Chapter 4
Electromagnetism

In this chapter, we will revise the pre-relativistic theory of electromagnetic phenomena, namely the theory of electromagnetism formulated before the advent of special relativity that the reader is expected to know, and we will write a fully relativistic and manifestly covariant theory. For *manifestly covariant* (or, equivalently, *manifestly Lorentz-invariant*) formulation, we mean that the theory is written in terms of tensors; that is, the laws of physics are written as equalities between two tensors or equalities to zero of a tensor. Since we know how tensors change when we move from a Cartesian inertial reference frame to another Cartesian inertial reference frame, it is straightforward to realize that the equations are invariant under Lorentz transformations. In this and the following chapters, we will always employ Gaussian units to describe electromagnetic phenomena. In the present chapter, unless stated otherwise, we will use Cartesian coordinates; the generalization to arbitrary coordinate systems will be discussed in Sect. 6.7.

The basic equations governing electromagnetic phenomena are the Lorentz force law and Maxwell's equations. The equation of motion of a particle of mass m and electric charge e moving in an electromagnetic field is

$$m\ddot{\mathbf{x}} = e\mathbf{E} + \frac{e}{c}\dot{\mathbf{x}} \times \mathbf{B}\,, \tag{4.1}$$

where c is the speed of light and $\mathbf{E}$ and $\mathbf{B}$ are, respectively, the electric and magnetic fields. Equation (4.1) is just Newton's Second Law, $m\ddot{\mathbf{x}} = \mathbf{F}$, when $\mathbf{F}$ is the Lorentz force. The equations of motion (field equations) for the electric and magnetic fields are the Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho\,, \tag{4.2}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{4.3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,, \tag{4.4}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\,, \tag{4.5}$$

C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4_4

where ρ is the electric charge density and $\mathbf{J}$ is the electric current density.

From Eq. (4.3), we see that we can introduce the *vector potential* $\mathbf{A}$ defined as

$$\mathbf{B} = \nabla \times \mathbf{A} \,. \tag{4.6}$$

Indeed the divergence of the curl of any vector field vanishes (see Appendix B.3). If we plug Eq. (4.6) into Eq. (4.4), we find

$$\nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = 0 \,. \tag{4.7}$$

We can thus introduce the *scalar potential* ϕ as

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi \,, \tag{4.8}$$

because $\nabla \times (\nabla \phi) = 0$ (see Appendix B.3). The electric field can thus be written in terms of the scalar potential ϕ and the vector potential $\mathbf{A}$

$$\mathbf{E} = -\nabla \phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \,. \tag{4.9}$$

4.1 Action

Now we want to write the action of a system composed of a point-like particle of mass m and electric charge e and an electromagnetic field. The action should be made of three parts: the action of the free point-like particle, $S_{\rm m}$, the action of the electromagnetic field, $S_{\rm em}$, and the action describing the interaction between the particle and the electromagnetic field, $S_{\rm int}$. The total action should thus have the following form

$$S = S_{\rm m} + S_{\rm int} + S_{\rm em} \,. \tag{4.10}$$

In the absence of the particle, we would be left with $S_{\rm em}$. In the absence of the electromagnetic field, we would be left with $S_{\rm m}$. If we turn the value of the electric charge of the particle off, we eliminate $S_{\rm int}$, and the particle and the electromagnetic field do not interact, so the equations of motion of the particle are independent of the electromagnetic field and, vice versa, those of the electromagnetic field do not depend on the particle.

Once we apply the Least Action Principle to the action in (4.10) we have to recover the Maxwell equations and the relativistic version of the Lorentz force law (4.1). As we will see in the next sections, this can be achieved with $S_{\rm m}$, $S_{\rm int}$, and $S_{\rm em}$ given, respectively, by

$$S_{\rm m} = -mc \int_\Gamma \sqrt{-ds^2}\,,$$
$$S_{\rm int} = \frac{e}{c}\int_\Gamma A_\mu dx^\mu\,,$$
$$S_{\rm em} = -\frac{1}{16\pi c}\int_\Omega F^{\mu\nu}F_{\mu\nu}\, d^4\Omega\,. \tag{4.11}$$

$F_{\mu\nu}$ is the *Faraday tensor* defined as[1]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{4.13}$$

and A_μ is the *4-vector potential* of the electromagnetic field. The latter is a 4-vector in which the time component is the scalar potential ϕ and the space components are the components of the 3-vector potential $\mathbf{A}$

$$A^\mu = (\phi, \mathbf{A})\,. \tag{4.14}$$

From Eqs. (4.6), (4.9), (4.13), and (4.14), we can write the Faraday tensor in terms of the electric and magnetic fields. In Cartesian coordinates (ct, x, y, z), we have

$$||F_{\mu\nu}|| = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}\,, \quad ||F^{\mu\nu}|| = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}\,, \tag{4.15}$$

where E_i and B_i are the i components of, respectively, the electric and the magnetic fields. We write them with lower indices, but they are not the spatial components of a dual vector. We remind that $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}$ and note that the ti and it components change sign, while the ij components are unchanged, when we pass from $F_{\mu\nu}$ to $F^{\mu\nu}$.

