# Chapter 8
# Schwarzschild Spacetime

The Einstein equations (7.6) relate the geometry of the spacetime, encoded in the Einstein tensor $G^{\mu\nu}$, to the matter content, described by the matter energy-momentum tensor $T^{\mu\nu}$. If we know the matter content, in principle we can solve the Einstein equations and find the spacetime metric $g_{\mu\nu}$ in some coordinate system. However, in general it is highly non-trivial to solve the Einstein equations, because they are second order non-linear partial differential equations for ten components of the metric tensor $g_{\mu\nu}$. Analytic solutions of the Einstein equations can be found when the spacetime has some special symmetries.

The *Schwarzschild metric* is a relevant example of an exact solution of the Einstein equations with important physical applications. It is the only spherically symmetric vacuum solution of the Einstein equations and usually it can well approximate the gravitational field of slowly-rotating astrophysical objects like stars and planets.

## 8.1 Spherically Symmetric Spacetimes

First, we want to find the most general form for the line element of a spherically symmetric spacetime. Note that at this stage we are not assuming the Einstein equations, which means that the same form of the line element holds in any theory of gravity in which the spacetime geometry is described by the metric tensor of the spacetime. To achieve our goal, we choose a particular coordinate system in which the metric tensor $g_{\mu\nu}$ clearly shows the symmetries of the spacetime.

As our starting point, we employ *isotropic coordinates* $(ct, x, y, z)$, we choose the origin of the coordinate system of the 3-space $x = y = z = 0$ at the center of symmetry, and we require that the line element of the 3-space, $dl$, only depends on the time $t$ and on the distance from the origin. $dl^2$ should thus have the following form



C. Bambi, *Introduction to General Relativity*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-981-13-1090-4_8

$$dl^2 = g\left(t, \sqrt{x^2+y^2+z^2}\right)\left(dx^2+dy^2+dz^2\right) , \tag{8.1}$$

where $g$ is an unknown function of $t$ and $\sqrt{x^2+y^2+z^2}$. Note that, in general, $\sqrt{x^2+y^2+z^2}$ is not the proper distance from the origin. However, points with the same value of $\sqrt{x^2+y^2+z^2}$ have the same proper distance from the origin, which is enough for us because we have not yet specified the function $g$.

We move to spherical-like coordinates $(t, r, \theta, \phi)$ with the following coordinate transformation

$$\begin{aligned} x &= r \sin\theta \cos\phi , \\ y &= r \sin\theta \sin\phi , \\ x &= r \cos\theta . \end{aligned} \tag{8.2}$$

The line element of the 3-space is now

$$dl^2 = g(t, r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) . \tag{8.3}$$

Let us now construct the form of the 4-metric $g_{\mu\nu}$. For $dt \neq 0$, $ds^2$ should be separately invariant under the following coordinate transformations

$$\theta \rightarrow \tilde{\theta} = -\theta , \quad \phi \rightarrow \tilde{\phi} = -\phi , \tag{8.4}$$

which implies $g_{t\theta} = g_{t\phi} = 0$. The line element of the 4-dimensional spacetime can thus be written as

$$ds^2 = -f(t,r)c^2dt^2 + g(t,r)dr^2 + h(t,r)dtdr + g(t,r)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{8.5}$$

where $f$ and $h$ are unknown functions of $t$ and $r$ only.

The expression in Eq. (8.5) can be further simplified. We can still consider a coordinate transformation

$$t \rightarrow \tilde{t} = \tilde{t}(t, r) , \quad r \rightarrow \tilde{r} = \tilde{r}(t, r) , \tag{8.6}$$

such that

$$\tilde{r}^2 = g(t, r)r^2 , \quad g_{\tilde{t}\tilde{r}} = 0 . \tag{8.7}$$

Note that the coordinate $\tilde{r}$ has a clear geometrical meaning. It corresponds to the value of the radial coordinate defining the 2-dimensional spherical surface of area $4\pi\tilde{r}^2$. Note also that, in general, $\tilde{r}$ does not describe the real distance from the center $\tilde{r} = 0$ (see Sect. 8.3 later). Eventually, the line element of the spacetime can be written as

$$ds^2 = -f(t,r)c^2dt^2 + g(t,r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{8.8}$$

Since we are interested in gravitational fields generated by a matter distribution with a finite extension, the spacetime must be asymptotically flat, namely we must recover the Minkowski metric at large radii. The boundary conditions are thus

$$\lim_{r\to\infty} f(t,r) = \lim_{r\to\infty} g(t,r) = 1\,. \tag{8.9}$$

As we have already stressed at the beginning of this section, the expression in (8.8) is the most general form for the line element of a spherically symmetric spacetime. If we assume to be in Einstein's gravity, we can solve the Einstein equations with the ansatz in (8.8) to find the explicit form of the functions $f$ and $g$ in Einstein's gravity for a certain matter distribution.

## 8.2 Birkhoff's Theorem

Birkhoff's theorem is an important uniqueness theorem valid in Einstein's gravity.

**Birkhoff's Theorem**. The only spherically symmetric solution of the vacuum Einstein equations is the *Schwarzschild metric*.

Let us first prove the theorem and then discuss its implications. We have to solve the Einstein equations with the ansatz (8.8) for the metric tensor and with $T^{\mu\nu} = 0$ on the right hand side. Since we are in vacuum, the scalar curvature vanishes, $R = 0$, and the Einstein equations reduce to $R^{\mu\nu} = 0$, see Eq. (7.10). The strategy is to calculate the Christoffel symbols and then the Ricci tensor from the formula

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda}\,. \tag{8.10}$$

The calculations may be somewhat long and boring, but they can be a good exercise to better understand the formalism of general relativity. The fastest way to calculate the Christoffel symbols is to write the geodesic equations from the Euler–Lagrange equations of the Lagrangian[1]

$$L = -\frac{f}{2}\left(\frac{dt}{d\lambda}\right)^2 + \frac{g}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{r^2}{2}\left(\frac{d\theta}{d\lambda}\right)^2 + \frac{1}{2}r^2\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\,, \tag{8.11}$$

[1]In these calculations we ignore the dimensional difference between $t$ and the space coordinates and we do not write the speed of light $c$ to simplify the equations. This is equivalent to employing units in which $c = 1$, which is a convention widely used among the gravity and particle physics communities.

