10

Energy methods

The basic equations of linear elasticity are derived in chapter 1 and are conveniently divided into three groups: the equilibrium equations, the strain-displacement relationships, and the constitutive laws, as illustrated in fig. 9.33. In chapter 9, two virtual work principles are derived. First, the principle of virtual work is established and shown to be entirely equivalent to the equilibrium equations of the system; this principle, however, provides no information about the other two sets of equations, the strain-displacement relationships and constitutive laws, which must be obtained in the traditional manner. Second, the principle of complementary virtual work is established and shown to be entirely equivalent to the strain-displacement relationships of the system; this principle, however, provides no information about the other two sets of equations, the equilibrium equations and constitutive laws, which must obtained in the traditional manner. To remedy this situation, new principles will be developed in this chapter that are entirely equivalent to two of the three groups of equation of linear elasticity. The main tool used to achieve this generalization of the principles presented in chapter 9 is the concept of *conservative forces*.

Types of forces

Newton's first law states that for static equilibrium to be achieved, the "sum of all forces must vanish." The power of this law resides in its generality, and all forces, without any distinction, play an equal role in this equilibrium condition, as underlined in example 9.3 on page 404, for instance.

With virtual work principles, however, various categories of forces are defined. For instance, internal and external forces and the virtual work they perform are clearly separated in the statement of both the principle of virtual work and its complementary counterpart, see principles 6 and 7 on pages 434 and 444, respectively. Externally applied forces and reaction forces also warrant a different treatment in the principle of virtual work. Reaction forces can be eliminated from the formulation because the work they perform vanishes when using kinematically admissible virtual displacements; on the other hand, when arbitrary virtual displacements are used, the virtual work they perform does not vanish, and they become an integral part of the formulation.

Conservative forces

The developments presented in this chapter are rooted in the crucial distinction between conservative and non-conservative forces. This fundamental concept is introduced in elementary physics courses, and it will be examined in much more depth in this chapter. Conservative forces enjoy many remarkable properties. For instance, the work they perform always vanishes when the force undergoes displacements that form a closed path; in other words, forces return to their initial magnitudes when displacements are returned to their initial values. Furthermore, when a dynamical system is subjected only to conservative forces, the total mechanical energy of the system is preserved in time, and hence, the term "conservative forces." From a more mathematical view point, conservative forces are characterized by the existence of a scalar quantity called a *potential* from which they can be derived.

The principle of virtual work considerably simplifies the analysis procedure for elastic structures because it involves only the computation and manipulation of scalar work quantities. If the externally applied forces acting on the system are conservative, they can be derived from a potential, and this fact can be used to further simplify the calculation of the virtual work done by these externally applied forces. Similarly, if the *strain energy* of an elastic component exists, the corresponding elastic forces can be derived from this strain energy, thus further simplifying the evaluation of the virtual work done by the internal forces.

The combination of the principle of virtual work and the concepts of strain energy and potential of the externally applied loads leads to the principle of minimum total potential energy, which will further simplify the analysis of elastic structures. This principle, however, is not as general as the principle of virtual work because it assumes that both internal and externally applied forces are conservative. Clearly, not all externally applied forces are conservative; for instance, friction or aerodynamic forces are not conservative. Similarly, if a material is deformed beyond its elastic limit and into the plastic regime, no strain energy function exists.

Whereas the principle of virtual work is always valid because is is equivalent to Newton's law, the applicability of the principle of minimum total potential energy is limited to systems involving conservative forces.

10.1 Conservative forces

Let $\underline{r}$ denote the position vector of a particle, and let $\underline{F}$ be a force acting on this particle. Conservative forces are a class of forces that depend only upon the position of the particles on which they act, $\underline{F} = \underline{F}(\underline{r})$. Although these forces may vary with time if the system moves, they do not depend explicitly on time or velocity. Figure 10.1 shows two arbitrary paths, denoted **ACB** and **ADB**, along which the particle moves in space from point **A** to point **B**.

Definition

By definition, force $\underline{F}$ is conservative if and only if the work it performs along any path joining the same initial and final points is identical. This is expressed by the following equation

$$W = \int_{\text{Path ACB}} \underline{F} \cdot \mathrm{d}\underline{r} = \int_{\text{Path ADB}} \underline{F} \cdot \mathrm{d}\underline{r}. \tag{10.1}$$

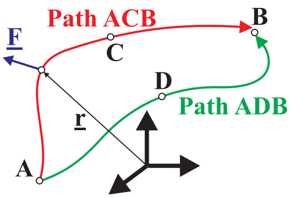

Fig. 10.1. Paths **ACB** and **ADB** join the same two points, **A** and **B**.

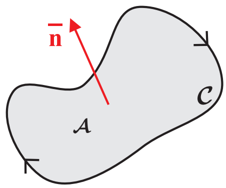

Fig. 10.2. Path enclosing a surface of area $\mathcal{A}$ with a normal $\bar{n}$.

Since reversing the limits of integration simply changes the sign of the integral, the work done by the force along path **ADB** is equal in magnitude and opposite in sign to that along path **BDA**. Equation (10.1) then implies the vanishing of the work done by the force over the closed path **ACBDA**. Because path **ACB** and **ADB** are arbitrary paths joining points **A** and **B**, it follows the a force is conservative if and only if the work it performs vanishes over any arbitrary closed path,

$$W = \oint_{\text{Any path}} \underline{F} \cdot \mathrm{d}\underline{r} = \oint_{\mathcal{C}} \underline{F} \cdot \mathrm{d}\underline{r} = 0, \tag{10.2}$$

where $\mathcal{C}$ is an arbitrary closed curve.

Potential of a conservative force

Based on the definition of conservative forces, eq. (10.2), Stokes' theorem [7] then implies that

$$\oint_{\mathcal{C}} \underline{F} \cdot \mathrm{d}\underline{r} = \int_{\mathcal{A}} \bar{n} \cdot \nabla \times \underline{F} \, \mathrm{d}\mathcal{A} = 0, \tag{10.3}$$

where $\mathcal{A}$ is an area enclosed by curve $\mathcal{C}$ and $\bar{n}$ the outward normal to area $\mathcal{A}$, as shown in fig. 10.2. If the force is conservative, the area integral must vanish for any area, $\mathcal{A}$, and this can only occur if the integrand vanishes, leading to $\nabla \times \underline{F} = 0$ for any curve, $\mathcal{C}$, and area, $\mathcal{A}$. Textbooks on vector algebra [7], prove the following identity: $\nabla \times \nabla \Phi = 0$, where Φ is an arbitrary scalar function. It can then be shown that the solution of equation $\nabla \times \underline{F} = 0$ is simply

$$\underline{F} = -\nabla \Phi, \tag{10.4}$$

where ∇ is the gradient operator.

If a vector field, $\underline{F}$, can be derived from a scalar function, Φ, this function is called a *potential*, and the vector function is said to "be derived from a potential." Because Φ is an arbitrary scalar function, the minus sign is redundant, but is, however, a convention that will be justified later.

It has now been established that if a force is conservative, it can be "derived from a potential." In more mathematical terms, a conservative force must be the gradient a scalar function, called the *potential of the force*. If $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ is an orthonormal basis, conservative forces can be expressed as

$$\underline{F} = -\nabla\Phi = -\frac{\partial\Phi}{\partial x_1}\bar{\imath}_1 - \frac{\partial\Phi}{\partial x_2}\bar{\imath}_2 - \frac{\partial\Phi}{\partial x_3}\bar{\imath}_3. \tag{10.5}$$

The work done by a conservative force over an arbitrary path joining point **1** to point **2**, with position vectors $\underline{r}_1$ and $\underline{r}_2$, respectively, is then

$$\begin{aligned} W &= \int_{\underline{r}_1}^{\underline{r}_2} \underline{F} \cdot \mathrm{d}\underline{r} = -\int_{\underline{r}_1}^{\underline{r}_2} \nabla\Phi \cdot \mathrm{d}\underline{r} \\ &= -\int_{\underline{r}_1}^{\underline{r}_2} \left(\frac{\partial\Phi}{\partial x_1}\mathrm{d}x_1 + \frac{\partial\Phi}{\partial x_2}\mathrm{d}x_2 + \frac{\partial\Phi}{\partial x_3}\mathrm{d}x_3 \right) = -\int_{\underline{r}_1}^{\underline{r}_2} \mathrm{d}\Phi = \Phi(\underline{r}_1) - \Phi(\underline{r}_2). \end{aligned}$$

Thus the work done by a conservative force *along any path* joining point **1** to point **2** depends only on the positions of these points and can be evaluated as the difference between the values of the potential function expressed at these two points,

$$W = \Phi(\underline{r}_1) - \Phi(\underline{r}_2) = -\Delta\Phi. \tag{10.6}$$

Summary

Conservative forces enjoy a number of remarkable properties. Initially, conservative forces are defined as forces that perform the same work along any path joining the same initial and final points, as expressed by eq. (10.1). Simple calculus reasoning is then used to prove that a force is conservative if and only if the work it performs vanishes over any arbitrary closed path, see eq. (10.2). Finally, conservative forces are shown to be derivable from a potential, as expressed by eq. (10.4). Consequently, the work done by a conservative force *along any path* joining two points can be evaluated as the difference between the potential function evaluated at these two points, see eq. (10.6).

Examples of conservative forces

To illustrate these concepts, consider the gravity force acting on a particle of mass m located in a gravity field characterized by an acceleration $-g\bar{\imath}_3$. It can easily be shown that an applied force that remains constant in magnitude and direction, such as a gravitational force, is conservative. Therefore, the scalar potential, Φ, of the gravity forces is $\Phi = mg\,\underline{r}\cdot\bar{\imath}_3 = mgx_3$, where $\underline{r} = x_1\bar{\imath}_1 + x_2\bar{\imath}_2 + x_3\bar{\imath}_3$ is the position vector of the particle. The gravity force, $\underline{F}_g$, acting on the particle can be obtained from this potential using eq. (10.5) to find $\underline{F}_g = -\nabla\Phi = -\partial\Phi/\partial x_3\;\bar{\imath}_3 = -mg\bar{\imath}_3$, and the gravity forces is said to be "derived from a potential." The work done by the gravity force as the particle moves from elevation x_{3a} to x_{3b} then becomes $W = \int_{x_{3a}}^{x_{3b}} \underline{F}_g \cdot \mathrm{d}\underline{r} = -\int_{x_{3a}}^{x_{3b}} \partial\Phi/\partial x_3\;\mathrm{d}x_3 = \Phi(x_{3a}) - \Phi(x_{3b})$. Clearly, this work depends only on the initial and final elevations but not on the particular path followed by the particle as it moved from the initial to the final elevation. If the particle moves along a closed path starting and ending at the same elevation, the work done by the gravity force vanishes.

As another example, consider the restoring force of an elastic spring of stiffness constant k. If the spring is stretched by an amount u, the restoring force is $-ku$, and can be derived from a potential of the form $A(u) = 1/2\;ku^2$. Indeed, using eq. (10.5), the elastic force in the spring becomes $F_s = -\partial A/\partial u = -ku$. This relationship is the constitutive law for the spring because it relates the force in the spring to its elongation. The quantity $A(u)$ is called the *strain energy* and it can be viewed as a "potential of the elastic forces" in the spring. Hence, the strain energy function implicitly defines the constitutive behavior of the component. Finally, the work done by the elastic restoring force as the spring stretches from u_a to u_b is $W = \int_{u_a}^{u_b} F_s\;\mathrm{d}u = -\int_{u_a}^{u_b} \partial A/\partial u\;\mathrm{d}u = A(u_a) - A(u_b)$. Here again, the work depends only on the initial and final positions.

