1

Basic equations of linear elasticity

Structural analysis is concerned with the evaluation of deformations and stresses arising within a solid object under the action of applied loads. If time is not explicitly considered as an independent variable, the analysis is said to be static; otherwise it is referred to as structural dynamic analysis, or simply structural dynamics. Under the assumption of small deformations and linearly elastic material behavior, three-dimensional formulations result in a set of fifteen linear first order partial differential equations involving the displacement field (three components), the stress field (six components) and the strain field (six components). This chapter presents the derivation of these governing equations. In many applications, this complex problem can be reduced to simpler, two-dimensional formulations called plane stress and plane strain problems.

For most situations, it is not possible to develop analytical solutions of these equations. Consequently, structural analysis is concerned with the analysis of *structural components*, such as bars, beams, plates, or shells, which will be addressed in subsequent chapters. In each case, assumptions are made about the behavior of these structural components, which considerably simplify the analysis process. For instance, given a suitable set of assumptions, the analysis of bar and beam problems reduces to the solution of one-dimensional equations for which analytical solutions are easily obtained.

1.1 The concept of stress

1.1.1 The state of stress at a point

The state of stress in a solid body is a measure of the intensity of forces acting within the solid. It can be visualized by cutting the solid by a plane normal to unit vector, $\bar{n}$, to create two free bodies which reveal the forces acting on the exposed surfaces. From basic statics, it is well-known that the distribution of forces and moments that will appear on the surface of the cut can be represented by an *equipollent* force, $\underline{F}$, acting at a point of the surface and a couple, $\underline{M}$. Newton's 3^{rd} law also requires a force and couple of equal magnitudes and opposite directions to act on the two surfaces created by the cut through the solid, as depicted in fig. 1.1. (See appendix A for a description of the vector, array and matrix notations used in this text.)

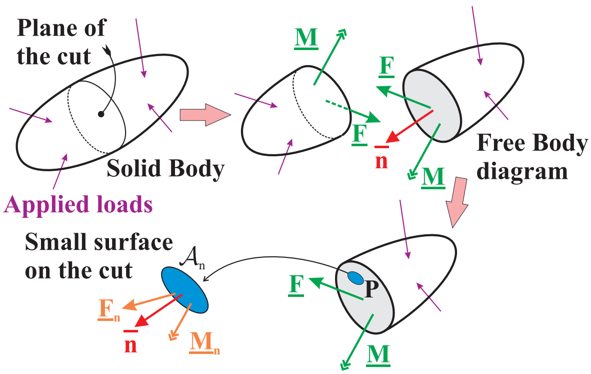

Fig. 1.1. A solid body cut by a plane to isolate a free body.

Consider now a small surface of area $\mathcal{A}_n$ located at point **P** on the surface generated by the cut in the solid. The forces and moments acting on this surface are equipollent to a force, $\underline{F}_n$, and couple, $\underline{M}_n$; note that these resultants are, in general, different, in both magnitude and orientation, from the corresponding resultants acting on the entire surface of the cut, as shown in fig. 1.1. Let the small surface be smaller and smaller until it becomes an element of infinitesimal area $\mathrm{d}\mathcal{A}_n \to 0$. As the surface shrinks to a differential size, the force and couple acting on the element keep decreasing in magnitude and changing in orientation whereas the normal to the surface remains the unit vector $\bar{n}$ of constant direction in space. This limiting process gives rise to the concept of *stress vector*, which is defined as

$$\underline{\tau}_n = \lim_{\mathrm{d}\mathcal{A}_n \to 0} \left(\frac{\underline{F}_n}{\mathrm{d}\mathcal{A}_n} \right). \tag{1.1}$$

The existence of the stress vector, *i.e.*, the existence of the limit in eq. (1.1), is a *fundamental assumption of continuum mechanics*. In this limiting process, it is assumed that the couple, $\underline{M}_n$, becomes smaller and smaller and, in the limit, $\underline{M}_n \to 0$ as $\mathrm{d}\mathcal{A}_n \to 0$; this is also an assumption of continuum mechanics which seems to be reasonable because in the limiting process, both forces and moment arms become increasingly small. Forces decrease because the area they act on decreases and moment arms decrease because the dimensions of the surface decrease. At the limit, the couple is the product of a differential element of force by a differential element of moment arm, giving rise to a negligible, second order differential quantity.

In conclusion, whereas an equipollent couple might act on the entire surface of the cut, the equipollent couple is assumed to vanish on a differential element of area of the same cut. The total force acting on a differential element of area, $\mathrm{d}\mathcal{A}_n$, is

$$\underline{F}_n = \mathrm{d}\mathcal{A}_n \, \underline{\tau}_n. \tag{1.2}$$

Clearly, the stress vector has units of force per unit area. In the SI system, this is measured in Newton per square meters, or Pascals (Pa).

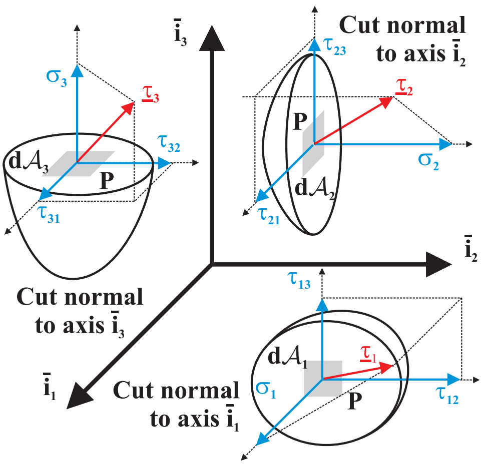

Fig. 1.2. A rigid body cut at point P by three planes orthogonal to the Cartesian axes.

During the limiting process described in the previous paragraph, the surface orientation, as defined by the normal to the surface, is kept constant in space. Had a different normal been selected, a different stress vector would have been obtained.

To illustrate this point, consider a solid body and a coordinate system, $\mathcal{I}$, consisting of three mutually orthogonal unit vectors, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$, as shown in fig. 1.2. First, the solid is cut at point **P** by a plane normal to axis $\bar{\imath}_1$; on the surface of the cut, at point **P**, a differential element of surface with an area $\mathrm{d}\mathcal{A}_1$ is defined and let $\underline{\tau}_1$ be the stress vector acting on this face. Next, the solid is cut at the same point by a plane normal to axis $\bar{\imath}_2$; at point **P**, let $\underline{\tau}_2$ be the stress vector acting on the differential element of surface with an area $\mathrm{d}\mathcal{A}_2$. Finally, the process is repeated a third time for a plane normal to axis $\bar{\imath}_3$; at point **P**, the stress vector $\underline{\tau}_3$ is acting on the differential element of surface with an area $\mathrm{d}\mathcal{A}_3$. Clearly, three stress vectors, $\underline{\tau}_1$, $\underline{\tau}_2$, and $\underline{\tau}_3$ are acting at the same point **P**, but on three mutually orthogonal faces normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively. Because these three stress vectors are acting on three faces with different orientations, there is no reason to believe that those stress vectors should be identical.

To further understand the state of stress at point **P**, the components of each stress vectors acting on the three faces are defined

$$\underline{\tau}_1 = \sigma_1 \, \bar{\imath}_1 + \tau_{12} \, \bar{\imath}_2 + \tau_{13} \, \bar{\imath}_3, \tag{1.3a}$$
$$\underline{\tau}_2 = \tau_{21} \, \bar{\imath}_1 + \sigma_2 \, \bar{\imath}_2 + \tau_{23} \, \bar{\imath}_3, \tag{1.3b}$$
$$\underline{\tau}_3 = \tau_{31} \, \bar{\imath}_1 + \tau_{32} \, \bar{\imath}_2 + \sigma_3 \, \bar{\imath}_3. \tag{1.3c}$$

The stress components σ_1, σ_2, and σ_3 are called *direct*, or *normal stresses*; they act on faces normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively, in directions along axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively. The stress components τ_{12} and τ_{13} are called *shearing* or *shear stresses*; both act on the face normal to axis $\bar{\imath}_1$, in directions of axes $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively. Similarly, stress components τ_{21} and τ_{23} both act on the face normal to axis $\bar{\imath}_2$, in directions of axes $\bar{\imath}_1$ and $\bar{\imath}_3$, respectively. Finally, stress components τ_{31} and τ_{32} both act on the face normal to axis $\bar{\imath}_3$, in directions along axes $\bar{\imath}_1$ and $\bar{\imath}_2$, respectively. The various stress components appearing in eq. (1.3) are referred to as the *engineering stress components*. The units of stress components are identical to those of the stress vector, force per unit area, or Pascal.

The stress components represented in fig. 1.2 are all defined as positive. Furthermore, the three faces depicted in this figure are positive faces. A face is *positive* when the outward normal to the face, *i.e.*, the normal pointing away from the body, is in the same direction as the axis to which the face is normal; a face is *negative* when its outward normal is pointing in the direction opposite to the axis to which the face is normal. The positive directions of stress components acting on negative faces are the opposite of those for stress components acting on positive faces. This sign convention is illustrated in fig. 1.3, which shows positive stress components acting on the six faces of a cube of differential size. Positive stress components are shown in solid lines on the three positive faces of the cube; positive stress components are shown in dotted lines on the three negative (hidden) faces of the cube.

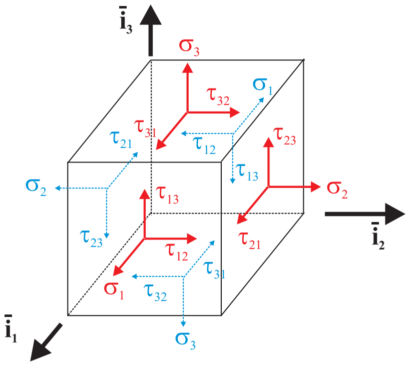

Fig. 1.3. Sign conventions for the stress components acting on a differential volume element. All stress components shown here are positive.

Taken together, the direct stress components σ_1, σ_2, and σ_3 and the shear stress components, τ_{12} and τ_{13}, τ_{21} and τ_{23}, and τ_{31} and τ_{32}, fully characterize the state of stress at point **P**. It will be shown in a later section that if the stress components acting on three orthogonal faces are known, it is possible to compute the stress components acting at the same point, on a face of arbitrary orientation. This discussion underlines the fact that the state of stress at a point is a complex concept: its complete definition requires the knowledge of nine stress components acting on three mutually orthogonal faces.

This should be contrasted with the concept of force. A force is vector quantity that is characterized by its magnitude and orientation. Alternatively, a force can be defined by the three components of the force vector in a given coordinate system. The definition of a force thus requires three quantities, whereas the definition of the stress state requires nine quantities.

A force is a vector, which is referred to as a *first order tensor*, whereas a state of stress is a *second order tensor*. Several quantities commonly used in solid mechanics are also second order tensors: the strain tensor, the bending stiffnesses of a beam, and the mass moments of inertia of a solid object. The first two of these quantities will be introduced in later sections and chapters. Much like the case for vectors, all second order tensors will be shown to possess certain common characteristics.

1.1.2 Volume equilibrium equations

In general, the state of stress varies throughout a solid body, and hence, stresses acting on two parallel faces located a small distance apart are not equal. Consider, for instance, the two opposite faces of a differential volume element that are normal to axis $\bar{\imath}_2$, as shown in fig. 1.4. The axial stress component on the negative face at coordinate x_2 is σ_2, but the stress components on the positive face at coordinate $x_2 + \mathrm{d}x_2$ will be slightly different and written as $\sigma_2(x_2 + \mathrm{d}x_2)$. If $\sigma_2(x_2)$ is an analytic function, it is then possible to express $\sigma_2(x_2 + \mathrm{d}x_2)$ in terms of $\sigma_2(x_2)$ using a Taylor series expansion to find

$$\sigma_2(x_2 + \mathrm{d}x_2) = \sigma_2(x_2) + \left.\frac{\partial \sigma_2}{\partial x_2}\right|_{x_2} \mathrm{d}x_2 + \ldots \text{higher order terms in } \mathrm{dx}_2.$$

This expansion is a fundamental step in the derivation of the differential equations governing the behavior of a continuum. The stress component on the positive face at coordinate $x_2 + \mathrm{d}x_2$ can be written as $\sigma_2(x_2 + \mathrm{d}x_2) \approx \sigma_2 + (\partial\sigma_2/\partial x_2)\mathrm{d}x_2$. The same Taylor series expansion technique can be applied to all other direct and shear stress components.

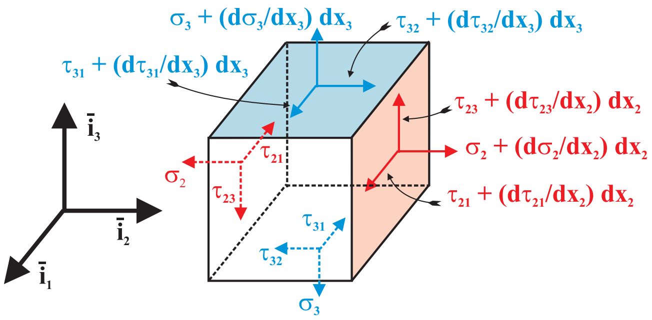

Fig. 1.4. Stress components acting on a differential element of volume. For clarity of the figure, the stress components acting on the faces normal to $\bar{\imath}_1$ are not shown.

Consider now the differential element of volume depicted in fig. 1.4. It is subjected to stress components acting on its six external faces and to body forces per unit volume, represented by a vector $\underline{b}$ acting at its centroid. These body forces could be gravity forces, inertial forces, or forces of an electric or magnetic origin; the components of this body force vector resolved in coordinate system $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ as $\underline{b} = b_1\,\bar{\imath}_1 + b_2\,\bar{\imath}_2 + b_3\,\bar{\imath}_3$. The units of the force vector are force per unit volume or Newton per cubic meter.

Force equilibrium

According to Newton's law, static equilibrium requires the sum of all the forces acting on this differential element to vanish. Considering all the forces acting along the direction of axis $\bar{\imath}_1$, the equilibrium condition is

$$
\begin{aligned}
&-\sigma_1\,\mathrm{d}x_2\mathrm{d}x_3 + \left(\sigma_1 + \frac{\partial\sigma_1}{\partial x_1}\mathrm{d}x_1\right)\,\mathrm{d}x_2\mathrm{d}x_3\\
&-\tau_{21}\,\mathrm{d}x_1\mathrm{d}x_3 + \left(\tau_{21} + \frac{\partial\tau_{21}}{\partial x_2}\mathrm{d}x_2\right)\,\mathrm{d}x_1\mathrm{d}x_3\\
&-\tau_{31}\,\mathrm{d}x_1\mathrm{d}x_2 + \left(\tau_{31} + \frac{\partial\tau_{31}}{\partial x_3}\mathrm{d}x_3\right)\,\mathrm{d}x_1\mathrm{d}x_2 + b_1\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = 0.
\end{aligned}
$$

This equation states an equilibrium of forces, and therefore the stress components must be multiplied by the area of the surface on which they act to yield the corresponding force. Similarly, the component of the body force per unit volume of the body is multiplied by the volume of the differential element, $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$, to give the body force acting on the element. After simplification, this equilibrium condition becomes

$$
\left[\frac{\partial\sigma_1}{\partial x_1} + \frac{\partial\tau_{21}}{\partial x_2} + \frac{\partial\tau_{31}}{\partial x_3} + b_1\right]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = 0.
$$

This equation is satisfied when the expression in brackets vanishes, and this yields the equilibrium equation in the direction of axis $\bar{\imath}_1$

$$
\frac{\partial\sigma_1}{\partial x_1} + \frac{\partial\tau_{21}}{\partial x_2} + \frac{\partial\tau_{31}}{\partial x_3} + b_1 = 0.
$$

For the same reasons, forces along axes $\bar{\imath}_2$ and $\bar{\imath}_3$ must vanish as well, and a similar reasoning yields the following three equilibrium equations

$$
\frac{\partial\sigma_1}{\partial x_1} + \frac{\partial\tau_{21}}{\partial x_2} + \frac{\partial\tau_{31}}{\partial x_3} + b_1 = 0, \tag{1.4a}
$$

$$
\frac{\partial\tau_{12}}{\partial x_1} + \frac{\partial\sigma_2}{\partial x_2} + \frac{\partial\tau_{32}}{\partial x_3} + b_2 = 0, \tag{1.4b}
$$

$$
\frac{\partial\tau_{13}}{\partial x_1} + \frac{\partial\tau_{23}}{\partial x_2} + \frac{\partial\sigma_3}{\partial x_3} + b_3 = 0, \tag{1.4c}
$$

which must be satisfied at all points inside the body.

The equilibrium conditions implied by Newton's law, eqs. (1.4), have been written by considering an differential element *of the undeformed body*. Of course, when forces are applied, the body deforms and so does every single differential element. Strictly speaking, equilibrium should be enforced *on the deformed configuration of the body*, rather than its undeformed configuration. Indeed, stresses are only present when external forces are applied and the body is deformed. When no forces are applied, the body is undeformed, but stresses all vanish.

Unfortunately, it is difficult to write equilibrium conditions on the deformed configuration of the body because this configuration is unknown; indeed, the goal of the theory of elasticity is to predict the deformation of elastic bodies under load. It is a basic assumption of the *linear theory of elasticity* developed here that the displacements of the body under the applied loads are very small, and hence, the difference between the deformed and undeformed configurations of the body is very small. Under this assumption, it is justified to impose equilibrium conditions to the undeformed configuration of the body, because it is nearly identical to its deformed configuration.

Moment equilibrium

To satisfy all equilibrium requirements, the sum of all the moments acting on the differential element of volume depicted in fig. 1.4 must also vanish. Consider first the moment equilibrium about axis $\bar{\imath}_1$. The contributions of the direct stresses and of the body forces can be eliminated by choosing an axis passing through the center of the differential element. The resulting moment equilibrium equation is

$$\begin{aligned}&\tau_{23}\ \mathrm{d}x_1\mathrm{d}x_3\frac{\mathrm{d}x_2}{2} + \left(\tau_{23} + \frac{\partial\tau_{23}}{\partial x_2}\mathrm{d}x_2\right)\ \mathrm{d}x_1\mathrm{d}x_3\frac{\mathrm{d}x_2}{2}\\ &-\tau_{32}\ \mathrm{d}x_1\mathrm{d}x_2\frac{\mathrm{d}x_3}{2} - \left(\tau_{32} + \frac{\partial\tau_{32}}{\partial x_3}\mathrm{d}x_3\right)\ \mathrm{d}x_1\mathrm{d}x_2\frac{\mathrm{d}x_3}{2}\\ &= \left[\tau_{23} - \tau_{32} + \frac{\partial\tau_{23}}{\partial x_2}\frac{\mathrm{d}x_2}{2} - \frac{\partial\tau_{32}}{\partial x_3}\frac{\mathrm{d}x_3}{2}\right]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = 0.\end{aligned}$$

The bracketed expression must vanish and after neglecting higher order terms, this reduces to the following equilibrium condition

$$\tau_{23} - \tau_{32} = 0.$$

Enforcing the vanishing of the sum of the moments about axes $\bar{\imath}_2$ and $\bar{\imath}_3$ leads to similar equations,

$$\tau_{23} = \tau_{32},\ \tau_{13} = \tau_{31},\ \tau_{12} = \tau_{21}. \quad (1.5)$$

The implication of these equalities is summarized by the principle of reciprocity of shear stresses, which is illustrated in fig. 1.5.