where $\lambda$ is the particle proper time (for time-like geodesics) or an affine parameter (for null geodesics). For $x^\mu = t$, the Euler–Lagrange equation is

$$
\begin{aligned}
&\frac{d}{d\lambda}\left(-f\frac{dt}{d\lambda}\right) + \frac{\dot{f}}{2}\left(\frac{dt}{d\lambda}\right)^2 - \frac{\dot{g}}{2}\left(\frac{dr}{d\lambda}\right)^2 = 0\,, \\
&f\frac{d^2t}{d\lambda^2} + \dot{f}\left(\frac{dt}{d\lambda}\right)^2 + f'\frac{dt}{d\lambda}\frac{dr}{d\lambda} - \frac{\dot{f}}{2}\left(\frac{dt}{d\lambda}\right)^2 + \frac{\dot{g}}{2}\left(\frac{dr}{d\lambda}\right)^2 = 0\,, \\
&\frac{d^2t}{d\lambda^2} + \frac{\dot{f}}{2f}\left(\frac{dt}{d\lambda}\right)^2 + \frac{f'}{f}\frac{dt}{d\lambda}\frac{dr}{d\lambda} + \frac{\dot{g}}{2f}\left(\frac{dr}{d\lambda}\right)^2 = 0\,,
\end{aligned}
\tag{8.12}
$$

where here and in what follows we use the dot $\dot{}$ to indicate the derivative with respect to the time $t$, i.e. $\dot{} = \partial_t$, and the prime $'$ to indicate the derivative with respect to the radial coordinate $r$, namely $' = \partial_r$. If we compare the geodesic equations with the last expression in (8.12), we see that the only non-vanishing Christoffel symbols of the type $\Gamma^t_{\mu\nu}$ are

$$
\Gamma^t_{tt} = \frac{\dot{f}}{2f}\,, \quad \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{f'}{2f}\,, \quad \Gamma^t_{rr} = \frac{\dot{g}}{2f}\,. \tag{8.13}
$$

For $x^\mu = r$, the Euler–Lagrange equation is

$$
\begin{aligned}
&\frac{d}{d\lambda}\left(g\frac{dr}{d\lambda}\right) + \frac{f'}{2}\left(\frac{dt}{d\lambda}\right)^2 - \frac{g'}{2}\left(\frac{dr}{d\lambda}\right)^2 - r\left(\frac{d\theta}{d\lambda}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0\,, \\
&g\frac{d^2r}{d\lambda^2} + \dot{g}\frac{dt}{d\lambda}\frac{dr}{d\lambda} + g'\left(\frac{dr}{d\lambda}\right)^2 + \frac{f'}{2}\left(\frac{dt}{d\lambda}\right)^2 - \frac{g'}{2}\left(\frac{dr}{d\lambda}\right)^2 \\
&\qquad - r\left(\frac{d\theta}{d\lambda}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0\,, \\
&\frac{d^2r}{d\lambda^2} + \frac{f'}{2g}\left(\frac{dt}{d\lambda}\right)^2 + \frac{\dot{g}}{g}\frac{dt}{d\lambda}\frac{dr}{d\lambda} + \frac{g'}{2g}\left(\frac{dr}{d\lambda}\right)^2 \\
&\qquad - \frac{r}{g}\left(\frac{d\theta}{d\lambda}\right)^2 - \frac{r\sin^2\theta}{g}\left(\frac{d\phi}{d\lambda}\right)^2 = 0\,,
\end{aligned}
\tag{8.14}
$$

and we find that the only non-vanishing Christoffel symbols of the type $\Gamma^r_{\mu\nu}$ are

$$
\begin{aligned}
&\Gamma^r_{tt} = \frac{f'}{2g}\,, \quad \Gamma^r_{tr} = \Gamma^r_{rt} = \frac{\dot{g}}{2g}\,, \\
&\Gamma^r_{rr} = \frac{g'}{2g}\,, \quad \Gamma^r_{\theta\theta} = -\frac{r}{g}\,, \quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{g}\,.
\end{aligned}
\tag{8.15}
$$

For $x^\mu = \theta$ we have

$$\frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0\,,$$
$$r^2\frac{d^2\theta}{d\lambda^2} + 2r\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0\,,$$
$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0\,, \tag{8.16}$$

and the non-vanishing Christoffel symbols of the type $\Gamma^\theta_{\mu\nu}$ are

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}\,, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,. \tag{8.17}$$

Lastly, for $x^\mu = \phi$ the Euler–Lagrange equation reads

$$\frac{d}{d\lambda}\left(r^2\sin^2\theta\frac{d\phi}{d\lambda}\right) = 0\,,$$
$$r^2\sin^2\theta\frac{d^2\phi}{d\lambda^2} + 2r\sin^2\theta\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2r^2\sin\theta\cos\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0\,,$$
$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0\,, \tag{8.18}$$

and we find

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}\,, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta\,. \tag{8.19}$$

Now that we have all the Christoffel symbols, we can calculate the non-vanishing components of the Ricci tensor $R_{\mu\nu}$. The $tt$-component is

$$\begin{aligned}
R_{tt} &= \frac{\partial\Gamma^t_{tt}}{\partial t} + \frac{\partial\Gamma^r_{tt}}{\partial r} - \frac{\partial}{\partial t}\left(\Gamma^t_{tt} + \Gamma^r_{tr}\right) + \Gamma^t_{tt}\left(\Gamma^t_{tt} + \Gamma^r_{tr}\right) \\
&\quad + \Gamma^r_{tt}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) - \Gamma^t_{tt}\Gamma^t_{tt} - \Gamma^t_{tr}\Gamma^r_{tt} - \Gamma^t_{tr}\Gamma^r_{tt} - \Gamma^r_{tr}\Gamma^r_{tr} \\
&= \frac{\partial\Gamma^r_{tt}}{\partial r} - \frac{\partial\Gamma^r_{tr}}{\partial t} + \Gamma^t_{tt}\Gamma^r_{tr} + \Gamma^r_{tt}\left(\Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) - \Gamma^t_{tr}\Gamma^r_{tt} - \Gamma^r_{tr}\Gamma^r_{tr} \\
&= \frac{f''}{2g} - \frac{f'g'}{2g^2} - \frac{\ddot{g}}{2g} + \frac{\dot{g}^2}{2g^2} + \frac{\dot{f}}{2f}\frac{\dot{g}}{2g} \\
&\quad + \frac{f'}{2g}\left(\frac{g'}{2g} + \frac{1}{r} + \frac{1}{r}\right) - \frac{f'}{2f}\frac{f'}{2g} - \frac{\dot{g}}{2g}\frac{\dot{g}}{2g} \\
&= \frac{f''}{2g} - \frac{f'}{4g}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{f'}{rg} + \frac{\dot{f}\dot{g}}{4fg} + \frac{\dot{g}^2}{4g^2} - \frac{\ddot{g}}{2g}\,.
\end{aligned} \tag{8.20}$$