The electric and magnetic fields can be written in terms of the Faraday tensor as follows

$$E_i = F_{it}\,, \quad B_i = \frac{1}{2}\varepsilon_{ijk}F^{jk}\,, \tag{4.16}$$

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[1] As it will be more clear later (see Sect. 6.7), the definition (4.13) is valid in any coordinate system, i.e. both Cartesian and non-Cartesian systems.

If the spacetime metric has signature $(+---)$, the Faraday tensor is still defined as in Eq. (4.13), but now the expression of $F_{\mu\nu}$ in Cartesian coordinates is

$$||F_{\mu\nu}|| = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}\,. \tag{4.12}$$

where ε_{ijk} is the *Levi–Civita symbol* (see Appendix B.2). The space-space components of the Faraday tensor can be written in terms of the magnetic field

$$F^{ij} = \varepsilon^{ijk} B_k \,. \tag{4.17}$$

Indeed

$$\begin{aligned} \varepsilon^{ijk} B_k &= \varepsilon^{kij} B_k = \frac{1}{2}\varepsilon^{kij}\varepsilon_{klm} F^{lm} = \frac{1}{2}\left(\delta^i_l \delta^j_m - \delta^i_m \delta^j_l\right) F^{lm} \\ &= \frac{1}{2}\left(F^{ij} - F^{ji}\right) = F^{ij} \,. \end{aligned} \tag{4.18}$$

If we write the electric charge e as the integral of the electric charge density over the 3-volume, we can write the interaction term in the action, $S_{\rm int}$, as follows (note that $d^4\Omega = c dt d^3V$)

$$\begin{aligned} S_{\rm int} &= \frac{1}{c}\int_V \rho d^3V \int_\Gamma A_\mu dx^\mu = \frac{1}{c}\int_V \rho d^3V \int_{t_1}^{t_2} \frac{dx^\mu}{dt} A_\mu \, dt \\ &= \frac{1}{c^2}\int_\Omega J^\mu A_\mu \, d^4\Omega \,, \end{aligned} \tag{4.19}$$

where J^μ is the *current 4-vector*[2]

$$J^\mu = (\rho c, \rho \mathbf{v}) \,. \tag{4.21}$$

In the case of a system with many electrically charged particles, the total action can be written as

$$S = -\sum_i m_i c \int_{\Gamma_i} \sqrt{-ds^2} - \int_\Omega \left(\frac{1}{16\pi c} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c^2} J^\mu A_\mu\right) d^4\Omega \,. \tag{4.22}$$

In the next sections, we will see that the action in (4.10), with $S_{\rm m}$, $S_{\rm int}$, and $S_{\rm em}$ given by Eq. (4.11) and A^μ given by Eq. (4.14), provides the relativistic version of the Lorentz force law (4.1) and the correct Maxwell equations. As we have already emphasized in the previous chapters, there is no fundamental recipe to obtain the action of a certain physical system. A specific action represents a specific theory. If the theoretical predictions agree well with observations, we have the right theory

[2] Note that the electric charge density ρ can be written as $\rho = \delta Q/\delta V$, where δQ is the electric charge in the infinitesimal volume δV. The latter depends on the reference frame and can be written in terms of the proper infinitesimal volume $\delta V = \delta V_0/\gamma$, see Eq. (2.44). Now $\delta Q/\delta V_0$ is an invariant and the current 4-vector can be written as

$$J^\mu = \frac{\delta Q}{\delta V_0}(\gamma c, \gamma \mathbf{v}) = \frac{\delta Q}{\delta V_0} u^\mu \,, \tag{4.20}$$

where u^μ is the 4-velocity. So $J^\mu \propto u^\mu$ and is clearly a 4-vector.

up to when we find a phenomenon that cannot be explained within our model. For the moment, the action above is our current theory for the description of a charged particle in an electromagnetic field.

4.2 Motion of a Charged Particle

As we did in Sect. 3.2, we can either employ the standard Lagrangian formalism of Newtonian mechanics with a time coordinate and a space, or a fully 4-dimensional approach in which all the spacetime coordinates are treated in the same way and we have some parameter (e.g. the particle proper time) to parametrize the particle trajectory.

4.2.1 3-Dimensional Formalism

The motion of the particle in a (pre-determined) electromagnetic field is described by the action of the free particle and by the interaction term. If the trajectory of the particle is parametrized by the time coordinate t, we have

$$S = \int_{\Gamma} \left(L_{\rm m} + L_{\rm int} \right) dt \, , \tag{4.23}$$

where the Lagrangians of the free particle and of the interaction term between the particle and the electromagnetic field are, respectively,

$$\begin{aligned} L_{\rm m} &= -mc\sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}} \, , \\ L_{\rm int} &= -e\phi + \frac{e}{c}\mathbf{A} \cdot \dot{\mathbf{x}} \, . \end{aligned} \tag{4.24}$$

Here the dot $\dot{}$ indicates the derivative with respect to the coordinate t, so $\dot{\mathbf{x}} = d\mathbf{x}/dt$ where $\mathbf{x} = (x, y, z)$.