At first glance, the potential, Φ, of a gravity force and the strain energy, A, of an elastic spring seem to be distinct, unrelated concepts. Both quantities, however, share a common property: forces can be derived from these scalar potentials. Consider a particle of mass m connected to an elastic spring of stiffness constant k and subjected to a gravity force acting in the direction of the spring. The downward displacement, u, of the mass measures both the spring stretch and the elevation of the particle. The externally applied gravity force can be derived from the potential, $\Phi = mgu$, as $F_g = -\partial\Phi/\partial u = -mg$; the restoring force in the spring can be derived from the strain energy, $A = 1/2\;ku^2$, which can also be viewed as the potential of the internal forces, as $F_s = -\partial A/\partial u = -ku$. The two forces acting on the particle can therefore be derived from a potential. Note that here again, a distinction is made between externally applied and internal forces, as is done for both the principle of virtual work and its complementary counterpart.

10.1.1 Potential for internal and external forces

In the development of the principle of virtual work, see section 9.4.1, a distinction is made between internal forces and externally applied loads. The same distinction will be made here: if external forces are conservative, they can be derived from a potential, called the "potential of the external loads," and if the internal forces are conservative, they can be derived from a potential, called the "potential of the internal forces."

When dealing with elastic systems, the internal forces are the stresses acting within the body, or the elastic forces acting in structural components such as springs or trusses. The potential of the internal forces is then more appropriately called *strain energy*, *deformation energy*, or *internal energy* and is denoted A. In view of eq. (10.6), it is possible to write

$$W_I = -\Delta A, \tag{10.7}$$

where W_I is the work done by the internal forces or stresses. Similarly, the potential of external forces is denoted Φ and eq. (10.6) then implies

$$W_E = -\Delta \Phi. \tag{10.8}$$

In future developments, it will be convenient to combine the strain energy and the potential of external forces into a single potential called the *total potential energy*, defined as

$$\Pi = A + \Phi. \tag{10.9}$$

The total work done by both internal and external forces then becomes

$$W = W_I + W_E = -\Delta A - \Delta \Phi = -\Delta \Pi. \tag{10.10}$$

In summary, if the internal forces in a body are conservative, a strain energy function exists, and the work done by these internal forces can be computed with the help of eq. (10.7). Similarly, if the loads externally applied to the body are conservative, a potential of the externally applied loads exists, and the work done by the externally applied loads can be computed with the help of eq. (10.8).

If both internal forces and externally applied loads are conservative, the system is called a *conservative system.* The result expressed by eq. (10.10) states: *for conservative systems, the work done by the internal and external forces equals the negative change in total potential energy of the system.* Note that since the work equals the change in the potential function, this potential function is defined only to within a constant, and so adding an arbitrary constant to the potential function will not alter the work done by the corresponding conservative force.

10.1.2 Calculation of the potential functions

To make use of potential functions, it is necessary to first evaluate them. To begin, consider the potential of the internal forces. The strain energy is a function of the deformation state in the body, $A = A(\underline{\epsilon})$, where the array of strain components is defined in eq. (2.11a). Because the strain energy is defined within a constant, it is convenient to select $A(\underline{\epsilon} = 0) = 0$, *i.e.*, the strain energy vanishes for the undeformed or unstrained state of the body. Equation (10.7) then becomes $W_I = -\Delta A = -\left[A(\underline{\epsilon}) - A(\underline{\epsilon} = 0)\right] = -A(\underline{\epsilon})$, and hence,

$$A(\underline{\epsilon}) = -W_I. \tag{10.11}$$

This formula provides a direct way to evaluate the strain energy by computing the work done by the internal forces. In many cases, however, it is cumbersome to compute the work done within a solid as the negative product of the internal stress component acting through strains or deformations. Consequently, an alternative approach is often used to determine the strain energy. In view of eq. (9.19), $W_I = -W_E$, and it follows that

$$A(\underline{\epsilon}) = W_E. \tag{10.12}$$

This result provides a convenient way of determining the strain energy stored in an elastic component by computing the work done by the externally applied loads as they deform the component.

Equation 10.12 can be interpreted as follows: if the internal forces in a solid are conservative, the work done by the externally applied forces is equal to the strain energy stored in the body. As the external forces are applied, the body deforms, the strain magnitudes increase, and so does the strain energy. Consequently, the work done by the externally applied loads is transformed into strain energy.

Of course, it is assumed that the forces are applied slowly, in a quasi-steady manner so that velocities remain very small and the associated kinetic energy is negligible. If the externally applied loads are slowly released, the body will return to its original, unstrained configuration. Because this corresponds to a motion of the internal forces along a closed path, the work done by the conservative internal forces will vanish for the entire cycle, and therefore, the work done during unloading will be the negative of that done during loading. This explains the term "conservative" used to characterize forces that can be derived from a potential.

The evaluation of the potential of the externally applied loads, Φ, is much more straightforward because it is simply the negative of the work done by the external forces acting through the displacements at their points of application. Consider a set of N_P forces, P_i, each of specified *constant magnitude* and each with a *line of action fixed in space*. Similarly, consider N_Q moments, Q_j, each of specified *constant magnitude* and each acting *about a fixed axis in space*. Such loads are sometimes called *dead loads*, because they remain unaffected by the motion of the body they act upon. The potential of these loads is then

$$\Phi = -W_E = -\sum_{i=1}^{N_P} P_i d_i - \sum_{j=1}^{N_Q} Q_j \phi_j, \tag{10.13}$$

where d_i and ϕ_j are the displacements and rotations, respectively, at the points of applications of the external forces and moments, respectively.

It is important to note that not all externally applied loads are conservative forces. For instance, aerodynamic loads vary with the motion of the structure they act upon. For thin airfoils at small angles of attack, the lift acting on the airfoil is proportional to this angle of attack, and therefore the lift depends on the rotation of the airfoil. Aerodynamic forces are non-conservative and cannot be derived from a potential. Another common class of non-conservative forces are follower forces. Such forces might be of constant magnitude, but the orientation of their line of action changes with the rotation of the structure upon which they act. Consider, for instance, the thrust of a rocket jet engine: if the rocket bends, the orientation of the engine thrust will change with the rotation of the structure at the point of attachment of the engine.

10.2 Principle of minimum total potential energy

Let a system be represented by N generalized coordinates, $\underline{q} = \left\{q_1, q_2, \ldots, q_N\right\}^T$, as discussed in section 9.4.1. If the system is conservative, the strain energy of the system can now be viewed as a function of these generalized coordinates, $A = A(\underline{q})$, and similarly, $\Phi = \Phi(\underline{q})$. Using the chain rule for derivatives, infinitesimal increments in strain energy and potential of the externally applied loads can be written as

$$\mathrm{d}A = \frac{\partial A}{\partial q_1}\,\mathrm{d}q_1 + \frac{\partial A}{\partial q_2}\,\mathrm{d}q_2 + \ldots + \frac{\partial A}{\partial q_N}\,\mathrm{d}q_N = \sum_{i=1}^{N} \frac{\partial A}{\partial q_i}\,\mathrm{d}q_i, \tag{10.14a}$$

$$\mathrm{d}\Phi = \frac{\partial \Phi}{\partial q_1}\,\mathrm{d}q_1 + \frac{\partial \Phi}{\partial q_2}\,\mathrm{d}q_2 + \ldots + \frac{\partial \Phi}{\partial q_N}\,\mathrm{d}q_N = \sum_{i=1}^{N} \frac{\partial \Phi}{\partial q_i}\,\mathrm{d}q_i. \tag{10.14b}$$

If the internal forces are conservative, eq. (10.11) relates the work they perform to the strain energy as $W_I = -A(\underline{\epsilon}) = -A(\underline{q})$ because the deformation field inside the body is a function of the generalized coordinates. Similarly, if the external forces are conservative, eq. (10.13) relates the work they perform to the potential of the externally applied loads as $W_E = -\Phi(\underline{q})$.

The virtual work done by the internal forces now becomes $\delta W_I = -\delta A(\underline{q})$, and for the external forces, $\delta W_E = -\delta \Phi(\underline{q})$. As discussed in section 9.3.1, operators "d" and "δ" are closely related, and by analogy with eqs. (10.14), it is possible to write

$$\delta W_I = -\delta A = -\sum_{i=1}^{N} \frac{\partial A}{\partial q_i}\,\delta q_i, \tag{10.15a}$$

$$\delta W_E = -\delta \Phi = -\sum_{i=1}^{N} \frac{\partial \Phi}{\partial q_i}\,\delta q_i. \tag{10.15b}$$

In section 9.4.1, the generalized forces associated with internal forces and externally applied loads, denoted Q_i^I and Q_i^E, respectively, are defined in eqs. (9.24a) and (9.24b), respectively. Identifying eq. (9.24a) with eq. (10.15a) and eq. (9.24b) with eq. (10.15b) then yields

$$Q_i^I = -\frac{\partial A}{\partial q_i}, \tag{10.16a}$$

$$Q_i^E = -\frac{\partial \Phi}{\partial q_i}. \tag{10.16b}$$

Since the internal forces in the body are assumed to be conservative, it follows that the internal generalized forces, Q_i^I, are themselves conservative because they can be derived from a potential, the strain energy of the structure. Similarly, the externally applied loads are assumed to be conservative, and their generalized counterparts are conservative as well because the can be derived from the potential of the externally applied loads.

The principle of virtual work, as expressed by eq. (9.25), implies $Q_i^I + Q_i^E = 0$, for all generalized coordinates. Introducing eqs. (10.16) then yields $-\partial A/\partial q_i - \partial \Phi/\partial q_i = \partial(A + \Phi)/\partial q_i = 0$. Finally, using the definition of the total potential energy, eq. (10.9), results in

$$\frac{\partial \Pi}{\partial q_i} = 0, \quad i = 1, 2, \ldots, N. \tag{10.17}$$

The same result can be obtained in a more expeditious manner by observing that when both internal forces and externally applied loads are conservative, the work done by these forces equals the negative change in total potential energy of the system, as expressed by eq. (10.10). Since the total potential energy, Π, is defined within a constant, it follows that the virtual work can be expressed as $\delta W = -\delta \Pi$. The principle of virtual work, principle 4, states that a system is in static equilibrium if and only if the sum of the virtual work done by the internal and external forces vanishes for all arbitrary virtual displacements, *i.e.*, $\delta W = -\delta \Pi = 0$, or

$$\delta \Pi = 0. \tag{10.18}$$

The total potential energy is a function of the generalized coordinates, $\Pi = \Pi(\underline{q})$, and hence, virtual changes in this quantity must vanish

$$\delta \Pi = \sum_{i=1}^{N} \left[\frac{\partial \Pi}{\partial q_i} \right] \delta q_i = 0. \tag{10.19}$$

Because the virtual changes in the generalized coordinates are arbitrary, the bracketed term must vanish, leading once again to eqs. (10.17). These observations lead to the following principle.

Principle 8 (Principle of stationary total potential energy) *A conservative system is in equilibrium if and only if virtual changes in the total potential energy vanish for all virtual displacements.*

Equation (10.17) expresses this principle in a somewhat more mathematical manner: *a conservative system is in equilibrium if and only if all partial derivatives of the total potential energy with respect to the generalized coordinates vanish.*

Because the principle of stationary total potential energy is derived directly from the principle of virtual work, it inherits many of its features. Sections 9.3.1 and 9.3.2 describe the use of the principle of virtual work with arbitrary virtual displacements and with kinematically admissible virtual displacements, respectively. When kinematically admissible virtual displacements are used, the virtual work done by the reaction forces vanishes, and these forces are eliminated from the formulation. On the other hand, when arbitrary virtual displacements are used, the virtual work done by the reaction forces does not vanish, and must be included in the virtual work done by the externally applied loads; the reaction forces must therefore be treated as externally applied loads.

The same distinction must be made when using the principle stationary total potential energy. If kinematically admissible virtual displacements are used, reaction forces are eliminated from the formulation, whereas if arbitrary virtual displacements are used, reaction forces must be treated as externally applied loads. In this latter case, reaction forces must be included in the potential of the externally applied loads when evaluating the total potential energy of the system.