The $tr$-component of $R_{\mu\nu}$ is given by

$$\begin{aligned}
R_{tr} &= \frac{\partial \Gamma^t_{tr}}{\partial t} + \frac{\partial \Gamma^r_{tr}}{\partial r} - \frac{\partial}{\partial r}\left(\Gamma^t_{tt} + \Gamma^r_{tr}\right) + \Gamma^t_{tr}\left(\Gamma^t_{tt} + \Gamma^r_{tr}\right) \\
&\quad + \Gamma^r_{tr}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) - \Gamma^t_{tt}\Gamma^t_{rt} - \Gamma^t_{tr}\Gamma^r_{rt} - \Gamma^r_{tt}\Gamma^t_{rr} - \Gamma^r_{tr}\Gamma^r_{rr} \\
&= \frac{\partial \Gamma^t_{tr}}{\partial t} - \frac{\partial \Gamma^t_{tt}}{\partial r} + \Gamma^r_{tr}\left(\Gamma^t_{rt} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) - \Gamma^r_{tt}\Gamma^t_{rr} \\
&= \frac{\dot{f}'}{2f} - \frac{f'\dot{f}}{2f^2} - \frac{\dot{f}'}{2f} + \frac{\dot{f}f'}{2f^2} \\
&\quad + \frac{\dot{g}}{2g}\left(\frac{f'}{2f} + \frac{1}{r} + \frac{1}{r}\right) - \frac{f'}{2g}\frac{\dot{g}}{2f} \\
&= \frac{\dot{g}^2}{rg}\,.
\end{aligned} \tag{8.21}$$

The $rr$-component is

$$\begin{aligned}
R_{rr} &= \frac{\partial \Gamma^t_{rr}}{\partial t} + \frac{\partial \Gamma^r_{rr}}{\partial r} - \frac{\partial}{\partial r}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) \\
&\quad + \Gamma^t_{rr}\left(\Gamma^t_{tt} + \Gamma^r_{tr}\right) + \Gamma^r_{rr}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) \\
&\quad - \Gamma^t_{rt}\Gamma^t_{rt} - \Gamma^t_{rr}\Gamma^r_{rt} - \Gamma^r_{rt}\Gamma^t_{rr} - \Gamma^r_{rr}\Gamma^r_{rr} - \Gamma^\theta_{r\theta}\Gamma^\theta_{r\theta} - \Gamma^\phi_{r\phi}\Gamma^\phi_{r\phi} \\
&= \frac{\partial \Gamma^t_{rr}}{\partial t} - \frac{\partial \Gamma^t_{rt}}{\partial r} - \frac{\partial \Gamma^\theta_{r\theta}}{\partial r} - \frac{\partial \Gamma^\phi_{r\phi}}{\partial r} + \Gamma^t_{rr}\Gamma^t_{tt} + \Gamma^r_{rr}\left(\Gamma^t_{rt} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) \\
&\quad - \Gamma^t_{rt}\Gamma^t_{rt} - \Gamma^t_{rr}\Gamma^r_{rt} - \Gamma^\theta_{r\theta}\Gamma^\theta_{r\theta} - \Gamma^\phi_{r\phi}\Gamma^\phi_{r\phi} \\
&= \frac{\ddot{g}}{2f} - \frac{\dot{f}\dot{g}}{2f^2} - \frac{f''}{2f} + \frac{f'^2}{2f^2} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{\dot{g}}{2f}\frac{\dot{f}}{2f} + \frac{g'}{2g}\left(\frac{f'}{2f} + \frac{1}{r} + \frac{1}{r}\right) \\
&\quad - \frac{f'}{2f}\frac{f'}{2f} - \frac{\dot{g}}{2f}\frac{\dot{g}}{2g} - \frac{1}{r^2} - \frac{1}{r^2} \\
&= -\frac{f''}{2f} + \frac{f'}{4f}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{g'}{rg} - \frac{\dot{f}\dot{g}}{4f^2} - \frac{\dot{g}^2}{4fg} + \frac{\ddot{g}}{2f}\,.
\end{aligned} \tag{8.22}$$

The $\theta\theta$-component reads

$$\begin{aligned}
R_{\theta\theta} &= \frac{\partial \Gamma^r_{\theta\theta}}{\partial r} - \frac{\partial \Gamma^\phi_{\theta\phi}}{\partial \theta} + \Gamma^r_{\theta\theta}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) - \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} - \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} - \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\theta\phi} \\
&= \frac{\partial \Gamma^r_{\theta\theta}}{\partial r} - \frac{\partial \Gamma^\phi_{\theta\phi}}{\partial \theta} + \Gamma^r_{\theta\theta}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\phi_{r\phi}\right) - \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} - \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\theta\phi} \\
&= -\frac{1}{g} + \frac{rg'}{g^2} + 1 + \cot^2\theta - \frac{r}{g}\left(\frac{f'}{2f} + \frac{g'}{2g} + \frac{1}{r}\right) + \frac{1}{r}\frac{r}{g} - \cot^2\theta \\
&= 1 - \frac{1}{g} + \frac{r}{2g}\left(\frac{g'}{g} - \frac{f'}{f}\right).
\end{aligned} \tag{8.23}$$

The $\phi\phi$-component is

$$\begin{aligned}
R_{\phi\phi} &= \frac{\partial \Gamma^r_{\phi\phi}}{\partial r} + \frac{\partial \Gamma^\theta_{\phi\phi}}{\partial \theta} + \Gamma^r_{\phi\phi}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta} + \Gamma^\phi_{r\phi}\right) + \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\theta\phi} \\
&\quad - \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r} - \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} - \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta} - \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} \\
&= \frac{\partial \Gamma^r_{\phi\phi}}{\partial r} + \frac{\partial \Gamma^\theta_{\phi\phi}}{\partial \theta} + \Gamma^r_{\phi\phi}\left(\Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^\theta_{r\theta}\right) - \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} - \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} \\
&= -\frac{\sin^2\theta}{g} + \frac{r\sin^2\theta g'}{g^2} + \sin^2\theta - \cos^2\theta - \frac{r\sin^2\theta}{g}\left(\frac{f'}{2f} + \frac{g'}{2g} + \frac{1}{r}\right) \\
&\quad + \frac{1}{r}\frac{r\sin^2\theta}{g} + \sin\theta\cos\theta\cot\theta \\
&= \sin^2\theta\left[1 - \frac{1}{g} + \frac{r}{2g}\left(\frac{g'}{g} - \frac{f'}{f}\right)\right], \qquad (8.24)
\end{aligned}$$

and it is equal to the $\theta\theta$-component multiplied by $\sin^2\theta$.

