_{\lambda\sigma} - \Gamma^\lambda_{\sigma\mu} A^\nu{}_{\rho\lambda} \, , \tag{7.34}$$

and Eq. (7.31) reduces to

$$\delta R_{\mu\nu} = \nabla_\rho \left(\delta \Gamma^\rho_{\mu\nu} \right) - \nabla_\nu \left(\delta \Gamma^\rho_{\mu\rho} \right) \, . \tag{7.35}$$

Equation (7.35) is called the *Palatini Identity*.

Employing Eqs. (7.28), (7.29), and (7.35), we can write the effect of the variation (7.22) on $\sqrt{-g}R$

$$\begin{aligned}
\delta \left(\sqrt{-g} R \right) &= \left(\delta \sqrt{-g} \right) R + \sqrt{-g} \left(\delta g^{\rho\sigma} \right) R_{\rho\sigma} + \sqrt{-g} g^{\mu\nu} \left(\delta R_{\mu\nu} \right) \\
&= \frac{1}{2} \sqrt{-g} g^{\mu\nu} \left(\delta g_{\mu\nu} \right) R - \sqrt{-g} g^{\rho\mu} g^{\sigma\nu} \left(\delta g_{\mu\nu} \right) R_{\rho\sigma} \\
&\quad + \sqrt{-g} g^{\mu\nu} \left[\nabla_\rho \left(\delta \Gamma^\rho_{\mu\nu} \right) - \nabla_\nu \left(\delta \Gamma^\rho_{\mu\rho} \right) \right] \\
&= \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) \sqrt{-g} \left(\delta g_{\mu\nu} \right) \\
&\quad + \sqrt{-g} \left\{ \nabla_\rho \left[g^{\mu\nu} \left(\delta \Gamma^\rho_{\mu\nu} \right) \right] - \nabla_\nu \left[g^{\mu\nu} \left(\delta \Gamma^\rho_{\mu\rho} \right) \right] \right\} \\
&= \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) \sqrt{-g} \left(\delta g_{\mu\nu} \right) \\
&\quad + \sqrt{-g} \left\{ \nabla_\rho \left[g^{\mu\nu} \left(\delta \Gamma^\rho_{\mu\nu} \right) \right] - \nabla_\rho \left[g^{\mu\rho} \left(\delta \Gamma^\nu_{\mu\nu} \right) \right] \right\} \, .
\end{aligned} \tag{7.36}$$

If we define

$$H^\rho = g^{\mu\nu} \left(\delta \Gamma^\rho_{\mu\nu} \right) - g^{\mu\rho} \left(\delta \Gamma^\nu_{\mu\nu} \right) \, , \tag{7.37}$$

we can rewrite the variation of the Einstein–Hilbert action as

$$\begin{aligned}
\delta S_{\rm EH} &= \frac{1}{2\kappa c} \int \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) \sqrt{-g} \left(\delta g_{\mu\nu} \right) d^4x \\
&\quad + \frac{1}{2\kappa c} \int \nabla_\rho H^\rho \sqrt{-g} \, d^4x \, .
\end{aligned} \tag{7.38}$$

Note that the last term in this expression is a divergence [see Eq. (5.64)]

$$\sqrt{-g} \nabla_\rho H^\rho = \sqrt{-g} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\rho} \left(\sqrt{-g} H^\rho \right) = \frac{\partial}{\partial x^\rho} \left(\sqrt{-g} H^\rho \right) \, , \tag{7.39}$$

and therefore we can apply Gauss's theorem to reduce the second integral on the right hand side of Eq. (7.38) into a surface integral. The Least Action Principle requires that the variations of the Lagrangian coordinates vanish at the boundary of the integration region, and therefore any surface integral vanishes too. At this point we have the following expression

$$\delta S = \frac{1}{2\kappa c} \int \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) \sqrt{-g} \left(\delta g_{\mu\nu} \right) d^4x + \delta S_{\rm m} \,, \tag{7.40}$$

In the next section, we will show that with the contribution from $\delta S_{\rm m}$ we recover the Einstein equations.