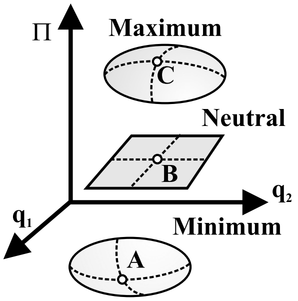

Fig. 10.3. Total potential energy.

For low dimensionality systems, it possible to give a graphical illustration of principle 8. Figure 10.3 shows the total potential energy as a function of two generalized coordinates, q_1 and q_2. Since it is always possible to select a virtual change in generalized coordinate as an actual, infinitesimal change in the same coordinate, eq. (10.18) implies $\mathrm{d}\Pi = 0$. This means that at an equilibrium point, the total potential energy is stationary as shown by points **A**, **B**, and **C**.

As illustrated in fig. 10.3, however, this stationary point could correspond to a minimum (point **A**), a maximum (point **C**), or even a saddle point. To make a distinction between these various cases, changes in the total potential energy in the neighborhood of the stationary point must be studied. Increments in this energy are expanded using a Taylor series

$$\mathrm{d}\Pi \approx \sum_{i=1}^{N} \frac{\partial \Pi}{\partial q_i} \mathrm{d}q_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \Pi}{\partial q_i \partial q_j} \mathrm{d}q_i \mathrm{d}q_j,$$

where the higher order terms are neglected. In the neighborhood of static equilibrium, the first term on the right-hand side vanishes in view of eq. (10.17), leaving

$$\mathrm{d}\Pi \approx \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \Pi}{\partial q_i \partial q_j} \mathrm{d}q_i \mathrm{d}q_j, \tag{10.20}$$

Based on this result, three different cases are possible.

1. If $\partial^2 \Pi/(\partial q_i \partial q_j) \mathrm{d}q_i \mathrm{d}q_j > 0$ for all $\mathrm{d}q_i$, the total potential energy is a minimum at equilibrium. The equilibrium is said to be *stable*, as illustrated by point **A** in fig. 10.3.
2. If $\partial^2 \Pi/(\partial q_i \partial q_j) \mathrm{d}q_i \mathrm{d}q_j = 0$ for all $\mathrm{d}q_i$, the total potential energy remains constant around the equilibrium point. The equilibrium is said to be *neutrally stable*, as illustrated by point **B** in fig. 10.3.
3. If $\partial^2 \Pi/(\partial q_i \partial q_j) \mathrm{d}q_i \mathrm{d}q_j < 0$ for any $\mathrm{d}q_i$, the total potential energy is a maximum at equilibrium. The equilibrium is said to be *unstable*, as illustrated by point **C** in fig. 10.3.

A minimum value of the total potential energy corresponds to a stable equilibrium configuration of the system because any perturbation from such an equilibrium configuration must increase the total potential energy. Since the work done by the externally applied loads is included in the total potential, the total potential cannot increase without an external source of energy, and the equilibrium configuration is stable. On the other hand, at a maximum point, any disturbance will decrease the total potential. Again, since the work done by the externally applied loads is included in the total potential, the released potential energy is converted into kinetic energy, leading to spontaneous motion of the system. This represents an unstable situation. The neutrally stable situation is the intermediate case for which a disturbance causes no change in the total potential.

Combining the principle of stationary total potential energy with the above discussion leads to the principle of minimum total potential energy.

Principle 9 (Principle of minimum total potential energy) *A conservative system is in a stable state of equilibrium if and only if the total potential energy is a minimum with respect to changes in the generalized coordinates.*

Practical applications of the principle of minimum total potential energy require the development of expressions for the strain energy stored in the structure and for the potential of the externally applied forces, the two quantities that make up the total potential energy. These will be described in sections 10.3 to 10.5. It is first useful, however, to consider the possibility that some of the external forces might be non-conservative, as discussed in the following section.

10.2.1 Non-conservative external forces

The principle of minimum total potential energy is based on two assumptions: first, the internal forces are conservative, and second, the externally applied loads are conservative. For important classes of problems, the first assumption is satisfied, but not the second. In this case, the principle of virtual work, principle 6 on page 434, implies

$$\delta W = \delta W_I + \delta W_E = -\delta A + \delta W_E^{nc} = 0,$$

where δW_E^{nc} represents the virtual work done by the *non-conservative forces*. This leads to the following principle.

Principle 10 *A system is in equilibrium if and only if virtual changes in the strain energy equal the virtual work done by the externally applied loads for all arbitrary virtual displacements.*

In other cases, externally applied forces are a mixture of conservative and non-conservative forces. It is then convenient to split the virtual work done by the externally applied forces, δW_E, into two parts: δW_E^c due to the conservative forces, and δW_E^{nc} due to the non-conservative forces, to find $\delta W_E = \delta W_E^c + \delta W_E^{nc}$. The principle of virtual work, principle 6, then implies $\delta W_I + \delta W_E = \delta W_I + \delta W_E^c + \delta W_E^{nc} = 0$. Introducing the strain energy and the potential of the conservative forces then yields

$$\delta(A + \Phi) = \delta W_E^{nc}.$$

It is important to note that the term δW_E^{nc} represents the virtual work done by the non-conservative forces, whereas $\delta(\Phi)$ represents the negative virtual work done by the conservative forces.

10.3 Strain energy in springs

The strain energy is a function of the deformation of the structure, $A = A(\underline{\epsilon})$; in turn, the deformation field is a function of the displacement field, or of the generalized coordinates, depending on the formulation of the problem. This section considers one of the simplest elastic structure: a spring. Two different types of springs will be considered. First, the *rectilinear spring* can be deformed in a rectilinear manner by a force that acts along the axis of the spring. Second, the *torsional* or *rotational spring* can be deformed in a rotation about its axis by a moment acting about this axis.

10.3.1 Rectilinear springs

Rectilinear springs are simple, elastic elements with two primary lumped properties: the stiffness constant and un-stretched length. For rectilinear springs, the applied force and the resulting deformation are along a common straight line, as depicted in fig. 10.4. The displacement of the spring is denoted u and its natural length, sometimes called the "un-stretched length," is denoted u_0. The force applied to the spring is denoted F and the force in the spring F_s. The constitutive behavior of the spring is typically given as $F = F(\Delta)$, where $\Delta = u - u_0$ is the extension of the spring, and $F(\Delta = 0) = F(u = u_0) = 0$.

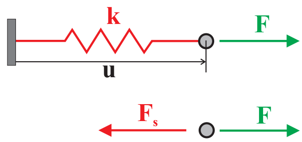

Fig. 10.4. Rectilinear spring subjected to a force F.

Linearly elastic springs

If the relationship between an applied load and the resulting extension is linear, *i.e.*, if $F = k\Delta$, where k is the spring's stiffness constant, the spring is said to be linear. It is unfortunate that the term "linear" is often used to describe both the rectilinear motion of the spring as well as the linearity of the force-extension relationship.

If the spring exhibits a linear constitutive behavior, $F = k\Delta$, it implies that the spring resists both tensile and compressive external forces, and the spring stiffness constant, k, is the same constant value for all forces or extension magnitudes. The stiffness has units of force per length, or N/m in the SI system.

The strain energy in the spring is evaluated with the help of eq. (10.12) to find

$$A = W_E = \int_{u_0}^{u} F \, \mathrm{d}u = \int_{u_0}^{u} k\Delta \, \mathrm{d}u = \int_{0}^{\Delta} k\Delta \, \mathrm{d}\Delta = \frac{1}{2}k\Delta^2 = \frac{1}{2}F\Delta. \quad (10.21)$$

The strain energy is a positive-definite function of the stretch, *i.e.*, $A > 0$ for any positive or negative value of the extension, Δ, and vanishes only when $\Delta = 0$. The internal force in the spring can be derived from the strain energy using eq. (10.16a) as $F_s = -\partial A/\partial u = -k\Delta$. The minus sign stems from the fact that the force in the spring opposes the externally applied force as shown in the free-body diagram in fig. 10.4.

The constitutive law for the spring is depicted as the straight line in the force versus extension plot shown in fig. 10.5. In view of eq. (10.21), the strain energy, A, is the shaded area under the curve.

The complementary strain energy, A', often called the stress energy, is defined as the shaded are to the left of the straight line and is computed as

$$A' = \int_0^F (u - u_0)\,\mathrm{d}F = \int_0^F \Delta\,\mathrm{d}F = \int_0^F \frac{F}{k}\,\mathrm{d}F = \frac{1}{2}\frac{F^2}{k} = \frac{1}{2}F\Delta. \qquad (10.22)$$

The complementary strain energy is naturally expressed in terms of forces, and hence, its name, "stress energy," or less often used "force energy."

Using the spring's constitutive law, it follows that $A' = 1/2\ F^2/k = 1/2\ F\Delta = 1/2\ k\Delta^2 = A$. Thus, the strain energy and its complementary counterpart are equal for linearly elastic springs. In fig. 10.5, the rectangle of area $F\Delta$ is separated by its diagonal into two triangles of equal areas $A = A' = F\Delta/2$. It follows that A and A' are related through

$$A + A' = F\Delta. \qquad (10.23)$$

This expression helps explain the term "complementary energy" used to denote this energy.

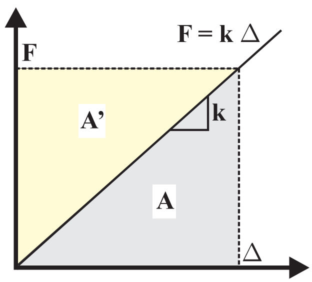

Fig. 10.5. Constitutive law for a linearly elastic spring.

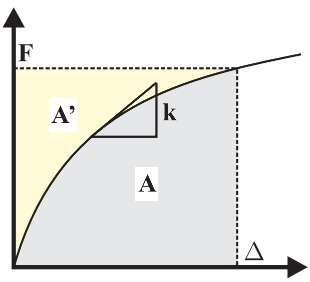

Fig. 10.6. Constitutive law for a nonlinearly elastic spring.

Nonlinearly elastic springs

The discussion has thus far focused on springs with a linear constitutive behavior. The concept of conservative forces, however, is not limited to elastic components presenting a linear behavior. Some metals, such as aluminum and copper, exhibit a slight amount of nonlinearly elastic behavior prior to their yield points. Many elastomers present quite obvious nonlinearly elastic behavior.

A number of analytical models have been developed to approximate the observed constitutive behavior, but perhaps the simplest is a law of the form

$$F = F_0 \tanh\left(\frac{\Delta}{u_0}\right), \tag{10.24}$$

where F_0 is a reference force and u_0 a reference displacement.

This type of law, which is shown in fig. 10.6, is representative of materials such as aluminum which do not exhibit a sharp transition from linear to nonlinear behavior. The stiffness of this spring is given by

$$k = \frac{\partial F}{\partial \Delta} = \frac{F_0}{u_0} \operatorname{sech}^2\left(\frac{\Delta}{u_0}\right) = k_0 \operatorname{sech}^2\left(\frac{\Delta}{u_0}\right),$$

where $k_0 = F_0/u_0$ is the stiffness of the spring at zero elongation.

The constitutive law, eq. (10.24), is now recast in a non-dimensional form by defining the non-dimensional force and extension as $\bar{F} = F/F_0$ and $\bar{\Delta} = \Delta/u_0$, respectively. The constitutive law then becomes $\bar{F} = \tanh(\bar{\Delta})$ and its inverse is $\bar{\Delta} = \operatorname{arctanh}(\bar{F})$.