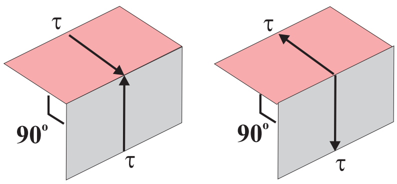

Fig. 1.5. Reciprocity of the shearing stresses acting on two orthogonal faces.

Principle 1 (Principle of reciprocity of shear stresses) *Shear stresses acting in the direction normal to the common edge of two orthogonal faces must be equal in magnitude and be simultaneously oriented toward or away from the common edge.*

Another implication of the reciprocity of the shear stresses is that of the nine components of stresses, six only are independent. It is common practice to arrange the stress tensor components in a 3×3 matrix format

$$\begin{bmatrix}\sigma_1 & \tau_{12} & \tau_{13}\\ \tau_{12} & \sigma_2 & \tau_{23}\\ \tau_{13} & \tau_{23} & \sigma_3\end{bmatrix}. \quad (1.6)$$

The principle of reciprocity implies the *symmetry of the stress tensor*.

1.1.3 Surface equilibrium equations

At the outer surface of the body, the stresses acting inside the body must be in equilibrium with the externally applied *surface tractions*. Surface tractions are represented by a stress vector, $\underline{t}$, that can be resolved in reference frame $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ as $\underline{t} = t_1\,\bar{\imath}_1 + t_2\,\bar{\imath}_2 + t_3\,\bar{\imath}_3$. Figure 1.6 shows a free body in the form of a differential tetrahedron bounded by three negative faces cut through the body in directions normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, and by a fourth face, **ABC**, of area $\mathrm{d}\mathcal{A}_n$, which is a differential element of the outer surface of the body. The unit normal to this element of area is denoted $\bar{n}$, and its components in coordinate system $\mathcal{I}$ are $\bar{n} = n_1\,\bar{\imath}_1 + n_2\,\bar{\imath}_2 + n_3\,\bar{\imath}_3$. Note that n_1, n_2, and n_3 are the cosines of the angle between $\bar{n}$ and $\bar{\imath}_1$, $\bar{n}$ and $\bar{\imath}_2$, and $\bar{n}$ and $\bar{\imath}_3$, respectively, also called the *direction cosines* of $\bar{n}$: $n_1 = \bar{n}\cdot\bar{\imath}_1 = \cos(\bar{n}, \bar{\imath}_1)$, $n_2 = \bar{n}\cdot\bar{\imath}_2 = \cos(\bar{n}, \bar{\imath}_2)$, and $n_3 = \bar{n}\cdot\bar{\imath}_3 = \cos(\bar{n}, \bar{\imath}_3)$.

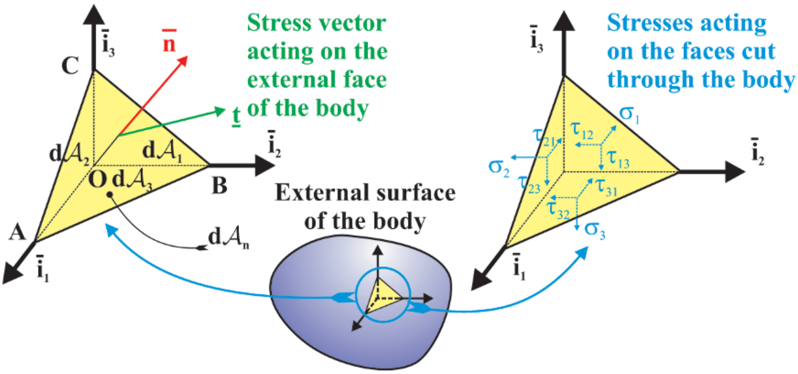

Fig. 1.6. A tetrahedron with one face along the outer surface of the body.

Equilibrium of forces acting along axis $\bar{\imath}_1$ implies

$$t_1 \mathrm{d}\mathcal{A}_n = \sigma_1 \mathrm{d}\mathcal{A}_1 + \tau_{21}\mathrm{d}\mathcal{A}_2 + \tau_{31}\mathrm{d}\mathcal{A}_3 - b_1 \frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3}{6}, \tag{1.7}$$

where $\mathrm{d}\mathcal{A}_1$, $\mathrm{d}\mathcal{A}_2$, and $\mathrm{d}\mathcal{A}_3$ are the areas of triangles **OBC**, **OAC** and **OAB**, respectively, and the last term represents the body force times the volume of the tetrahedron. The areas of the three faces normal to the axes are found by projecting face **ABC** onto planes normal to the axes using the direction cosines to find

$$\mathrm{d}\mathcal{A}_1 = n_1 \mathrm{d}\mathcal{A}_n, \quad \mathrm{d}\mathcal{A}_2 = n_2 \mathrm{d}\mathcal{A}_n, \quad \text{and} \quad \mathrm{d}\mathcal{A}_3 = n_3 \mathrm{d}\mathcal{A}_n. \tag{1.8}$$

Dividing eq. (1.7) by $\mathrm{d}\mathcal{A}_n$ then yields the first component of the surface traction vector

$$t_1 = \sigma_1 n_1 + \tau_{21} n_2 + \tau_{31} n_3,$$

where the body force term vanishes because it is a higher order differential term. The same procedure can be followed to express equilibrium conditions along the directions of axes $\bar{\imath}_2$ and $\bar{\imath}_3$. The three components of the surface traction vector then become

$$t_1 = \sigma_1\, n_1 + \tau_{12}\, n_2 + \tau_{13}\, n_3, \tag{1.9a}$$

$$t_2 = \tau_{12}\, n_1 + \sigma_2\, n_2 + \tau_{23}\, n_3, \tag{1.9b}$$

$$t_3 = \tau_{31}\, n_1 + \tau_{32}\, n_2 + \sigma_3\, n_3. \tag{1.9c}$$

A body is said to be in equilibrium if eqs. (1.4) are satisfied at all points inside the body, and eqs. (1.9) are satisfied at all points of its external surface.

1.2 Analysis of the state of stress at a point

The state of stress at a point is characterized in the previous section by the normal and shear stress components acting on the faces of a differential element of volume cut from the solid. The faces of this cube are cut normal to the axes of a Cartesian reference frame $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$, and the stress vector acting on these faces are resolved along the same axes. Clearly, another face at an arbitrary orientation with respect to these axes can be selected. In section 1.2.1, it will be shown that the stresses acting on this face can be related to the stresses acting on the faces normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$. This important result implies that once the stress components are known on three mutually orthogonal faces at a point, they are known on *any* face passing through that point. Hence, the state of stress at a point is fully defined once the stress components acting on three mutually orthogonal faces at a point are known.

1.2.1 Stress components acting on an arbitrary face

To establish relationships between stresses, it is necessary to consider force or moment equilibrium due to these stresses, and this must be done with reference to a specific free body diagram. Figure 1.7 shows a specific free body constructed from a tetrahedron defined by three faces cut normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, and a fourth face normal to unit vector $\bar{n} = n_1\, \bar{\imath}_1 + n_2\, \bar{\imath}_1 + n_3\, \bar{\imath}_3$, of arbitrary orientation. This tetrahedron is known as *Cauchy's tetrahedron.* The components, n_1, n_2, and n_3, of this unit vector are the *direction cosines* of unit vector $\bar{n}$, *i.e.*, the cosines of the angles between $\bar{n}$ and $\bar{\imath}_1$, $\bar{n}$ and $\bar{\imath}_2$, and $\bar{n}$ and $\bar{\imath}_3$, respectively.

Figure 1.7 shows the stress components acting on faces **COB**, **AOC** and **AOB**, of area $\mathrm{d}\mathcal{A}_1$, $\mathrm{d}\mathcal{A}_2$, and $\mathrm{d}\mathcal{A}_3$, respectively; the stress vector, τ_n, acts on face **ABC** of area $\mathrm{d}\mathcal{A}_n$. The body force vector, $\underline{b}$, is also acting on this tetrahedron. Equilibrium of forces acting on tetrahedron **OABC** requires

$$\underline{\tau}_1 \mathrm{d}\mathcal{A}_1 + \underline{\tau}_2 \mathrm{d}\mathcal{A}_2 + \underline{\tau}_3 \mathrm{d}\mathcal{A}_3 = \underline{\tau}_n \mathrm{d}\mathcal{A}_n + \underline{b}\, \mathrm{d}\mathcal{V},$$

where $\underline{\tau}_1$, $\underline{\tau}_2$ and $\underline{\tau}_3$ are the stress vectors acting on the faces normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively, and $\mathrm{d}\mathcal{V}$ is the volume of the tetrahedron.

Dividing this equilibrium equation by $\mathrm{d}\mathcal{A}_n$ and using eq. (1.8) gives the stress vector acting of the inclined face as

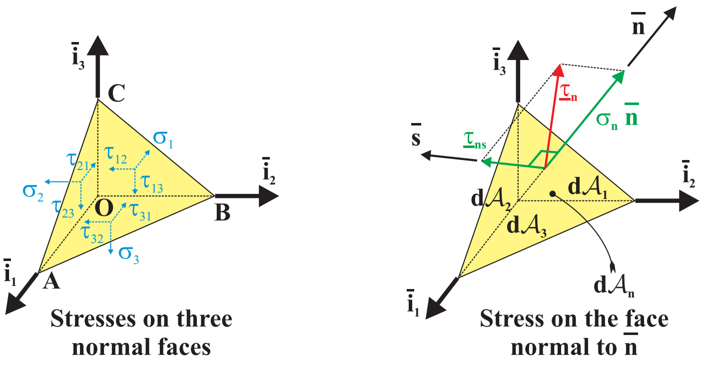

Fig. 1.7. Differential tetrahedron element with one face, **ABC**, normal to unit vector $\underline{n}$ and the other three faces normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively.

$$\underline{\tau}_n = \underline{\tau}_1 n_1 + \underline{\tau}_2 n_2 + \underline{\tau}_3 n_3 - \underline{b}\, \mathrm{d}\mathcal{V}/\mathrm{d}\mathcal{A}_n$$

The body force term is multiplied by a higher order term, $\mathrm{d}\mathcal{V}/\mathrm{d}\mathcal{A}_n$, which can be neglected in the equilibrium condition. Expanding the three stress vectors in terms on the stress components then yields

$$\underline{\tau}_n = (\sigma_1\bar{\imath}_1 + \tau_{12}\bar{\imath}_2 + \tau_{13}\bar{\imath}_3)\, n_1 + (\tau_{21}\bar{\imath}_1 + \sigma_2\bar{\imath}_2 + \tau_{23}\bar{\imath}_3)\, n_2 + (\tau_{31}\bar{\imath}_1 + \tau_{32}\bar{\imath}_2 + \sigma_3\bar{\imath}_3)\, n_3. \tag{1.10}$$

To determine the direct stress, σ_n, acting on face **ABC**, it is necessary to project this vector equation in the direction of unit vector $\bar{n}$. This can be achieved by taking the dot product of the stress vector by unit vector $\bar{n}$ to find

$$\bar{n} \cdot \underline{\tau}_n = \bar{n}\cdot \left[(\sigma_1\bar{\imath}_1 + \tau_{12}\bar{\imath}_2 + \tau_{13}\bar{\imath}_3)\, n_1 + (\tau_{21}\bar{\imath}_1 + \sigma_2\bar{\imath}_2 + \tau_{23}\bar{\imath}_3)\, n_2 + \right. \\ \left. (\tau_{31}\bar{\imath}_1 + \tau_{32}\bar{\imath}_2 + \sigma_3\bar{\imath}_3)\, n_3 \right].$$

Because $\bar{n} = n_1\, \bar{\imath}_1 + n_2\, \bar{\imath}_1 + n_3\, \bar{\imath}_3$, this yields

$$\sigma_n = (\sigma_1 n_1 + \tau_{12} n_2 + \tau_{13} n_3)\, n_1 + (\tau_{21} n_1 + \sigma_2 n_2 + \tau_{23} n_3)\, n_2 \\ + (\tau_{31} n_1 + \tau_{32} n_2 + \sigma_3 n_3)\, n_3,$$

and finally, after minor a rearrangement of terms,

$$\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 + 2\tau_{23} n_2 n_3 + 2\tau_{13} n_1 n_3 + 2\tau_{12} n_1 n_2. \tag{1.11}$$

The stress components acting in the plane of face **ABC** can be evaluated in a similar manner by projecting eq. (1.10) along a unit vector in the plane of face **ABC**. Consider a unit vector, $\bar{s} = s_1\, \bar{\imath}_1 + s_2\, \bar{\imath}_1 + s_3\, \bar{\imath}_3$, normal to $\bar{n}$, *i.e.*, such that $\bar{n} \cdot \bar{s} = 0$. The shear stress component acting on face **ABC** in the direction of unit vector $\bar{s}$ is denoted τ_{ns} and is obtained by projecting eq. (1.10) along vector $\bar{s}$ to find

$$\tau_{ns} = (\sigma_1 s_1 + \tau_{12} s_2 + \tau_{13} s_3)\, n_1 + (\tau_{21} s_1 + \sigma_2 s_2 + \tau_{23} s_3)\, n_2 \\ + (\tau_{31} s_1 + \tau_{32} s_2 + \sigma_3 s_3)\, n_3,$$

and finally, after minor a rearrangement of terms,

$$\tau_{ns} = \sigma_1 n_1 s_1 + \sigma_2 n_2 s_2 + \sigma_3 n_3 s_3 + \tau_{12}(n_2 s_1 + n_1 s_2) + \tau_{13}(n_1 s_3 + n_3 s_1) + \tau_{23}(n_2 s_3 + n_3 s_2). \tag{1.12}$$

Equations. (1.11) and (1.12) express an important result of continuum mechanics. They imply that once the stress components acting on three mutually orthogonal faces are known, the stress components on a face of arbitrary orientation can be readily computed. To evaluate the direct stress component acting on an arbitrary face, all that is required are the direction cosines of the normal to the face. Evaluation the shear stress component acting on the same face requires, in addition, the direction cosines of the direction of the shear stress component in that face.

Consider the following question: how much information is required to fully define the state of stress at point **P** of a solid? Clearly, the body can be cut at this point by a plane of arbitrary orientation. The stress vector acting on this face gives information about the state of stress at point **P**. The stress vector acting on a face with another orientation would give additional information about the state of stress at the same point. If additional faces are considered, each new stress vector provides additional information. This reasoning would seem to imply that the complete knowledge of the state of stress at a point requires an infinite amount of information, specifically, the stress vectors acting on *all* the possible faces passing through point **P**. Equations. (1.11) and (1.12), however, demonstrate the fallacy of this reasoning: once the stress vectors acting on three mutually orthogonal faces are known, the stress vector acting on *any* other face can be readily predicted. In conclusion, complete definition of the state of stress at a point only requires knowledge of the stress vectors, or equivalently of the stress tensor components, acting on three mutually orthogonal faces.

1.2.2 Principal stresses

As discussed in the previous section, eqs. (1.11) and (1.12) enable the computation of the stress components acting on a face of arbitrary orientation, based on the knowledge of the stress components acting on three mutually orthogonal faces. As illustrated in fig. 1.7, the stress vector acting on a face of arbitrary orientation has, in general, a component $\sigma_n\,\bar{n}$, acting in the direction normal to the face, and a component $\tau_{ns}\,\bar{s}$, acting within the plane of the face.

This discussion raises the following question: is there a face orientation for which the stress vector is exactly normal to the face? In other words, does a particular orientation, $\bar{n}$, exist for which the stress vector acting on this face *consists solely* of $\underline{\tau}_n = \sigma_p\,\bar{n}$, where σ_p is the yet unknown magnitude of this direct stress component?

Introducing this expression into eq. (1.10) results in

$$\sigma_p \bar{n} = (\sigma_1 \bar{\imath}_1 + \tau_{12} \bar{\imath}_2 + \tau_{13} \bar{\imath}_3)\, n_1 + (\tau_{21} \bar{\imath}_1 + \sigma_2 \bar{\imath}_2 + \tau_{23} \bar{\imath}_3)\, n_2 + (\tau_{31} \bar{\imath}_1 + \tau_{32} \bar{\imath}_2 + \sigma_3 \bar{\imath}_3)\, n_3.$$

This equation alone does not allow the determination of both σ_p and of unit vector $\bar{n}$. Projecting this vector relationship along axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$ leads to the following three scalar equations

$$
\begin{aligned}
(\sigma_1 - \sigma_p)\, n_1 + \tau_{12}\, n_2 + \tau_{13}\, n_3 &= 0, \\
\tau_{12}\, n_1 + (\sigma_2 - \sigma_p)\, n_2 + \tau_{23}\, n_3 &= 0, \\
\tau_{13}\, n_1 + \tau_{23}\, n_2 + (\sigma_3 - \sigma_p)\, n_3 &= 0,
\end{aligned}
$$

respectively. The unknowns of the problem are the direction cosines, n_1, n_2, and n_3 that define the orientation of the face on which shear stresses vanish, and the magnitude, σ_p, of the direct stress component acting on this face.

These equations are recast as a homogeneous system of linear equations for the unknown direction cosines

$$
\begin{bmatrix} \sigma_1 - \sigma_p & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 - \sigma_p & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 - \sigma_p \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0. \tag{1.13}
$$

Since this is a homogeneous system of equations, the trivial solution, $n_1 = n_2 = n_3 = 0$, is, in general, the solution of this system. When the determinant of the system vanishes, however, non-trivial solutions will exist. The vanishing of the determinant of the system leads to the cubic equation for the magnitude of the direct stress

$$
\sigma_p^3 - I_1 \sigma_p^2 + I_2 \sigma_p - I_3 = 0, \tag{1.14}
$$

where the quantities I_1, I_2, and I_3 are defined as

$$
I_1 = \sigma_1 + \sigma_2 + \sigma_3, \tag{1.15a}
$$

$$
I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 - \tau_{12}^2 - \tau_{13}^2 - \tau_{23}^2, \tag{1.15b}
$$

$$
I_3 = \sigma_1\sigma_2\sigma_3 - \sigma_1\tau_{23}^2 - \sigma_2\tau_{13}^2 - \sigma_3\tau_{12}^2 + 2\tau_{12}\tau_{13}\tau_{23}, \tag{1.15c}
$$

are called the three *stress invariants*.

The solutions of eq. (1.14) are called the *principal stresses*. Since this is a cubic equation, three solutions exist, denoted σ_{p1}, σ_{p2}, and σ_{p3}. For each of these three solutions, the matrix of the system of equations defined by eq. (1.13) has a zero determinant, and a non-trivial solution exists for the directions cosines that now define the direction of a face on which the shear stresses vanish. This direction is called a *principal stress direction*. Because the equations to be solved are homogeneous, their solution will include an arbitrary constant, which can be determined by enforcing the normality condition for unit vector $\bar{n}$, $n_1^2 + n_2^2 + n_3^2 = 1$.