7.4 Matter Energy-Momentum Tensor

7.4.1 Definition

Let us now consider the effect of a variation of the metric coefficients on the action of the matter sector

$$S_{\rm m} = \frac{1}{c} \int \mathscr{L}_{\rm m} \sqrt{-g}\, d^4x \,. \tag{7.41}$$

At this point we *define* as the matter energy-momentum tensor appearing in the Einstein equations the tensor $T^{\mu\nu}$ given by

$$\delta S_{\rm m} = \frac{1}{2c} \int T^{\mu\nu} \sqrt{-g} \left(\delta g_{\mu\nu} \right) d^4x \,. \tag{7.42}$$

Such a tensor is symmetric by construction, since $g_{\mu\nu}$ is symmetric. We can also check a posteriori that the definition (7.42) of $T^{\mu\nu}$ provides the same matter energy-momentum tensor that we obtain starting from that of special relativity (Sect. 3.9) and proceeding as discussed in Sect. 6.7.

Eventually the variation of the total action (gravity and matter sectors) gives

$$\delta S = \frac{1}{2\kappa c} \int \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} + \kappa T^{\mu\nu} \right) \sqrt{-g} \left(\delta g_{\mu\nu} \right) d^4x \,. \tag{7.43}$$

$\delta S = 0$ for any choice of $\delta g_{\mu\nu}$ only if the Einstein equations hold.

7.4.2 Examples

As a first example, let us consider the action of the electromagnetic field in curved spacetime

$$S = -\frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x \tag{7.44}$$

When we consider a variation of $g_{\mu\nu}$, we have[3]

$$\begin{aligned}\delta\left(F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}\right) &= F_{\mu\nu}F_{\rho\sigma}\left(\delta g^{\mu\rho}\right)g^{\nu\sigma}\sqrt{-g} + F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}\left(\delta g^{\nu\sigma}\right)\sqrt{-g}\\ &\quad + F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\left(\delta\sqrt{-g}\right)\\ &= -F_{\mu\nu}F_{\rho\sigma}g^{\mu\alpha}g^{\rho\beta}\left(\delta g_{\alpha\beta}\right)g^{\nu\sigma}\sqrt{-g}\\ &\quad - F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\alpha}g^{\sigma\beta}\left(\delta g_{\alpha\beta}\right)\sqrt{-g}\\ &\quad + \frac{1}{2}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}\,g^{\alpha\beta}\left(\delta g_{\alpha\beta}\right) . \end{aligned} \tag{7.47}$$

Changing the indices, we can rewrite the last expression as

$$\delta\left(F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\right) = \left(-F^{\mu\rho}F^{\nu}{}_{\rho} - F^{\rho\mu}F_{\rho}{}^{\nu} + \frac{1}{2}F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu}\right)\sqrt{-g}\left(\delta g_{\mu\nu}\right) . \tag{7.48}$$

The variation of the action is thus

$$\delta S = \frac{1}{2c}\int\left(\frac{1}{4\pi}F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{16\pi}F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu}\right)\sqrt{-g}\left(\delta g_{\mu\nu}\right)d^4x , \tag{7.49}$$

and the energy-momentum tensor of the electromagnetic field is

$$T^{\mu\nu} = \frac{1}{4\pi}F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{16\pi}F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu} , \tag{7.50}$$

in agreement with the expression found in Sect. 4.5 when the spacetime metric is $\eta_{\mu\nu}$.