The strain energy in the spring can be found by direct integration of the force over a differential displacement as

$$A = \int_0^{\Delta} F \, \mathrm{d}\Delta = F_0 u_0 \int_0^{\Delta} \tanh \bar{\Delta} \, \mathrm{d}\bar{\Delta} = F_0 u_0 \ln(\cosh \bar{\Delta}),$$

and the complementary strain energy, A', is given in a similar manner by

$$A' = \int_0^{F} \Delta \, \mathrm{d}F = F_0 u_0 \int_0^{\bar{F}} \operatorname{arctanh}\bar{F} \, \mathrm{d}\bar{F} = u_0 F_0 \left(\bar{F} \operatorname{arctanh}\bar{F} + \ln \sqrt{1 - \bar{F}^2}\right).$$

The strain energy is the shaded area under the force versus extension curve, see fig. 10.6, whereas the shaded area to the left of the same curve is the complementary strain energy. In contrast to the linearly elastic spring shown fig. 10.5, the two energies are not equal, $A \neq A'$, for a nonlinearly elastic spring. It is still true, however, that $A + A' = F\Delta$, as can be seen graphically as well as shown by using the non-dimensional forms for the strain and complementary strain energy as

$$\begin{aligned}
\frac{A}{u_0 F_0} + \frac{A'}{u_0 F_0} &= \ln\left(\cosh \bar{\Delta}\right) + \bar{F} \operatorname{arctanh} \bar{F} + \ln \sqrt{1 - \bar{F}^2} \\
&= \ln \frac{1}{\sqrt{1 - \tanh^2 \bar{\Delta}}} + \bar{F} \operatorname{arctanh} \bar{F} + \ln \sqrt{1 - \bar{F}^2} \\
&= -\ln \sqrt{1 - \bar{F}^2} + \bar{F} \operatorname{arctanh} \bar{F} + \ln \sqrt{1 - \bar{F}^2} = \bar{F}\bar{\Delta},
\end{aligned}$$

where the hyperbolic function identity, $\cosh^2 a = 1/(1 - \tanh^2 a)$, is used along with the non-dimensional constitutive law itself. This result shows that A and A' are truly complementary in the same way that they are for a linearly elastic spring.

The strain energy function incorporates the constitutive law for the material. Indeed, the elastic force in the spring can be derived from the strain energy,

$$F = \frac{\partial A}{\partial \Delta} = \frac{1}{u_0} \frac{\partial}{\partial \bar{\Delta}} \left[F_0 u_0 \ln(\cosh \bar{\Delta}) \right] = F_0 \tanh \left(\frac{\Delta}{u_0} \right). \tag{10.25}$$

Figure 10.7 illustrates the difference between the nonlinearly and linearly elastic springs. The upper figure shows the strain energy or potential for both springs, the middle figure the force-extension relationship and the bottom figure the spring stiffness defined as the local tangent to the constitutive law curve; all three figures are plotted against the normalized spring extension. The spring studied here is a "softening spring," because it presents a decreasing stiffness at higher extensions and its strain energy is less than that of a linearly elastic spring at all extension magnitudes.

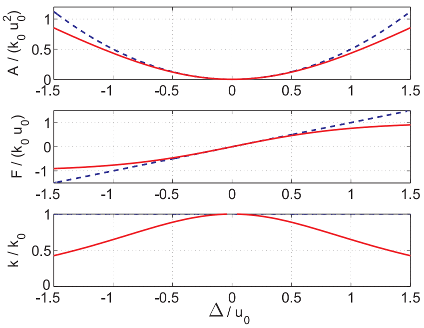

Fig. 10.7. Nonlinear spring with the constitutive law given by eq. (10.24). Top figure: strain energy; middle figure: force; bottom figure: stiffness. Solid line: nonlinear spring; dashed line: linear spring.

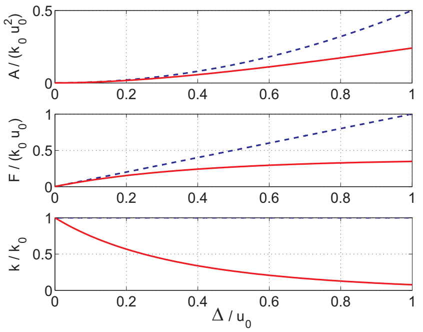

Fig. 10.8. Nonlinear "bungee" spring with the potential given by eq. (10.27). Top figure: strain energy; middle figure: force; bottom figure: stiffness. Solid line: nonlinear spring; dashed line: linear spring.

Example 10.1. Nonlinearly elastic "bungee cords"

The mathematical form of the force-extension relationship must reflect as accurately as possible the experimentally observed behavior of the spring. Typically, the force-extension curve is first obtained experimentally, then a curve fitting procedure is used to approximate the data using a carefully chosen analytical representation of the constitutive law.

An interesting example is a "bungee cord," which can undergo very large deformations without failing. The following equation uses the logarithmic function to approximate the experimentally measured behavior of bungee cords

$$F = \begin{cases} k_0 u_0 \dfrac{\ln(1+\bar{\Delta})}{1+\bar{\Delta}}, & \text{for } 0 \le \bar{\Delta} < 1, \\ 0, & \text{for } \quad \bar{\Delta} < 0, \end{cases} \tag{10.26}$$

where $\bar{\Delta} = (u - u_0)/u_0$ is the non-dimensional bungee stretch, u_0 its natural length, and k_0 is the initial elastic stiffness (*i.e.*, at $\Delta = 0$).

The relationship in eq. (10.26) closely approximates the experimental data for $\bar{\Delta} < 1$, *i.e.*, when the bungee cord extends to less than twice its natural length. When the bungee cord is in its un-stretched state, a force is required to increase its length, and this force is a nonlinear function of the extension. Another nonlinearity in the constitutive law is the unsymmetrical behavior of the bungee in tension and compression: the cord cannot support a compressive force, and therefore, the above constitutive law is not valid for negative extensions, $\bar{\Delta} < 0$.

The potential of the bungee cord for $\bar{\Delta} > 0$ is

$$A = \int_0^u F \mathrm{d}u = \int_0^u k_0 u_0 \frac{\ln(1+\bar{\Delta})}{1+\bar{\Delta}} \, \mathrm{d}u = \frac{k_0 u_0^2}{2} \ln^2(1+\bar{\Delta}). \tag{10.27}$$

The characteristics of the bungee cord are illustrated in fig. 10.8. The upper figure shows the spring's strain energy given by eq. (10.27), the middle figure the force-stretch relationship given by eq. (10.26) and the bottom figure the spring's apparent stiffness. For reference, the corresponding quantities for a linear spring with equal stiffness constant, k_0, are also depicted. The apparent stiffness, k, of the bungee cord is the tangent to the force-extension curve,

$$k = \frac{\mathrm{d}F}{\mathrm{d}\Delta} = k_0 \frac{1 - \ln(1+\bar{\Delta})}{(1+\bar{\Delta})^2}. \tag{10.28}$$

As the stretch of the cord increases, its stiffness decreases and vanishes when $\ln(1 + \bar{\Delta}) = 1$, or $\bar{\Delta} \approx 1.718$. Clearly, this is not realistic, and therefore, the approximation to the force-stretch behavior given by eq. (10.26) is only valid for $\bar{\Delta} < 1$.

For the constitutive law expressed by eq. (10.26), the complementary strain energy cannot be easily computed. Indeed, it would be necessary to express the spring stretch in terms of the applied force, $\bar{\Delta} = \bar{\Delta}(F)$, but in view of the logarithmic function appearing in eq. (10.26), it is not easy to obtain this expression.

10.3.2 Torsional springs

Torsional springs are also simple elastic elements with lumped elastic properties. Instead of the rectilinear motion that characterizes the springs described in the previous section, torsional springs undergo an angular motion, θ, under the action of an externally applied torque, M, as depicted in fig. 10.9.

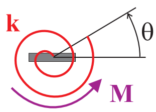

Fig. 10.9. Torsional spring subjected to a moment M.

For a linearly elastic torsional spring, the constitutive law is $M = k\theta$, where k is the stiffness constant of the spring. Note that although the same symbol, k, is often used to denote the stiffness constants of both rectilinear and torsional springs, their units are not the same: for a torsional spring the stiffness constant has units of moment per rotation, N·m/rad, or sometimes N·m/deg. The un-stretched rotation, θ_0, of the spring is not necessarily zero, and in such cases, the constitutive relationship should be written as $M = k(\theta - \theta_0)$.

Of course, the elastic behavior of the torsional spring could be nonlinear, as discussed in the previous section for rectilinear springs. In either case, if the force in the spring is conservative, it is possible to obtain an expression for its potential.

10.3.3 Bars

As discussed in section 9.5, each bar of a truss is assumed to behave like a rectilinear spring. It then follows from eq. (10.21) that the strain energy in a bar can be written as

$$A = \frac{1}{2} k e^2 = \frac{1}{2} \frac{E\mathcal{A}}{L} e^2, \tag{10.29}$$

where e is the bar elongation, and $k = E\mathcal{A}/L$ its stiffness.

Example 10.2. Spring-mass system
Consider the rectilinear spring with a weight, mg, attached as depicted in fig. 10.10. The spring is assumed to behave linearly, its natural length is denoted u_0, its final length is u, and its extension is $\Delta = u - u_0$.

The strain energy in the linear spring is given by eq. (10.21) as $A = 1/2\ k\Delta^2$, and the potential of the gravity force is $\Phi = -mg\Delta$. Note that since the potential is defined within a constant, it is also correct to write $\Phi = -mg(u_0 + \Delta) = -mgu$.

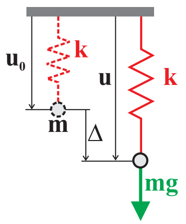

Fig. 10.10. Rectilinear spring supporting a weight.

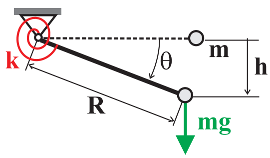

Fig. 10.11. Torsional spring supporting a weight.

The total potential energy of the system is $\Pi = A + \Phi = 1/2\ k\Delta^2 - mg\Delta$. Because the problem presents a single generalized coordinate, Δ, the principle of minimum total potential energy, eq. (10.17), implies

$$\frac{\partial \Pi}{\partial \Delta} = k\Delta - mg = 0.$$

The solution of this linear equation gives the extension of the spring, $\Delta = mg/k$.

In the neighborhood of this equilibrium configuration, variation in the total potential energy is given by eq. (10.20) as $\mathrm{d}\Pi \approx (\partial^2\Pi/\partial\Delta^2)\ \mathrm{d}\Delta^2 = k\ \mathrm{d}\Delta^2$. Because the spring stiffness constant is a positive number, it follows that $\mathrm{d}\Pi > 0$ and the equilibrium configuration is stable.

Example 10.3. Rotational spring-mass system
Consider next a rigid bar of length R carrying a weight, mg, at its tip and pinned to the ground at the other end, as shown in fig. 10.11. At the pivot point, a linear torsional spring of stiffness constant k restrains the rotation of the bar. The spring is unstretched when $\theta = 0$, which corresponds to the horizontal position of the bar.

The strain energy of the spring is simply $A = 1/2\ k\theta^2$. The potential of the externally applied load is computed with the help of eq. (10.13) as $\Phi = -W_E = -mgh$, where h is the motion of the point of application of the tip weight projected along the direction of the load. In this case, $h = R\sin\theta$, leading to $\Phi = -mgR\sin\theta$. The total potential energy of the system is now $\Pi = A + \Phi = 1/2\, k\theta^2 - mgR\sin\theta$.

Because the problem presents a single generalized coordinate, θ, the principle of minimum total potential energy, eq. (10.17), implies

$$\frac{\partial \Pi}{\partial \theta} = k\theta - mgR\cos\theta = 0.$$

The equilibrium configuration is the solution of this transcendental equation for θ, which can be recast as $\bar{k}\theta = \cos\theta$, where $\bar{k} = k/(mgR)$ is the non-dimensional stiffness constant of the spring. The transcendental equation does not admit a closed form solution, and it must be solved graphically or iteratively using a procedure such as Newton's method. For $\bar{k} \rightarrow 0$, that is, for a spring of very low stiffness or for a very large tip mass, the equilibrium angle $\theta \approx \pi/[2(1+\bar{k})]$. For $\bar{k} \rightarrow \infty$, that is, for a very stiff spring or very small tip mass, the equilibrium angle $\theta \approx 1/\bar{k}$.