The 3-momentum is[3]

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{m\dot{\mathbf{x}}}{\sqrt{1 - \dot{\mathbf{x}}^2/c^2}} + \frac{e}{c}\mathbf{A} \, , \tag{4.26}$$

[3] As already pointed out in Sect. 3.2, the conjugate 3-momentum is

$$\mathbf{P}^* = \frac{\partial L}{\partial \dot{\mathbf{x}}} \, . \tag{4.25}$$

In Cartesian coordinates, the metric tensor is δ_{ij}, and therefore $\mathbf{P}^* = \mathbf{P}$, where $\mathbf{P}$ is the 3-momentum.

which can also be written as

$$\mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A}\,, \tag{4.27}$$

where $\mathbf{p} = m\gamma\dot{\mathbf{x}}$ is the 3-momentum of the free particle (see Sect. 3.2). The energy of the particle is

$$E = \frac{\partial L}{\partial \dot{\mathbf{x}}}\dot{\mathbf{x}} - L = \frac{mc^2}{\sqrt{1 - \dot{\mathbf{x}}^2/c^2}} + e\phi\,. \tag{4.28}$$

Let us now derive the equations of motion. The first term in the Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{d}{dt}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right) = \frac{d\mathbf{p}}{dt} + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{e}{c}\left(\dot{\mathbf{x}} \cdot \nabla\right)\mathbf{A}\,. \tag{4.29}$$

The second term in the Euler-Lagrange equation is

$$\frac{\partial L}{\partial \mathbf{x}} = -e\nabla\phi + \frac{e}{c}\nabla\left(\mathbf{A} \cdot \dot{\mathbf{x}}\right)\,. \tag{4.30}$$

From the identity

$$\nabla\left(\mathbf{V} \cdot \mathbf{W}\right) = \left(\mathbf{W} \cdot \nabla\right)\mathbf{V} + \left(\mathbf{V} \cdot \nabla\right)\mathbf{W} + \mathbf{W} \times \left(\nabla \times \mathbf{V}\right) + \mathbf{V} \times \left(\nabla \times \mathbf{W}\right)\,, \tag{4.31}$$

we can rewrite Eq. (4.30) as

$$\frac{\partial L}{\partial \mathbf{x}} = -e\nabla\phi + \frac{e}{c}\left(\dot{\mathbf{x}} \cdot \nabla\right)\mathbf{A} + \frac{e}{c}\dot{\mathbf{x}} \times \left(\nabla \times \mathbf{A}\right)\,, \tag{4.32}$$

because the differentiation with respect to $\mathbf{x}$ is carried out for constant $\dot{\mathbf{x}}$. Eventually, we get the following equation of motion

$$\frac{d\mathbf{p}}{dt} = -\frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} - e\nabla\phi + \frac{e}{c}\dot{\mathbf{x}} \times \left(\nabla \times \mathbf{A}\right)\,. \tag{4.33}$$

In terms of the electric and magnetic fields, we have

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}\dot{\mathbf{x}} \times \mathbf{B}\,, \tag{4.34}$$

In the non-relativistic limit, $\mathbf{p} = m\dot{\mathbf{x}}$, and we recover Eq. (4.1).

4.2.2 4-Dimensional Formalism

Let us now parametrize the particle trajectory with the particle proper time τ. The action reads

$$S = \int_{\Gamma} \left(L_{\rm m} + L_{\rm int}\right) d\tau \,, \tag{4.35}$$

where τ is the proper time of the particle, the Lagrangian terms are

$$\begin{aligned} L_{\rm m} &= \frac{1}{2} m \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \,, \\ L_{\rm int} &= \frac{e}{c} A_\mu \dot{x}^\mu \,, \end{aligned} \tag{4.36}$$

and now the dot $\dot{}$ indicates the derivative with respect to τ, i.e. $\dot{x}^\mu = dx^\mu/d\tau$. From the Euler-Lagrange equations, we find

$$\begin{aligned} & \frac{d}{d\tau}\frac{\partial L_{\rm m}}{\partial \dot{x}^\mu} + \frac{d}{d\tau}\frac{\partial L_{\rm int}}{\partial \dot{x}^\mu} - \frac{\partial L_{\rm int}}{\partial x^\mu} = 0 \,, \\ & \frac{d}{d\tau}\left(m \eta_{\mu\nu} \dot{x}^\nu\right) + \frac{d}{d\tau}\left(\frac{e}{c} A_\mu\right) - \frac{e}{c}\frac{\partial A_\nu}{\partial x^\mu}\dot{x}^\nu = 0 \,, \\ & m \ddot{x}_\mu + \frac{e}{c}\frac{\partial A_\mu}{\partial x^\nu}\dot{x}^\nu - \frac{e}{c}\frac{\partial A_\nu}{\partial x^\mu}\dot{x}^\nu = 0 \,, \\ & m \ddot{x}_\mu - \frac{e}{c} F_{\mu\nu} \dot{x}^\nu = 0 \,. \end{aligned} \tag{4.37}$$

The equations of motion of the particle are

$$\ddot{x}^\mu = \frac{e}{mc} F^{\mu\nu} \dot{x}_\nu \,. \tag{4.38}$$

If we do not use Cartesian coordinates, in Eq. (4.36) we have $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$, $L_{\rm m}$ provides the geodesic equations, and Eq. (4.38) reads