This solution process can be repeated for each of the three principal stresses. This will result in three different principal stress directions. It can be shown that these three directions are mutually orthogonal.

1.2.3 Rotation of stresses

In the previous sections, free body diagrams are formed with faces cut in directions normal to axes of the orthonormal basis $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$, and the stress vectors are resolved into stress components along the same directions. The orientation of this basis is entirely arbitrary: basis $\mathcal{I}^* = (\bar{\imath}_1^*, \bar{\imath}_2^*, \bar{\imath}_3^*)$ could also have been selected, and an analysis identical to that of the previous sections would have led to the definition of normal stresses σ_1^*, σ_2^*, σ_3^*, and shear stresses τ_{23}^*, τ_{13}^*, τ_{12}^*. A typical equilibrium equation at a point of the body would be written as

$$\frac{\partial \sigma_1^*}{\partial x_1^*} + \frac{\partial \tau_{21}^*}{\partial x_2^*} + \frac{\partial \tau_{31}^*}{\partial x_3^*} + b_1^* = 0, \tag{1.16}$$

where the notation $(.)^*$ is used to indicate the components of the corresponding quantity resolved in basis $\mathcal{I}^*$. A typical surface traction would be defined as

$$t_1^* = n_1^* \, \sigma_1^* + n_2^* \, \tau_{21}^* + n_3^* \, \tau_{31}^*. \tag{1.17}$$

Although expressed in different reference frames, eqs. (1.4) and (1.16), or (1.9) and (1.17) express the same equilibrium conditions for the body. Two orthonormal bases, $\mathcal{I}$ and $\mathcal{I}^*$, are involved in this problem. The orientation of basis $\mathcal{I}^*$ relative to basis $\mathcal{I}$ is discussed in section A.3.1 and leads to the definition of the matrix of direction cosines, or rotation matrix, $\underline{\underline{R}}$, given by eq. (A.36).

Consider the stress component σ_1^*: it represents the magnitude of the direct stress component acting on the face normal to axis $\bar{\imath}_1^*$. Equation (1.11) can now be used to express this stress component in terms of the stress components resolved in axis system $\mathcal{I}$ to find

$$\sigma_1^* = \sigma_1 \ell_1^2 + \sigma_2 \ell_2^2 + \sigma_3 \ell_3^2 + 2\tau_{23}\ell_2\ell_3 + 2\tau_{13}\ell_1\ell_3 + 2\tau_{12}\ell_1\ell_2, \tag{1.18}$$

where ℓ_1, ℓ_2, and ℓ_3, are the direction cosines of unit vector $\bar{\imath}_1^*$. Similar equations can be derived to express the stress components σ_2^* and σ_3^* in terms of the stress components resolved in axis system $\mathcal{I}$. For σ_2^*, the direction cosines ℓ_1, ℓ_2, and ℓ_3 appearing in eq. (1.18) are replaced by direction cosines m_1, m_2, and m_3, respectively, whereas direction cosines n_1, n_2, and n_3 will appear in the expression for σ_3^*. Coordinate rotations are defined in appendix A.3.

The shear stress components follow from eq. (1.12) as

$$\begin{aligned} \tau_{12}^* = \; & \sigma_1 \ell_1 m_1 + \sigma_2 \ell_2 m_2 + \sigma_3 \ell_3 m_3 + \tau_{12}(\ell_2 m_1 + \ell_1 m_2) \\ & + \tau_{13}(\ell_1 m_3 + \ell_3 m_1) + \tau_{23}(\ell_2 m_3 + \ell_3 m_2). \end{aligned} \tag{1.19}$$

Here again, similar relationships can be derived for the remaining shear stress components, τ_{13}^* and τ_{23}^*, through appropriate cyclic permutation of the indices.

All these relationships can be combined into the following compact matrix equation

$$\begin{bmatrix} \sigma_1^* & \tau_{12}^* & \tau_{13}^* \\ \tau_{21}^* & \sigma_2^* & \tau_{23}^* \\ \tau_{31}^* & \tau_{32}^* & \sigma_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \underline{\underline{R}}, \tag{1.20}$$

where $\underline{\underline{R}}$ is the rotation matrix defined by eq. (A.36). This equation concisely encapsulates the relationship between the stress components resolved in two different coordinate systems, and it can be used to compute the stress components resolved in basis $\mathcal{I}^*$ in terms of the stress components resolved in basis $\mathcal{I}$.

Finally, since the principal stresses at a point are independent of the particular coordinate system used to define the stress state, the coefficients of the cubic equation that determines the principal stresses, eq. (1.14), must be invariant with respect to reference frames. This is the very reason why quantities I_1, I_2, and I_3 defined by eq. (1.15) are called the *stress invariants*. The word "invariant" refers to the fact that these quantities are *invariant with respect to a change of coordinate system*. Let $\mathcal{I}^*$ and $\mathcal{I}$ be two different orthonormal bases,

$$I_1 = \sigma_1^* + \sigma_2^* + \sigma_3^* = \sigma_1 + \sigma_2 + \sigma_3, \tag{1.21a}$$

$$\begin{aligned} I_2 &= \sigma_1^*\sigma_2^* + \sigma_2^*\sigma_3^* + \sigma_3^*\sigma_1^* - \tau_{12}^{*2} - \tau_{13}^{*2} - \tau_{23}^{*2} \\ &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 - \tau_{12}^2 - \tau_{13}^2 - \tau_{23}^2, \end{aligned} \tag{1.21b}$$

$$\begin{aligned} I_3 &= \sigma_1^*\sigma_2^*\sigma_3^* - \sigma_1^*\tau_{23}^{*2} - \sigma_2^*\tau_{13}^{*2} - \sigma_3^*\tau_{12}^{*2} + 2\tau_{12}^*\tau_{13}^*\tau_{23}^* \\ &= \sigma_1\sigma_2\sigma_3 - \sigma_1\tau_{23}^2 - \sigma_2\tau_{13}^2 - \sigma_3\tau_{12}^2 + 2\tau_{12}\tau_{13}\tau_{23}. \end{aligned} \tag{1.21c}$$

Tedious algebra using eqs. (1.20) to write the stress components resolved in basis $\mathcal{I}^*$ in terms of the stresses components resolved in basis $\mathcal{I}$ will reveal that the above relationships are correct.

Example 1.1. Computing principal stresses
Consider the following stress tensor

$$\underline{\underline{S}} = \begin{bmatrix} -5 & -4 & 0 \\ -4 & 1 & 0 \\ & 0 & 1 \end{bmatrix}.$$

Compute the principal stresses and the principal stress directions. The stress invariants defined by eq. (1.15) are computed as $I_1 = -3$, $I_2 = -25$ and $I_3 = -21$. The principal stress equation, eq. (1.14), now becomes

$$\sigma_p^3 + 3\sigma_p^2 - 25\sigma_p + 21 = (\sigma_p - 1)(\sigma_p^2 + 4\sigma_p - 21) = 0,$$

The solutions of this cubic equations yield the principal stresses as $\sigma_{p1} = 3$, $\sigma_{p2} = 1$ and $\sigma_{p3} = -7$.

Next, the principal direction associated with $\sigma_{p1} = 3$ is computed. The homogeneous system defined by eq. (1.13) becomes

$$\begin{bmatrix} -8 & -4 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0.$$

The determinant of this system vanishes because the first two equations are a multiple of each other. The first equation yields $n_1 = \alpha$ and $n_2 = -2\alpha$, where α is an arbitrary constant, whereas the third equation gives $n_3 = 0$. Since the principal direction must be unit vector, $n_1^2 + n_2^2 + n_3^2 = 1$, or $5\alpha^2 = 1$; finally $n_1 = 1/\sqrt{5}$, $n_2 = -2/\sqrt{5}$ and $n_3 = 0$. Proceeding in a similar manner for the other two principal stresses, the three principal directions are found to be

$$\bar{n}_1 = \frac{1}{\sqrt{5}} \begin{Bmatrix} 1 \\ -2 \\ 0 \end{Bmatrix}; \quad \bar{n}_2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}; \quad \bar{n}_3 = \frac{1}{\sqrt{5}} \begin{Bmatrix} -2 \\ -1 \\ 0 \end{Bmatrix}.$$

It is easily verified that the principal directions are orthogonal to each other; indeed, $\bar{n}_1 \cdot \bar{n}_2 = \bar{n}_2 \cdot \bar{n}_3 = \bar{n}_3 \cdot \bar{n}_1 = 0$.

Example 1.2. Principal stresses as an eigenproblem
Consider the following stress tensor

$$\underline{\underline{S}} = \begin{bmatrix} 5.0 & 2.5 & -1.3 \\ 2.5 & 7.8 & -3.4 \\ -1.3 & -3.4 & -4.5 \end{bmatrix}.$$

Compute the principal stresses and the principal stress directions. Rather than following the procedure described in the previous examples, the homogeneous system of linear equations, eq. (1.13), that govern the problem is recast as

$$\begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \sigma_p \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}. \tag{1.22}$$

In this form, it becomes clear (see appendix A.2.4) that the determination of the principal stresses and principal stress directions is equivalent to the determination of the three eigenvalues, σ_{p1}, σ_{p2} and σ_{p3}, of the stress tensor, and determination of the corresponding three eigenvectors, $\bar{n}_1$, $\bar{n}_2$, and $\bar{n}_3$. Using a standard linear algebra software package, the three eigenpairs of the above stress tensor are found to be

$$\sigma_{p1} = -5.4180, \ \bar{n}_1 = \begin{Bmatrix} -0.064 \\ -0.237 \\ -0.969 \end{Bmatrix}; \quad \sigma_{p2} = 3.5693, \ \bar{n}_2 = \begin{Bmatrix} 0.879 \\ -0.473 \\ 0.058 \end{Bmatrix};$$
$$\sigma_{p3} = 10.1487, \ \bar{n}_3 = \begin{Bmatrix} 0.472 \\ 0.849 \\ -0.239 \end{Bmatrix}.$$

Here again, it is easily verified that the principal directions are orthogonal to each other by computing $\bar{n}_i \cdot \bar{n}_j$ for any combination of i and j. This can be represented in a more compact way by creating a matrix, denoted $\underline{\underline{P}}$, that is constructed by arranging the principal stress direction vectors as the columns

$$\underline{\underline{P}} = [\bar{n}_1, \bar{n}_2, \bar{n}_3] = \begin{bmatrix} -0.0640 & 0.8791 & 0.4723 \\ -0.2372 & -0.4731 & 0.8485 \\ -0.9693 & 0.0577 & -0.2388 \end{bmatrix}.$$

Because the principal directions are mutually orthogonal unit vectors, this matrix is orthogonal, that is: $\underline{\underline{P}}^T \underline{\underline{P}} = \underline{\underline{I}}$, where $\underline{\underline{I}}$ is the 3×3 identity matrix. Furthermore, since matrix $\underline{\underline{P}}$ stores the eigenvectors of the stress tensor $\underline{\underline{S}}$, it follows that the transformation $\underline{\underline{P}}^T \underline{\underline{S}}\, \underline{\underline{P}}$ will diagonalize the stress tensor. That is,

$$\underline{\underline{P}}^T \underline{\underline{S}}\, \underline{\underline{P}} = \begin{bmatrix} \sigma_{p1} & 0 & 0 \\ 0 & \sigma_{p2} & 0 \\ 0 & 0 & \sigma_{p3} \end{bmatrix} = \begin{bmatrix} -5.4180 & 0 & 0 \\ 0 & 3.5693 & 0 \\ 0 & 0 & 10.1487 \end{bmatrix},$$

and this can easily be verified by direct computation.

Example 1.3. Stresses acting on the octahedral face
Figure 1.8 shows a tetrahedron cut along three faces normal to the principal stress directions defined by axes $\bar{\imath}_1^*$, $\bar{\imath}_2^*$ and $\bar{\imath}_3^*$. The three mutually orthogonal edges of the tetrahedron each are of unit length. The fourth face of the tetrahedron is the *octahedral face* which is, by definition, the face that is equally inclined with respect to the principal stress directions. The normal to the octahedral face is $\bar{n}^T = \left\{1, 1, 1\right\}/\sqrt{3}$, *i.e.*, the direction cosines of this unit vector are $1/\sqrt{3}$ with respect to each of the three principal stress directions. Find the stress components acting on the octahedral face.

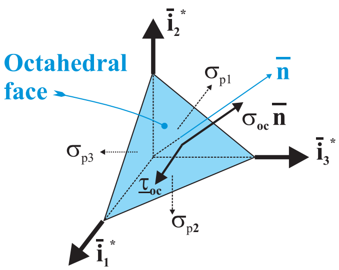

Fig. 1.8. The octahedral face.

By definition, the principal stress directions are such that on the corresponding faces, the shear stresses vanish. Hence, fig. 1.8 shows only the principal stress acting on each face. The stress vector acting on the octahedral face can be resolved into the octahedral direct stress vector, $\sigma_{\rm oc}\,\bar{n}$, acting in the direction normal to the octahedral face, and octahedral shear stress vector, $\underline{\tau}_{\rm oc}$, acting in the plane of the octahedral face. Using eq. (1.11), the magnitude of the direct octahedral stress is

$$\sigma_{\rm oc} = \sigma_{p1}\left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_{p2}\left(\frac{1}{\sqrt{3}}\right)^2 + \sigma_{p3}\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{\sigma_{p1} + \sigma_{p2} + \sigma_{p3}}{3}. \tag{1.23}$$

The direct stress acting on the octahedral face is the *average of the principal stresses*.

The equilibrium condition for the tetrahedron in fig. 1.8 is now

$$\frac{1}{2}\sigma_{p1}\,\bar{\imath}_1^* + \frac{1}{2}\sigma_{p2}\,\bar{\imath}_2^* + \frac{1}{2}\sigma_{p3}\,\bar{\imath}_3^* = \frac{\sqrt{3}}{2}(\sigma_{\rm oc}\,\bar{n} + \underline{\tau}_{\rm oc}), \tag{1.24}$$

where the factor of 1/2 represents the area of each of the three faces normal to the principal axes directions and $\sqrt{3}/2$ the area of the octahedral face which is an equilateral triangle with sides of length $\sqrt{2}$. The octahedral shear stress vector now becomes

$$\sqrt{3}\,\underline{\tau}_{\rm oc} = (\sigma_{p1} - \sigma_{\rm oc})\bar{\imath}_1^* + (\sigma_{p2} - \sigma_{\rm oc})\bar{\imath}_2^* + (\sigma_{p3} - \sigma_{\rm oc})\bar{\imath}_3^*.$$

The magnitude of the octahedral shear stress, $\tau_{\rm oc} = \|\underline{\tau}_{\rm oc}\|$, is

$$\tau_{\rm oc} = \frac{1}{\sqrt{3}}\left[(\sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2) - \frac{1}{3}(\sigma_{p1} + \sigma_{p2} + \sigma_{p3})^2\right]^{1/2}. \tag{1.25}$$

The first two invariants of the stress state, see eqs. (1.21a) and (1.21b), are easily expressed in terms of principal stresses as $I_1 = \sigma_{p1} + \sigma_{p2} + \sigma_{p3}$ and $I_2 = \sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p3}\sigma_{p1}$. The octahedral stresses are now expressed in terms of these invariants as

$$\sigma_{\rm oc} = \frac{I_1}{3}, \quad \tau_{\rm oc} = \frac{\sqrt{2}}{3}\sqrt{I_1^2 - 3I_2}.$$

1.2.4 Problems

Problem 1.1. Stresses on an inclined face

Consider the tetrahedron shown in fig. 1.7. A set of three mutually orthogonal unit vectors will be defined: $\bar{\ell}$ is a unit vector parallel to vector **AB**, $\bar{m}$ is such that $\bar{m} = \bar{n} \times \bar{\ell}$, and $\bar{n}$ is the normal to face **ABC**. Let the stress vector acting on face **ABC** be resolved along these axes, *i.e.*, let $\underline{\tau}_n = \tau_{nl}\,\bar{\ell} + \tau_{nm}\,\bar{m} + \sigma_n\,\bar{n}$. *(1)* Find the stress components, $\tau_{n\ell}$, τ_{nm} and σ_n, in terms of the stress components acting on the faces normal to axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$.

Problem 1.2. Principal stresses

Given a state of stress defined by: σ_1=200 MPa, σ_2=300 MPa, $\sigma_3 = -100$ MPa, $\tau_{12} =$ 50 MPa, $\tau_{13} = -80$ MPa and $\tau_{23} =$ 100 MPa, *(1)* Determine the principal stresses. *(2)* Determine the principal stress directions. Note: you should consider using a software package to handle the computations.

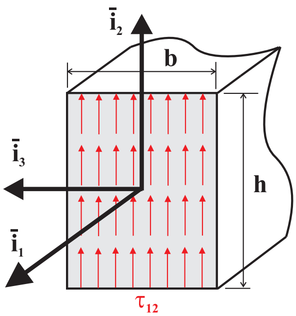

Fig. 1.9. Uniform distribution of shear stresses over the cross-section of a beam.

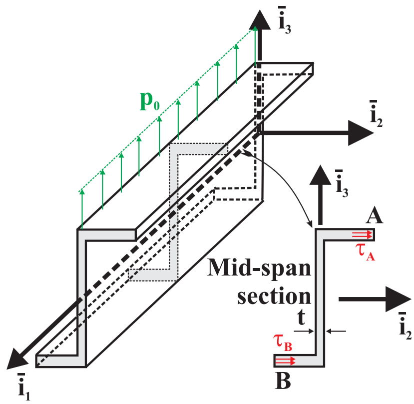

Fig. 1.10. Shear stresses at points **A** and **B** on cross-section.

Problem 1.3. Shear stress distribution over the cross-section of a beam
Figure 1.9 depicts a beam with a rectangular cross-section of a width b and height h. This beam is subjected to a vertical shear force, V_2, and the resulting shear stress distribution is assumed to be uniformly distributed over the cross-section, *i.e.*, $\tau_{12} = V_3/(bh)$. *(1)* Is this assumption reasonable? Explain your answer.

Problem 1.4. Shear stresses in a "Z" section
Figure 1.10 depicts a cantilevered beam with a "Z" cross-section subjected to a distributed transverse load p_0. Due to this loading, direct and shear stresses will develop in the beam. *(1)* Evaluate the shear stresses, denoted τ_A and τ_B, acting in the plane of the beam's mid-span cross-section at points **A** and **B**, respectively. Explain your answer.

1.3 The state of plane stress

A particular state of stress of great practical importance is the *plane state of stress*. In this case, all stress components acting along the direction of axis $\bar{\imath}_3$ are assumed to vanish, or to be negligible compared to the stress components acting in the other two directions. The only non-vanishing stress components are σ_1, σ_2, and τ_{12}, and furthermore, these stress components are assumed to be independent of x_3. This state of stress occurs, for instance, in a very thin plate or sheet subjected to loads applied in its own plane. This type of situation is illustrated by the thin sheet shown in fig. 1.11. For the plane stress state, the two flat surfaces of the thin sheet must be stress free.