The action for a free point-like particle is [see Eq. (3.22)]

$$S = \frac{1}{2}\int m g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\,d\tau . \tag{7.51}$$

The definition of the energy-momentum tensor requires an action as in Eq. (7.41). We thus have to rewrite the mass of the particle m as a mass density ρ integrated over the 4-volume of the spacetime. Since the particle is point-like, the mass density

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[3]Remember that the fundamental variables of the electromagnetic sector are A_μ and $\partial_\nu A_\mu$, while A^μ and $\partial^\nu A^\mu$ have the metric tensor inside. For this reason, in Eq. (7.47) we consider the variation of

$$F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-g} \tag{7.45}$$

and not of

$$F^{\mu\nu}F^{\rho\sigma}g_{\mu\rho}g_{\nu\sigma}\sqrt{-g} . \tag{7.46}$$

The variation of Eq. (7.46) would provide a different (and wrong) result.

is

$$\begin{aligned}\rho &= m\delta^4\left[x^\sigma - \tilde{x}^\sigma(\tau)\right] \\ &= \frac{m}{\sqrt{-g}}\delta\left[x^0 - \tilde{x}^0(\tau)\right]\delta\left[x^1 - \tilde{x}^1(\tau)\right]\delta\left[x^2 - \tilde{x}^2(\tau)\right]\delta\left[x^3 - \tilde{x}^3(\tau)\right] \\ &= \frac{m}{\sqrt{-g}}\prod_\sigma \delta\left[x^\sigma - \tilde{x}^\sigma(\tau)\right] .\end{aligned} \tag{7.52}$$

where $\{\tilde{x}^\mu(\tau)\}$ are the coordinates of the particle trajectory. The action of the free point-like particle becomes

$$\begin{aligned}S &= \frac{1}{2c}\int m\delta^4\left[x^\sigma - \tilde{x}^\sigma(\tau)\right] g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\sqrt{-g}\, d^4x\, d\tau \\ &= \frac{1}{2c}\int m\left[\prod_\sigma \delta\left[x^\sigma - \tilde{x}^\sigma(\tau)\right]\right] g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\, d^4x\, d\tau .\end{aligned} \tag{7.53}$$

When we consider a variation of the metric tensor, we find

$$\begin{aligned}\delta S &= \frac{1}{2c}\int m\left[\prod_\sigma \delta\left[x^\sigma - \tilde{x}^\sigma(\tau)\right]\right]\dot{x}^\mu\dot{x}^\nu\left(\delta g_{\mu\nu}\right)\, d^4x\, d\tau \\ &= \frac{1}{2c}\int\left\{\int \frac{m}{\sqrt{-g}}\left[\prod_\sigma \delta\left[x^\sigma - \tilde{x}^\sigma(\tau)\right]\right]\dot{x}^\mu\dot{x}^\nu\, d\tau\right\}\sqrt{-g}\left(\delta g_{\mu\nu}\right)\, d^4x ,\end{aligned} \tag{7.54}$$

and therefore the energy-momentum tensor of the free point-like particle is

$$T^{\mu\nu} = \int \frac{m}{\sqrt{-g}}\left[\prod_\sigma \delta\left[x^\sigma - \tilde{x}^\sigma(\tau)\right]\right]\dot{x}^\mu\dot{x}^\nu\, d\tau . \tag{7.55}$$

Let us consider the special case of an inertial reference frame in the Minkowski spacetime. For Cartesian coordinates $\sqrt{-g} = 1$ and we can write $d\tau$ as

$$d\tau = \frac{d\tau}{dt}dt = \frac{dt}{\gamma} , \tag{7.56}$$

where γ is the Lorentz factor of the particle. Integrating over dt, Eq. (7.55) becomes

$$T^{\mu\nu} = m\delta^3\left(\mathbf{x} - \tilde{\mathbf{x}}(\tau(t))\right)\frac{\dot{x}^\mu\dot{x}^\nu}{\gamma} , \tag{7.57}$$

and we recover the result of Eq. (3.101).