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = \frac{e}{mc} F^{\mu\nu} \dot{x}_\nu \,. \tag{4.39}$$

4.3 Maxwell's Equations in Covariant Form

4.3.1 Homogeneous Maxwell's Equations

The covariant form of the homogeneous Maxwell equations, Eqs. (4.3) and (4.4), is

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0\,. \tag{4.40}$$

Equation (4.40) directly follows from the definition of $F_{\mu\nu}$. If we plug Eq. (4.13) into Eq. (4.40) we find

$$\partial_\mu \left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) + \partial_\nu \left(\partial_\rho A_\mu - \partial_\mu A_\rho\right) + \partial_\rho \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = 0\,. \tag{4.41}$$

Let us now check that Eq. (4.40) is equivalent to Eqs. (4.3) and (4.4). Equation (4.40) is non-trivial only when there are no repeated indices. So we have four independent equations, which correspond to the cases in which (μ, ν, ρ) is (t, x, y), (t, x, z), (t, y, z), and (x, y, z). Permutations of these indices provide the same equations.

For $(\mu, \nu, \rho) = (t, x, y)$ we have

$$\frac{1}{c}\frac{\partial F_{xy}}{\partial t} + \frac{\partial F_{yt}}{\partial x} + \frac{\partial F_{tx}}{\partial y} = 0\,. \tag{4.42}$$

Employing the relations in (4.16), we replace the components of the Faraday tensor with those of the electric and magnetic fields

$$\frac{1}{c}\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0\,. \tag{4.43}$$

Equation (4.43) can be rewritten with the Levi–Civita symbol ε_{ijk} as

$$\frac{1}{c}\frac{\partial B_z}{\partial t} + \varepsilon_{zjk}\frac{\partial E_k}{\partial x^j} = 0\,. \tag{4.44}$$

This is the z component of Eq. (4.4). For $(\mu, \nu, \rho) = (t, x, z)$, we recover the y component of Eq. (4.4), and for $(\mu, \nu, \rho) = (t, y, z)$ we get the x component.

For $(\mu, \nu, \rho) = (x, y, z)$ (no t component), we have

$$\frac{\partial F_{yz}}{\partial x} + \frac{\partial F_{zx}}{\partial y} + \frac{\partial F_{xy}}{\partial z} = 0\,. \tag{4.45}$$

Employing Eq. (4.16), we find

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial B_i}{\partial x^i} = 0\,, \tag{4.46}$$

and we recover Eq. (4.3).

4.3.2 Inhomogeneous Maxwell's Equations

The covariant form of the inhomogeneous Maxwell equations, Eqs. (4.2) and (4.5), is[4]

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} J^\nu \,. \tag{4.48}$$

Equation (4.48) is the field equation from the Least Action Principle, so it follows from

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu A_\nu\right)} - \frac{\partial \mathscr{L}}{\partial A_\nu} = 0 \,. \tag{4.49}$$

$S_{\rm m}$ does not provide any contribution, so we only have to consider

$$S_{\rm em} + S_{\rm int} = \frac{1}{c} \int_\Omega \left(\mathscr{L}_{\rm em} + \mathscr{L}_{\rm in}\right) d^4\Omega \,, \tag{4.50}$$

where

$$\mathscr{L}_{\rm em} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \,, \quad \mathscr{L}_{\rm int} = \frac{1}{c} J^\mu A_\mu \,. \tag{4.51}$$

$\mathscr{L}_{\rm em}$ only depends on $\partial_\mu A_\nu$ and we have

$$\begin{aligned}
\frac{\partial \mathscr{L}_{\rm em}}{\partial \left(\partial_\mu A_\nu\right)} &= -\frac{\partial}{\partial \left(\partial_\mu A_\nu\right)} \frac{1}{16\pi} \left[\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) \left(\partial_\tau A_\upsilon - \partial_\upsilon A_\tau\right) \eta^{\rho\tau} \eta^{\sigma\upsilon}\right] \\
&= -\frac{1}{16\pi} \left[\left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho\right) F^{\rho\sigma} + F^{\tau\upsilon} \left(\delta^\mu_\tau \delta^\nu_\upsilon - \delta^\mu_\upsilon \delta^\nu_\tau\right)\right] \\
&= -\frac{1}{16\pi} \left(F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu}\right) \,.
\end{aligned} \tag{4.52}$$

Since the Faraday tensor is antisymmetric, i.e. $F^{\mu\nu} = -F^{\nu\mu}$, we have

$$\frac{\partial \mathscr{L}_{\rm em}}{\partial \left(\partial_\mu A_\nu\right)} = -\frac{1}{4\pi} F^{\mu\nu} \,. \tag{4.53}$$

[4] Since $F^{\mu\nu}$ is antisymmetric, we can also write

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \,. \tag{4.47}$$

$\mathscr{L}_{\rm int}$ depends on A_ν and is independent of $\partial_\mu A_\nu$. We have

$$\frac{\partial \mathscr{L}_{\rm int}}{\partial A_\nu} = \frac{1}{c} J^\nu \, . \tag{4.54}$$

When we combine Eqs. (4.53) and (4.54), we find Eq. (4.48).