The other components of the Ricci tensor vanish. This can be seen noting that the transformations in (8.4) have the following effect:

$$\begin{aligned}
&R_{\theta\mu} \to R_{\tilde{\theta}\mu} = \frac{\partial\theta}{\partial\tilde{\theta}}\frac{\partial x^\nu}{\partial x^\mu}R_{\theta\nu} = -R_{\theta\mu} \quad (\text{for } \mu \neq \theta), \\
&R_{\phi\mu} \to -R_{\phi\mu} \quad (\text{for } \mu \neq \phi). \qquad (8.25)
\end{aligned}$$

Since the metric is invariant under such transformations, the components of the Ricci tensor should be invariant too, and therefore they must vanish.

Eventually, we have four independent equations

$$R_{tt} = \frac{f''}{2g} - \frac{f'}{4g}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{f'}{rg} + \frac{\dot{f}\dot{g}}{4fg} + \frac{\dot{g}^2}{4g^2} - \frac{\ddot{g}}{2g} = 0, \qquad (8.26)$$

$$R_{tr} = \frac{\dot{g}^2}{rg} = 0, \qquad (8.27)$$

$$R_{rr} = -\frac{f''}{2f} + \frac{f'}{4f}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{g'}{rg} - \frac{\dot{f}\dot{g}}{4f^2} - \frac{\dot{g}^2}{4fg} + \frac{\ddot{g}}{2f} = 0, \qquad (8.28)$$

$$R_{\theta\theta} = 1 - \frac{1}{g} + \frac{r}{2g}\left(\frac{g'}{g} - \frac{f'}{f}\right) = 0, \qquad (8.29)$$

From Eq. (8.27) we see that $g = g(r)$. Equation (8.29) can be written as

$$\frac{f'}{f} = \frac{2g}{r} - \frac{2}{r} + \frac{g'}{g}, \qquad (8.30)$$

and we thus see that $f'/f$ is a function of $r$ only. This means that $f$ can be written as

$$f(t,r) = f_1(t) f_2(r) \,. \tag{8.31}$$

With the following coordinate transformation

$$dt \to d\tilde{t} = \sqrt{f_1(t)} dt \,, \tag{8.32}$$

we can always absorb $f_1(t)$ in the temporal coordinate. Eventually, the line element of the spacetime can be written in the following form

$$ds^2 = -f(r)c^2dt^2 + g(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{8.33}$$

which shows that the metric is independent of $t$.

We combine Eqs. (8.26) and (8.28) in the following way

$$gR_{tt} + fR_{rr} = 0 \,, \tag{8.34}$$

and we find

$$\begin{aligned}
&\frac{f''}{2} - \frac{f'}{4}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{f'}{r} - \frac{f''}{2} + \frac{f'}{4}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{fg'}{rg} = 0 \,, \\
&\frac{f'}{r} + \frac{fg'}{rg} = 0 \,, \\
&\frac{1}{rg}\frac{d}{dr}(fg) = 0 \\
&fg = \text{constant} \,.
\end{aligned} \tag{8.35}$$

Imposing the condition in Eq. (8.9), which is necessary because far from the source we want to recover the Minkowski spacetime, we find

$$g = \frac{1}{f} \,. \tag{8.36}$$

We can now rewrite Eq. (8.29) in terms of $f(r)$ only and solve the new differential equation

$$\begin{aligned}
&1 - f + \frac{rf}{2}\left(-f\frac{f'}{f^2} - \frac{f'}{f}\right) = 0 \,, \\
&1 - f - rf' = 0 \,, \\
&\frac{d}{dr}(rf) = 1 \,,
\end{aligned}$$

$$f = 1 + \frac{C}{r}\,, \tag{8.37}$$

where $C$ is a constant. The constant $C$ can be inferred from the Newtonian limit. From Eq. (6.14), we know that

$$g_{tt} = -f = -\left(1 + \frac{2\Phi}{c^2}\right)\,. \tag{8.38}$$

For a spherically symmetric distribution of mass, the Newtonian potential reads

$$\Phi = -\frac{G_{\rm N}M}{r}\,, \tag{8.39}$$

where $M$ is the mass of the body generating the gravitational field. We thus find $C = -2G_{\rm N}M/c^2$ and the line element of the spacetime reads

$$ds^2 = -\left(1 - \frac{2G_{\rm N}M}{c^2 r}\right)c^2dt^2 + \frac{dr^2}{1 - \frac{2G_{\rm N}M}{c^2 r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{8.40}$$

This concludes the proof of the theorem. The solution is called the Schwarzschild metric. It is remarkable that $M$ is the only parameter that characterizes the spacetime metric in the exterior region; that is, the gravitational field in the exterior region is independent of the internal structure and composition of the massive body. As shown in Appendix F, $M$ can be associated to the actual mass of the body only in the Newtonian limit. As discussed in Sect. 3.7 within the framework of special relativity, the total mass of a physical system is lower than the sum of the masses of its constituents because there is also a binding energy. The same is true here.

Note that the Schwarzschild metric has been derived only assuming that the spacetime is spherically symmetric and solving the vacuum Einstein equations. The fact that the metric is independent of $t$ is a consequence, it is not an assumption. This implies that the matter distribution does not have to be static, but it can move while maintaining the spherical symmetry, for example pulsating, and the vacuum solution is still described by the Schwarzschild metric. This implies that a spherically symmetric pulsating distribution of matter does not emit gravitational waves (this point will be discussed better in Chap. 12).

## 8.3 Schwarzschild Metric

In the Schwarzschild metric, the coordinates $(ct, r, \theta, \phi)$ can assume the following values

$$t \in (-\infty, \infty)\,, \quad r \in [r_0, \infty)\,, \quad \theta \in (0, \pi)\,, \quad \phi \in [0, 2\pi)\,, \tag{8.41}$$

**Table 8.1** Mass $M$, Schwarzschild radius $r_S$, and physical radius $r_0$ of the Sun, Earth, and a proton

| Object | $M$ (g) | $r_S$ | $r_0$ |
|---|---|---|---|
| Sun | $1.99 \cdot 10^{33}$ | 2.95 km | $6.97 \cdot 10^5$ km |
| Earth | $5.97 \cdot 10^{27}$ | 8.87 mm | $6.38 \cdot 10^3$ km |
| Proton | $1.67 \cdot 10^{-24}$ | $2.48 \cdot 10^{-39}$ fm | 0.8 fm |

where $r_0$ is the radius of the body. The Schwarzschild metric is indeed valid in the vacuum only, namely in the "exterior" region. A different solution will describe the "interior" region $r < r_0$, where $T^{\mu\nu} \neq 0$. Note that the metric is ill-defined at the so-called *Schwarzschild radius* $r_S$

$$r_S = \frac{2G_N M}{c^2} \,. \tag{8.42}$$

This requires $r_0 > r_S$. In general, this is not a problem: as shown in Table 8.1, the Schwarzschild radius is typically much smaller than the radius of an object.