The action for a real scalar field in curved spacetime is

$$S = -\frac{\hbar}{2c} \int \left[g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) + \frac{m^2 c^2}{\hbar^2} \phi^2 \right] \sqrt{-g} \, d^4 x \, . \tag{7.58}$$

When we consider a variation of the metric coefficients, we find

$$\begin{aligned} \delta S = & -\frac{\hbar}{2c} \int \Bigg\{ \left(\delta g^{\mu\nu}\right) \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) \sqrt{-g} \\ & + \left[g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) + \frac{m^2 c^2}{\hbar^2} \phi^2 \right] \left(\delta \sqrt{-g}\right) \Bigg\} \, d^4 x \\ = & \frac{1}{2c} \int \hbar \Bigg\{ g^{\mu\rho} g^{\nu\sigma} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) \\ & - \frac{1}{2} g^{\rho\sigma} \left[g^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) + \frac{m^2 c^2}{\hbar^2} \phi^2 \right] \Bigg\} \sqrt{-g} \left(\delta g_{\rho\sigma}\right) \, d^4 x \, . \end{aligned} \tag{7.59}$$

The resulting energy-momentum tensor is

$$T^{\mu\nu} = \hbar \left(\partial^\mu \phi\right) \left(\partial^\nu \phi\right) - \frac{\hbar}{2} g^{\mu\nu} \left[g^{\rho\sigma} \left(\partial_\rho \phi\right) \left(\partial_\sigma \phi\right) + \frac{m^2 c^2}{\hbar^2} \phi^2 \right] . \tag{7.60}$$

7.4.3 Covariant Conservation of the Matter Energy-Momentum Tensor

As discussed in Sect. 3.9, in the Minkowski spacetime in Cartesian coordinates, the equation $\partial_\mu T^{\mu\nu} = 0$ is associated with the conservation of the 4-momentum of the system. In curved spacetime, the conservation equation $\partial_\mu T^{\mu\nu} = 0$ becomes

$$\nabla_\mu T^{\mu\nu} = 0 \, , \tag{7.61}$$

which also follows from the Einstein equations. Note, however, that Eq. (7.61) does not imply the conservation of the 4-momentum. With the formula in Eqs. (5.65), (7.61) can be written as

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(T^{\mu\nu} \sqrt{-g}\right) + \Gamma^\nu_{\sigma\mu} T^{\mu\sigma} = 0 \, . \tag{7.62}$$

This is not a conservation law because it cannot be written as a partial derivative and therefore we cannot apply the Gauss theorem and proceed as in Sect. 3.9. The physical reason is that, in the presence of a gravitational field, we do not have the conservation of the matter 4-momentum, but the conservation of the 4-momentum of the whole system, including both that of matter and that of the gravitational field.

7.5 Pseudo-Tensor of Landau–Lifshitz

As discussed in the previous section, the covariant conservation of the matter energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, is not a conservation equation. Since it is often necessary to evaluate energy balances of physical processes, it would be useful to find a non-covariant formulation of the theory in such a way that we can write something like

$$\partial_\mu \mathscr{T}^{\mu\nu} = 0\,, \tag{7.63}$$

where $\mathscr{T}^{\mu\nu}$ is a quantity connected to the 4-momentum of the whole system and associated with conserved physical quantities. This issue has been studied since the advent of general relativity and solved in different ways. In this section, we will follow the approach proposed by Landau and Lifshitz [1].

Let us consider a locally inertial frame (LIF) at the point x_0 (see Sect. 6.4.2 for the definition of locally inertial frame). Here all the first derivatives of the metric vanish, the Christoffel symbols vanish as well, and Eq. (7.61) becomes

$$\partial_\mu T^{\mu\nu}_{\rm LIF} = 0\,. \tag{7.64}$$

The matter energy-momentum tensor $T^{\mu\nu}_{\rm LIF}$ can be obtained from the Einstein equations

$$T^{\mu\nu}_{\rm LIF} = \frac{c^4}{8\pi G_{\rm N}} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{\rm LIF}\,. \tag{7.65}$$