Let us now check that we recover the inhomogeneous Maxwell equations (4.2) and (4.5). For $\nu = t$ we have

$$\partial_\mu F^{\mu t} = \partial_i F^{it} = -\partial_i E_i = -\frac{4\pi}{c} J^t \, , \tag{4.55}$$

and it is Eq. (4.2) because $J^t = \rho c$.

For $\nu = i$ we have

$$\partial_\mu F^{\mu i} = \frac{1}{c} \partial_t F^{ti} + \partial_j F^{ji} = -\frac{4\pi}{c} J^i \, . \tag{4.56}$$

We replace the components of the Faraday tensor with those of the electric and magnetic fields and we find

$$\frac{1}{c} \frac{\partial E_i}{\partial t} - \varepsilon^{ijk} \frac{\partial B_k}{\partial x^j} = -\frac{4\pi}{c} J^i \, . \tag{4.57}$$

For a 3-dimensional vector **V**, we have

$$(\nabla \times \mathbf{V})^i = \varepsilon^{ijk} \partial_j V_k \, , \tag{4.58}$$

and we thus recover Eq. (4.5) for $i = x$, y, and z.

Let us note that Eq. (4.48) implies the conservation of the electric current. If we apply the differential operator ∂_ν to both sides of this equation, we find

$$\partial_\nu \partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} \partial_\nu J^\nu \, . \tag{4.59}$$

Since $F^{\mu\nu}$ is an antisymmetric tensor, $\partial_\nu \partial_\mu F^{\mu\nu} = 0$, and thus we find the conservation of the electric current

$$\partial_\mu J^\mu = 0 \, . \tag{4.60}$$

Equation (4.60) is a continuity equation, as we can easily see if we rewrite it as

$$\frac{1}{c} \frac{\partial J^t}{\partial t} = -\frac{\partial J^i}{\partial x^i} \, , \tag{4.61}$$

and we integrate both sides over the 3-dimensional space volume

$$\frac{1}{c}\frac{d}{dt}\int_V J^t \, d^3V = -\int_V \left(\partial_i J^i\right) \, d^3V = -\int_\Sigma J^i d^2\sigma_i \, , \tag{4.62}$$

where in the last passage we applied Gauss's theorem. The total electric charge in the volume V is

$$Q = \int_V J^t \, d^3V \, . \tag{4.63}$$

Equation (4.62) thus tells us that any variation of the total electric charge can be calculated from the outflow/inflow of the electric current at the boundary of the region. If there is no outflow/inflow of the electric current, we have the conservation of the electric charge in that region

$$\frac{d}{dt}Q = 0 \, . \tag{4.64}$$

4.4 Gauge Invariance

If we plug Eq. (4.13) into Eq. (4.48), we find

$$\Box A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = -\frac{4\pi}{c} J^\mu \, . \tag{4.65}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembertian. Equation (4.65) is invariant under the following transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \, , \tag{4.66}$$

where $\Lambda = \Lambda(x)$ is a generic function. Indeed we have

$$\begin{aligned} \Box A'^\mu - \Box\partial^\mu \Lambda - \partial^\mu \left(\partial_\nu A'^\nu - \Box\Lambda\right) &= -\frac{4\pi}{c} J^\mu \, , \\ \Box A'^\mu - \partial^\mu \left(\partial_\nu A'^\nu\right) &= -\frac{4\pi}{c} J^\mu \, , \end{aligned} \tag{4.67}$$

which is the same as Eq. (4.65) with A'_μ replacing A_μ. We can thus always choose A_μ such that it satisfies the following condition

$$\partial_\mu A^\mu = 0 \, . \tag{4.68}$$

If our initial A_μ does not meet this condition, we can perform the transformation (4.66) such that

$$\Box \Lambda = \partial_\mu A^\mu \, , \tag{4.69}$$

and our new equation is

$$\Box A^\mu = -\frac{4\pi}{c} J^\mu \, . \tag{4.70}$$

The condition (4.68) is called the *Lorentz gauge* and it is quite a common choice because it sometimes simplifies the calculations. The formal solution of Eq. (4.70) is

$$A^\mu = \frac{1}{c} \int d^3\mathbf{x}' \, \frac{J^\mu(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, . \tag{4.71}$$

4.5 Energy-Momentum Tensor of the Electromagnetic Field

From Eq. (3.88), we can compute the energy-momentum tensor of the electromagnetic field

$$\begin{aligned} T^\nu_\mu &= -\frac{\partial \mathscr{L}_{\rm em}}{\partial \left(\partial_\nu A_\rho\right)} \left(\partial_\mu A_\rho\right) + \delta^\nu_\mu \mathscr{L}_{\rm em} \\ &= \frac{1}{4\pi} F^{\nu\rho} \left(\partial_\mu A_\rho\right) - \frac{1}{16\pi} \delta^\nu_\mu F^{\rho\sigma} F_{\rho\sigma} \, , \end{aligned} \tag{4.72}$$

which we can rewrite as

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\nu\rho} \left(\partial^\mu A_\rho\right) - \frac{1}{16\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \, . \tag{4.73}$$