Although the spring is assumed to be linear, the equilibrium equations of the problem are nonlinear because the motion of the bar is finite, *i.e.*, no limit is set on the magnitude of angle θ. When $\bar{k} \rightarrow \infty$, system deflections remain small, the equilibrium equation of the problem becomes linear, $\bar{k}\theta \approx 1$, and the solution is easily found as $\theta = 1/\bar{k} = mgR/k$. The nonlinearities introduced by large deflections are called *geometric nonlinearities*.

Example 10.4. Buckling of a rigid bar under compressive load
The study of the behavior of the total potential energy in the vicinity of an equilibrium state provides important information about the stability of the system. To illustrate this concept, consider the rigid bar of length L, connected to the ground through a pivot point and subjected to a tip compressive load, P, as shown in fig. 10.12. A rectilinear spring is attached at the tip of the bar and is assumed to remain horizontal at all times; the spring is unstretched when the bar is in the vertical position, *i.e.*, when $\theta = 0$.

The strain energy for the spring is $A = 1/2\ k\Delta^2$, where the extension of the spring is $\Delta = L\sin\theta$. The potential of the externally applied load is computed with the help of eq. (10.13) as $\Phi = -W_E = -Ph$, where $h = L(1-\cos\theta)$ is the motion of the point of application of the tip force projected along its line of action. The total potential energy of the system is $\Pi = A + \Phi = 1/2\, kL^2\sin^2\theta - PL(1-\cos\theta)$.

Since the problem presents a single generalized coordinate, θ, the principle of minimum total potential energy, eq. (10.17), implies

$$\frac{\partial \Pi}{\partial \theta} = kL^2\sin\theta\cos\theta - PL\sin\theta = L\sin\theta\,(kL\cos\theta - P) = 0.$$

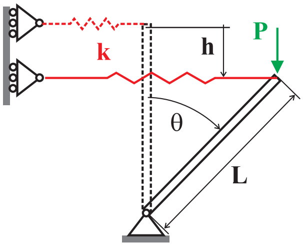

Fig. 10.12. Rigid bar with tip spring under compressive load. Original and deformed configurations.

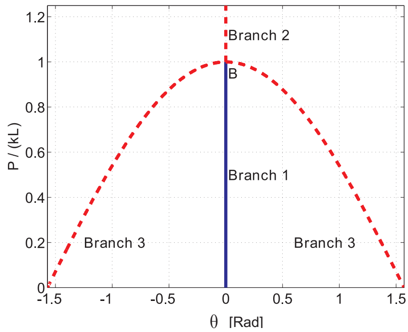

Fig. 10.13. Response of the system: stable branch: solid line; unstable branches: dotted lines.

This equation possesses two distinct solutions: $\sin\theta = 0$ and $P = kL\cos\theta$. The first solution, $\sin\theta = 0$, yields $\theta = 0$ or $\theta = \pm\pi$, for any value of load, P. This corresponds to the configuration in which the bar remains vertical, straight up or down with respect to the pivot, for any applied load. The second solution, $P = kL\cos\theta$, can be solved for the deflection of the bar as a function of the applied load: $\theta = \arccos(P/kL)$. These two solutions are depicted in fig. 10.13.

To study the stability of these solutions, the variation of the total potential energy in the vicinity of an equilibrium configuration is evaluated with the help of eq. (10.20) as

$$\mathrm{d}\Pi \approx \frac{\partial^2 \Pi}{\partial \theta^2}\mathrm{d}\theta^2 = \left[kL^2\left(\cos^2\theta - \sin^2\theta\right) - PL\cos\theta\right]\mathrm{d}\theta^2.$$

Consider the first solution, $\theta = 0$. Along this branch, labeled "branch 1" in fig. 10.13, the variation of the total potential energy is $\mathrm{d}\Pi \approx L(kL - P)\mathrm{d}\theta^2$. For $P < kL$, it follows that $\mathrm{d}\Pi > 0$, leading to a stable solution. A solid line is used in fig. 10.13 to indicate that this branch is stable. For $P > kL$, however, $\mathrm{d}\Pi < 0$, and the equilibrium solution becomes unstable; this unstable branch, labeled "branch 2," is shown as a dotted line.

Consider now the second solution, $P = kL\cos\theta$. In the vicinity of this equilibrium configuration, it is clear that $\mathrm{d}\Pi \approx -kL^2\sin^2\theta\,\mathrm{d}\theta^2 < 0$, and the solution, labeled "branch 3," is unstable. Finally, it is easily verified that the last equilibrium solution, $\theta = \pm\pi$, is stable for all $P > 0$.

It now becomes possible to describe the behavior of the system under an increasing load, P. For $P < kL$, the bar is in stable equilibrium in the vertical configuration. As load P increases, point **B** in fig. 10.13 is reached. At this point three distinct equilibrium solutions now become possible: $\theta = 0$, and $\theta = \arccos(P/kL)$, for either positive or negatives values of θ. These equilibrium solutions are labeled "branch 2" and "branch 3," and are all unstable, as indicated in fig. 10.13. Point **B** is called a *bifurcation point* because three equilibrium solutions emanate from it.

From a purely mathematical perspective, although all are unstable, each of the three solutions is a correct equilibrium solution of the problem. In practice, the branch to be followed by the bar depends on the imperfection present in the system. Indeed, a rigid bar is never "perfectly straight" nor "perfectly homogeneous," and load P can never be "exactly aligned" with the axis of the bar. If the bar is slightly canted to the right or load P leaning in that direction, the system will collapse to the right; conversely, a left leaning imperfection or loading will cause the bar to collapse to the left. Because imperfections are always present, "branch 2" is never observed in practice.

From this discussion, it is clear that the system can only sustain loads $P < kL$ with $\theta = 0$, at which point the system collapses. Point **B** is called the *buckling point*, and $P_{\rm cr} = kL$ is the *critical load* or *buckling load*. More details about the buckling phenomenon can be found in chapter 14.

As a final point, it should be noted that the branch $\theta = 0$ is stable for any *negative* value of P; this correspond to the case when the load is pulling the vertical bar upward, an obviously stable configuration.

Example 10.5. Rigid aircraft suspended by "bungee cords"

For the purpose of dynamic testing, an aircraft is suspended from a hangar's roof by means of three bungee cords attached to the aircraft's left wing at point **L**, right wing at point **R**, and tail at point **T**, as depicted in fig. 9.16 on page 424. The aircraft's total mass is M and the center of mass is located at point **C**. For simplicity, the aircraft is assumed to be rigid and the displacements under the load are assumed to remain small. The generalized coordinates of the problem are selected as Δ_L, Δ_R, and Δ_T, the downward vertical distance of points **L**, **R**, and **T**, respectively, from the bungee cord attachment points. The strain energy for each of the bungee cords is given by eq. (10.27), and hence, the total strain energy of the system is

$$A = \frac{k_L u_L^2}{2} \ln^2(1 + \bar{\Delta}_L) + \frac{k_R u_R^2}{2} \ln^2(1 + \bar{\Delta}_R) + \frac{k_T u_T^2}{2} \ln^2(1 + \bar{\Delta}_T), \quad (10.30)$$

where $\bar{\Delta}_L = (\Delta_L - u_L)/u_L$, $\bar{\Delta}_R = (\Delta_R - u_R)/u_R$, and $\bar{\Delta}_T = (\Delta_T - u_T)/u_T$, are the non-dimensional extensions of the three bungee cords, u_L, u_R, and u_T are their natural lengths, and k_L, k_R, and k_T, are their apparent stiffness for small extensions.

Equation (10.30) illustrates one of the fundamental advantages of energy methods: because strain energy is an additive scalar quantity, *the strain energy of a system is simply the sum of the strain energies stored in each of its elastic components*. In this example, the total strain energy of the system is the sum of the strain energies in each of the three bungee cords. Because the aircraft is assumed to be rigid, it stores no strain energy.

The potential of the gravity load acting on the aircraft is $\Phi = -Mgh$, where h is the distance of the mass center below a reference plane and is given by, $h = (a/d)\Delta_T + (1 - a/d)(\Delta_L + \Delta_R)/2$. This expression is a direct consequence of the assumption of a rigid aircraft. The potential of the externally applied loads now becomes

$$\Phi = -Mg \left[\Delta_T \frac{a}{d} + \frac{\Delta_L + \Delta_R}{2} \left(1 - \frac{a}{d}\right) \right]. \quad (10.31)$$

The total potential energy of the system is $\Pi = A + \Phi$, and the equilibrium equations of the system, eqs. (10.17), become

$$k_L u_L \frac{\ln(1+\bar{\Delta}_L)}{1+\bar{\Delta}_L} = \frac{Mg}{2}\left(1 - \frac{a}{d}\right), \tag{10.32a}$$

$$k_R u_R \frac{\ln(1+\bar{\Delta}_R)}{1+\bar{\Delta}_R} = \frac{Mg}{2}\left(1 - \frac{a}{d}\right), \tag{10.32b}$$

$$k_T u_T \frac{\ln(1+\bar{\Delta}_T)}{1+\bar{\Delta}_T} = Mg\frac{a}{d}. \tag{10.32c}$$

These are three nonlinear equations to be solved for the non-dimensional extensions of the three bungee cords.

It is interesting to compare the present solution with the solution presented in example 9.12 on page 423. Nonlinear springs are used in the present example, but linear springs are used in example 9.12. The present example illustrates the concept of *material nonlinearities*, *i.e.*, nonlinear relationships for the material constitutive laws, as opposed to the *geometric nonlinearities* encountered in examples 10.3 and 10.4.

In example 9.12, the generalized coordinates are selected as u, ϕ_1, and ϕ_2, the vertical translation and two rotations of the rigid aircraft, whereas in the present case, the non-dimensional extensions of the three bungee cords are used. The use of the former generalized coordinates results in a system of coupled equations as shown in eq. (9.26), whereas the use of the latter yields the uncoupled equations (10.32). Both sets of generalized coordinates are equally valid because both uniquely define the configuration of the aircraft. The final form of the governing equations of the problem, however, does depend on the choice of a specific set of generalized coordinates. An interesting exercise would be to repeat the solution of the present problem by selecting u, ϕ_1, and ϕ_2 as the generalized coordinates.

Finally, it must be repeated that the solution presented here assumes the movement of the aircraft to remain small. If that assumption is violated, the kinematics of the problem will be more complicated. For instance, if the generalized coordinates are selected to be u, ϕ_1, and ϕ_2, the development of large displacements, and hence, of finite rotations, will introduce trigonometric functions of angles ϕ_1 and ϕ_2, which leads to geometric nonlinearities.

10.3.4 Problems

Problem 10.1. Rotating disk with spring restraint
Work problem 9.1 using the principle of minimum total potential energy.

Problem 10.2. Lever with sliding pivots
Bar **ABC** is of length $b + a$ and is constrained to move vertically at point **A** and horizontally at **B**, while a horizontal force, P, is applied at point **C**, as depicted in fig. 10.14. Point **A** is restrained by a vertical spring of stiffness constant k, which is relaxed when angle $\theta = 0$. Use the principle of minimum total potential energy to determine the equilibrium configurations of the system.

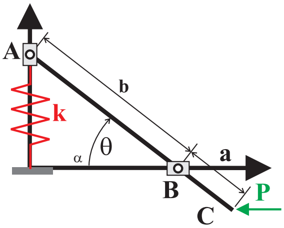

Fig. 10.14. Lever with spring-restrained sliding pivots.

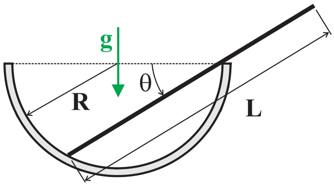

Fig. 10.15. Rod in frictionless hemispherical bowl ($L > 2R$).