The relation between the temporal coordinate of the Schwarzschild metric, $t$, and the proper time of an observer at a point with fixed $(r, \theta, \phi)$ is

$$d\tau = \sqrt{1 - \frac{r_S}{r}}dt < dt \,. \tag{8.43}$$

The relation between the space coordinates and the proper distance is more tricky. For simplicity, we consider the case of a distance in the radial direction. For a light signal, $ds^2 = 0$, and therefore in the case of pure radial motion we have

$$dt = \pm\frac{1}{c}\frac{dr}{1 - \frac{r_S}{r}} \,, \tag{8.44}$$

where the sign is $+$ $(-)$ if the light signal is moving to larger (smaller) radii. Instead of the temporal coordinate $t$, we rewrite Eq. (8.44) in terms of the proper time of an observer at a point with fixed $(r, \theta, \phi)$

$$d\tau = \pm\frac{1}{c}\frac{dr}{\sqrt{1 - \frac{r_S}{r}}} \,. \tag{8.45}$$

We can thus define the infinitesimal proper distance $d\rho$ as

$$d\rho = \frac{dr}{\sqrt{1 - \frac{r_S}{r}}} > dr \,. \tag{8.46}$$

This is indeed the infinitesimal distance along the radial direction measured by an observer at that point with the help of a light signal. The proper distance between the point $(r_1, \theta, \phi)$ and the point $(r_2, \theta, \phi)$ is obtained by integrating over the radial direction

$$\begin{aligned} \Delta\rho &= \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{r_S}{r}}} \approx \int_{r_1}^{r_2} \left(1 + \frac{1}{2}\frac{r_S}{r}\right) dr \\ &= (r_2 - r_1) + \frac{r_S}{2} \ln \frac{r_2}{r_1} \, . \end{aligned} \tag{8.47}$$

In the case of the Minkowski spacetime, $r_S = 0$, and we recover the standard distance $r_2 - r_1$. For $r_S \neq 0$, there is a correction proportional to $r_S$.

Note that $d\tau \to dt$ and $d\rho \to dr$ for $r \to \infty$, which can be interpreted as the fact that the Schwarzschild coordinates correspond to the coordinates of an observer at infinity.

## 8.4 Motion in the Schwarzschild Metric

Let us now study the motion of test-particles in the Schwarzschild metric. The Lagrangian of the system is (for simplicity, we set $m = 0, 1$ the mass of, respectively, massless and massive particles)

$$L = \frac{1}{2}\left(-fc^2\dot{t}^2 + g\dot{r}^2 + r^2\dot{\theta} + r^2\sin^2\theta\dot{\phi}^2\right) \, , \tag{8.48}$$

where here the dot $\dot{}$ indicates the derivative with respect to the proper time/affine parameter $\lambda$ and

$$f = \frac{1}{g} = 1 - \frac{r_S}{r} \, . \tag{8.49}$$

The Euler–Lagrange equation for the $\theta$ coordinate is equal to that in Newton's gravity met in Sect. 1.8. Without loss of generality, we can study the case of a particle moving in the equatorial plane $\theta = \pi/2$. The Lagrangian (8.48) thus simplifies to

$$L = \frac{1}{2}\left(-fc^2\dot{t}^2 + g\dot{r}^2 + r^2\dot{\phi}^2\right) \, . \tag{8.50}$$

There are three constants of motion: the energy (as measured at infinity) $E$, the angular momentum (as measured at infinity) $L_z$,[2] and the mass of the test-particle.

[2]As in Sect. 1.8, we use the notation $L_z$ because this is also the axial component of the angular momentum (since $\theta = \pi/2$) and we do not want to call it $L$ because it may generate confusion with the Lagrangian.

The conservation of the energy of the test-particle follows from the fact that the Lagrangian (8.50) is independent of the time coordinate $t$[3]

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{t}} = 0 \quad \Rightarrow \quad p_t = \frac{1}{c}\frac{\partial L}{\partial \dot{t}} = -fc\dot{t} = -\frac{E}{c} = \text{constant}\,. \quad (8.51)$$

The Lagrangian (8.50) is also independent of the coordinate $\phi$, and we thus have the conservation of the angular momentum

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} = L_z = \text{constant}\,. \quad (8.52)$$

The conservation of the mass comes from the equation

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -fc^2\dot{t}^2 + g\dot{r}^2 + r^2\dot{\phi}^2 = -kc^2\,, \quad (8.53)$$

where $k = 0$ (for massless particles) and $k = 1$ (for massive particles).

From Eqs. (8.51) and (8.52) we find, respectively,

$$\dot{t} = \frac{E}{c^2 f}\,, \quad \dot{\phi} = \frac{L_z}{r^2}\,. \quad (8.54)$$

We plug these expressions of $\dot{t}$ and $\dot{\phi}$ into Eq. (8.53) and we find

$$g\dot{r}^2 + \frac{L_z^2}{r^2} - \frac{E^2}{c^2 f} = -kc^2\,. \quad (8.55)$$

If we multiply Eq. (8.55) by $f/2$ and we write the explicit form of $f$ and $g$, we obtain

$$\frac{1}{2}\dot{r}^2 + \left(1 - \frac{r_S}{r}\right)\frac{L_z^2}{2r^2} - \frac{1}{2}\frac{E^2}{c^2} = -\frac{1}{2}\left(1 - \frac{r_S}{r}\right)kc^2\,. \quad (8.56)$$

This equation can be rewritten as

$$\frac{1}{2}\dot{r}^2 = \frac{E^2 - kc^4}{2c^2} - V_{\rm eff}\,, \quad (8.57)$$

where

[3] Note that for a massive particle we choose $\lambda = \tau$ the particle proper time. In such a case, for a static particle at infinity we have $\dot{t} = 1$ and $E = c^2$ (because we are assuming $m = 1$, otherwise we would have $E = mc^2$); that is, the particle energy is just the rest mass. For a static particle at smaller radii, $E < c^2$ because the (Newtonian) gravitational potential energy is negative. Note also that $p_t = -E/c$ is conserved while the temporal component of the 4-momentum, $p^t$, is not a constant of motion. The same is true for $p_\phi$ and $p^\phi$: only $p_\phi$ is conserved.