Let us now evaluate the terms on the right hand side of this equation at x_0. $R^{\mu\nu}_{\rm LIF}$ is given by (remember that at x_0 all the Christoffel symbols vanish)

$$\begin{aligned} R^{\mu\nu}_{\rm LIF} &= g^{\mu\rho} g^{\nu\sigma} \left(\frac{\partial \Gamma^\lambda_{\rho\sigma}}{\partial x^\lambda} - \frac{\partial \Gamma^\lambda_{\rho\lambda}}{\partial x^\sigma} \right) \\ &= g^{\mu\rho} g^{\nu\sigma} \frac{\partial}{\partial x^\lambda} \left[\frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\sigma}}{\partial x^\rho} + \frac{\partial g_{\rho\kappa}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\kappa} \right) \right] \\ &\quad - g^{\mu\rho} g^{\nu\sigma} \frac{\partial}{\partial x^\sigma} \left(\frac{1}{2g} \frac{\partial g}{\partial x^\rho} \right)\,, \end{aligned} \tag{7.66}$$

where in the last passage we used Eq. (5.63). Since the first derivatives of the metric tensor vanish, we can write

$$
\begin{aligned}
R^{\mu\nu}_{\rm LIF} = &\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left[g^{\mu\rho}g^{\nu\sigma}g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\sigma}}{\partial x^\rho}+\frac{\partial g_{\rho\kappa}}{\partial x^\sigma}-\frac{\partial g_{\rho\sigma}}{\partial x^\kappa}\right)\right] \\
&-\frac{\partial}{\partial x^\sigma}\left(g^{\mu\rho}g^{\nu\sigma}\frac{1}{2g}\frac{\partial g}{\partial x^\rho}\right) \\
=&-\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left(g^{\mu\rho}g^{\nu\sigma}\frac{\partial g^{\lambda\kappa}}{\partial x^\rho}g_{\kappa\sigma}+g^{\mu\rho}g^{\nu\sigma}\frac{\partial g^{\lambda\kappa}}{\partial x^\sigma}g_{\rho\kappa}-\frac{\partial g^{\mu\rho}}{\partial x^\kappa}g^{\nu\sigma}g^{\lambda\kappa}g_{\rho\sigma}\right) \\
&-\frac{\partial}{\partial x^\sigma}\left[\frac{1}{2g}\frac{\partial}{\partial x^\rho}\left(gg^{\mu\rho}g^{\nu\sigma}\right)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^\sigma\partial x^\rho}\left(g^{\mu\rho}g^{\nu\sigma}\right) \\
=&-\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left[g^{\mu\rho}\frac{\partial g^{\lambda\nu}}{\partial x^\rho}+g^{\nu\sigma}\frac{\partial g^{\lambda\mu}}{\partial x^\sigma}-\frac{\partial g^{\mu\nu}}{\partial x^\kappa}g^{\lambda\kappa}\right] \\
&-\frac{\partial}{\partial x^\sigma}\left[\frac{1}{2g}\frac{\partial}{\partial x^\rho}\left(gg^{\mu\rho}g^{\nu\sigma}\right)\right]+\frac{1}{2}\frac{\partial^2 g^{\mu\rho}}{\partial x^\sigma\partial x^\rho}g^{\nu\sigma}+\frac{1}{2}\frac{\partial^2 g^{\nu\sigma}}{\partial x^\sigma\partial x^\rho}g^{\mu\rho} \\
=&-\frac{1}{2}g^{\mu\rho}\frac{\partial^2 g^{\lambda\nu}}{\partial x^\lambda\partial x^\rho}-\frac{1}{2}g^{\nu\sigma}\frac{\partial^2 g^{\lambda\mu}}{\partial x^\lambda\partial x^\sigma}+\frac{1}{2}\frac{\partial^2 g^{\mu\nu}}{\partial x^\lambda\partial x^\kappa}g^{\lambda\kappa} \\
&-\frac{\partial}{\partial x^\sigma}\left[\frac{1}{2g}\frac{\partial}{\partial x^\rho}\left(gg^{\mu\rho}g^{\nu\sigma}\right)\right]+\frac{1}{2}\frac{\partial^2 g^{\mu\rho}}{\partial x^\sigma\partial x^\rho}g^{\nu\sigma}+\frac{1}{2}\frac{\partial^2 g^{\nu\sigma}}{\partial x^\sigma\partial x^\rho}g^{\mu\rho} \\
=&\frac{1}{2}\frac{\partial^2 g^{\mu\nu}}{\partial x^\lambda\partial x^\kappa}g^{\lambda\kappa}-\frac{1}{2}\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(gg^{\mu\rho}g^{\nu\sigma}\right)\right]. \qquad (7.67)
\end{aligned}
$$