Problem 10.3. Rod in frictionless hemispherical bowl
A uniform rod of mass, m, and length, L, rests from inside to across the rim of a frictionless hemispherical bowl of radius, R, as shown in the cross-sectional view in fig. 10.15. Assume that $L > 2R$. Using the principle of minimum total potential energy, find the equilibrium angle of inclination, θ, of the rod to the horizontal plane.

Problem 10.4. Geometrically nonlinear spring-mass
Problem 9.6 describes a spring-mass system with a nonlinear geometry which arises from the large displacements that are developed, as shown in fig. 9.22. Use the principle of minimum total potential energy to compute the equilibrium configuration of the system.

Problem 10.5. Rigid aircraft suspended by bungee cords
Work example 10.5 with the generalized coordinates u, ϕ_1, and ϕ_2 defined in example 9.12.

10.4 Strain energy in beams

10.4.1 Beam under axial loads

Consider a beam subjected only to axial loads as discussed in section 5.4 and depicted in fig. 5.6 on page 179. Material constitutive laws are assumed to be linear and elastic. Focus now on an infinitesimal slice of the beam of span-wise length $\mathrm{d}x_1$, acted upon by an axial force N_1. The left face of this differential element undergoes an axial displacement $\bar{u}_1$, whereas the displacement of its right face is $\bar{u}_1 + (\mathrm{d}\bar{u}_1/\mathrm{d}x_1)\mathrm{d}x_1$. As the axial force acting on the left face increases from zero to its final value, N_1, the work it performs is $-1/2\ N_1\bar{u}_1$; the minus sign is due to the fact on the left face, displacement and force are counted positive in opposite directions. For linear constitutive laws, see fig. 10.5, the area under the force-displacement curve is the area of a triangle, $1/2\ N_1\bar{u}_1$.

The work done by the axial force acting on the right face as it increases from zero to N_1 is $1/2\ N_1\,[\bar{u}_1 + (\mathrm{d}\bar{u}_1/\mathrm{d}x_1)\mathrm{d}x_1]$. The total work done by the axial force is found by adding the contributions from the two faces to find $1/2\ N_1(\mathrm{d}\bar{u}_1/\mathrm{d}x_1)\mathrm{d}x_1 = 1/2\ N_1\bar{\epsilon}_1\mathrm{d}x_1$, where $\bar{\epsilon}_1$ is the sectional axial strain. The work done by the externally applied force, N_1, on a differential element of the beam is

$$\mathrm{d}W_E = \frac{1}{2}\, N_1 \bar{\epsilon}_1 \mathrm{d}x_1 = \frac{1}{2}\, S\bar{\epsilon}_1^2 \,\mathrm{d}x_1, \tag{10.33}$$

where the linear sectional constitutive law, eq. (5.16), is used to obtain the last equality.

The quantity

$$a(\bar{\epsilon}_1) = \frac{1}{2}\, S\bar{\epsilon}_1^2, \tag{10.34}$$

is known as the *strain energy density function* and gives the strain energy per unit length of the beam. This strain energy density can be viewed as the potential of the axial force, which can be derived from this potential as $N_1 = -\partial a(\bar{\epsilon}_1)/\partial\bar{\epsilon}_1 = -S\bar{\epsilon}_1$. Again, the minus sign indicates that this is the internal force in the beam, not the axial force externally applied to the differential element.

The total strain energy developed by the axial force distribution over the beam's span is now obtained by integration of the strain energy density

$$A(\bar{\epsilon}_1) = \int_0^L a(\bar{\epsilon}_1)\,\mathrm{d}x_1 = \frac{1}{2}\int_0^L S\bar{\epsilon}_1^2 \,\mathrm{d}x_1. \tag{10.35}$$

Sometimes, it is preferable to express the strain energy stored in the beam in terms of the axial force by using eq. (5.6), to find

$$A(\bar{\epsilon}_1) = \int_0^L \frac{N_1^2}{2S}\,\mathrm{d}x_1 = A'(N_1). \tag{10.36}$$

Here, $a'(N_1) = N_1^2/2S$ is known as the *stress energy density function*, or *complementary strain energy density*. $A'(N_1)$ is the total stress energy or complementary energy stored in the beam expressed in terms of the axial force distribution. As observed earlier, in the case of a linear constitutive law, the strain energy and its complementary counterpart are equal.

To illustrate these concepts, consider a bar fixed at its root end and subjected to only an axial tip force, P. Static equilibrium implies the $N_1 = P$ at all points along the beam's span, and hence, the axial strain is a constant, $\bar{\epsilon}_1 = \Delta/L$, where Δ is the bar's tip deflection and L its length. The strain energy, eq. (10.35), is then $A(\bar{\epsilon}_1) = 1/2\ \int_0^L S\bar{\epsilon}_1^2\,\mathrm{d}x_1 = 1/2\ S\Delta^2/L$. Clearly, a beam subjected to a tip axial load is equivalent to a rectilinear spring of stiffness constant $k = S/L$.

10.4.2 Beam under transverse loads

Beams subjected to transverse loads are discussed in section 5.5 and are depicted in fig. 5.14 on page 187. Material constitutive laws are assumed to be linear. Consider now an infinitesimal slice of the beam of span-wise length $\mathrm{d}x_1$, acted upon by a bending moment M_3. The left face of this differential element rotates by an angle $\mathrm{d}\bar{u}_2/\mathrm{d}x_1$, whereas the rotation of its right face is $\mathrm{d}\bar{u}_2/\mathrm{d}x_1 + (\mathrm{d}^2\bar{u}_2/\mathrm{d}x_1^2)\mathrm{d}x_1$. When the bending moment acting on the left face increases from zero to its final value, M_3, the work it performs is $-1/2\ M_3\mathrm{d}\bar{u}_2/\mathrm{d}x_1$; the minus sign is due to the fact on the left face, rotation and moment are counted positive in opposite directions. The work done by the bending moment acting on the right face is $1/2\ M_3\left[\mathrm{d}\bar{u}_2/\mathrm{d}x_1 + (\mathrm{d}^2\bar{u}_2/\mathrm{d}x_1^2)\mathrm{d}x_1\right]$. The total work done by the bending moment is found by adding the contributions from the two faces to find $1/2\ M_3(\mathrm{d}^2\bar{u}_2/\mathrm{d}x_1^2)\mathrm{d}x_1 = 1/2\ M_3\kappa_3\mathrm{d}x_1$, where κ_3 is the sectional curvature defined by eq. (5.6). The work done by the externally applied bending moment, M_3, on a differential element of the beam is

$$\mathrm{d}W_E = \frac{1}{2}\ M_3\kappa_3\ \mathrm{d}x_1 = \frac{1}{2}\ H_{33}^c\kappa_3^2\ \mathrm{d}x_1. \tag{10.37}$$

where the linear sectional constitutive law, eq. (5.37), is used to obtain the last equality.

The quantity

$$a(\kappa_3) = \frac{1}{2}\ H_{33}^c\kappa_3^2, \tag{10.38}$$

is known as the strain energy density function and gives the strain energy per unit length of the beam. This strain energy density can be viewed as the potential of the bending moment, which can be derived from this potential as $M_3 = -\partial a(\kappa_3)/\partial\kappa_3 = -H_{33}^c\kappa_3$. Again, the minus sign indicates that this is the internal moment in the beam, not the bending moment externally applied to the differential element.

The total strain energy developed by the bending moment distribution in the beam is then obtained by integration of the strain energy density

$$A(\kappa_3) = \int_0^L a(\kappa_3)\ \mathrm{d}x_1 = \frac{1}{2}\int_0^L H_{33}^c\kappa_3^2\ \mathrm{d}x_1. \tag{10.39}$$

The curvature can also be expressed in terms of the transverse deflection using eq. (5.6) so that

$$A(u_2(x_1)) = \frac{1}{2}\int_0^L H_{33}^c\left(\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right)^2\ \mathrm{d}x_1. \tag{10.40}$$

The strain energy stored in the beam can also be expressed in terms of the bending moment by using eq. (5.37) in eq. (10.39) to find

$$A(M_3) = \int_0^L \frac{M_3^2}{2H_{33}^c}\ \mathrm{d}x_1 = A'(M_3). \tag{10.41}$$

In this case, $a'(M_3) = M_3^2/2H_{33}^c$ is known as the stress energy density function. $A'(M_3)$ is the total complementary strain energy stored in the beam expressed in terms of the bending moment distribution.

10.4.3 Beam under torsional loads

Consider a circular cylindrical beam subjected to torsion as discussed in section 7.1. Material constitutive laws are assumed to be linear. An infinitesimal slice of the cylinder of span-wise length $\mathrm{d}x_1$ is acted upon by a torque M_1. The left face of this differential element undergoes a rotation, ϕ_1, whereas the rotation of its right face is $\phi_1 + (\mathrm{d}\phi_1/\mathrm{d}x_1)\mathrm{d}x_1$. As the torque acting on the left face increases from zero to its final value, M_1, the work it performs is $-1/2\ M_1\phi_1$; the minus sign is due to the fact on the left face, rotation and torque are counted positive in opposite directions. The work done by the torque acting on the right face as it increases from zero to M_1 is $1/2\ M_1\ [\phi_1 + (\mathrm{d}\phi_1/\mathrm{d}x_1)\mathrm{d}x_1]$. The total work done by the torque is found by adding the contributions from the two faces to find $1/2\ M_1(\mathrm{d}\phi_1/\mathrm{d}x_1)\mathrm{d}x_1 = 1/2\ M_1\kappa_1\mathrm{d}x_1$, where κ_1 is the sectional twist rate. The work done by the externally applied torque, M_1, on a differential element of the beam is

$$\mathrm{d}W_E = \frac{1}{2}\ M_1\kappa_1\mathrm{d}x_1 = \frac{1}{2}\ H_{11}\kappa_1^2\ \mathrm{d}x_1. \tag{10.42}$$

where the linear sectional constitutive law, eq. (7.13), is used to obtain the last equality.

The quantity

$$a(\kappa_1) = \frac{1}{2}\ H_{11}\kappa_1^2, \tag{10.43}$$

is known as the strain energy density function and gives the strain energy per unit length of the cylinder. This strain energy density can be viewed as the potential of the torque, which can be derived from this potential as $M_1 = -\partial a(\kappa_1)/\partial\kappa_1 = -H_{11}\kappa_1$. Again, the minus sign indicates that this is the internal torque in the beam, not the torque externally applied to the differential element.

The total strain energy developed in the cylindrical beam by the torque distribution over the cylinder's span is then obtained by integration

$$A(\kappa_1) = \int_0^L a(\kappa_1)\ \mathrm{d}x_1 = \frac{1}{2}\int_0^L H_{11}\kappa_1^2\ \mathrm{d}x_1. \tag{10.44}$$

Sometimes, it is preferable to express the strain energy stored in the cylindrical beam in terms of the torque by using eq. (7.13) to find

$$A(M_1) = \int_0^L \frac{M_1^2}{2H_{11}}\ \mathrm{d}x_1 = A'(M_1). \tag{10.45}$$

$a'(M_1) = M_1^2/2H_{11}$ is known as the stress energy density function. $A'(M_1)$ is the total complementary strain energy stored in the cylinder expressed in terms of the torque distribution.