This tensor is not symmetric. It can be made symmetric as shown in Sect. 3.9

$$T^{\mu\nu} \to T^{\mu\nu} - \frac{1}{4\pi} \frac{\partial}{\partial x^\rho} \left(A^\mu F^{\nu\rho}\right) \, , \tag{4.74}$$

where $A^\mu F^{\nu\rho} = -A^\mu F^{\rho\nu}$. Let us note that

$$\frac{1}{4\pi} \frac{\partial}{\partial x^\rho} \left(A^\mu F^{\nu\rho}\right) = \frac{1}{4\pi} F^{\nu\rho} \left(\partial_\rho A^\mu\right) \, , \tag{4.75}$$

because $\partial_\rho F^{\nu\rho} = 0$ in the absence of electric currents. The energy-momentum tensor of the electromagnetic field thus becomes

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\nu\rho} F^\mu{}_\rho - \frac{1}{16\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \, , \tag{4.76}$$

and it is symmetric, so we can also write

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\mu\rho} F^{\nu}_{\ \ \rho} - \frac{1}{16\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \, . \tag{4.77}$$

For the moment, Eq. (4.74) may look an *ad hoc* recipe. We will see in Sect. 7.4 that it is possible to define the energy-momentum tensor without ambiguities.

Lastly, it is worth noting that the energy-momentum tensor of the electromagnetic field has vanishing trace

$$\begin{aligned} T^{\mu}_{\mu} &= \frac{1}{4\pi} F^{\mu\rho} F_{\mu\rho} - \frac{1}{16\pi} \delta^{\mu}_{\mu} F^{\rho\sigma} F_{\rho\sigma} \\ &= \frac{1}{4\pi} F^{\mu\rho} F_{\mu\rho} - \frac{1}{4\pi} F^{\rho\sigma} F_{\rho\sigma} = 0 \, . \end{aligned} \tag{4.78}$$

4.6 Examples

4.6.1 Motion of a Charged Particle in a Constant Uniform Electric Field

Let us consider a particle of mass m and electric charge e in a constant uniform electric field $\mathbf{E} = (E, 0, 0)$. At the time $t = 0$, the position of the particle is $\mathbf{x} = (x_0, y_0, z_0)$ and its velocity is $\dot{\mathbf{x}} = (\dot{x}_0, \dot{y}_0, 0)$.

In the non-relativistic theory, the equation of motion is given by Eq. (4.1). For the x component we have

$$m\ddot{x} = eE \quad \rightarrow \quad \dot{x}(t) = \dot{x}_0 + \frac{eE}{m} t \, , \quad x(t) = x_0 + \dot{x}_0 t + \frac{eE}{2m} t^2 \, . \tag{4.79}$$

For the y component we have

$$m\ddot{y} = 0 \quad \rightarrow \quad \dot{y}(t) = \dot{y}_0 \, , \quad y(t) = y_0 + \dot{y}_0 t \, . \tag{4.80}$$

From Eq. (4.80) we can write t in terms of y, y_0, and $\dot{y}_0$. We plug this expression into Eq. (4.79) and we find

$$x(y) = \left(\frac{eE}{2m\dot{y}_0^2} \right) y^2 + \left(\frac{\dot{x}_0}{\dot{y}_0} - \frac{eEy_0}{m\dot{y}_0^2} \right) y + \left(x_0 - \frac{\dot{x}_0 y_0}{\dot{y}_0} + \frac{eEy_0^2}{2m\dot{y}_0^2} \right) \, . \tag{4.81}$$

This is the equation of a parabola.

In the relativistic theory, the equation governing the motion of the particle is (4.34). Let us assume that at the time $t = 0$ the position of the particle is $\mathbf{x} = (x_0, y_0, z_0)$ and its 3-momentum is $\mathbf{p} = (q_x, q_y, 0)$. From Eq. (4.34) we have

$$\dot{p}_x = eE \rightarrow p_x = q_x + eEt\,, \quad \dot{p}_y = 0 \rightarrow p_y = q_y\,. \tag{4.82}$$

The particle energy is thus (here we use ε to indicate the particle energy because E is the electric field)

$$\varepsilon = \sqrt{m^2c^4 + p_x^2c^2 + p_y^2c^2}\,, \tag{4.83}$$

as found in Sect. 3.2. The 3-velocity is $\dot{\mathbf{x}} = \mathbf{p}c^2/\varepsilon$. For the x component we have

$$\begin{aligned} \dot{x} &= \frac{p_x c^2}{\varepsilon} = \frac{q_x c + eEct}{\sqrt{m^2c^2 + (q_x + eEt)^2 + q_y^2}} \\ \rightarrow \quad x(t) &= x_0 + \frac{\sqrt{m^2c^4 + (q_x + eEt)^2\, c^2 + q_y^2c^2}}{eE}\,. \end{aligned} \tag{4.84}$$

For the y component we have

$$\begin{aligned} \dot{y} &= \frac{p_y c^2}{\varepsilon} = \frac{q_y c}{\sqrt{m^2c^2 + (q_x + eEt)^2 + q_y^2}} \\ \rightarrow y(t) &= y_0 + \frac{q_y c}{eE} \ln\left[q_x + eEt + \sqrt{m^2c^2 + (q_x + eEt)^2 + q_y^2}\right]. \end{aligned} \tag{4.85}$$