1.3.1 Equilibrium equations

The equations of equilibrium derived for the general, three-dimensional case, see eq. (1.4), considerably simplify in the plane stress case. The equation in the $\bar{\imath}_3$ direction is satisfied, and the remaining two equations reduce to

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + b_1 = 0; \quad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + b_2 = 0. \tag{1.26}$$

Similar simplifications take place for the definition of surface tractions in eq. (1.9),

$$t_1 = n_1\,\sigma_1 + n_2\,\tau_{21}; \quad t_2 = n_1\,\tau_{12} + n_2\,\sigma_2. \tag{1.27}$$

For this two-dimensional problem, the boundary of the thin sheet on which externally applied stresses and forces may act is the thin edge defined by the curve $\mathcal{C}$ as shown in fig. 1.11. The outer normal to this curve is the unit vector $\bar{n} = n_1\,\bar{\imath}_1 + n_2\,\bar{\imath}_2$ and the tangent direction is the unit vector $\bar{s} = s_1\,\bar{\imath}_1 + s_2\,\bar{\imath}_2$. If θ is the angle between the normal and axis $\bar{\imath}_1$, it follows that $n_1 = \cos\theta$, $n_2 = \sin\theta$, $n_3 = 0$ and $s_1 = -\sin\theta$, $s_2 = \cos\theta$, $s_3 = 0$. The surface traction component in the direction of vector $\bar{n}$ then follows from eq. (1.11) as

$$t_n = \cos^2\theta\,\sigma_1 + \sin^2\theta\,\sigma_2 + 2\sin\theta\,\cos\theta\,\tau_{12}, \tag{1.28}$$

and eq. (1.12) yields the surface traction component in the direction of the tangent $\bar{s}$ to curve $\mathcal{C}$ as

$$t_s = \sin\theta\cos\theta(\sigma_2 - \sigma_1) + (\cos^2\theta - \sin^2\theta)\,\tau_{12}. \tag{1.29}$$

Thus, for plane stress problems, the equilibrium equations, eq. (1.26), must be satisfied at all points within the body, and along curve $\mathcal{C}$, the surface equilibrium equations, eq. (1.27), or eqs. (1.28) and (1.29), must be satisfied.

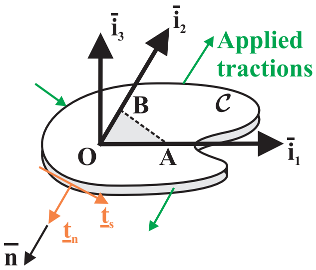

Fig. 1.11. Plane stress problem in thin sheet with in-plane tractions.

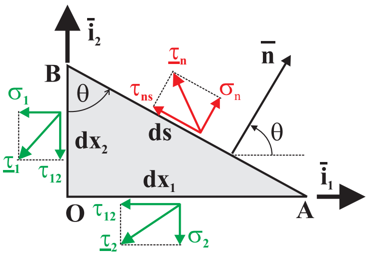

Fig. 1.12. Differential element with a face at an angle θ.

1.3.2 Stresses acting on an arbitrary face within the sheet

Figure 1.12 shows a free body **OAB** taken from within the thin sheet in fig. 1.11. It is a differential triangle with two sides cut normal to axes $\bar{\imath}_1$ and $\bar{\imath}_2$, and the third side cut normal to a unit vector, $\bar{n} = n_1\,\bar{\imath}_1 + n_2\,\bar{\imath}_2$, at an arbitrary orientation angle θ with respect to axis $\bar{\imath}_1$. Clearly, $n_1 = \cos\theta$ and $n_2 = \sin\theta$.

Triangle **OAB** is the two-dimensional version of Cauchy's tetrahedron presented in section 1.2.1 and depicted in fig. 1.7. Hence, the results derived in section 1.2 are directly applicable to the present case. Figure 1.12 shows the stress components acting on sides **OA** and **OB**, of length $\mathrm{d}x_1$, and $\mathrm{d}x_2$, respectively. On side **AB**, of length $\mathrm{d}s$, the stress vector $\underline{\tau}_n$ is acting. Finally, the body force vector, $\underline{b}$, is also acting on this triangle. For convenience, the thickness of the body in the direction of axis $\bar{\imath}_3$ is taken to be unity.

Equilibrium of forces acting on triangle **OAB** can be expressed by multiplying each of the stress vectors by the area over which they acts, *i.e.*, the length times the unit thickness, and this yields

$$\underline{\tau}_2\mathrm{d}x_1 + \underline{\tau}_1\mathrm{d}x_2 = \underline{\tau}_n\mathrm{d}s + \underline{b}\,\mathrm{d}x_1\mathrm{d}x_2/2,$$

where $\underline{\tau}_1$ and $\underline{\tau}_2$ are the stress vectors acting on the faces normal to axes $\bar{\imath}_1$ and $\bar{\imath}_2$, respectively. Dividing this equilibrium equation by $\mathrm{d}s$ gives the stress vector acting on the inclined face as

$$\underline{\tau}_n = \underline{\tau}_1 n_1 + \underline{\tau}_2 n_2 - \underline{b}\, \mathrm{d}x_1 \mathrm{d}x_2 / 2\mathrm{d}s$$

The body force term is multiplied by a higher order differential term, which can neglected. Expanding the stress vectors in terms of the stress components then yields

$$\underline{\tau}_n = (\sigma_1 \bar{\imath}_1 + \tau_{12} \bar{\imath}_2)\ \cos\theta + (\tau_{21} \bar{\imath}_1 + \sigma_2 \bar{\imath}_2)\ \sin\theta. \tag{1.30}$$

The three-dimensional equivalent of this relationship is given by eq. (1.10).

Projecting this vector equation in the direction of unit vector $\bar{n}$ yields the direct stress component, σ_n, acting on this face as $\sigma_n = (\sigma_1 \cos\theta + \tau_{12} \sin\theta)\ \cos\theta + (\tau_{21} \cos\theta + \sigma_2 \sin\theta)\ \sin\theta$, or after rearrangement,

$$\sigma_n = \sigma_1\ \cos^2\theta + \sigma_2\ \sin^2\theta + 2\tau_{12}\ \cos\theta \sin\theta. \tag{1.31}$$

Next, eq. (1.30) is projected in the direction normal to unit vector $\bar{n}$. This is in the direction of edge **AB**, and the direction cosines of this vector with axes $\bar{\imath}_1$ and $\bar{\imath}_2$ are $-\sin\theta$ and $\cos\theta$, respectively. The shear stress component, τ_{ns}, acting on side **AB** then becomes $\tau_{ns} = (-\sigma_1 \sin\theta + \tau_{12} \cos\theta)\ \cos\theta + (-\tau_{21} \sin\theta + \sigma_2 \cos\theta)\ \sin\theta$ which, after rearrangement, becomes

$$\tau_{ns} = -\sigma_1\ \cos\theta \sin\theta + \sigma_2\ \sin\theta \cos\theta + \tau_{12}(\cos^2\theta - \sin^2\theta). \tag{1.32}$$

Equations (1.31) and (1.32) could have been directly derived from their three-dimensional equivalent, eqs. (1.11) and (1.12), respectively, by noting that for the plane stress case, $n_1 = \cos\theta$, $n_2 = \sin\theta$, $n_3 = 0$ and $s_1 = -\sin\theta$, $s_2 = \cos\theta$, $s_3 = 0$.

These important results show that knowledge of the stress components σ_1, σ_2, and τ_{12} on two orthogonal faces allows computation of the stress components acting on a face with an arbitrary orientation. In other words, the knowledge of the stress components on two orthogonal faces fully defines the state of stress at a point.

1.3.3 Principal stresses

Principal stresses and their directions can also be determined for plane stress situations. It is a straightforward process to simply write eqs. (1.13), (1.14) and (1.15) with $\sigma_3 = \tau_{23} = \tau_{13} = 0$. This yields a vanishing principal stress along axis $\bar{\imath}_3$ and a quadratic equation for the remaining two principal stresses, which must lie in plane $(\bar{\imath}_1, \bar{\imath}_2)$. The computational procedure is otherwise unchanged.

It is more interesting, however, to consider eq. (1.31) as defining the direct stress, σ_n, acting on side **AB** of triangle **OAB**, see fig. 1.12. The magnitude of this direct stress is a function of θ, the orientation angle of this face. The particular orientation, θ_p, that maximizes (or minimizes) the magnitude of this stress component is determined by requiring the vanishing of the derivative of σ_n with respect to angle θ, to find

$$\frac{\mathrm{d}\sigma_n}{\mathrm{d}\theta} = -2\sigma_1 \cos\theta_p \sin\theta_p + 2\sigma_2 \cos\theta_p \sin\theta_p + 2\tau_{12}(\cos^2\theta_p - \sin^2\theta_p) = 0.$$

Using the elementary double-angle trigonometric identities, the orientation of the side that gives the extreme direct stress is found to be

$$\tan 2\theta_p = \frac{2\tau_{12}}{\sigma_1 - \sigma_2}. \tag{1.33}$$

This equation possesses two solutions θ_p and $\theta_p + \pi/2$ corresponding to two mutually orthogonal principal stress directions. The maximum axial stress is found along one direction, and the minimum along the other.

To determine these axes unambiguously, it is convenient to develop separate equations for both $\sin 2\theta_p$ and $\cos 2\alpha_p$ as follows. If eq. (1.33) is rewritten as

$$\tan 2\theta_p = \frac{2\tau_{12}}{\sigma_1 - \sigma_2} = \frac{\sin 2\theta_p}{\cos 2\theta_p},$$

it is then possible to identify $\sin 2\theta_p = \tau_{12}/\Delta$ and $\cos 2\theta_p = (\sigma_1 - \sigma_2)/2\Delta$, where Δ is determined by the following trigonometric identity, $\sin^2 2\theta_p + \cos^2 2\theta_p = 1$, to find

$$\Delta = \left[\left(\frac{\sigma_1 - \sigma_2}{2} \right)^2 + (\tau_{12})^2 \right]^{1/2}.$$

Thus, the sine and cosine of angle $2\theta_p$ can be expressed as follows

$$\sin 2\theta_p = \frac{\tau_{12}}{\Delta}, \quad \cos 2\theta_p = \frac{\sigma_1 - \sigma_2}{2\Delta}, \tag{1.34}$$

where

$$\Delta = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2} \right)^2 + \tau_{12}^2}. \tag{1.35}$$

This result is equivalent to eq. (1.33), but it gives a unique solution for θ_p because both the sine and cosine of the angle are known. The maximum and minimum axial stresses, denoted σ_{p1} and σ_{p2}, respectively, act in the directions θ_p and $\theta_p + \pi/2$, respectively. These maximum and minimum axial stresses, called the *principal stresses*, are evaluated by introducing eq. (1.34) into eq. (1.31) to find

$$\sigma_{p1} = \frac{\sigma_1 + \sigma_2}{2} + \Delta; \quad \sigma_{p2} = \frac{\sigma_1 + \sigma_2}{2} - \Delta. \tag{1.36}$$

The principal stresses are maximum and minimum values of the axial stress in an algebraic sense. Note that it is possible, however, to have $|\sigma_{p2}| > |\sigma_{p1}|$.

The shear stress acting on the faces normal to the principal stress directions vanishes, as expected. This can be verified by introducing eq. (1.34) into eq. (1.32)

$$\tau_{ns} = -\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta_p + \tau_{12} \cos 2\theta_p = -\frac{\sigma_1 - \sigma_2}{2} \frac{\tau_{12}}{\Delta} + \tau_{12} \frac{\sigma_1 - \sigma_2}{2\Delta} = 0.$$

It is also interesting to find the orientation of the faces leading to the maximum value of the shear stress. Indeed, in view of eq. (1.32), the shear stress is also a function of the face orientation angle. The orientation, θ_s, of the face on which the maximum (or minimum) shear stress acts satisfies the following extremal condition

$$\frac{\mathrm{d}\tau_{ns}}{\mathrm{d}\theta} = -\frac{\sigma_1 - \sigma_2}{2} 2\cos 2\theta_s - \tau_{12}\, 2\sin 2\theta_s = 0, \tag{1.37}$$

or

$$\tan 2\theta_s = -\frac{\sigma_1 - \sigma_2}{2\tau_{12}} = -\frac{1}{\tan 2\theta_p}, \tag{1.38}$$

where the last equality follows from eq. (1.34). Here again, this equation presents two solutions, θ_s and $\theta_s + \pi/2$, corresponding to two mutually orthogonal faces. To define these orientations unequivocally, separate definitions of the sine and cosines of angle $2\theta_s$ are given as follows

$$\sin 2\theta_s = -\frac{\sigma_1 - \sigma_2}{2\Delta}; \quad \cos 2\theta_s = \frac{\tau_{12}}{\Delta}, \tag{1.39}$$

where Δ is again given by eq. (1.35).

The maximum shear stress acting on these faces results from introducing eq. (1.39) into eq. (1.32) to find

$$\tau_{\max} = \Delta = \frac{\sigma_{p1} - \sigma_{p2}}{2}. \tag{1.40}$$

Since $\tan 2\theta_s = -1/\tan 2\theta_p$, trigonometric identities reveal that

$$\theta_s = \theta_p - \frac{\pi}{4}. \tag{1.41}$$

This means that the faces on which the maximum shear stresses occur are inclined at a 45° angle with respect to the principal stress directions.

The axial stresses acting on these faces are found by introducing eq. (1.39) into eq. (1.31) and using the first stress invariant property to find

$$\sigma_{1s} = \sigma_{2s} = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_{p1} + \sigma_{p2}}{2}. \tag{1.42}$$

1.3.4 Rotation of stresses

In the previous sections, faces are cut in planes normal to the two axes of an orthonormal basis $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$, and the stress vectors are resolved into stress components along the same directions. It is clear that the orientation of this basis is entirely arbitrary: an orthonormal basis $\mathcal{I}^* = (\bar{\imath}_1^*, \bar{\imath}_2^*)$ could have been selected, and an analysis identical to that of the previous sections would have led to the definition of axial stresses σ_1^* and σ_2^*, and shear stress τ_{12}^*. A typical equilibrium equation at a point of the body would be written as

$$\frac{\partial \sigma_1^*}{\partial x_1^*} + \frac{\partial \tau_{21}^*}{\partial x_2^*} + b_1^* = 0; \tag{1.43}$$

where the notation $(\cdot)^*$ is used to indicate the components of the corresponding quantity resolved in $\mathcal{I}^*$. A typical surface traction is be defined as

$$t_1^* = n_1^* \, \sigma_1^* + n_2^* \, \tau_{21}^*. \tag{1.44}$$

Although expressed in different reference frames, eqs. (1.26) and (1.43), or (1.27) and (1.44) express the same equilibrium conditions for the body. The problem at hand involves two distinct orthonormal bases, $\mathcal{I}$ and $\mathcal{I}^*$, and the relationship between these two basis is developed in appendix A.3.3.

Consider the stress component σ_1^*: it represents the magnitude of the direct stress component acting on the face normal to axis $\bar{\imath}_1^*$. Let θ be the angle between unit vector $\bar{\imath}_1^*$ and axis $\bar{\imath}_1$. Equation (1.31) can now be used to express the stress component σ_1^* in terms of the stress components resolved in axis system $\mathcal{I}$ to find

$$\sigma_1^* = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau_{12} \sin\theta \cos\theta. \tag{1.45}$$

A similar equation can be derived to express σ_2^* in terms of the stress components resolved in axis system $\mathcal{I}$ by replacing angle θ by $\theta + \pi/2$ in the above equation; $\theta + \pi/2$ is the angle between unit vector $\bar{\imath}_2^*$ and axis $\bar{\imath}_1$.

Finally, the shear stress component can be computed from eq. (1.32) as

$$\tau_{12}^* = -\sigma_1 \, \sin\theta\cos\theta + \sigma_2 \, \sin\theta\cos\theta + \tau_{12}(\cos^2\theta - \sin^2\theta). \tag{1.46}$$

These results can be combined into a compact matrix form as

$$\left\{ \begin{matrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{matrix} \right\} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \left\{ \begin{matrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{matrix} \right\}. \tag{1.47}$$

This relationship can be easily inverted by recognizing that the inverse transformation is obtained simply by replacing θ by $-\theta$ to find

$$\left\{ \begin{matrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{matrix} \right\} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \left\{ \begin{matrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{matrix} \right\}. \tag{1.48}$$

With the help of double-angle trigonometric identities, the transformation rules for stress components, eq. (1.47), can also be written in the following useful form

$$\sigma_1^* = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau_{12} \, \sin 2\theta, \tag{1.49a}$$

$$\sigma_2^* = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta - \tau_{12} \, \sin 2\theta, \tag{1.49b}$$

$$\tau_{12}^* = \qquad - \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta + \tau_{12} \, \cos 2\theta. \tag{1.49c}$$

These important results show that knowledge of the stress components σ_1, σ_2, and τ_{12} on two orthogonal faces allows computation of the stress components acting on a face with an arbitrary orientation. In other words, the knowledge of the stress components on two orthogonal faces fully defines the state of stress at a point.

1.3.5 Special states of stress

Two plane stress states are of particular interest. One is called the *hydrostatic stress state* and the other is called the *pure shear state*. A third special state of plane stress is the stress developed in a thin-walled cylindrical pressure vessel.

Hydrostatic stress state. A stress state of practical importance is the *hydrostatic* state of stress. In this case, the principal stresses are equal, *i.e.*, $\sigma_{p1} = \sigma_{p2} = p$, where p is the *hydrostatic pressure*. It follows from eq. (1.49) that the stresses acting on a face with any arbitrary orientation are

$$\sigma_1 = \sigma_2 = p, \quad \tau_{12} = 0. \tag{1.50}$$

Pure shear state. A stress state of great practical importance is the state of *pure shear* characterized by principal stresses of equal magnitude but opposite signs, *i.e.*, $\sigma_{p2} = -\sigma_{p1}$, as depicted in fig. 1.13. Equations (1.45) and (1.46) then reveal the direct and shear stresses, respectively, acting on a face inclined at a 45° angle with respect to the principal stress directions as

$$\tau^*_{12} = -\sigma_{p1}; \quad \sigma^*_1 = \sigma^*_2 = 0. \tag{1.51}$$

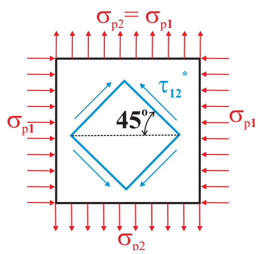

Fig. 1.13. A differential plane stress element in pure shear.

On faces oriented at 45° angles with respect to the principal stress directions, the direct stresses vanish and the shear has a maximum value, equal in magnitude to the common magnitudes of the two principal stresses.