10.4.4 Relationship with virtual work

It is interesting to compare the results obtained in this section with those developed is section 9.7. The internal work done by a constant bending moment, M_3, undergoing a curvature, κ_3, is given by eq. (9.69) as $\mathrm{d}W_I = -M_3\kappa_3\ \mathrm{d}x_1$, for a slice of the beam of infinitesimal size, $\mathrm{d}x_1$. Next, eq. (9.19) yields $\mathrm{d}W_E = -\mathrm{d}W_I = M_3\kappa_3\ \mathrm{d}x_1$. This result seems to contradict eq. (10.37) in section 10.4.2, which states that $\mathrm{d}W_E = 1/2\ M_3\kappa_3\ \mathrm{d}x_1$. Fortunately, this is only an apparent contradiction. In section 9.7, the bending moment is assumed to remain constant in magnitude while undergoing a curvature; in section 10.4.1, however, the bending moment is assumed grow in proportion to the curvature. If the bending moment is kept constant, the work it performs as the curvature increases is

$$\mathrm{d}W_E = \left[\int_0^{\kappa_3} M_3\ \mathrm{d}\kappa_3\right] \mathrm{d}x_1 = \left[M_3 \int_0^{\kappa_3} \mathrm{d}\kappa_3\right] \mathrm{d}x_1 = M_3\kappa_3\ \mathrm{d}x_1.$$

In contrast, if the bending moment is proportional to the curvature, *i.e.*, if $M_3 = k\kappa_3$, where k is the constant of proportionality between the two quantities, the work becomes

$$\mathrm{d}W_E = \left[\int_0^{\kappa_3} M_3\ \mathrm{d}\kappa_3\right] \mathrm{d}x_1 = \left[\int_0^{\kappa_3} k\kappa_3\ \mathrm{d}\kappa_3\right] \mathrm{d}x_1 = \frac{1}{2}k\kappa_3^2 \mathrm{d}x_1 = \frac{1}{2}M_3\kappa_3\ \mathrm{d}x_1.$$

Clearly, the difference between the two results can be directly attributed to the nature of the bending moment: if the bending moment *remains constant during the deformation*, the work it performs is $\mathrm{d}W_E = M_3\kappa_3\ \mathrm{d}x_1$, whereas if the bending moment *increases in proportion to the deformation*, the work becomes $\mathrm{d}W_E = 1/2\ M_3\kappa_3\ \mathrm{d}x_1$.

The same reasoning applies to bars under torsion. In section 9.7.2, the work done by a constant torque, M_1, undergoing a twist rate, κ_1, is found to be $\mathrm{d}W_E = -\mathrm{d}W_I = M_1\kappa_1\ \mathrm{d}x_1$, see eq. (9.71). In section 10.4.3, the work done by a torque that increases in proportion to the twist rate is found as $\mathrm{d}W_E = 1/2\ H_{11}\kappa_1^2\ \mathrm{d}x_1$, see eq. (10.42). Here again, the two results differ by a factor of one half, which is directly related to the nature of the torque.

All the results derived in section 9.7 for the work done by internal stresses or forces of constant magnitude in various types of structures can be readily used to obtain the work done by internal stresses or forces of magnitude proportional to the deformation in the same structures by simply multiplying the expression by a factor of one half.

The discussion of the previous paragraphs begs the following question: why is the moment assumed to be constant in the developments of section 9.7, whereas it is assumed to increase in proportion to the deformation in the present section? In section 9.7, the goal is to derive expressions for the virtual work and complementary virtual work. When computing the virtual work, virtual displacements do not affect the forces or stresses in the system, *i.e.*, the internal forces or stress remain constant, unaffected by virtual displacements. For instance, the work done by a constant moment in a beam is $W_I = -\int_0^L M_3\kappa_3\ \mathrm{d}x_1$, see eq. (9.69), where the bending moment, M_3, remains constant, unaffected by the curvature, κ_3. The virtual work is then $\delta W_I = -\int_0^L M_3\delta\kappa_3\ \mathrm{d}x_1$, see eq. (9.70a), where the bending moment remains constant, unaffected by the virtual curvature.

In contrast, the present study focuses on the determination of the strain energy stored in a structure. As the external loads are slowly applied to the solid, internal forces and moments increase in proportion to the deformation. Because the system is assumed to be conservative, the work done by the externally applied forces is now stored in the elastic body in the form of strain energy. Consequently, when the strain energy is computed from the evaluation of the work done by the externally applied loads, it must be assumed that the internal forces increase in proportion to the deformation, as is done in the present section.

10.5 Strain energy in solids

In this section, expressions for the strain energy in three-dimensional solids will be derived. The starting point of the development is the expression developed in section 9.7.3. Expressions for the strain energy in beams undergoing three-dimensional deformations will also be developed.

10.5.1 Three-dimensional solid

In section 9.7.3, the internal work done by constant stresses undergoing general, three-dimensional deformation is found to be $W_I = -\int_{\mathcal{V}} \underline{\sigma}^T \underline{\epsilon} \, \mathrm{d}\mathcal{V}$, see eq. 9.76, where $\underline{\epsilon}$ and $\underline{\sigma}$ are the arrays of strain and stress components defined by eqs. (2.11a) and (2.11b), respectively. It follows that the work done by the constant, external stresses is $W_E = \int_{\mathcal{V}} \underline{\sigma}^T \underline{\epsilon} \, \mathrm{d}\mathcal{V}$. Finally, if the stresses increase in proportion to the deformations, the work becomes

$$W_E = \frac{1}{2} \int_{\mathcal{V}} \underline{\sigma}^T \underline{\epsilon} \, \mathrm{d}\mathcal{V}. \tag{10.46}$$

If the material behaves according to Hooke's law given by eqs. (2.4) and (2.9), the work can be expressed in terms of the strain components only as

$$W_E = \frac{1}{2} \int_{\mathcal{V}} \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) + 2\nu(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) \right.$$
$$\left. + \frac{1-2\nu}{2}(\gamma_{23}^2 + \gamma_{31}^2 + \gamma_{12}^2) \right] \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} a(\underline{\epsilon}) \, \mathrm{d}\mathcal{V} = A(\underline{\epsilon}).$$

From this, the strain energy density function for a three-dimensional solid behaving according to Hooke's law becomes

$$a(\underline{\epsilon}) = \frac{1}{2} \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) + 2\nu(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3) \right.$$
$$\left. + \frac{1-2\nu}{2}(\gamma_{23}^2 + \gamma_{31}^2 + \gamma_{12}^2) \right]. \tag{10.47}$$

This expression can be written in a more compact form as follows

$$a(\underline{\epsilon}) = \frac{1}{2} \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu) I_1^2 - 2(1-2\nu) I_2], \tag{10.48}$$

where I_1 and I_2 are the first two invariants of the strain tensor defined by eqs. (1.86a) and (1.86b), respectively. It is also possible to write the strain energy density function in term of the strain array as

$$a(\underline{\epsilon}) = \frac{1}{2}\,\underline{\epsilon}^T \underline{\underline{C}}\,\underline{\epsilon}, \tag{10.49}$$

where $\underline{\underline{C}}$ is the 6×6 stiffness matrix of the material defined by eq. (2.14).

Because Hooke's law is a linear stress-strain relationship, the strain energy and its complementary counterpart are equal, $a(\underline{\epsilon}) = a'(\underline{\sigma})$. The complementary strain energy density is expressed in terms of the stress components as

$$\begin{aligned} a'(\underline{\sigma}) = \frac{1}{2E}\Big[&\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) \\ &+ 2(1+\nu)\left(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2\right)\Big]. \end{aligned} \tag{10.50}$$

A more compact expression can be obtained by making use of the invariants of the stress tensor, I_1 and I_2, given by eqs. (1.15a) and (1.15b), respectively, to find

$$a'(\underline{\sigma}) = \frac{1}{2E}\left[I_1^2 - 2(1+\nu)I_2\right] = \frac{1}{2}\left[\frac{I_1^2}{E} - \frac{I_2}{G}\right]. \tag{10.51}$$

Finally, it is also possible to write the complementary strain energy density function in term of the stress array as

$$a'(\underline{\sigma}) = \frac{1}{2}\,\underline{\sigma}^T \underline{\underline{S}}\,\underline{\sigma}, \tag{10.52}$$

where $\underline{\underline{S}}$ is the 6×6 compliance matrix of the material defined by eq. (2.12).

10.5.2 Three-dimensional beams

The internal work done by constant stress resultants in three-dimensional beams undergoing deformation is derived in section 9.7.4, see eq. (9.78). From this result, the work done by the same stress resultants when they increase in proportion to the deformations becomes

$$W_E = \frac{1}{2}\int_0^L \left(N_1\bar{\epsilon}_1 + M_2\kappa_2 + M_3\kappa_3\right)\,\mathrm{d}x_1. \tag{10.53}$$

If the beam is made of a linearly elastic material obeying Hooke's law, the sectional constitutive laws are given by eq. (6.12), assuming that the origin of the axis system is selected to be at the section's centroid. Eliminating the stress resultants from eq. (10.53) with the help of the sectional constitutive laws yields the strain energy in the beam as

$$A = \frac{1}{2}\int_0^L \left(S\bar{\epsilon}_1^2 + H_{22}^c\kappa_2^2 - 2H_{23}^c\kappa_2\kappa_3 + H_{33}^c\kappa_3^2\right)\,\mathrm{d}x_1. \tag{10.54}$$

Similarly, the complementary strain energy is obtained from eq. (10.53), where the sectional strains are expressed in terms of the stress resultants using the compliance form of the sectional constitutive laws, eqs. (6.13), to find

$$A' = \frac{1}{2} \int_0^L \left(\frac{N_1^2}{S} + \frac{H_{33}^c}{\Delta_H} M_2^2 + 2\frac{H_{23}^c}{\Delta_H} M_2 M_3 + \frac{H_{22}^c}{\Delta_H} M_3^2 \right) \mathrm{d}x_1, \quad (10.55)$$

where $\Delta_H = H_{22}^c H_{33}^c - H_{23}^c H_{23}^c$.

Equations (10.54) and (10.55) are general expression for the strain energy in three-dimensional beams and its complementary counterpart, respectively. They assume a linearly elastic material behavior characterized by Hooke's law, and the origin of the axis system must be located at the section's centroid.

These expressions can be simplified for specific applications. For instance, if the beam is undergoing axial deformations only, the first term only is kept and $A = 1/2 \int_0^L S\bar{\epsilon}_1^2 \,\mathrm{d}x_1$ whereas $A' = 1/2 \int_0^L N_1^2/S \,\mathrm{d}x_1$. If the axis system is selected to coincide with the principal centroidal axes of bending, $H_{23}^c = 0$, and

$$A = \frac{1}{2} \int_0^L \left(S\bar{\epsilon}_1^2 + H_{22}^c \kappa_2^2 + H_{33}^c \kappa_3^2 \right) \mathrm{d}x_1, \quad (10.56a)$$

$$A' = \frac{1}{2} \int_0^L \left(\frac{N_1^2}{S} + \frac{M_2^2}{H_{22}^c} + \frac{M_3^2}{H_{33}^c} \right) \mathrm{d}x_1. \quad (10.56b)$$

10.6 Applications to trusses and beams

The principle of minimum total potential energy leads to an elegant solution procedure for truss and beam problems, both of which will be addressed in the sections below.

10.6.1 Applications to trusses

To illustrate the application of the principle of minimum total potential energy to truss problems, a simple problem will be solved first, then, the general approach will be presented more formally, leading to a step-by-step procedure.

Consider the three-bar, hyperstatic truss depicted in fig. 10.16. All bars have the same cross-sectional area, $\mathcal{A}$, and modulus, E. Determine both the joint displacements and the member forces. In fig. 10.16, the three bars are labeled by a number indicated in a square box. The bar lengths are $L_1 = L_3 = L/\cos\theta$ and $L_2 = L$.