Let us note the difference between the non-relativistic and relativistic theories. In the non-relativistic theory, the velocity diverges

$$\lim_{t\rightarrow\infty} \dot{x}(t) = \infty\,, \quad \lim_{t\rightarrow\infty} \dot{y}(t) = \dot{y}_0\,. \tag{4.86}$$

In the relativistic theory, the velocity of the charged particle asymptotically tends to the speed of light

$$\lim_{t\rightarrow\infty} \dot{x}(t) = c\,, \quad \lim_{t\rightarrow\infty} \dot{y}(t) = 0\,, \tag{4.87}$$

but it never reaches it

$$\dot{x}^2 + \dot{y}^2 = c^2\left[\frac{(q_x + eEt)^2 + q_y^2}{m^2c^2 + (q_x + eEt)^2 + q_y^2}\right] < c^2. \tag{4.88}$$

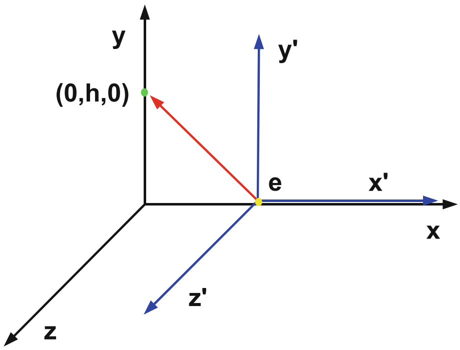

Fig. 4.1 The particle with electric charge e is moving with 3-velocity $\dot{\mathbf{x}} = (v, 0, 0)$ with respect to the reference frame with the Cartesian coordinates (x, y, z). In the reference frame with the Cartesian coordinates (x', y', z'), the particle is at rest at the point $\mathbf{x}' = (0, 0, 0)$. We want to calculate the electric and magnetic fields at the point $\mathbf{x} = (0, h, 0)$ measured by the observer with the Cartesian coordinates (x, y, z)

4.6.2 Electromagnetic Field Generated by a Charged Particle

Let us consider a particle with electric charge e. In the reference frame with the Cartesian coordinates (x, y, z), the particle has 3-velocity $\dot{\mathbf{x}} = (v, 0, 0)$. We want to calculate the electric and magnetic fields at the point $\mathbf{x} = (0, h, 0)$. The system is sketched in Fig. 4.1.

In the system with the Cartesian coordinates (x', y', z'), the particle is at rest at the point $\mathbf{x}' = (0, 0, 0)$. In this coordinate system, it is straightforward to write the intensities of the electric and magnetic fields. The magnetic field vanishes, because there is no electric current. The electric field at every point is simply that of a static charge. So we have

$$\mathbf{E}' = \frac{e}{r'^3}\mathbf{r}' = \frac{e}{\left(h^2 + v^2 t'^2\right)^{3/2}} \begin{pmatrix} -vt' \\ h \\ 0 \end{pmatrix} , \quad \mathbf{B}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{4.89}$$

where $r' = \sqrt{h^2 + v^2 t'^2}$ is the distance between the particle and the point where we want to evaluate the electric and magnetic fields. The Faraday tensor in the reference frame (x', y', z') is thus

$$||F'^{\mu\nu}|| = \begin{pmatrix} 0 & E'_x & E'_y & 0 \\ -E'_x & 0 & 0 & 0 \\ -E'_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.90}$$

The Faraday tensor in the reference frame (x, y, z) can be obtained with the coordinate transformation

$$F^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F'^{\rho\sigma}\,, \tag{4.91}$$

where $\Lambda^{\mu}{}_{\rho}$ is the Lorentz boost

$$||\Lambda^{\mu}{}_{\nu}|| = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.92}$$

Since $F^{\mu\nu}$ is antisymmetric, the diagonal components vanish and we have only six independent components. Let us start from F^{tx}. We have

$$\begin{aligned} F^{tx} &= \Lambda^{t}{}_{\rho}\Lambda^{x}{}_{\sigma}F'^{\rho\sigma} = \Lambda^{t}{}_{t}\Lambda^{x}{}_{x}F'^{tx} + \Lambda^{t}{}_{x}\Lambda^{x}{}_{t}F'^{xt} \\ &= \gamma^2 E'_x - \gamma^2\beta^2 E'_x = E'_x\,, \end{aligned} \tag{4.93}$$

because $1 - \beta^2 = 1/\gamma^2$. For F^{ty}, we have

$$F^{ty} = \Lambda^{t}{}_{\rho}\Lambda^{y}{}_{\sigma}F'^{\rho\sigma} = \Lambda^{t}{}_{\rho}\Lambda^{y}{}_{y}F'^{\rho y} = \Lambda^{t}{}_{t}\Lambda^{y}{}_{y}F'^{ty} = \gamma E'_y\,, \tag{4.94}$$

because the only non-vanishing $\Lambda^{y}{}_{\rho}$ is $\Lambda^{y}{}_{y} = 1$, and then the only non-vanishing $F'^{\rho y}$ is F'^{ty}. For F^{xy}, we have