Stress state in thin-walled pressure vessels. The stress state in the walls of thin-walled tanks, called pressure vessels, of certain shapes consists of two in-plane normal stresses and an in-plane shear stress. Although the pressure vessel may be subjected to a large internal pressure that will produce a pressure loading on the interior wall in the transverse direction, the magnitude of this stress often is orders of magnitude smaller than the in-plane stress components and is therefore usually neglected. The spherical pressure vessel and a long cylindrical pressure vessel (ignoring the effect of the ends) are two useful examples.

A thin-walled ($t \ll R$) cylindrical pressure vessel subjected to an internal pressure, p_i, is depicted in fig. 1.14, where it is assumed that the only stresses are the two in-plane stress components, σ_a in the axial direction, and σ_h in the circumferential or "hoop" direction, and possibly a shear stress, τ_{ah}. In the central portion of the cylinder, it is possible to create the simple free body shown in the figure, which will allow direct calculation of these stresses. From axial force equilibrium, it follows that $\sigma_a \pi R t = p_i \pi R^2/2$, and hence, $\sigma_a = p_i R/2t$. Equilibrium in the tangential (hoop) direction implies $2\sigma_h bt = p_i 2Rb$, and hence, $\sigma_h = p_i R/t$. Finally, it should be clear that $\tau_{ah} = 0$ for this axis orientation.

It is left as an exercise to show that by a similar free body analysis of a spherical thin-walled pressure vessel, $\sigma_a = \sigma_h = p_i R/t$ in any direction and the shear stress

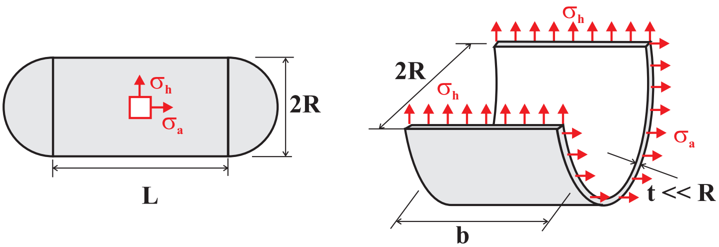

Fig. 1.14. Long, thin-walled cylindrical pressure vessel (left) and free body diagram (right) used to calculate in-plane stresses σ_h and σ_a.

vanishes. This is a special case of two-dimensional hydrostatic stress. A more formal analysis of pressure vessels is presented in section 4.4.

1.3.6 Mohr's circle for plane stress

Equation (1.49) expresses the direct and shear stresses acting on a face oriented at an arbitrary angle θ with respect the axis $\bar{\imath}_1$, but the presence of trigonometric functions involving the angle 2θ makes it difficult to give a simple, geometric interpretation of these formulæ. A useful geometric interpretation, however, called *Mohr's circle*, can be developed. Let the state of stress at a point be defined by its principal stresses, σ_{p1} and σ_{p2}. Equation (1.49) then implies that the stresses acting on a face oriented at an angle θ with respect to the principal stress directions can be written as

$$\sigma^* = \sigma_a + R\ \cos 2\theta; \quad \tau^* = -R\ \sin 2\theta, \tag{1.52}$$

where $\sigma_a = (\sigma_{p1} + \sigma_{p2})/2$ and $R = (\sigma_{p1} - \sigma_{p2})/2$. With this notation and the help of basic trigonometric identities, eq. (1.52) becomes

$$(\sigma^* - \sigma_a)^2 + (\tau^*)^2 = R^2. \tag{1.53}$$

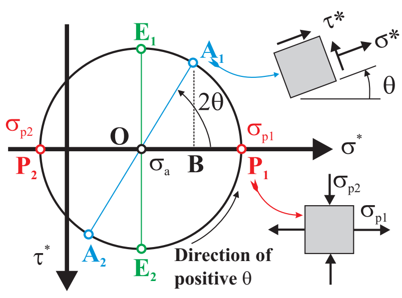

Fig. 1.15. Mohr's circle for visualizing plane stress state.

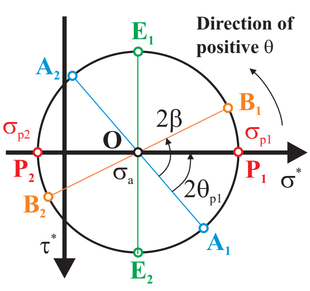

Fig. 1.16. Mohr's circle construction procedure.

This equation clearly represents the *equation of a circle*, known as *Mohr's circle* in which σ^* is plotted along the horizontal axis, τ^* is plotted along the vertical axis, and the circle is centered at a coordinate σ_a on the horizontal axis with a radius of R, as depicted in fig. 1.15[1]. The reason for plotting τ^* with an inverted axis will become clear in the next paragraphs.

Consider point $\mathbf{A_1}$ on Mohr's circle such that segment $\mathbf{OA_1}$ makes an angle 2θ with the horizontal. The coordinates of this point are $\sigma^* = \sigma_a + R\cos 2\theta$ and $\tau^* = -R\sin 2\theta$; hence, in view of eq. (1.52), the coordinates of point $\mathbf{A_1}$ represent the state of stress on a face oriented at an angle θ. In fact, each point on Mohr's circle represents the state of stress acting on a face at a specific orientation.

An **important sign convention** must be defined: on Mohr's circle, a positive angle θ is measured in the **counterclockwise direction**, see fig. 1.15, to match the positive direction of angle θ that identifies the orientation of a face in fig. 1.12. Given the sign convention for angle θ, the shear stress must be positive downward on the ordinate of Mohr's circle depicted in fig. 1.15.[2]

The following observations are made.

- At point $\mathbf{P_1}$, the stress state is $\sigma^* = \sigma_{p1}$ and $\tau^* = 0$; this corresponds, as expected, to the stress components acting in the principal stress direction. Similar results are found at point $\mathbf{P_2}$ which represents the stress components acting in the second principal direction.
- At point $\mathbf{E_1}$, associated with an angle $\theta = \pi/4$, the stress components are $\tau^*_{\max} = R = (\sigma_{p1} - \sigma_{p2})/2$ and $\sigma^* = \sigma_a = (\sigma_{p1} + \sigma_{p2})/2$. These results are identical to those expressed by eqs. (1.40) and (1.42), respectively. In the graphical representation of stress states given by Mohr's circle, it becomes obvious that the maximum shear stress is found on faces oriented at $\pm 45°$ angles with respect to the principal stress directions, and this is defined by points $\mathbf{E_1}$ and $\mathbf{E_2}$ in fig. 1.15.
- Points $\mathbf{A_1}$ and $\mathbf{A_2}$ represent the stress components acting on two faces oriented $90°$ apart. The shear stresses acting on those two faces are equal in magnitude and of opposite sign, as required by the principle of reciprocity of shear stresses illustrated in fig. 1.5. The direct stresses correspond to stresses σ_1^* and σ_2^* in eqs. (1.49).

In the above discussion, Mohr's circle is constructed based on the knowledge of the principal stresses represented by points $\mathbf{P_1}$ and $\mathbf{P_2}$ in figs. 1.15 and 1.16. In practice, it is often the case that the state of stress at a point is defined by known stress components σ_1, σ_2 and τ_{12}. These three stress components define two diametrically opposed points, $\mathbf{A_1}$ and $\mathbf{A_2}$, on Mohr's circle depicted in fig. 1.16. Once this circle is constructed with the help of the procedure described below, the stress components acting on any face rotated by an angle β in a counterclockwise direction, represented

[1] A Mohr's circle representation that describes the rotation of a three-dimensional second order tensor can be constructed but it involves three interdependent circles and is quite tedious to construct and to use.

[2] An equivalent construction of Mohr's circle has the shear stress positive upwards along the ordinate, but angle θ is then positive in the clockwise direction.

by points $\mathbf{B_1}$ and $\mathbf{B_2}$ in fig. 1.16, can be directly obtained from simple geometric constructions.

1. Draw a first point, identified as point $\mathbf{A_1}$, at coordinates (σ_1, τ_{12}). This represents the direct and shear stresses acting on one face of the solid.
2. Draw a second point, identified as point $\mathbf{A_2}$, at coordinates $(\sigma_2, -\tau_{12})$. This represents the direct and shear stresses acting on a face of the solid at a $90°$ angle counterclockwise with respect to the first face. Since the two faces are $90°$ apart, these two points must define diametrically opposite points on Mohr's circle.
3. Draw a straight line segment joining points $\mathbf{A_1}$ and $\mathbf{A_2}$; the intersection of this segment with the horizontal axis defines the center of the Mohr's circle of diameter $\mathbf{A_1OA_2}$ at point $\mathbf{O}$. Points $\mathbf{A_1}$ and $\mathbf{A_2}$ represent the stress components on two orthogonal faces, that is, on faces of relative orientation $\theta = 90°$, since the angle between segments $\mathbf{OA_1}$ and $\mathbf{OA_2}$ is $2\theta = 180°$.
4. Once Mohr's circle is drawn, the stress state on faces at any orientation angle can be computed. For instance, the stress components acting on a face oriented at an angle β from the face on which stress components σ_1 and τ_{12} act can be computed by constructing a new diameter $\mathbf{B_1OB_2}$ rotated 2β degrees from the reference diameter $\mathbf{A_1OA_2}$. The coordinates of point $\mathbf{B_1}$ yield the new stress components.

Mohr's circle displays in a graphical manner many important features characterizing the state of stress at a point.

1. The principal stresses, σ_{p1} and σ_{p2}, shown in figs. 1.15 and 1.16, are represented by the points $\mathbf{P_1}$ and $\mathbf{P_2}$ at the intersection of Mohr's circle with the horizontal axis. Clearly, these points define the orientation of the faces on which the direct stresses take on maximum and minimum values and for which the shear stress vanishes.
2. The faces on which the maximum shear stresses occurs are represented by the points at the intersection of Mohr's circle with a vertical line passing through its center. It is clear that the magnitude of the maximum shear stress equals the radius of Mohr's circle: $\tau_{\max} = (\sigma_{p1} - \sigma_{p2})/2$, see eq. (1.40). The angle between the principal stress directions and those of the face of maximum shear is $45°$, because the angle $\mathbf{P_1OE_1}$ is $90°$, see eq. (1.41). Finally, the direct stresses acting on the faces of maximum shear equal the average of the principal stresses, $\sigma_{1s} = \sigma_{2s} = (\sigma_{p1} + \sigma_{p2})/2$, see eq. (1.42).
3. The stress components acting on two mutually orthogonal faces are represented by two diametrically opposite points on Mohr's circle. Since the center of the circle is on the horizontal axis, the shear stresses on those two faces are equal in magnitude and opposite in sign, as required by the principle of reciprocity of shear stresses illustrated in fig. 1.5.
4. Finally, note that *all the points on Mohr's circle represent the same state of stress* at one point of the solid. Of course, this state of stress is represented by stress components that depend on the orientation of the face on which they act.

Mohr's circle is a graphical representation of all the possible stress components corresponding to a single state of stress.

1.3.7 Lamé's ellipse

Lamé's ellipse provides an elegant geometric interpretation of the state of stress at a point. Consider a material in a plane state of stress and let $\underline{\tau}_n$ be the stress vector acting on the face with a unit normal $\bar{n}$ at an angle θ with respect to axes $\bar{\imath}_1^*$, as depicted in fig. 1.17. As angle θ varies, the tip of the stress vector, $\underline{\tau}_n$, draws an ellipse, called *Lamé's ellipse*, with its center at **O** and its semi-axes given by the absolute value of the principal stresses, $|\sigma_{p1}|$ and $|\sigma_{p2}|$, respectively. The minor and major axes of the ellipse are aligned with the principal stress directions so that axes $\bar{\imath}_1^*$ and $\bar{\imath}_2^*$ are the principal stress directions.

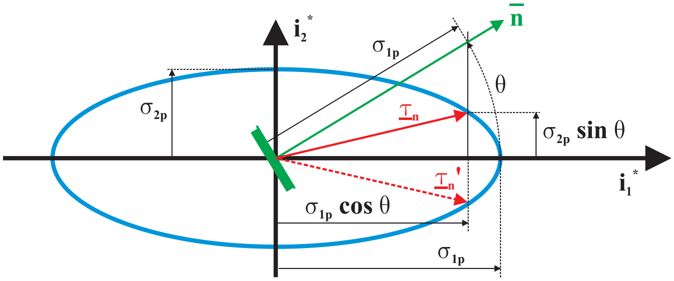

Fig. 1.17. Lamé's ellipse. Stress vector $\underline{\tau}_n$ corresponds to positive principal stresses whereas stress vector $\underline{\tau}_n'$ corresponds to $\sigma_{p1} > 0$ and $\sigma_{p2} < 0$.

To prove the stated claim that the locus of the tip of the stress vector draws the ellipse shown in fig. 1.17, the stress vector acting on the face at an angle θ with respect to axis $\bar{\imath}_1^*$ can be expressed with the help of eq. (1.30) as $\underline{\tau}_n = \sigma_{1p} \cos\theta \, \bar{\imath}_1^* + \sigma_{2p} \sin\theta \, \bar{\imath}_2^*$, where it is noted that $\sigma_1 = \sigma_{1p}$, $\sigma_2 = \sigma_{2p}$, and $\tau_{12} = 0$ because the selected axis system coincides with the principal stress directions. Let x_1 and x_2 be the coordinates of the tip of the stress vector, hence, $\underline{\tau}_n = x_1 \, \bar{\imath}_1^* + x_2 \, \bar{\imath}_2^*$. It then follows that $x_1 = \sigma_{1p} \cos\theta$ and $x_2 = \sigma_{2p} \sin\theta$, and elimination of the angle θ using the elementary trigonometric identity leads to

$$\left(\frac{x_1}{\sigma_{1p}}\right)^2 + \left(\frac{x_2}{\sigma_{2p}}\right)^2 = 1. \tag{1.54}$$

This is the equation of an ellipse with semi-axes equal to $|\sigma_{p1}|$ and $|\sigma_{p2}|$, respectively, proving the stated claim.

As the orientation of the face changes, the tip of the stress vector sweeps around Lamé's ellipse. Note that while the shape of the ellipse is not affected by the sign of the principal stresses, the orientation of the stress vector does depend on their sign. For instance, the stress vector $\underline{\tau}_n$ shown in fig. 1.17 corresponds to $\sigma_{p1} > 0$ and

$\sigma_{p2} > 0$; for the case where $\sigma_{p1} > 0$ and $\sigma_{p2} < 0$, however, the stress vector acting on the same face is now represented by vector $\underline{\tau}'_n$.

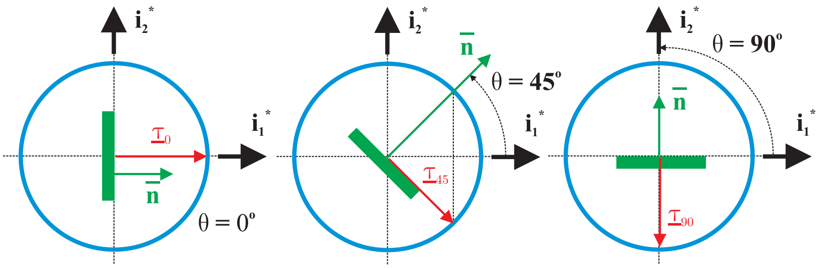

Fig. 1.18. Lamé's ellipse for the case of pure shear; the three figures illustrate the stress vectors acting on faces at 0, 45, and 90 degrees with respect to axis $\bar{\imath}_1^*$.

An interesting case is provided by the pure shear state of stress discussed in section 1.3.5. This is defined by the principal stresses $\sigma_{1p} = \tau$ and $\sigma_{2p} = -\tau$: the principal stresses are equal in magnitude but of opposite sign. Since the two semi-axes of Lamé's ellipse are equal, it becomes the circle depicted in fig. 1.18, and hence, the norm of the stress vector remains constant as the face on which it act rotates. When the face is oriented at a 45 degree angle, the stress vector acts at a -45 degree angle with respect to axis $\bar{\imath}_1^*$ and the face is subjected to only a shear stress, as expected. Finally, note that while the face rotates counterclockwise, the stress vector describes Lamé's ellipse in the clockwise direction.

1.3.8 Problems

Problem 1.5. Stress states on two sets of faces
The plane stress state at a point is known and characterized by the following stress components: $\sigma_1 = 250$ MPa, $\sigma_2 = 250$ MPa, and $\tau_{12} = 0$ MPa in a coordinate system $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$. Find the stress components σ_1^*, σ_2^*, and τ_{12}^* in a coordinate system $\mathcal{I}^* = (\bar{\imath}_1^*, \bar{\imath}_2^*)$, where $\bar{\imath}_1^*$ is at a 25 degree angle with respect to $\bar{\imath}_1$.

Problem 1.6. Stress invariants for plane stress state
The stress invariants defined in eq. (1.15) for three-dimensional problems. *(1)* Show that for plane stress problems, the following two quantities are invariants

$$I_1 = \sigma_1 + \sigma_2; \quad I_2 = \sigma_1 \sigma_2 - \tau_{12}^2. \tag{1.55}$$

(2) Prove your claim of invariance by showing that these quantities are identical when computed in terms of the principal stresses and in terms of stresses acting on a face at an arbitrary orientation.

Problem 1.7. Stress rotation formulæ in matrix form
Show that the plane stress stress rotation formulae given by eq. (1.47) can be recast in the following compact matrix form

$$\begin{bmatrix} \sigma_1^* & \tau_{12}^* \\ \tau_{12}^* & \sigma_2^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1 & \tau_{12} \\ \tau_{12} & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Problem 1.8. Mohr's circle

Draw Mohr's circle for the state of stress defined by σ_1 = 80 MPa, σ_2 = -20 MPa and τ_{12} = 40 MPa. Using this circle, *(1)* calculate the stress on axes rotated 60 degrees counterclockwise from the reference axes, and *(2)* determine the principal stresses and the corresponding directions. Do these results agree with the results in section 1.3.3?

Problem 1.9. Mohr's circle for the state of pure shear

Draw Mohr's circle for the state of pure shear defined in section 1.3.5. Show how eq. (1.51) can be readily derived from Mohr's circle.

Problem 1.10. Mohr's circle for the hydrostatic state of stress

Draw Mohr's circle for the state of hydrostatic stress defined in section 1.3.5. Show how eq. (1.50) can be readily derived from Mohr's circle.

Problem 1.11. Stresses in a pressure vessel

A cylindrical pressure vessel of radius R and thickness t is subjected to an internal pressure p_i. At any point in the cylindrical portion of vessel wall, two stress components are acting: the hoop stress, $\sigma_h = Rp_i/t$ and the axial stress, $\sigma_a = Rp_i/(2t)$. The radial stress, acting in the direction perpendicular to the wall, is very small, $\sigma_r \approx 0$. The pressure vessel features a weld line at a 45 degree angle with respect to the axis of the cylinder, as shown in fig. 1.19. *(1)* Find the direct stress acting in the direction perpendicular to the weld line. *(2)* Find the shear stress acting along the weld line.

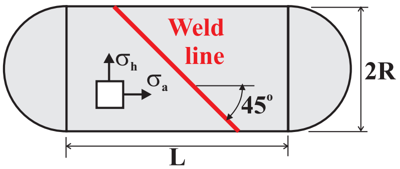

Fig. 1.19. Pressure vessel with a weld line.