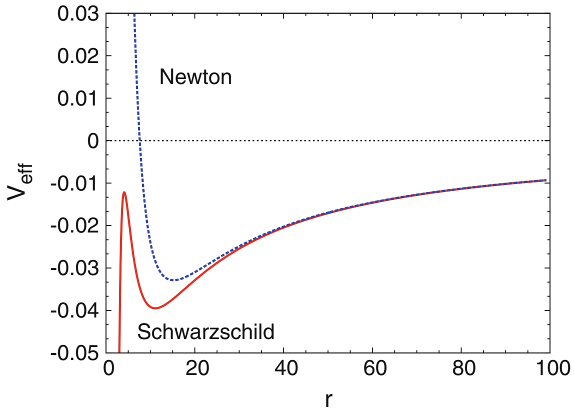


**Fig. 8.1** Comparison between the effective potential in Eq. (8.58) for $k = 1$ valid in the Schwarzschild metric (red solid line) and the effective potential in Eq. (1.92) valid in Newton's gravity for the gravitational field generated by a point-like massive body (blue dashed line). We assume $G_{\rm N}M = 1$, $L_z = 3.9$, and $c = 1$

$$V_{\rm eff} = -k\frac{G_{\rm N}M}{r} + \frac{L_z^2}{2r^2} - \frac{G_{\rm N}ML_z^2}{c^2r^3}\,. \tag{8.58}$$

Equation (8.57) is the counterpart of Eq. (1.91) in Newton's gravity. The effective potential $V_{\rm eff}$ in Eq. (8.58) can be compared with the Newtonian effective potential in Eq. (1.92). For $k = 1$, we see that the first and the second terms on the right hand side in Eq. (8.58) are exactly those in Eq. (1.92). The third term is the correction to the Newtonian case and becomes important only at very small radii, because it scales as $1/r^3$. Figure 8.1 shows the difference between the effective potential in Eq. (8.58) and that in Eq. (1.92).

Let us now proceed as we did in Sect. 1.9 for the derivation of Kepler's Laws. We write

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \frac{L_z}{r^2}\frac{dr}{d\phi}\,, \tag{8.59}$$

and we remove the parameter $\lambda$ in Eq. (8.57)

$$\frac{L_z^2}{2r^4}\left(\frac{dr}{d\phi}\right)^2 - k\frac{G_{\rm N}M}{r} + \frac{L_z^2}{2r^2} - \frac{G_{\rm N}ML_z^2}{c^2r^3} = \frac{E^2 - kc^4}{2c^2}\,. \tag{8.60}$$

We introduce the variable $u = u(\phi)$

$$r = \frac{1}{u}\,, \quad u' = \frac{du}{d\phi}\,. \tag{8.61}$$

Equation (8.60) becomes

$$u'^2 - k\frac{2G_{\rm N}M}{L_z^2}u + u^2 - \frac{2G_{\rm N}M}{c^2}u^3 = \frac{E^2 - kc^4}{c^2L_z^2}\,. \tag{8.62}$$

We derive this equation with respect to $\phi$ and we obtain

$$2u'\left(u'' - k\frac{G_\mathrm{N}M}{L_z^2} + u - \frac{3G_\mathrm{N}M}{c^2}u^2\right) = 0\,. \tag{8.63}$$

The equations of the orbits are thus

$$u' = 0\,, \tag{8.64}$$

$$u'' - k\frac{G_\mathrm{N}M}{L_z^2} + u - \frac{3G_\mathrm{N}M}{c^2}u^2 = 0\,. \tag{8.65}$$

From Eq. (8.64), we find circular orbits as in the Newtonian case from Eq. (1.100). Equation (8.65) is the relativistic generalization of Eq. (1.101).

## 8.5 Schwarzschild Black Holes

The Schwarzschild metric can describe the exterior region of a massive body, namely the region $r > r_0$, where $r_0 > r_\mathrm{S}$ is the radius of the object. A different metric holds in the interior region $r < r_0$. As already discussed in Sect. 8.3, for typical bodies $r_0 \gg r_\mathrm{S}$ and therefore it is irrelevant that the metric is not well-defined at $r = r_\mathrm{S}$. If this is not the case and there is no interior solution, we have a black hole and the surface $r = r_\mathrm{S}$ is the black hole *event horizon*. We will see in Sect. 10.5 how a similar object can be formed.

From Eq. (8.51), we can write

$$dt = \frac{E}{c^2 f}d\lambda = \frac{E}{c^2 f}\frac{d\lambda}{dr}dr = \frac{E}{c^2 f}\frac{dr}{\dot{r}}\,. \tag{8.66}$$

From Eq. (8.57), we have

$$\frac{1}{\dot{r}} = -\frac{c}{\sqrt{E^2 - kc^4 - 2V_\mathrm{eff}c^2}}\,, \tag{8.67}$$

where the sign $-$ is chosen because we are interested in a particle moving to smaller radii. We replace $1/\dot{r}$ in Eq. (8.66) with the expression on the right hand side in Eq. (8.67). In the case of a massive particle ($k = 1$) with vanishing angular momentum ($L_z = 0$), we find

$$dt = -\frac{1}{c}\frac{E}{1 - \frac{r_\mathrm{S}}{r}}\frac{dr}{\sqrt{E^2 - c^4 + \frac{r_\mathrm{S}c^4}{r}}}\,, \tag{8.68}$$

which describes the motion of a massive particle falling onto the massive object with vanishing angular momentum. If we integrate both the left and the right hand sides in Eq. (8.68), we find that, according to our coordinate system corresponding to that of a distant observer, the particle takes the time $\Delta t$ to move from the radius $r_2$ to the radius $r_1 < r_2$

$$\Delta t = \frac{1}{c} \int_{r_1}^{r_2} \frac{E}{1 - \frac{r_S}{r}} \frac{dr}{\sqrt{E^2 - c^4 + \frac{r_S c^4}{r}}} \, . \tag{8.69}$$

For $r_1 \to r_S$, $\Delta t \to \infty$ regardless of the value of $E$ (for an explicit example, we can consider the case $E = c^2$ corresponding to a particle at rest at infinity); that is, for an observer at very large radii the particle takes an infinite time to reach the radial coordinate $r = r_S$.