We can proceed in a similar way to calculate $g^{\mu\nu}R$ in the locally inertial frame. We have

$$
\begin{aligned}
\left(g^{\mu\nu}R\right)_{\rm LIF} &= g^{\mu\nu}g^{\rho\sigma}R_{\rho\sigma} \\
&= g^{\mu\nu}g^{\rho\sigma}\left(\frac{\partial \Gamma^\lambda_{\rho\sigma}}{\partial x^\lambda}-\frac{\partial \Gamma^\lambda_{\rho\lambda}}{\partial x^\sigma}\right) \\
&= g^{\mu\nu}g^{\rho\sigma}\frac{\partial}{\partial x^\lambda}\left[\frac{1}{2}g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\sigma}}{\partial x^\rho}+\frac{\partial g_{\rho\kappa}}{\partial x^\sigma}-\frac{\partial g_{\rho\sigma}}{\partial x^\kappa}\right)\right] \\
&\quad -g^{\mu\nu}g^{\rho\sigma}\frac{\partial}{\partial x^\sigma}\left(\frac{1}{2g}\frac{\partial g}{\partial x^\rho}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x^\lambda}\left[g^{\mu\nu}g^{\rho\sigma}g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\sigma}}{\partial x^\rho}+\frac{\partial g_{\rho\kappa}}{\partial x^\sigma}-\frac{\partial g_{\rho\sigma}}{\partial x^\kappa}\right)\right] \\
&\quad -\frac{\partial}{\partial x^\sigma}\left(g^{\mu\nu}g^{\rho\sigma}\frac{1}{2g}\frac{\partial g}{\partial x^\rho}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x^\lambda}\left(g^{\mu\nu}g^{\rho\sigma}g^{\lambda\kappa}\frac{\partial g_{\kappa\sigma}}{\partial x^\rho}+g^{\mu\nu}g^{\rho\sigma}g^{\lambda\kappa}\frac{\partial g_{\rho\kappa}}{\partial x^\sigma}-g^{\mu\nu}g^{\rho\sigma}g^{\lambda\kappa}\frac{\partial g_{\rho\sigma}}{\partial x^\kappa}\right) \\
&\quad -\frac{1}{2}\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(gg^{\mu\nu}g^{\rho\sigma}\right)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^\sigma\partial x^\rho}\left(g^{\mu\nu}g^{\rho\sigma}\right). \qquad (7.68)
\end{aligned}
$$