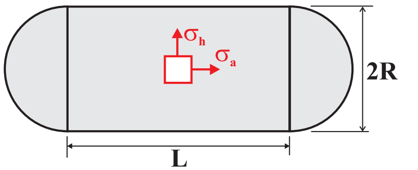

Fig. 1.20. Stresses acting in a pressure vessel.

Problem 1.12. Maximum stresses in a pressure vessel

Figure 1.20 shows a cylindrical pressure vessel of radius R and thickness t subjected to an internal pressure p_i. At any point in the cylindrical portion of vessel wall, two stress components are acting: the hoop stress, $\sigma_h = Rp_i/t$ and the axial stress, $\sigma_a = Rp_i/(2t)$. The radial stress, acting in the direction perpendicular to the wall, is very small, $\sigma_r \approx 0$. *(1)* Find the orientation of the face on which the maximum direct stress is acting. What is the value of the maximum direct stress? *(2)* Find the orientation of the face on which the maximum shear stress is acting acting. What is the value of the maximum shear stress?

Problem 1.13. Stresses in a composite material layer

A layer of unidirectional composite material is subjected to a state of stress $\sigma_1 = 245$ MPa, $\sigma_2 = -175$ MPa, and $\tau_{12} = 95$ MPa. As depicted in fig. 1.21, the fibers in the unidirectional composite material layer run at an angle $\theta = 25$ degrees with respect to axis $\bar{\imath}_1$. *(1)* Find the direct stress acting in the direction of the fiber. *(2)* Find the direct stress acting in the direction perpendicular to the fiber.

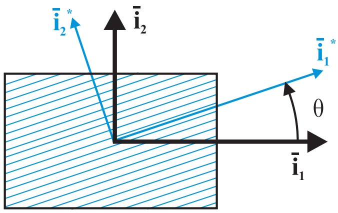

Fig. 1.21. Layer of unidirectional composite material with fiber direction.

1.4 The concept of strain

The state of strain at a point is a characterization of the deformation in the neighborhood of a material point in a solid. The description of the state of strain at a point is a great deal more complicated than that of the stress state, and the presence of nonlinear terms is much more obvious. The state of strain is concerned with the deformation of a solid in the neighborhood of a given point, say point **P**, located by a position vector $\underline{r} = x_1 \bar{\imath}_1 + x_2 \bar{\imath}_2 + x_3 \bar{\imath}_3$, as depicted in fig. 1.22.

To visualize this deformation, a small rectangular parallelepiped **PQRST** of differential size $\mathrm{d}x_1$ by $\mathrm{d}x_2$ by $\mathrm{d}x_3$ is cut in the neighborhood of point **P**. The *reference configuration* is the configuration of the solid in its undeformed state. Under the action of applied loads, the body deforms and assumes a new configuration, called the *deformed configuration*. All the material particles that formed the rectangular parallelepiped **PQRST** in the reference configuration now form the parallelepiped **PQRST** in the deformed configuration. The state of strain at a point characterizes the deformation of the parallelepiped without any consideration for the loads that created the deformation.

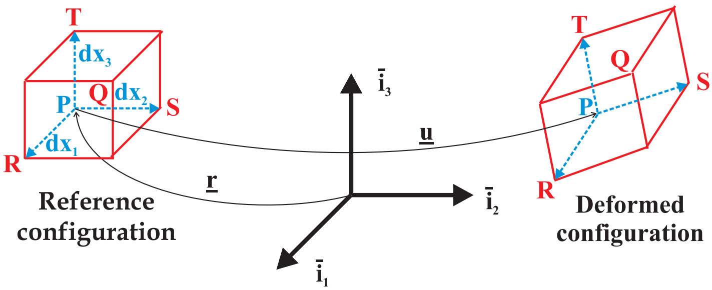

Fig. 1.22. The neighborhood of point **P** in the reference and deformed configurations.

While position vector, $\underline{r}$, locates material point **P**, the *displacement vector,* $\underline{u}$, is a measure of how much a material point moves from the reference to the deformed configuration. The components of the displacement vector resolved in coordinate system $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ can be expressed as

$$\underline{u}(x_1, x_2, x_3) = u_1(x_1, x_2, x_3)\,\bar{\imath}_1 + u_2(x_1, x_2, x_3)\,\bar{\imath}_2 + u_3(x_1, x_2, x_3)\,\bar{\imath}_3. \quad (1.56)$$

This displacement field describes the displacement of a point at position (x_1, x_2, x_3) within the solid and consists of two parts: a rigid body motion and a *deformation* or *straining* of the solid. The rigid body motion itself consists of two parts: a rigid body translation and a rigid body rotation. By definition, a rigid body motion does not produce strain in the body. Consequently, the strain-displacement equations must extract from the displacement field the information that describes only the deformation of the body while ignoring its rigid body motion.

1.4.1 The state of strain at a point

A *material line* is the ensemble of material particles that form a straight line in the reference configuration of the body. For instance, segments **PR**, **PS** and **PT** of the reference configuration are material lines. Due to the deformation of the body, all the material particles forming material line **PR** will move to segment **PR** in the deformed configuration. Due to the differential nature of this segment, it can be assumed to remain straight in the deformed configuration.

When comparing segment **PR** in the reference and deformed configurations, the motion consists of two parts: a change in orientation and a change in length. Clearly, the change in length is a deformation or stretching of the material line. Similarly, segments **PR** and **PS** form a rectangle in the reference configuration, but they form a parallelogram in the deformed configuration. The angular distortion of the rectangle into a parallelogram represents a deformation of the body. Stretching of a material line and angular distortion between two material lines will be selected as measures of the state of strain at a point.

The stretching or *relative elongations* of material lines **PR**, **PS** and **PT** will be denoted as ϵ_1, ϵ_2 and ϵ_3, respectively. The *angular distortions* between segments **PS** and **PT**, **PR** and **PT**, and **PR** and **PS** will be denoted γ_{23}, γ_{13}, and γ_{12}, respectively.

Relative elongations or extensional strains

The relative elongation, ϵ_1, of material line **PR** is defined as

$$\epsilon_1 = \frac{||\mathbf{PR}||_{\rm def} - ||\mathbf{PR}||_{\rm ref}}{||\mathbf{PR}||_{\rm ref}}, \tag{1.57}$$

where the subscripts $(\cdot)_{\rm ref}$ and $(\cdot)_{\rm def}$ are used to indicate the reference and deformed configurations, respectively, and $||\cdot||$ means magnitude of a vector. The relative elongation is a non-dimensional quantity. The length of the material line in the reference configuration is

$$||\mathbf{PR}||_{\rm ref} = \|{\rm d}x_1\, \bar{\imath}_1\| = {\rm d}x_1, \tag{1.58}$$

whereas in the deformed configuration, it is

$$\begin{aligned}\|\mathbf{PR}\|_{\rm def} =&\|{\rm d}x_1\bar{\imath}_1 + \underline{u}(x_1+{\rm d}x_1) - \underline{u}(x_1)\|\\ =&\|{\rm d}x_1\bar{\imath}_1 + \bar{u}(x_1) + \frac{\partial \underline{u}}{\partial x_1}{\rm d}x_1 - \underline{u}(x_1)\| = \|{\rm d}x_1\bar{\imath}_1 + \frac{\partial \underline{u}}{\partial x_1}\,{\rm d}x_1\|\\ =&\|\bar{\imath}_1{\rm d}x_1 + \left(\frac{\partial u_1}{\partial x_1}\bar{\imath}_1 + \frac{\partial u_2}{\partial x_1}\bar{\imath}_2 + \frac{\partial u_3}{\partial x_1}\bar{\imath}_3\right){\rm d}x_1\|\\ =&\sqrt{1+2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2}\,{\rm d}x_1,\end{aligned} \tag{1.59}$$

where the higher order differential terms in the Taylor series expansion of the displacement field are neglected. The relative elongation now becomes

$$\epsilon_1 = \sqrt{1+2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2} - 1. \tag{1.60}$$

A fundamental assumption of linear elasticity is that all displacement components remain very small so that *all second order terms can be neglected.* This can be stated as requiring

$$\left|\frac{\partial u_1}{\partial x_1}\right| \ll 1, \quad \left|\frac{\partial u_2}{\partial x_1}\right| \ll 1, \quad \left|\frac{\partial u_3}{\partial x_1}\right| \ll 1, \quad \left|\frac{\partial u_1}{\partial x_2}\right| \ll 1, \quad \text{etc.} \tag{1.61}$$

With these assumptions and by making use of the binomial expansion[3], the expression for the relative elongation given in eq. (1.60) reduces to

$$\epsilon_1 \approx 1 + \frac{\partial u_1}{\partial x_1} - 1 = \frac{\partial u_1}{\partial x_1}. \tag{1.62}$$

A similar reasoning applied to material lines **PS** and **PT** yields expressions for the three components of relative elongation

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3}. \tag{1.63}$$

Angular distortions or shear strains

The angular distortion, γ_{23}, between two material lines **PT** and **PS** is defined as the change of the initially right angle

$$\gamma_{23} = \langle TPS\rangle_{\rm ref} - \langle TPS\rangle_{\rm def} = \frac{\pi}{2} - \langle TPS\rangle_{\rm def}, \tag{1.64}$$

where the notation $\langle TPS\rangle$ is used to indicate the angle between segments **PT** and **PS**. Both relative elongation and angular distortion are non-dimensional quantities. To eliminate the difference between the two angles, basic properties of the sine function are used: the sine of the angular distortion becomes

[3] When $|a| \ll 1$, it is possible to expand $(1 \pm a)^n \approx 1 \pm na$.

$$\sin\gamma_{23} = \sin\left(\frac{\pi}{2} - \langle TPS\rangle_{\rm def}\right) = \cos\langle TPS\rangle_{\rm def}. \tag{1.65}$$

The cosine of the angle between the two material lines is computed from the law of cosines applied to triangle TPS in the deformed configuration

$$\|\mathbf{TS}\|^2_{\rm def} = \|\mathbf{PT}\|^2_{\rm def} + \|\mathbf{PS}\|^2_{\rm def} - 2\cos\langle TPS\rangle_{\rm def}\|\mathbf{PT}\|_{\rm def}\|\mathbf{PS}\|_{\rm def}. \tag{1.66}$$

The angular distortion thus becomes

$$\gamma_{23} = \arcsin\frac{\|\mathbf{PT}\|^2_{\rm def} + \|\mathbf{PS}\|^2_{\rm def} - \|\mathbf{TS}\|^2_{\rm def}}{2\|\mathbf{PT}\|_{\rm def}\|\mathbf{PS}\|_{\rm def}}. \tag{1.67}$$

The same procedure as used above in determining ϵ_1 can be used to compute $\|\mathbf{PR}\|_{\rm def}$ and $\|\mathbf{PS}\|_{\rm def}$ but since the present computations are a bit more tedious, it will be convenient to introduce two temporary vectors, $\underline{A}$ an $\underline{B}$, defined as follows

$$\mathbf{PT}_{\rm def} = \left(\bar{\imath}_3 + \frac{\partial\underline{u}}{\partial x_3}\right){\rm d}x_3 = \underline{A}, \quad \mathbf{PS}_{\rm def} = \left(\bar{\imath}_2 + \frac{\partial\underline{u}}{\partial x_2}\right){\rm d}x_2 = \underline{B},$$

and hence

$$\mathbf{TS}_{\rm def} = \mathbf{PS}_{\rm def} - \mathbf{PT}_{\rm def} = \underline{B} - \underline{A}.$$

With the help of this notation, the numerator, N, of eq. (1.67) becomes

$$\begin{aligned} N &= \underline{A}\cdot\underline{A} + \underline{B}\cdot\underline{B} - (\underline{B}-\underline{A})\cdot(\underline{B}-\underline{A}) = 2\underline{A}\cdot\underline{B} \\ &= 2\left(\bar{\imath}_2 + \frac{\partial\underline{u}}{\partial x_2}\right)\left(\bar{\imath}_3 + \frac{\partial\underline{u}}{\partial x_3}\right){\rm d}x_2{\rm d}x_3 = 2\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial\underline{u}}{\partial x_2}\frac{\partial\underline{u}}{\partial x_3}\right){\rm d}x_2{\rm d}x_3 \\ &= 2\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right){\rm d}x_2{\rm d}x_3. \end{aligned} \tag{1.68}$$

The denominator, D, can be expressed in the same manner to find

$$\begin{aligned} D =& 2\|\underline{A}\|\ \|\underline{B}\| = 2\sqrt{\underline{A}\cdot\underline{A}}\ \sqrt{\underline{B}\cdot\underline{B}} \\ =& 2{\rm d}x_2{\rm d}x_3\sqrt{1 + 2\frac{\partial u_2}{\partial x_2} + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_2}\right)^2} \\ & \sqrt{1 + 2\frac{\partial u_3}{\partial x_3} + \left(\frac{\partial u_1}{\partial x_3}\right)^2 + \left(\frac{\partial u_2}{\partial x_3}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2}. \end{aligned} \tag{1.69}$$

Finally, these results can be combined in eq. (1.67) to yield the rather cumbersome expression $\gamma_{23} = \arcsin N/D$. With the help of the small displacement assumption, see eq. (1.61), the numerator simplifies to $N \approx 2(\partial u_2/\partial x_3 + \partial u_3/\partial x_2)\ {\rm d}x_2{\rm d}x_3$, whereas the denominator reduces to $D \approx 2(1+\partial u_2/\partial x_2+\partial u_3/\partial x_3)\ {\rm d}x_2{\rm d}x_3$. With these simplifications, the shearing strain component becomes

$$\gamma_{23} \approx \frac{\partial u_2/\partial x_3 + \partial u_3/\partial x_2}{1 + \partial u_2/\partial x_2 + \partial u_3/\partial x_3} \approx \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}. \tag{1.70}$$

A similar reasoning applies to the other material lines to yield the three angular distortions or shear strains as

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}, \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \tag{1.71}$$

Summary

The relative elongations, eqs. (1.63), and angular distortions, eqs. (1.71) characterize the state of deformation at a point. The relative elongations are also called *direct strains* or *axial strains*, whereas the angular distortions are called *shearing strains* or simply *shear strains*. It is important to note that the *strain-displacement relationships*, eqs. (1.63) and (1.71) are obtained under the small displacement assumption defined in eq. (1.61). If the displacements become large, expressions (1.60) and (1.67) should be used instead. It is clear that the small displacement assumption implies that all strain components also remain very small, *i.e.*,

$$|\epsilon_1| \ll 1, \ |\epsilon_2| \ll 1, \ |\epsilon_3| \ll 1, \ |\gamma_{23}| \ll 1, \ |\gamma_{13}| \ll 1, \ |\gamma_{12}| \ll 1. \tag{1.72}$$

Rigid body rotation

In general, the motion of a solid body can be decomposed into a rigid body motion and straining or deformation. The previous sections are focused on the deformation of the solid, but the rigid body motion can also be extracted from the displacement field. The components of the rotation vector associated with the displacement field are

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \tag{1.73a}$$

$$\omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \tag{1.73b}$$

$$\omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right). \tag{1.73c}$$

Each components of the rotation vector $\underline{\omega}^T = \left\{\omega_1, \omega_2, \omega_3\right\}$ represent the rotation of the solid about axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively.

1.4.2 The volumetric strain

Consider the block of material defined by the three segments **PR**, **PS**, and **PT**. The volume of this block in the reference configuration is $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$. After deformation, this volume becomes

$$\mathcal{V} \approx (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)\ \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \approx (1+\epsilon_1+\epsilon_2+\epsilon_3)\ \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \tag{1.74}$$

where the higher order strain quantities are neglected in view of eq. (1.72). The relative change in volume is now

$$e = \frac{(1+\epsilon_1+\epsilon_2+\epsilon_3)\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 - \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3}{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3} = \epsilon_1+\epsilon_2+\epsilon_3. \qquad (1.75)$$

The quantity e is known as the *volumetric strain* and measures the relative change in volume of the material.

1.5 Analysis of the state of strain at a point

The state of strain at a point is characterized in the previous section by the relative elongations of three material lines and their relative angular distortions. The orientations of three material lines are selected parallel to the axes of the Cartesian reference frame $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$. It is clear that the orientation of this reference frame is entirely arbitrary: a reference frame $\mathcal{I}^* = (\bar{\imath}_1^*, \bar{\imath}_2^*, \bar{\imath}_3^*)$ could have been selected, and an analysis identical to that of the previous section would have led to the definition of relative elongations

$$\epsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*}, \quad \epsilon_2^* = \frac{\partial u_2^*}{\partial x_2^*}, \quad \epsilon_3^* = \frac{\partial u_3^*}{\partial x_3^*}, \qquad (1.76)$$

and angular distortions

$$\gamma_{23}^* = \frac{\partial u_2^*}{\partial x_3^*} + \frac{\partial u_3^*}{\partial x_2^*}, \quad \gamma_{13}^* = \frac{\partial u_1^*}{\partial x_3^*} + \frac{\partial u_3^*}{\partial x_1^*}, \quad \gamma_{12}^* = \frac{\partial u_1^*}{\partial x_2^*} + \frac{\partial u_2^*}{\partial x_2^*}. \qquad (1.77)$$

Although expressed in different reference frames, the strain displacements equations, eq. (1.63) and (1.71), or (1.76) and (1.77) both characterize the state of deformation at a point of the body. Therefore, the strain components resolved in the two reference frames should be closely related. Because strain components are purely geometric in nature, it should not be unexpected that the relationship between the strain components resolved in two different coordinate systems is also purely geometric in nature.

1.5.1 Rotation of strains

In this section, the strain components resolved two different bases, $\mathcal{I}$ and $\mathcal{I}^*$, will be related to each other. The orientation of basis $\mathcal{I}^*$ relative to basis $\mathcal{I}$ is discussed in appendix A.3.1 and leads to the definition of the matrix of direction cosines, or rotation matrix, $\underline{\underline{R}}$, given by eq. (A.36).

With the help of the chain rule for derivatives, the first component of strain given by eq. (1.76) becomes

$$\epsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\frac{\partial x_1}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial x_2}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_3}\frac{\partial x_3}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\ell_1 + \frac{\partial u_1^*}{\partial x_2}\ell_2 + \frac{\partial u_1^*}{\partial x_3}\ell_3, \qquad (1.78)$$

where eq. (A.39) is used to express the derivatives of x_1, x_2, and x_3 with respect to x_1^*.