$$F^{xy} = \Lambda^{x}{}_{\rho}\Lambda^{y}{}_{\sigma}F'^{\rho\sigma} = \Lambda^{x}{}_{\rho}\Lambda^{y}{}_{y}F'^{\rho y} = \Lambda^{x}{}_{t}\Lambda^{y}{}_{y}F'^{ty} = \gamma\beta E'_y\,, \tag{4.95}$$

because, again, the only non-vanishing $\Lambda^{y}{}_{\rho}$ is $\Lambda^{y}{}_{y} = 1$, and then the only non-vanishing $F'^{\rho y}$ is F'^{ty}. Lastly, all $F^{\mu z}$s vanish, because $F^{\mu z} = \Lambda^{\mu}{}_{\rho}\Lambda^{z}{}_{\sigma}F'^{\rho\sigma}$, the only non-vanishing $\Lambda^{z}{}_{\sigma}$ is $\Lambda^{z}{}_{z} = 1$, but $F'^{\rho z} = 0$.

In the end, we have

$$||F^{\mu\nu}|| = \begin{pmatrix} 0 & E'_x & \gamma E'_y & 0 \\ -E'_x & 0 & \gamma\beta E'_y & 0 \\ -\gamma E'_y & -\gamma\beta E'_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.96}$$

Since we are considering the point $\mathbf{x} = (0, h, 0)$, $t' = \gamma t$ (because $x = 0$), and the electric and magnetic fields measured in the reference frame (x, y, z) can be written as

$$\mathbf{E} = \frac{e\gamma}{\left(h^2 + v^2\gamma^2 t^2\right)^{3/2}}\begin{pmatrix} -vt \\ h \\ 0 \end{pmatrix}, \quad \mathbf{B} = \frac{e\gamma\beta h}{\left(h^2 + v^2\gamma^2 t^2\right)^{3/2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.97}$$

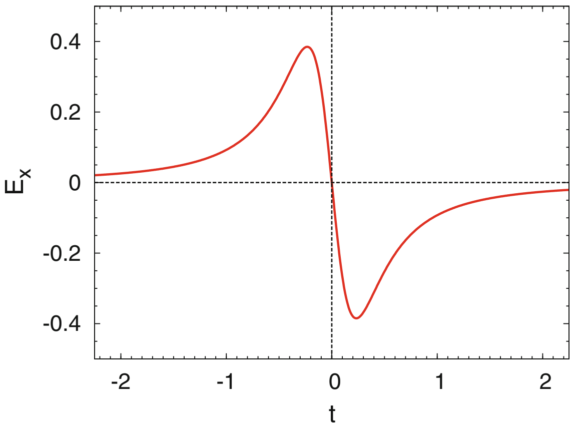

Fig. 4.2 E_x as a function of time at the position $(0, h, 0)$ for $\beta = 0.95$. Units in which $e = h = 1$ are used. See the text for more details

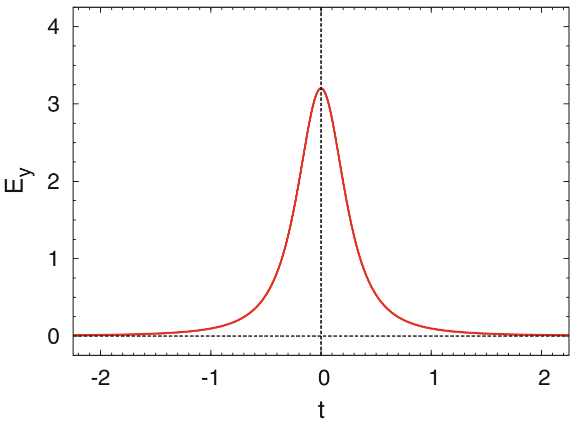

Fig. 4.3 As in Figure 4.2 for E_y

The x component of the electric field in the reference frame (x, y, z) at the point $\mathbf{x} = (0, h, 0)$ is shown in Fig. 4.2 as a function of the time t. The maximum value of E_x is

$$(E_x)_{\max} = \frac{2}{3\sqrt{3}} \frac{e}{h^2}, \tag{4.98}$$

which is reached at the time $t = h/(\sqrt{2}v\gamma)$. As shown in Fig. 4.3, the y component of the electric field looks like a pulse. The maximum intensity and the time width of the pulse are

$$\left(E_y\right)_{\max} = \frac{e\gamma}{h^2}, \quad (\Delta t)_{\text{half}-\max} = 2\sqrt{4^{1/3} - 1}\, \frac{h}{\gamma v}. \tag{4.99}$$

Problems

4.1 Compute the invariants $F^{\mu\nu}F_{\mu\nu}$ and $\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ in terms of the electric and magnetic fields.

4.2 Verify the vector identity in Eq. (4.31).

4.3 Consider a Cartesian coordinate system of an inertial reference frame in which we measure a constant uniform electric field $\mathbf{E} = (E, 0, 0)$. Calculate the electric field measured in the inertial reference frame moving with constant velocity $\mathbf{v} = (v, 0, 0)$ with respect to the former.

4.4 Let us consider a constant uniform electric field $\mathbf{E} = (E, 0, 0)$. Calculate the associated energy-momentum tensor.