Using the formula in Eq. (5.57), we write

$$
\begin{aligned}
(g^{\mu\nu}R)_{\mathrm{LIF}} &= -\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left(g^{\mu\nu}g^{\rho\sigma}\frac{\partial g^{\lambda\kappa}}{\partial x^\rho}g_{\kappa\sigma} + g^{\mu\nu}g^{\rho\sigma}\frac{\partial g^{\lambda\kappa}}{\partial x^\sigma}g_{\rho\kappa} + g^{\mu\nu}g^{\lambda\kappa}\frac{1}{g}\frac{\partial g}{\partial x^\kappa}\right) \\
&\quad -\frac{1}{2}\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(g g^{\mu\nu}g^{\rho\sigma}\right)\right] + \frac{1}{2}\frac{\partial^2 g^{\mu\nu}}{\partial x^\sigma \partial x^\rho}g^{\rho\sigma} + \frac{1}{2}g^{\mu\nu}\frac{\partial^2 g^{\rho\sigma}}{\partial x^\sigma \partial x^\rho} \\
&= -\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left(g^{\mu\nu}\frac{\partial g^{\lambda\rho}}{\partial x^\rho} + g^{\mu\nu}\frac{\partial g^{\lambda\sigma}}{\partial x^\sigma} + g^{\mu\nu}g^{\lambda\kappa}\frac{1}{g}\frac{\partial g}{\partial x^\kappa}\right) \\
&\quad -\frac{1}{2}\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(g g^{\mu\nu}g^{\rho\sigma}\right)\right] + \frac{1}{2}\frac{\partial^2 g^{\mu\nu}}{\partial x^\sigma \partial x^\rho}g^{\rho\sigma} + \frac{1}{2}g^{\mu\nu}\frac{\partial^2 g^{\rho\sigma}}{\partial x^\sigma \partial x^\rho} \\
&= -g^{\mu\nu}\frac{\partial^2 g^{\lambda\rho}}{\partial x^\lambda \partial x^\rho} - \frac{1}{2}\frac{\partial}{\partial x^\lambda}\left[\frac{1}{g}\frac{\partial}{\partial x^\kappa}\left(g g^{\mu\nu}g^{\lambda\kappa}\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^\lambda \partial x^\kappa}\left(g^{\mu\nu}g^{\lambda\kappa}\right) \\
&\quad -\frac{1}{2}\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(g g^{\mu\nu}g^{\rho\sigma}\right)\right] + \frac{1}{2}\frac{\partial^2 g^{\mu\nu}}{\partial x^\sigma \partial x^\rho}g^{\rho\sigma} + \frac{1}{2}g^{\mu\nu}\frac{\partial^2 g^{\rho\sigma}}{\partial x^\sigma \partial x^\rho} \\
&= -\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(g g^{\mu\nu}g^{\rho\sigma}\right)\right] + \frac{\partial^2 g^{\mu\nu}}{\partial x^\sigma \partial x^\rho}g^{\rho\sigma} \, . \qquad (7.69)
\end{aligned}
$$

The matter energy-momentum tensor in the locally inertial reference frame turns out to be

$$
\begin{aligned}
T^{\mu\nu}_{\mathrm{LIF}} &= \frac{c^4}{8\pi G_{\mathrm{N}}}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{\mathrm{LIF}} \\
&= \frac{c^4}{8\pi G_{\mathrm{N}}}\left\{\frac{1}{2}\frac{\partial^2 g^{\mu\nu}}{\partial x^\lambda \partial x^\kappa}g^{\lambda\kappa} - \frac{1}{2}\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(g g^{\mu\rho}g^{\nu\sigma}\right)\right]\right. \\
&\qquad \left. + \frac{1}{2}\frac{\partial}{\partial x^\sigma}\left[\frac{1}{g}\frac{\partial}{\partial x^\rho}\left(g g^{\mu\nu}g^{\rho\sigma}\right)\right] - \frac{1}{2}\frac{\partial^2 g^{\mu\nu}}{\partial x^\sigma \partial x^\rho}g^{\rho\sigma}\right\} \\
&= \frac{\partial}{\partial x^\sigma}\left\{\frac{c^4}{16\pi G_{\mathrm{N}}}\frac{1}{(-g)}\frac{\partial}{\partial x^\rho}\left[(-g)\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right]\right\} \\
&= \frac{1}{(-g)}\frac{\partial}{\partial x^\sigma}\left\{\frac{c^4}{16\pi G_{\mathrm{N}}}\frac{\partial}{\partial x^\rho}\left[(-g)\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right]\right\} , \qquad (7.70)
\end{aligned}
$$

which we can rewrite as

$$
(-g)\, T^{\mu\nu}_{\mathrm{LIF}} = \frac{\partial}{\partial x^\sigma}\tau^{\mu\nu\sigma} \, , \qquad (7.71)
$$

where we have introduced

$$
\tau^{\mu\nu\sigma} = \frac{c^4}{16\pi G_{\mathrm{N}}}\frac{\partial}{\partial x^\rho}\left[(-g)\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right] . \qquad (7.72)
$$