Next the displacement component u_1^* is expressed in terms of the displacement components in coordinate system $\mathcal{I}$ with the help of eq. (A.39), to find

$$\begin{aligned}\epsilon_1^* = \ell_1 \frac{\partial}{\partial x_1}\left(\ell_1 u_1 + \ell_2 u_2 + \ell_3 u_3\right) + \ell_2 \frac{\partial}{\partial x_2}\left(\ell_1 u_1 + \ell_2 u_2 + \ell_3 u_3\right)\\ + \ell_3 \frac{\partial}{\partial x_3}\left(\ell_1 u_1 + \ell_2 u_2 + \ell_3 u_3\right).\end{aligned} \tag{1.79}$$

The last step is to use the strain-displacement relationships, eqs. (1.63) and (1.71), to find

$$\epsilon_1^* = \epsilon_1 \ell_1^2 + \epsilon_2 \ell_2^2 + \epsilon_3 \ell_3^2 + \gamma_{12}\ell_1\ell_2 + \gamma_{13}\ell_1\ell_3 + \gamma_{23}\ell_2\ell_3.$$

A similar analysis for the other direct strain components results in the following expressions for the extensional strain in system $\mathcal{I}^*$,

$$\epsilon_1^* = \epsilon_1 \ell_1^2 + \epsilon_2 \ell_2^2 + \epsilon_3 \ell_3^2 + 2\frac{\gamma_{12}}{2}\ell_1\ell_2 + 2\frac{\gamma_{13}}{2}\ell_1\ell_3 + 2\frac{\gamma_{23}}{2}\ell_2\ell_3, \tag{1.80a}$$

$$\epsilon_2^* = \epsilon_1 m_1^2 + \epsilon_2 m_2^2 + \epsilon_3 m_3^2 + 2\frac{\gamma_{12}}{2}m_1 m_2 + 2\frac{\gamma_{13}}{2}m_1 m_3 + 2\frac{\gamma_{23}}{2}m_2 m_3, \tag{1.80b}$$

$$\epsilon_3^* = \epsilon_1 n_1^2 + \epsilon_2 n_2^2 + \epsilon_3 n_3^2 + 2\frac{\gamma_{12}}{2}n_1 n_2 + 2\frac{\gamma_{13}}{2}n_1 n_3 + 2\frac{\gamma_{23}}{2}n_2 n_3. \tag{1.80c}$$

Proceeding in a similar manner yields the shear strain components expressed in basis $\mathcal{I}^*$

$$\begin{aligned}\frac{\gamma_{12}^*}{2} &= \epsilon_1 \ell_1 m_1 + \epsilon_2 \ell_2 m_2 + \epsilon_3 \ell_3 m_3 + \frac{\gamma_{12}}{2}(\ell_1 m_2 + \ell_2 m_1)\\ &\quad + \frac{\gamma_{13}}{2}(\ell_1 m_3 + \ell_3 m_1) + \frac{\gamma_{23}}{2}(\ell_2 m_3 + \ell_3 m_2),\end{aligned} \tag{1.81a}$$

$$\begin{aligned}\frac{\gamma_{13}^*}{2} &= \epsilon_1 \ell_1 n_1 + \epsilon_2 \ell_2 n_2 + \epsilon_3 \ell_3 n_3 + \frac{\gamma_{12}}{2}(\ell_1 n_2 + \ell_2 n_1)\\ &\quad + \frac{\gamma_{13}}{2}(\ell_1 n_3 + \ell_3 n_1) + \frac{\gamma_{23}}{2}(\ell_2 n_3 + \ell_3 n_2),\end{aligned} \tag{1.81b}$$

$$\begin{aligned}\frac{\gamma_{23}^*}{2} &= \epsilon_1 m_1 n_1 + \epsilon_2 m_2 n_2 + \epsilon_3 m_3 n_3 + \frac{\gamma_{12}}{2}(m_1 n_2 + m_2 n_1)\\ &\quad + \frac{\gamma_{13}}{2}(m_1 n_3 + m_3 n_1) + \frac{\gamma_{23}}{2}(m_2 n_3 + m_3 n_2).\end{aligned} \tag{1.81c}$$

Expressions (1.80) and (1.81) are quite tedious, but the permutations of indices are readily observed. In these equations, it should be noted that the shear strain components are divided by a factor of 2. In this form, eqs. (1.80) and (1.81) become similar to eqs. (1.11) and (1.12), respectively; the axial strain take the place of the axial stresses and the shear strain that of the shear stresses.

The shearing strain components γ_{23}, γ_{13} and γ_{12} are called the *engineering shear strain components*, whereas the *tensor shear strain components*, ϵ_{23}, ϵ_{13} and ϵ_{12}, are defined as

$$\epsilon_{23} = \frac{\gamma_{23}}{2}, \quad \epsilon_{13} = \frac{\gamma_{13}}{2}, \quad \epsilon_{12} = \frac{\gamma_{12}}{2}. \tag{1.82}$$

When using the tensor strain components, the strain rotation expressions, eqs. (1.80) and (1.81), can be written in a compact matrix form as

$$\begin{bmatrix} \epsilon_1^* & \epsilon_{12}^* & \epsilon_{13}^* \\ \epsilon_{12}^* & \epsilon_2^* & \epsilon_{23}^* \\ \epsilon_{13}^* & \epsilon_{23}^* & \epsilon_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \underline{\underline{R}}, \tag{1.83}$$

where $\underline{\underline{R}}$ is the rotation matrix defined by eq. (A.36).

Comparing this result with eq. (1.20) for the rotation of stress components, it becomes clear that the transformation equations for these second order tensors are identical. Equation (1.20) expresses the transformation rules for the components of the second order stress tensor, whereas eq. (1.83) expresses the same rule for the second order strain tensor. In fact, a second order tensor is defined as a mathematical entity whose components measured in two different coordinate systems transform according to the ruled expressed by eqs. (1.20) or (1.83).

1.5.2 Principal strains

Because it has been established that stress and strain components are the components of the second order stress and strain tensors, respectively, it should not be unexpected that the concept of principal stresses, discussed in section 1.2.2 for the stress tensor, has its equivalent when it comes to the strain tensor.

To introduce the concept of principal strains, the following question is asked: is there a coordinate system $\mathcal{I}^*$ for which the shear strains vanish? If such a coordinate system exists, eq. (1.83) implies that

$$\begin{bmatrix} \epsilon_1^* & 0 & 0 \\ 0 & \epsilon_2^* & 0 \\ 0 & 0 & \epsilon_3^* \end{bmatrix} = \underline{\underline{R}}^T \begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \underline{\underline{R}},$$

where $\epsilon_1^* = \epsilon_{p1}$, $\epsilon_2^* = \epsilon_{p2}$, and $\epsilon_3^* = \epsilon_{p3}$ are the principal strains. By pre-multiplying by $\underline{\underline{R}}$ and reversing the equality, this equation can be written in the following form

$$\begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \underline{\underline{R}} = \underline{\underline{R}} \begin{bmatrix} \epsilon_{p1} & 0 & 0 \\ 0 & \epsilon_{p2} & 0 \\ 0 & 0 & \epsilon_{p3} \end{bmatrix},$$

where the orthogonality of the direction cosine matrix, eq. (A.37), is used. It follows that the principal strains, ϵ_{p1}, ϵ_{p2} and ϵ_{p3}, are the solutions of three systems of three equations of the form

$$\begin{bmatrix} \epsilon_1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \epsilon_p \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix},$$

where ϵ_p represents each of the three principal strains and n_1, n_2, and n_3 the principal strain directions. These equations can be recast as a homogeneous system of linear equations

$$\begin{bmatrix} \epsilon_1 - \epsilon_p & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_2 - \epsilon_p & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_3 - \epsilon_p \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0. \tag{1.84}$$

Clearly, this homogeneous system is equivalent to system (1.13) that defines the principal stresses.

Since this is a homogeneous system of equations, the trivial solution, $n_1 = n_2 = n_3 = 0$, is, in general, the solution of this system. When the determinant of the system vanishes, however, non-trivial solutions will exist. The vanishing of the determinant of the system leads to a cubic equation for the magnitude of the principal strains given by

$$\epsilon_p^3 - I_1\epsilon_p^2 + I_2\epsilon_p - I_3 = 0, \tag{1.85}$$

where the quantities, I_1, I_2, and I_3, defined as

$$I_1 = \epsilon_1 + \epsilon_2 + \epsilon_3, \tag{1.86a}$$

$$I_2 = \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 - \epsilon_{12}^2 - \epsilon_{13}^2 - \epsilon_{23}^2, \tag{1.86b}$$

$$I_3 = \epsilon_1\epsilon_2\epsilon_3 - \epsilon_1\epsilon_{23}^2 - \epsilon_2\epsilon_{13}^2 - \epsilon_3\epsilon_{12}^2 + 2\epsilon_{12}\epsilon_{13}\epsilon_{23}, \tag{1.86c}$$

are called the three *strain invariants*.

The solutions of eq. (1.85) are called the *principal strains*. Because this is a cubic equation, there will be three solutions, denoted ϵ_{p1}, ϵ_{p2}, and ϵ_{p2}. For each of these three solutions, the matrix of the system of equations defined by eq. (1.84) has a zero determinant, and a non-trivial solution exists for the directions cosines that now define the direction for which the shear strains vanish. Such direction is called a *principal strain direction*. Because the equations to be solved are homogeneous, their solution will include an arbitrary constant which can be determined by enforcing the normality condition for unit vector $\bar{n}$, $n_1^2 + n_2^2 + n_3^2 = 1$.

Since there exist three principal strains, three principal strain directions must also exist. It can be shown that these three directions are mutually orthogonal.

1.6 The state of plane strain

A particular state of strain of great practical importance is the *plane state of strain*. In this case, the displacement component along the direction of axis $\bar{\imath}_3$ is assumed to vanish, or to be negligible compared to the displacement components in the other two directions. This means that the only non-vanishing strain components are ϵ_1, ϵ_2, and γ_{12}, and furthermore, these strain components are assumed to be independent of x_3.

Unlike the plane state of stress considered in section 1.3, plane strain problems are not characterized by having one dimension much thinner than the others. Instead, displacement in one direction is zero. An example of a plane strain problem is that of a very long buried pipe aligned with the $\bar{\imath}_3$ direction. Such a problem is clearly three-dimensional in its overall geometry, but if the displacement along the direction of axis $\bar{\imath}_3$ is small or negligible, the pipe is in a plane state of strain.

1.6.1 Strain-displacement relations for plane strain

If the material is in a plane state of strain, *i.e.*, if $u_3 = 0$ and $\partial/\partial x_3 = 0$, eqs (1.63) and (1.71) reduce to

$$\epsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \tag{1.87}$$

1.6.2 Rotation of strains

Next, the strain components measured two different orthonormal bases, $\mathcal{I}$ and $\mathcal{I}^*$, will be related to each other. Since this problem involves two distinct orthonormal bases, the relationship between these two basis, as explored in appendix A.3.3, is relevant to this development.

With the help of the chain rule for derivatives, the first component of strain given by eq. (1.76) becomes

$$\epsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\frac{\partial x_1}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial x_2}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1}\cos\theta + \frac{\partial u_1^*}{\partial x_2}\sin\theta,$$

where eq. (A.43) is used to express the derivatives of x_1 and x_2 with respect to x_1^*. Next, the displacement component u_1^* is expressed in terms of the displacement components resolved in coordinate system $\mathcal{I}$ with the help of eq. (A.43) to yield

$$\epsilon_1^* = \cos\theta \frac{\partial}{\partial x_1}\left(u_1\cos\theta + u_2\sin\theta\right) + \sin\theta\frac{\partial}{\partial x_2}\left(u_1\cos\theta + u_2\sin\theta\right). \tag{1.88}$$

The last step is to use the strain-displacement relationships, eqs. (1.87), to find

$$\epsilon_1^* = \cos^2\theta\,\epsilon_1 + \sin^2\theta\,\epsilon_2 + \sin\theta\cos\theta\,\gamma_{12}. \tag{1.89}$$

Proceeding in a similar manner yields the shear strain components in the $\mathcal{I}^*$ coordinate system

$$\frac{\gamma_{12}^*}{2} = -\epsilon_1\cos\theta\sin\theta + \epsilon_2\sin\theta\cos\theta + \frac{\gamma_{12}}{2}(\cos^2\theta - \sin^2\theta). \tag{1.90}$$

Here again, it is convenient to use the tensor component of shearing strain, $\epsilon_{12} = \gamma_{12}/2$, see eq. (1.82).

These results can be written in a matrix form as

$$\left\{\begin{matrix}\epsilon_1^* \\ \epsilon_2^* \\ \epsilon_{12}^*\end{matrix}\right\} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta\end{bmatrix}\left\{\begin{matrix}\epsilon_1 \\ \epsilon_2 \\ \epsilon_{12}\end{matrix}\right\}. \tag{1.91}$$

This relationship can be readily inverted by recognizing that the inverse transformation is obtained by replacing θ by $-\theta$ to find

$$\left\{\begin{matrix}\epsilon_1 \\ \epsilon_2 \\ \epsilon_{12}\end{matrix}\right\} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta\end{bmatrix}\left\{\begin{matrix}\epsilon_1^* \\ \epsilon_2^* \\ \epsilon_{12}^*\end{matrix}\right\}. \tag{1.92}$$

Note that these transformation formulæ are identical to those derived for the stress tensor, see eqs. (1.47) and (1.48), respectively.

With the help of double-angle trigonometric identities, the transformation equations for tensor strain components, eq. (1.91), can also be written as

$$\epsilon_1^* = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\theta + \epsilon_{12} \sin 2\theta, \quad (1.93a)$$

$$\epsilon_2^* = \frac{\epsilon_1 + \epsilon_2}{2} - \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\theta - \epsilon_{12} \sin 2\theta, \quad (1.93b)$$

$$\epsilon_{12}^* = - \frac{\epsilon_1 - \epsilon_2}{2} \sin 2\theta + \epsilon_{12} \cos 2\theta. \quad (1.93c)$$

While use of the strain tensor components, ϵ_{ij}, renders the treatment of stress and strain component rotation formulæ identical, it is customary to use the engineering shear strain components, γ_{ij}, instead of their tensor counterparts, and hence, the previous equations become

$$\epsilon_1^* = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\theta + \frac{\gamma_{12}}{2} \sin 2\theta, \quad (1.94a)$$

$$\epsilon_2^* = \frac{\epsilon_1 + \epsilon_2}{2} - \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\theta - \frac{\gamma_{12}}{2} \sin 2\theta, \quad (1.94b)$$

$$\gamma_{12}^* = - (\epsilon_1 - \epsilon_2) \sin 2\theta + \gamma_{12} \cos 2\theta. \quad (1.94c)$$

This important result shows that knowledge of the plane strain components ϵ_1, ϵ_2, and γ_{12} in two orthogonal directions allows the computation of the strain components in an arbitrary orientation. In other words, the knowledge of the plane strain components in two orthogonal directions fully defines the state of strain at that point.

1.6.3 Principal strains

The relative elongation in an arbitrary direction, θ, can be computed with the help of eq. (1.94). The orientation, θ_p, in which the maximum (or minimum) elongation occurs is determined by requiring the derivative of ϵ_1^* with respect to θ to vanish, and this yields

$$\frac{\mathrm{d}\epsilon_1^*}{\mathrm{d}\theta} = -\frac{\epsilon_1 - \epsilon_2}{2} 2 \sin 2\theta_p + \frac{\gamma_{12}}{2} 2 \cos 2\theta_p = 0. \quad (1.95)$$

This can be solved for $2\theta_p$ to find the orientation of extreme elongation as

$$\tan 2\theta_p = \frac{\gamma_{12}/2}{(\epsilon_1 - \epsilon_2)/2}, \quad (1.96)$$

where the factor 2 has not been canceled out in order to retain the similarity with eq. (1.33) if τ_{12} is replaced with $\gamma_{12}/2$ and σ_1 and σ_2 with ϵ_1 and ϵ_2, respectively. This equation presents two solutions, θ_{p1} and $\theta_{p2} = \theta_{p1} + \pi/2$, corresponding to two mutually orthogonal principal strain directions. The maximum axial strain is found along one direction, and the minimum is found along the other.

To define these orientations unequivocally, it is convenient to separately define the sine and cosines of angle $2\theta_p$ as follows

$$\sin 2\theta_p = \frac{\gamma_{12}}{2\Delta}, \quad \cos 2\theta_p = \frac{\epsilon_1 - \epsilon_2}{2\Delta}, \tag{1.97}$$

where

$$\Delta = \sqrt{\left(\frac{\epsilon_1 - \epsilon_2}{2}\right)^2 + \left(\frac{\gamma_{12}}{2}\right)^2}. \tag{1.98}$$

This result is equivalent to eq. (1.96), but it gives a unique solution for θ_p because both the sine and cosine of the angle are known. The maximum and minimum axial strains, denoted ϵ_{p1} and ϵ_{p2}, respectively, act in the directions θ_{p1} and $\theta_{p2} = \theta_{p1} + \pi/2$, respectively. These maximum and minimum axial strains, called the *principal strains*, are evaluated by introducing eq. (1.97) into eq. (1.94) to find

$$\epsilon_{p1} = \frac{\epsilon_1 + \epsilon_2}{2} + \Delta; \quad \epsilon_{p2} = \frac{\epsilon_1 + \epsilon_2}{2} - \Delta. \tag{1.99}$$

Finally, the shear strain in the principal directions vanishes as can be verified by introducing eq. (1.97) into eq. (1.94).

The development of the equations for the state of strain at a point yield equations that are very similar to those developed in section 1.2.2 for the state of stress at a point. In particular, the transformation equations are similar in form (identical if the strain tensor components, $\epsilon_{12} = \gamma_{12}/2$, are used to define the shear strain) and lead in both cases to the existence of principal stresses and principal strains. The orientations of the principal stresses and principal strains are not necessarily identical.

1.6.4 Mohr's circle for plane strain

Equations (1.94) express the direct and shear strains along an arbitrary direction defined by angle θ with respect the axis $\bar{\imath}_1$, but the presence of trigonometric functions involving the angle 2θ makes it difficult to give a geometric interpretation of these formulae. Let the state of strain at a point be defined by its principal strains, ϵ_{p1} and ϵ_{p2}; eq. (1.94) then implies that the strains along a direction defined by angle θ with respect to the principal strain directions can be written as

$$\epsilon^* = \epsilon_a + R\ \cos 2\theta, \quad \frac{\gamma^*}{2} = -R\ \sin 2\theta, \tag{1.100}$$

where $\epsilon_a = (\epsilon_{p1} + \epsilon_{p2})/2$, and $R = (\epsilon_{p1} - \epsilon_{p2})/2$. With this notation and the help of trigonometric identities, eq. (1.100) becomes

$$(\epsilon^* - \epsilon_a)^2 + \left(\frac{\gamma^*}{2}\right)^2 = R^2. \tag{1.101}$$

This equation represents the equation of a circle which is known as *Mohr's circle*. When ϵ^* is plotted along the horizontal axis and $\gamma^*/2$ along the vertical axis, the center of the circle is at a coordinate ϵ_a on the horizontal axis, and the radius of the circle is R, as depicted in fig. 1.23.

Consider now point $\mathbf{A_1}$ on Mohr's circle such that segment $\mathbf{OA_1}$ makes an angle 2θ with the horizontal. The coordinates of this point are $\epsilon^* = \epsilon_a + R\cos 2\theta$ and $\gamma^*/2 = -R\sin 2\theta$; hence, in view of eq. (1.100), the coordinates of point $\mathbf{A_1}$ represent the strain components along a direction defined by angle θ. In fact, each point on Mohr's circle represents the strain components along a specific orientation.