Let us now calculate the time measured by the same particle. The relation between the proper time of the particle, $\tau$, and the coordinate time, $t$, is

$$d\tau = dt \sqrt{1 - \frac{r_S}{r}} \, . \tag{8.70}$$

and Eq. (8.68) becomes

$$d\tau = -\frac{1}{c} \frac{E}{\sqrt{1 - \frac{r_S}{r}}} \frac{dr}{\sqrt{E^2 - c^4 + \frac{r_S c^4}{r}}} \, . \tag{8.71}$$

Integrating we find

$$\Delta \tau = \frac{1}{c} \int_{r_1}^{r_2} \frac{E}{\sqrt{1 - \frac{r_S}{r}}} \frac{dr}{\sqrt{E^2 - c^4 + \frac{r_S c^4}{r}}} \, . \tag{8.72}$$

For $r_1 \to r_S$, $\Delta \tau$ remains finite; that is, the particle can cross the surface at $r = r_S$, but the coordinate system of the distant observer can only describe the motion of the particle for $r > r_S$.

The radius $r = r_S$ is the black hole event horizon and causally disconnects the black hole ($r < r_S$) from the exterior region ($r > r_S$). A particle in the exterior region can cross the event horizon and enter the black hole (indeed it takes a finite time to reach and cross the surface at $r = r_S$) but then it cannot communicate with the exterior region any longer (this point will be more clear in the next section).

Note that the metric is ill-defined at $r = r_S$, but the spacetime is regular there. For instance, the *Kretschmann scalar* (as a scalar, it is an invariant) is

$$\mathscr{K} \equiv R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48 G_N^2 M^2}{c^4 r^6} = \frac{12 r_S^2}{r^6} \, , \tag{8.73}$$

and does not diverge at $r = r_S$.

The singularity of the metric at $r = r_S$ depends on the choice of the coordinate system and can be removed by a coordinate transformation.[4] For instance, the Lemaitre coordinates $(cT, R, \theta, \phi)$ are defined as

$$cdT = cdt + \left(\frac{r_S}{r}\right)^{1/2}\left(1 - \frac{r_S}{r}\right)^{-1} dr \, ,$$
$$dR = cdt + \left(\frac{r}{r_S}\right)^{1/2}\left(1 - \frac{r_S}{r}\right)^{-1} dr \, . \tag{8.74}$$

In the Lemaitre coordinates, the line element of the Schwarzschild metric reads

$$ds^2 = -c^2 dT^2 + \frac{r_S}{r} dR^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, , \tag{8.75}$$

where

$$r = (r_S)^{1/3}\left[\frac{3}{2}(R - cT)\right]^{2/3} \, . \tag{8.76}$$

The Schwarzschild radius $r = r_S$ in the new coordinates is

$$\frac{3}{2}(R - cT) = r_S \, , \tag{8.77}$$

and the metric is regular there. The Lemaitre coordinates can well describe both the black hole region $0 < r < r_S$ and the exterior region $r > r_S$.

The maximal analytic extension of the Schwarzschild spacetime is found when we employ the Kruskal–Szekeres coordinates. In these coordinates, the line element reads

$$ds^2 = -\frac{4r_S^3}{r} e^{-r/r_S} dV dU + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, , \tag{8.78}$$

where $U = \tilde{t} - \tilde{r}$ and $V = \tilde{t} + \tilde{r}$ are (dimensionless) light-cone coordinates and

$$\tilde{t} = \begin{cases} \left(\frac{r}{r_S} - 1\right)^{1/2} e^{r/(2r_S)} \sinh\left(\frac{ct}{2r_S}\right) & \text{if } \; r > r_S \, , \\ \left(1 - \frac{r}{r_S}\right)^{1/2} e^{r/(2r_S)} \cosh\left(\frac{ct}{2r_S}\right) & \text{if } \; 0 < r < r_S \, . \end{cases}$$
$$\tilde{r} = \begin{cases} \left(\frac{r}{r_S} - 1\right)^{1/2} e^{r/(2r_S)} \cosh\left(\frac{ct}{2r_S}\right) & \text{if } \; r > r_S \, , \\ \left(1 - \frac{r}{r_S}\right)^{1/2} e^{r/(2r_S)} \sinh\left(\frac{ct}{2r_S}\right) & \text{if } \; 0 < r < r_S \, . \end{cases} \tag{8.79}$$

[4] Note that the metric is ill-defined even at $r = 0$, which is a true spacetime singularity and cannot be removed by a coordinate transformation. The Kretschmann scalar diverges at $r = 0$.

The Schwarzschild solution in Kruskal–Szekeres coordinates includes also a white hole and a parallel universe, which are not present in the Schwarzschild spacetime in Schwarzschild coordinates. This will be briefly shown in the next section.

## 8.6 Penrose Diagrams

*Penrose diagrams* are 2-dimensional spacetime diagrams particularly suitable to studying the global properties and the causal structure of asymptotically flat spacetimes. Every point represents a 2-dimensional sphere of the original 4-dimensional spacetime. Penrose diagrams are obtained by a conformal transformation of the original coordinates such that the entire spacetime is transformed into a compact region. Since the transformation is conformal, angles are preserved. In this section, we employ units in which $c = 1$ and therefore null geodesics are lines at 45°. Time-like geodesics are inside the light-cone, space-like geodesics are outside. A more detailed discussion on the topic can be found, for instance, in [1, 2].

### 8.6.1 *Minkowski Spacetime*

The simplest example is the Penrose diagram of the Minkowski spacetime. In spherical coordinates $(t, r, \theta, \phi)$, the line element of the Minkowski spacetime is $(c = 1)$

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,. \tag{8.80}$$

With the following conformal transformation

$$\begin{aligned} t &= \frac{1}{2}\tan\frac{T+R}{2} + \frac{1}{2}\tan\frac{T-R}{2} \,, \\ r &= \frac{1}{2}\tan\frac{T+R}{2} - \frac{1}{2}\tan\frac{T-R}{2} \,, \end{aligned} \tag{8.81}$$

the line element becomes

$$\begin{aligned} ds^2 = {} & \left(4\cos^2\frac{T+R}{2}\cos^2\frac{T-R}{2}\right)^{-1}\left(-dT^2 + dR^2\right) \\ & + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \,. \end{aligned} \tag{8.82}$$

Note that the transformation in (8.81) employs the tangent function, tan, in order to bring points at infinity to points at a finite value in the new coordinates.

The Penrose diagram for the Minkowski spacetime is shown in Fig. 8.2. The semi-infinite $(t, r)$ plane is now a triangle. The dashed vertical line is the origin $r = 0$.

Every point corresponds to the 2-sphere $(\theta, \phi)$. There are five different asymptotic regions. Without a rigorous treatment, they can be defined as follows[5]:

*Future time-like infinity* $i^+$: the region toward which time-like geodesics extend. It corresponds to the points at $t \to \infty$ with finite $r$.

*Past time-like infinity* $i^-$: the region from which time-like geodesics come. It corresponds to the points at $t \to -\infty$ with finite $r$.