Equation (7.71) has been obtained in a locally inertial reference frame. In a generic reference frame, the equality between the left and right hand sides does not hold. We will have another piece that we call $(-g)\, t^{\mu\nu}$

$$(-g)\,(T^{\mu\nu} + t^{\mu\nu}) = \frac{\partial}{\partial x^\sigma} \tau^{\mu\nu\sigma} \,. \tag{7.73}$$

By construction, $\partial_\mu \partial_\sigma \tau^{\mu\nu\sigma} = 0$, because $\tau^{\mu\nu\sigma}$ is antisymmetric in the indices μ and σ, and therefore

$$\frac{\partial}{\partial x^\mu} \left[(-g)\,(T^{\mu\nu} + t^{\mu\nu})\right] = 0 \,. \tag{7.74}$$

$t^{\mu\nu}$ is called the *pseudo-tensor of Landau–Lifshitz*. From the Einstein equations in a generic reference frame, we can obtain the expression of $t^{\mu\nu}$

$$\begin{aligned} t^{\mu\nu} = \frac{c^4}{16\pi G_\mathrm{N}} \Big[& \left(2\Gamma^\rho_{\kappa\lambda}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\kappa\sigma}\Gamma^\sigma_{\lambda\rho} - \Gamma^\rho_{\kappa\rho}\Gamma^\sigma_{\lambda\sigma}\right)\left(g^{\mu\kappa} g^{\nu\lambda} - g^{\mu\nu} g^{\kappa\lambda}\right) \\ & + g^{\mu\kappa} g^{\lambda\rho} \left(\Gamma^\nu_{\kappa\sigma}\Gamma^\sigma_{\lambda\rho} + \Gamma^\nu_{\lambda\rho}\Gamma^\sigma_{\kappa\sigma} - \Gamma^\nu_{\rho\sigma}\Gamma^\sigma_{\kappa\lambda} - \Gamma^\nu_{\kappa\lambda}\Gamma^\sigma_{\rho\sigma}\right) \\ & + g^{\nu\kappa} g^{\lambda\rho} \left(\Gamma^\mu_{\kappa\sigma}\Gamma^\sigma_{\lambda\rho} + \Gamma^\mu_{\lambda\rho}\Gamma^\sigma_{\kappa\sigma} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\kappa\lambda} - \Gamma^\mu_{\kappa\lambda}\Gamma^\sigma_{\sigma\rho}\right) \\ & + g^{\kappa\lambda} g^{\rho\sigma} \left(\Gamma^\mu_{\kappa\rho}\Gamma^\nu_{\lambda\sigma} - \Gamma^\mu_{\kappa\lambda}\Gamma^\nu_{\rho\sigma}\right) \Big] . \end{aligned} \tag{7.75}$$

$t^{\mu\nu}$ is not a tensor being a combination of Christoffel symbols that are not tensors. However, it transforms as a tensor under a linear coordinate transformation (hence the name "pseudo-tensor").

Problems

7.1 Let us consider the following action of a scalar field ϕ non-minimally coupled to gravity

$$S = -\frac{1}{c} \int \left[\frac{\hbar}{2} g^{\mu\nu} \left(\partial_\mu \phi\right)(\partial_\nu \phi) + \frac{1}{2}\frac{m^2 c^2}{\hbar}\phi^2 - \xi R \phi^2\right] \sqrt{-g}\, d^4x \,. \tag{7.76}$$

Evaluate the energy-momentum tensor of the scalar field.

7.2 Consider the action in Eq. (7.76). Write the equations of motion.

7.3 Rewrite the Einstein–Hilbert action in Eq. (7.24) in order to obtain the Einstein equations in (7.8) with a cosmological constant Λ.

Reference

1. L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, Oxford, 1980)