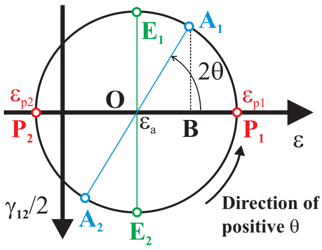

Fig. 1.23. Mohr's circle for visualizing plane strain state.

An **important sign convention** must be defined: on Mohr's circle, a positive angle θ is measured in the **counterclockwise direction**, see fig. 1.23, to match the positive direction of angle θ that identifies the orientation of a face in fig. 1.12. Given the sign convention for angle θ, the shear strain must be positive downward on the ordinate of Mohr's circle depicted in fig. 1.23.[4]

All the developments presented in section 1.3.6 for visualizing a plane state of stress using Mohr's circle also apply to the present problem of visualizing the plane state of strain, provided however, that the strain tensor is used. This means that $\gamma_{12}/2$ must be plotted on the vertical axis.

1.7 Measurement of strains

The goal of the theory of elasticity is to predict the state of stress at any point of an elastic body, given the applied loading. Such predictions must be validated by measuring the state of stress at specific points of a body, then comparing these measurements with the corresponding predictions. Unfortunately, no practical experimental device can measure stresses directly. An indirect measurement must therefore be made by first measuring the state of strain, then computing the corresponding state of stress using the constitutive laws for the material.

Strain gauges

Measurement of the state of strain itself is not an entirely straightforward process. First, it is relatively difficult to measure the strain state at an interior point of a solid body, so most measurement methods focus on measuring strains on the body's external surface. As noted in previous sections, the two-dimensional strain state is characterized by both direct and shear components. Measurements of the very small angular changes associated with shear strains are difficult to perform, but measurements of extensional strains on a surface are surprisingly easy to acquire.

The relative elongation at the surface of a body can be measured with the help of what are called electrical resistance strain gauges, or more simply *strain gauges*. This device consists of a very thin electric wire, or an etched foil pattern, which is glued to the surface of the solid. When the solid experiences an extensional strain, this strain is transferred through the glue to the gauge, hence increasing the length

[4] An equivalent construction of Mohr's circle has the shear strain positive upwards along the ordinate, but angle θ is then positive in the clockwise direction.

of the wire. In turn, the wire's cross-section is reduced by Poisson's effect, thereby slightly increasing its electrical resistance. The reverse happens for compressional strains, and the electrical resistance is slightly reduced in this case.

An accurate electrical measurement of this resistance change, using a Wheatstone bridge circuit for instance, yields an estimate of the length change, which in turn, allows an accurate estimate of the relative elongation and finally, the extensional strain in the direction of the wire. Because strain quantities are very small, strain measurements are often labeled in *micro-strains*, which indicates a relative elongation of μ m/m = 10^{-6} m/m. Because strains are non-dimensional quantities, the units employed for measurement of elongation can be any units of length.

Chevron strain gauge

Figure 1.24 shows the external surface of a body with two strain gauges forming at a 90 degree angle with respect to each other; this configuration is sometimes called *chevron strain gauge*. This device is of finite size, and hence, the two extensional strain measurements are not made exactly at the same point, but if the chevron strain gauge is very small and the strain gradients are small compared to its size, it can be assumed that the two gauges experience the same strain state.

Let e_{+45} and e_{-45} be the experimentally measured relative elongations in the two gauge directions. The two gauges of the chevron are oriented at $\pm$ 45 degrees with respect to a triad, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$, as shown in fig. 1.24. The state of strain at that point is defined by the three strain components, ϵ_1, ϵ_2, and γ_{12}, resolved in triad $\mathcal{I}$. With the help of eq. (1.94a), these measurements can be expressed as follows

$$e_{+45} = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2}\cos(2 \times \ \ 45^\circ) + \frac{\gamma_{12}}{2}\sin(2 \times \ \ 45^\circ) = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\gamma_{12}}{2},$$
$$e_{-45} = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2}\cos(2 \times 135^\circ) + \frac{\gamma_{12}}{2}\sin(2 \times 135^\circ) = \frac{\epsilon_1 + \epsilon_2}{2} - \frac{\gamma_{12}}{2}.$$

Clearly, the two measurements, e_{+45} and e_{-45}, are not sufficient to determine the strain state at the chevron's location. Indeed, three measurements would be required to determine the three strain components, ϵ_1, ϵ_2, and γ_{12}. It is possible, however, to unequivocally determine the shear strain by subtracting the above equations from each other to find

$$\gamma_{12} = e_{+45} - e_{-45}. \tag{1.102}$$

Adding the two equations yields $\epsilon_1 + \epsilon_2 = e_{+45} + e_{-45}$, but the two normal strain components, ϵ_1 and ϵ_2, cannot be determined individually.

The complete state of strain at the surface of the body is specified by three independent quantities, *i.e.*, either two extensional and a shear strain, or two principal strains and a principal direction. These can be computed from the measurement of relative elongation in three distinct directions on the surface.

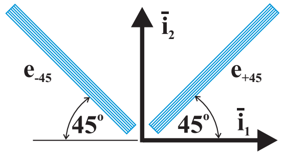

Fig. 1.24. Two strain gauges at the surface of a solid.

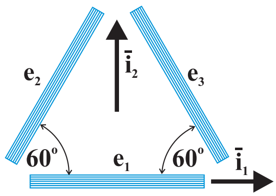

Fig. 1.25. Three strain gauges forming a rosette at the surface of a solid.

Strain gauge rosette

The experimental determination of the strain state at the surface of a body requires three independent measurements. One approach is to locate three strain gauges forming an equilateral triangle at the external surface of a body, as depicted in fig. 1.25. This type of device is commonly referred to as a *strain gauge rosette*; the configuration shown in the figure is often called a "delta rosette." Once again, this rosette is of finite size, and hence, the three extensional strain measurements are not made exactly at the same point, but if the rosette is very small and the strain gradients are small compared to the size of the rosette, it can be assumed that the three gauges experience the same strain state.

Let e_1, e_2, and e_3 be the experimentally measured relative elongations in the three gauge directions. With the help of eq. (1.94a), these measurements can be related to the strain components measured in triad $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$ as follows

$$e_1 = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2},$$
$$e_2 = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos(+2 \times 60^\circ) + \frac{\gamma_{12}}{2} \sin(+2 \times 60^\circ),$$
$$e_3 = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos(-2 \times 60^\circ) + \frac{\gamma_{12}}{2} \sin(-2 \times 60^\circ).$$

These relationships can be inverted to yield the strain components in terms of the measured axial strains

$$\epsilon_1 = e_1, \quad \epsilon_2 = \frac{2}{3}\left(e_2 + e_3 - \frac{e_1}{2}\right), \quad \gamma_{12} = \frac{2}{\sqrt{3}}\left(e_2 - e_3\right). \tag{1.103}$$

The principal strain directions then follow from (1.97)

$$\sin 2\theta_p = \frac{e_2 - e_3}{\sqrt{3}\Delta}, \quad \cos 2\theta_p = \frac{2e_1 - e_2 - e_3}{3\Delta}, \tag{1.104}$$

and the principal strains are

$$\epsilon_{p1} = \bar{e} + \Delta, \quad \epsilon_{p2} = \bar{e} - \Delta, \tag{1.105}$$

where $\bar{e} = (e_1 + e_2 + e_3)/3$ and $\Delta = 2/3\,\sqrt{e_1^2 + e_2^2 + e_3^2 - e_2 e_3 - e_1 e_3 - e_1 e_2}$.

Various commonly used strain gauge arrangements are depicted in fig. 1.26. Note that a complete evaluation of the state of strain requires the knowledge of three strain components, and thus requires three independent measurements in three distinct directions. Combinations (a) and (c) of fig. 1.26 provide three independent measurements from which the strain state can be evaluated using a similar approach to that developed above for the delta strain gauge rosette shown in fig. 1.25.

Combinations (B) and (D) allow four independent measurements to be made to provide enough information in the event when one of the gauges is damaged. If the four gauges are properly working, the redundant information can be used to compensate for experimental errors, as illustrated in example 1.4 for the T-Delta rosette.

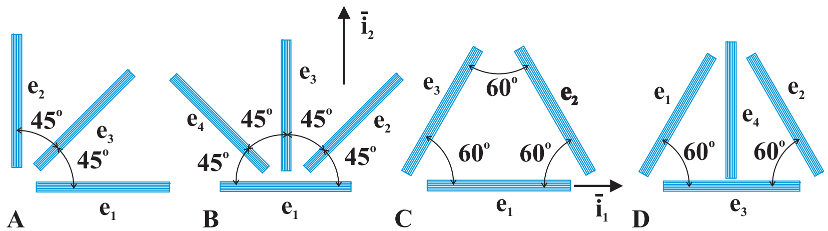

Fig. 1.26. Various commonly used strain gauge arrangements.

Example 1.4. Data reduction for the T-Delta rosette

Consider the T-Delta rosette shown in fig. 1.26 D. Given the output of the four gauges, e_1, e_2, e_3 and e_4, find the state of strain at the location of the rosette. First, the four measurements are expressed in terms of the three strain components with the help of eq. (1.94a) to find

$$
\begin{aligned}
e_1 &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{\epsilon_1-\epsilon_2}{2}\cos 120 + \frac{\gamma_{12}}{2}\sin 120 = \frac{\epsilon_1+\epsilon_2}{2} - \frac{\epsilon_1-\epsilon_2}{4} + \frac{\sqrt{3}}{4}\gamma_{12},\\
e_2 &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{\epsilon_1-\epsilon_2}{2}\cos 240 + \frac{\gamma_{12}}{2}\sin 240 = \frac{\epsilon_1+\epsilon_2}{2} - \frac{\epsilon_1-\epsilon_2}{4} - \frac{\sqrt{3}}{4}\gamma_{12},\\
e_3 &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{\epsilon_1-\epsilon_2}{2} = \epsilon_1,\\
e_4 &= \frac{\epsilon_1+\epsilon_2}{2} + \frac{\epsilon_1-\epsilon_2}{2}\cos 180 + \frac{\gamma_{12}}{2}\sin 180 = \epsilon_2.
\end{aligned}
$$

These relationships form a set of four equations for three unknowns, the strain components ϵ_1, ϵ_2, and γ_{12}, which can be written in a compact matrix form as

$$
\begin{bmatrix} 1/4 & 3/4 & \sqrt{3}/4 \\ 1/4 & 3/4 & -\sqrt{3}/4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}
=
\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix}.
$$

These equations form an over-determined set of equations to evaluate the three components of strain. Since the strain measurement are likely to involve experimental errors, it seems appropriate to solve the over-determined system in a least squares sense, as explained in appendix A.2.10. For this problem, the least-squares solution given by eq. (A.33) becomes

$$\frac{1}{8}\begin{bmatrix}9 & 3 & 0\\ 3 & 17 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \gamma_{12}\end{Bmatrix} = \begin{Bmatrix}(e_1+e_2)/4+e_3\\ 3(e_1+e_2)/4+e_4\\ \sqrt{3}\,(e_1-e_2)/4\end{Bmatrix}.$$

The solution of this 3×3 linear system then yields the desired strain components as

$$\epsilon_1 = \frac{2e_1+2e_2+17e_3-3e_4}{18};\ \epsilon_2 = \frac{6e_1+6e_2-3e_3+9e_4}{18};\ \gamma_{12} = \frac{2(e_1-e_2)}{\sqrt{3}}.$$

1.7.1 Problems

Problem 1.14. Data reduction for the delta rosette
Consider the delta rosette shown in fig. 1.26 C. The measured data are $e_1 = 410\mu$, $e_2 = -290\mu$, and $e_3 = 610\mu$. *(1)* Find the state of strain at this location. *(2)* Draw Mohr's circle for this state of strain. *(3)* Find the orientation of the principal strain directions, and *(4)* find the principal strains. Use a software package to carry out these calculations.

Problem 1.15. Data reduction for the rectangular rosette
Consider the rectangular rosette shown in fig. 1.26 A. The measured data are $e_1 = -510\mu$, $e_2 = 780\mu$, $e_3 = 340\mu$. *(1)* Develop expressions similar to eq. (1.103) for the state of strain with respect to a surface axis system aligned with gauges #1 and #2. *(2)* Find the state of strain at this location for the given data. *(3)* Draw Mohr's circle for this state of strain. *(4)* Find the orientation of the principal strain directions, and *(5)* the principal strains.

Problem 1.16. Data reduction for the T-V rosette
Consider the T-V rosette shown in fig. 1.26 B. The measured data is $e_1 = 910\mu$, $e_2 = 990\mu$, $e_3 = 310\mu$ and $e_4 = 190\mu$. Use a least square approach to solve this problem. *(1)* Find the state of strain at this location. *(2)* Draw Mohr's circle for this state of strain. *(3)* Find the orientation of the principal strain directions, and *(4)* the principal strains.

Problem 1.17. Correlating rosette strain measurements
Consider the strain gauge arrangements shown in fig. 1.26 B. If the strain measurements e_1, e_2 and e_3 are given find the strain e_4.

Problem 1.18. Correlating rosette strain measurements
Consider the strain gauge arrangements shown in fig. 1.26 D. If the strain measurements e_1, e_2 and e_4 are given, find the strain e_3.

Problem 1.19. Misaligned Delta rosette
The delta rosette depicted in fig. 1.27 has been improperly installed on a solid: instead of aligning the rosette with axes $\bar{\imath}_1$ and $\bar{\imath}_2$, as desired, the gage was installed at an angle θ with respect to the desired directions. This implies that the gauge measurements will be e_1^*, e_2^* and e_3^*, instead of the desired e_1, e_2 and e_3. Since the misalignment is unintentional, the experimentalist will use the measurements, e_1^*, e_2^* and e_3^*, *as if they were* e_1, e_2 and e_3, respectively.

In other words, he will use the measurements e_1^*, e_2^* and e_3^* to extract the strain state, thinking that $\theta = 0$. *(1)* If the strain state is $\epsilon_1 = 1245\mu$, $\epsilon_2 = -780\mu$ and $\gamma_{12} = 675\mu$, determine the state of strain that the experimentalist will erroneously extract, denoted $\hat{\epsilon}_1$, $\hat{\epsilon}_2$ and $\hat{\gamma}_{12}$, as function of the misalignment angle. *(2)* On one graph, plot the relative errors $(\hat{\epsilon}_1 - \epsilon_1)/\epsilon_1$, $(\hat{\epsilon}_2 - \epsilon_2)/\epsilon_2$ and $(\hat{\gamma}_{12} - \gamma_{12})/\gamma_{12}$, as functions of $\theta \in [-10, 10]$ degrees.

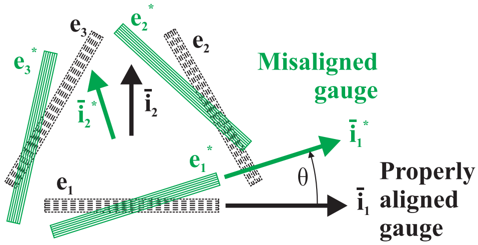

Fig. 1.27. Delta rosette with an angular misalignment of θ.

Problem 1.20. Transverse shear strain in beams
In beam theory, it is assumed that the planar cross-section of the beam remains planar and remains perpendicular to the axis of the beam as it bends. This implies that two material lines, the axis of the beam and a material line in the plane of the cross-section, remain perpendicular to each other. In view of this assumption, what is the transverse shear strain along the axis of the beam?

1.8 Strain compatibility equations

The displacement field uniquely defines the deformation of a solid body. Six strain components, however, are defined to characterize the state of deformation at a point. Hence, the strain components are not independent and must satisfy a set of relationships called the *strain compatibility equations*. Consider the following derivatives of the shear strain components

$$\frac{\partial^2 \gamma_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2}{\partial x_2 \partial x_3}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{\partial^3 u_2}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u_3}{\partial^2 x_2 \partial x_3} = \frac{\partial^2 \epsilon_2}{\partial x_3^2} + \frac{\partial^2 \epsilon_3}{\partial x_2^2}.$$

This implies that the shear and axial strain components are not independent. Consider now a different set of derivatives

$$\frac{\partial^2 \epsilon_1}{\partial x_2 \partial x_3} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2 \partial x_3}, \quad \frac{\partial \gamma_{23}}{\partial x_1} = \frac{\partial^2 u_2}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_2};$$

$$\frac{\partial \gamma_{13}}{\partial x_2} = \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_2}, \quad \frac{\partial \gamma_{12}}{\partial x_3} = \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3},$$

which imply

$$2\frac{\partial^2 \epsilon_1}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial \gamma_{23}}{\partial x_1} + \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{12}}{\partial x_3}\right).$$

This is another relationship between the shear and axial strain components.

Similar relationships can be obtained through cyclical permutations of the indices to yield *Saint-Venant's strain compatibility equations*

$$\frac{\partial^2 \gamma_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 \epsilon_2}{\partial x_3^2} + \frac{\partial^2 \epsilon_3}{\partial x_2^2}, \tag{1.106a}$$

$$\frac{\partial^2 \gamma_{13}}{\partial x_1 \partial x_3} = \frac{\partial^2 \epsilon_1}{\partial x_3^2} + \frac{\partial^2 \epsilon_3}{\partial x_1^2}, \tag{1.106b}$$

$$\frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \epsilon_1}{\partial x_2^2} + \frac{\partial^2 \epsilon_2}{\partial x_1^2}, \tag{1.106c}$$

$$2\frac{\partial^2 \epsilon_1}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial \gamma_{23}}{\partial x_1} + \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{12}}{\partial x_3}\right), \tag{1.106d}$$

$$2\frac{\partial^2 \epsilon_2}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_2}\left(+\frac{\partial \gamma_{23}}{\partial x_1} - \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{12}}{\partial x_3}\right), \tag{1.106e}$$

$$2\frac{\partial^2 \epsilon_3}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(+\frac{\partial \gamma_{23}}{\partial x_1} + \frac{\partial \gamma_{13}}{\partial x_2} - \frac{\partial \gamma_{12}}{\partial x_3}\right). \tag{1.106f}$$

Some reflection is needed to fully understand the need for stating the compatibility equations. Clearly, if the state of deformation is defined by the three components of the displacement vector, *i.e.*, if the displacement field is given, it is a simple matter to compute the six strain components using eqs. (1.63) and (1.71). The inverse problem, however, is not so simple: if the state of deformation is defined by six components of strain, *i.e.*, given the strain field, it is not obvious to determine the displacement components that give rise to this strain field. Indeed, the six strain components are generated based on three displacement components only. Furthermore, some strain states could possibly be associated with displacement fields that include discontinuities or jumps corresponding to gaps or tears in the continuous body. In summary, if the six components of the strain field are derived from the three components of the displacement field, they are not independent and must satisfy Saint-Venant's strain compatibility equations. Three only of the six compatibility equations are independent.