*Spatial infinity* $i^0$: the region toward which space-like slices extend. It corresponds to the points at $r \to \infty$ with finite $t$.

*Future null infinity* $\mathscr{I}^+$: the region toward which outgoing null geodesics extend. It corresponds to the points at $t + r \to \infty$ with finite $t - r$.

*Past null infinity* $\mathscr{I}^-$: the region from which ingoing null geodesics come. It corresponds to the points at $t - r \to -\infty$ with finite $t + r$.

These five asymptotic regions are points or segments in the Penrose diagram. Their $T$ and $R$ coordinates are:

$$\begin{aligned} &i^+ \qquad T = \pi\,, \quad R = 0\,. \\ &i^- \qquad T = -\pi\,, \quad R = 0\,. \\ &i^0 \qquad T = 0\,, \quad R = \pi\,, \end{aligned} \tag{8.83}$$

and

$$\begin{aligned} &\mathscr{I}^+ \qquad T + R = \pi\,, \quad T - R \in (-\pi;\pi)\,. \\ &\mathscr{I}^- \qquad T - R = -\pi\,, \quad T + R \in (-\pi;\pi)\,. \end{aligned} \tag{8.84}$$

### 8.6.2 Schwarzschild Spacetime

Penrose diagrams become a powerful tool to explore the global properties and the causal structure of more complicated spacetimes.

Let us now consider the Schwarzschild spacetime in Kruskal–Szekeres coordinates. The line element is given in Eq. (8.78). With the following coordinate transformation

[5] The symbol $\mathscr{I}$ is usually pronounced "scri".

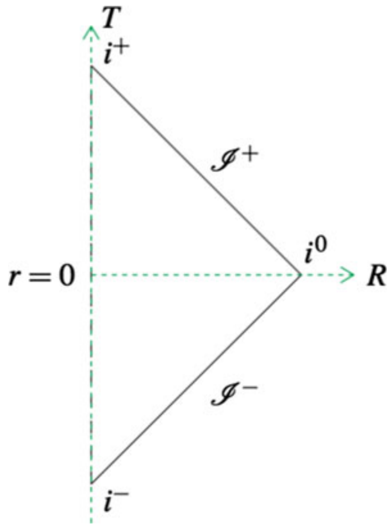


**Fig. 8.2** Penrose diagram for the Minkowski spacetime

$$
\begin{aligned}
V &= \frac{1}{2}\tan\frac{T+R}{2} + \frac{1}{2}\tan\frac{T-R}{2}\,, \\
U &= \frac{1}{2}\tan\frac{T+R}{2} - \frac{1}{2}\tan\frac{T-R}{2}\,,
\end{aligned}
\tag{8.85}
$$

the line element becomes

$$
\begin{aligned}
ds^2 = &\frac{32G_\mathrm{N}^3M^3}{r}e^{-r/(2G_\mathrm{N}M)}\left(4\cos^2\frac{T+R}{2}\cos^2\frac{T-R}{2}\right)^{-1}\left(-dT^2+dR^2\right) \\
&+r^2d\theta^2+r^2\sin^2\theta d\phi^2\,.
\end{aligned}
\tag{8.86}
$$

Figure 8.3 shows the Penrose diagram for the maximal extension of the Schwarzschild spacetime with its asymptotic regions $i^+$, $i^-$, $i^0$, $\mathscr{I}^+$, and $\mathscr{I}^-$. We can distinguish four regions, indicated, respectively, by I, II, III, and IV in the figure.

Region I corresponds to our universe, namely the exterior region of the Schwarzschild spacetime in Schwarzschild coordinates. Region II is the black hole, so the Schwarzschild spacetime in Schwarzschild coordinates has only regions I and II. The central singularity of the black hole at $r = 0$ is represented by the line with wiggles above region II. The event horizon of the black hole at $r = r_\mathrm{S}$ is the red line at 45° separating regions I and II. Any ingoing light ray in region I is captured by the black hole, while any outgoing light ray in region I reaches future null infinity $\mathscr{I}^+$. Null and time-like geodesics in region II cannot exit the black hole and they necessarily fall to the singularity at $r = 0$.

Regions III and IV emerge from the extension of the Schwarzschild spacetime. Region III corresponds to another universe. The red line at 45° separating regions II and III is the event horizon of the black hole at $r = r_\mathrm{S}$. Like in region I, any light ray in

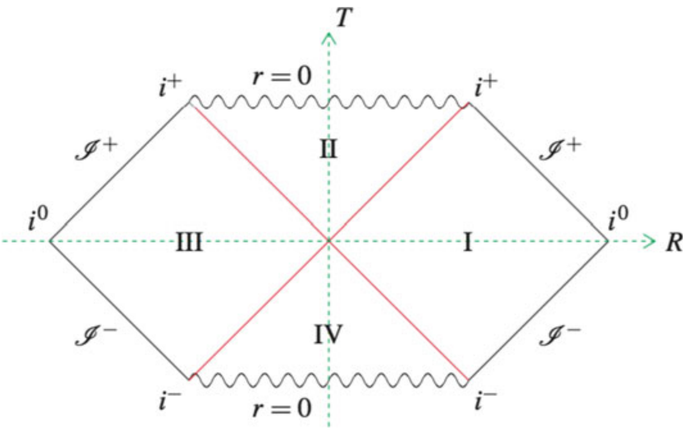


**Fig. 8.3** Penrose diagram for the maximal extension of the Schwarzschild spacetime

region III can either cross the event horizon or escape to infinity. No future-oriented null or time-like geodesics can escape from region II. Our universe in region I and the other universe in region III cannot communicate: no null or time-like geodesic can go from one region to another.

Region IV is a *white hole*. If a black hole is a region of the spacetime where null and time-like geodesics can only enter and never exit, a white hole is a region where null and time-like geodesics can only exit and never enter. The red lines at 45° separating region IV from regions I and III are the horizons at $r = r_S$ of the white hole.

## Problems

**8.1** Let us consider a massive particle orbiting a geodesic circular orbit in the Schwarzschild spacetime. Calculate the relation between the particle proper time and the coordinate time $t$ of the Schwarzschild metric.

**8.2** Let us consider the Penrose diagram for the Minkowski spacetime, Fig. 8.2. We have a massive particle that emits an electromagnetic pulse at $t = 0$. Show the trajectories of the massive particle and of the electromagnetic pulse in the Penrose diagram.

**8.3** Let us consider the Penrose diagram for the maximal extension of the Schwarzschild spacetime, Fig. 8.3. Show the future light-cone of an event in region I, of an event inside the black hole, and of an event inside the white hole.

# References

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2. P.K. Townsend, gr-qc/9707012