First, eq. (9.27) is used to find the bar elongations as $e_1 = u_1 \cos\theta + u_2 \sin\theta$, $e_2 = u_2$, and $e_3 = -u_1 \cos\theta + u_2 \sin\theta$. In view of eq. (10.29), the bar strain energy is written as $A = 1/2\, ke^2$, where e is the bar elongation and $k = E\mathcal{A}/L$ its stiffness. The strain energy in the truss is then the sum of the bar strain energies

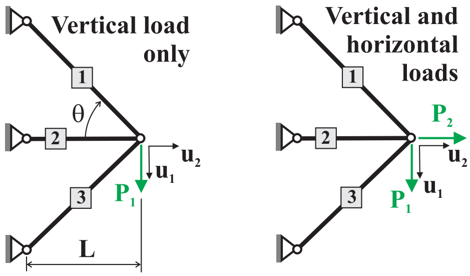

Fig. 10.16. Simple 3-bar truss.

$$\begin{aligned} A =& \frac{1}{2}\left(\frac{E\mathcal{A}\cos\theta}{L}e_1^2 + \frac{E\mathcal{A}}{L}e_2^2 + \frac{E\mathcal{A}\cos\theta}{L}e_3^2\right) \\ =& \frac{1}{2}\frac{E\mathcal{A}}{L}\left[(u_1\cos\theta + u_2\sin\theta)^2\cos\theta + u_2^2 + (-u_1\cos\theta + u_2\sin\theta)^2\cos\theta\right] \\ =& \frac{1}{2}\frac{E\mathcal{A}}{L}\left[2u_1^2\cos^3\theta + (1 + 2\sin^2\theta\cos\theta)u_2^2\right]. \end{aligned}$$

Based on eq. (10.13), the potential of the externally applied load, P_1, is given by $\Phi = -P_1 u_1$. The total potential, Π, then becomes $\Pi = A + \Phi = A - P_1 u_1$.

This problem has two degrees of freedom, u_1 and u_2, and the principle of minimum total potential energy, eq. (10.17), then requires

$$\frac{\partial \Pi}{\partial u_1} = \frac{E\mathcal{A}}{L} 2u_1\cos^3\theta - P_1 = 0,$$
$$\frac{\partial \Pi}{\partial u_2} = \frac{E\mathcal{A}}{L}(1 + 2\sin^2\theta\cos\theta)u_2 = 0.$$

It is convenient to recast these equations in a matrix form to underline the fact that they form a set of two linear equations for the two generalized coordinates of the problem, u_1 and u_2,

$$\begin{bmatrix} 2\cos^3\theta & 0 \\ 0 & 1 + 2\sin^2\theta\cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L}{E\mathcal{A}} \begin{Bmatrix} P_1 \\ 0 \end{Bmatrix}.$$

Solving these equations then yields $u_1 = P_1 L/(2E\mathcal{A}\cos^3\theta)$ and $u_2 = 0$.

Once the displacements of the system have been evaluated, the elongation-displacement equations yield the non-dimensional elongations in each bar as

$$\frac{e_1}{L} = \frac{1}{2\cos^2\theta}\frac{P_1}{E\mathcal{A}}, \quad e_2 = 0, \quad \frac{e_3}{L} = -\frac{1}{2\cos^2\theta}\frac{P_1}{E\mathcal{A}}.$$

Next, the non-dimensional bar forces are obtained from the constitutive laws as

$$\frac{F_1}{P_1} = \frac{1}{2\cos\theta}, \quad F_2 = 0, \quad \frac{F_3}{P_1} = -\frac{1}{2\cos\theta}.$$

The approach presented here first finds the joint displacements, then evaluates bar elongations based on the elongation-displacement equations, and finally determines the bar forces with the help of the constitutive laws. The principle of minimum total potential energy enforces the equilibrium equations of the problem. The solution process does not make special provisions for the fact that the three-bar truss is a hyperstatic structure: it is equally applicable to both iso- and hyperstatic structures.

The response of a particular structure must often be evaluated under various loading conditions. The right portion of fig. 10.16 depicts the same three-bar truss subjected to two loads, P_1 and P_2, both applied at the common joint of the three bars. The only change in the above analysis is that the expression for the potential of the externally applied loads now becomes $\Phi = -P_1 u_1 - P_2 u_2$. Repeating the steps of the analysis leads to the following set of linear equations

$$\begin{bmatrix} 2\cos^3\theta & 0 \\ 0 & 1 + 2\sin^2\theta\cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L}{E\mathcal{A}} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix},$$

and yields the joint displacements: $u_1 = P_1 L/(2E\mathcal{A}\cos^3\theta)$ and $u_2 = P_2 L/[E\mathcal{A}(1 + 2\sin^2\theta\cos\theta)]$.

General procedure

The general procedure for the solution of truss problems using the principle of minimum total potential energy can be summarized in the following steps.

1. Based on the geometry of the problem, find the length, L_i, of each of the N_b bars of the truss. The Young's modulus, E_i, and cross-section area, $\mathcal{A}_i$, are given for each bar. Compute the stiffness, $k_i = (E\mathcal{A})_i/L_i$, of each bar.
2. Select the generalized coordinates of the problem to be the N joint displacements. Do not include the displacements at the supports because these are constrained to be zero.
3. Find the bar extensions, e_i, in terms of the joint displacements using eq. (9.27).
4. Determine the total strain energy of the system by adding up the contributions from the N_b bars,
$$A = \frac{1}{2}\sum_{i=1}^{N_b} k_i e_i^2.$$
5. Write the potential of the externally applied loads, Φ, using eq. (10.13). Because the externally applied loads, $P_j, j = 1, 2, \ldots N_P$, are assumed to act at the joints, the contribution of each load is $-P_j d_j$, where d_j is the displacement along the line of action of the force. The total potential of the externally applied loads is then
$$\Phi = -\sum_{j=1}^{N_P} P_j d_j.$$
6. The governing equations of the system are found by invoking the principle of minimum total potential energy expressed by eq. (10.17). Because the strain energy is a quadratic function of the joint displacements, and the potential of the externally applied loads a linear function of the same variables, the resulting equations form a linear set of N equations for the N generalized coordinates.

7. Solve the equations for the joint displacements.
8. Determine the bar elongations from the elongation-displacement equations.
9. Determine the bar forces from the constitutive laws, $F_i = k_i e_i$.

The procedure is unaffected by the nature of the truss: both iso- and hyperstatic problems can be solved in the same manner. For large trusses, the number of generalized coordinates increases, and the size of the set of linear equations for the generalized coordinates increases. Clearly, the approach is not suitable for hand calculations, but the solution of large systems of linear equations is easily obtained with the help of computers.

Example 10.6. Pentagonal truss
Consider the ten-bar pentagonal truss depicted in fig. 10.17. All bars have the same modulus, E, and cross-sectional area, $\mathcal{A}$, and a single vertical load, P, is applied at the top joint of the truss. Because both structure and loading are symmetric with respect to the vertical axis, the response of the truss must exhibit the same symmetry. The numbering of the bars reflects the symmetry of the problem: the behavior of the identically numbered bars must be identical. Only four independent joint displacement components are needed and are indicated in fig. 10.17; these will be selected as the generalized coordinates of the problem.

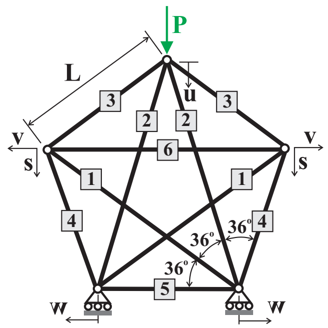

Fig. 10.17. A pentagonal truss.

The geometry of a regular pentagon implies that the angles between the diagonals of the pentagon and the sides are all 36°. The bar lengths are found as $L_1 = L_2 = L_6 = L/(2\sin 18°) = 1.618\ L$, and $L_3 = L_4 = L_5 = L$. Equation (9.27) then yields the elongations of the bars as

$$
\begin{aligned}
e_1 &= (v + w)\cos(36°) - s\sin(36°) = 0.809(v + w) - 0.588s \\
e_2 &= -u\sin(72°) + w\cos(72°) = -0.951u + 0.309w \\
e_3 &= (s - u)\sin(36°) + v\cos(36°) = 0.588(s - u) + 0.809v \\
e_4 &= (v - w)\cos(72°) - s\sin(72°) = 0.309(v - w) - 0.951s \\
e_5 &= 2w \\
e_6 &= 2v.
\end{aligned}
\tag{10.57}
$$

The truss strain energy is found by summing up the individual bar contributions

$$A = \frac{1}{2}\frac{E\mathcal{A}}{L}\left(\frac{2e_1^2}{1.618} + \frac{2e_2^2}{1.618} + 2e_3^2 + 2e_4^2 + e_5^2 + \frac{e_6^2}{1.618}\right).$$

Due to symmetry, the contributions of bars 1, 2, 3, and 4 are doubled. The total potential of the externally applied loads reduces to a single term, $\Phi = -Pu$.

The principle of minimum total potential energy expressed by eq. (10.17) then implies the vanishing of the derivatives of the total potential energy, $\Pi(u, v, w, s)$, with respect to each of the generalized coordinates,

$$\frac{\partial \Pi}{\partial u} = \frac{\partial \Pi}{\partial v} = \frac{\partial \Pi}{\partial w} = \frac{\partial \Pi}{\partial s} = 0.$$

Tedious algebraic manipulations yield a set of four simultaneous algebraic equations for the unknown joint displacements

$$\begin{bmatrix} 1.809 & -0.9511 & -0.3633 & -0.6910 \\ -0.9511 & 4.781 & 0.6180 & -0.2245 \\ -0.3633 & 0.6180 & 5.118 & 0 \\ -0.6910 & -0.2245 & 0 & 2.927 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ s \end{Bmatrix} = \frac{PL}{E\mathcal{A}} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

These equations can solved numerically using a computer, and the result is

$$\frac{u}{L} = 0.703\,\frac{P}{E\mathcal{A}}, \quad \frac{v}{L} = 0.144\,\frac{P}{E\mathcal{A}}, \quad \frac{w}{L} = 0.0325\,\frac{P}{E\mathcal{A}}, \quad \frac{s}{L} = 0.177\,\frac{P}{E\mathcal{A}}.$$

Once the joint displacements are determined, the bar non-dimensional elongation are obtained from the elongation-displacement relationships, eqs. (10.57),

$$\begin{aligned} \frac{e_1}{L} &= 0.0387\,\frac{P}{E\mathcal{A}}, & \frac{e_2}{L} &= -0.6580\,\frac{P}{E\mathcal{A}}, & \frac{e_3}{L} &= -0.1930\,\frac{FL}{EA} \\ \frac{e_4}{L} &= -0.1340\,\frac{P}{E\mathcal{A}}, & \frac{e_5}{L} &= 0.0650\,\frac{P}{E\mathcal{A}}, & \frac{e_6}{L} &= 0.2880\,\frac{P}{E\mathcal{A}}. \end{aligned}$$

Finally, the non-dimensional bar forces are evaluated with the help of the constitutive laws to find

$$\begin{aligned} \frac{F_1}{P} &= 0.0239, & \frac{F_2}{P} &= -0.4070, & \frac{F_3}{P} &= -0.1930, \\ \frac{F_4}{P} &= -0.1340, & \frac{F_5}{P} &= 0.0650, & \frac{F_6}{P} &= 0.1780. \end{aligned}$$

10.6.2 Problems

Problem 10.6. Planar 3-bar truss

The hyperstatic, three bar truss depicted in fig. 10.18 is subjected to a load, P, applied at joint **A**, with a line of action at an angle $\theta = 45$ degrees with respect to the horizontal. All bars have the same Young's modulus, E, and cross-sectional area, $\mathcal{A}$. *(1)* Determine the displacement components, u_1 and u_2, of joint **A**. *(2)* Find the elongations in each bar. *(3)* Evaluate the forces in each bar.

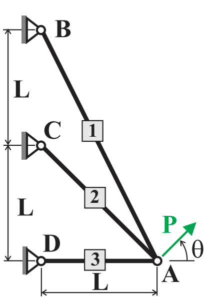

Fig. 10.18. Planar 3-bar truss with load applied at joint A.

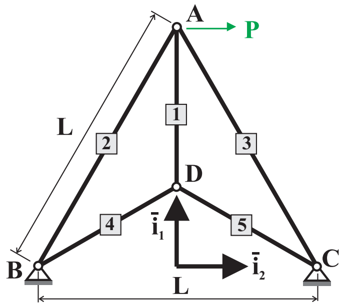

Fig. 10.19. Hyperstatic planar triangular truss.

Problem 10.7. Hyperstatic planar triangular truss

The five-bar, hyperstatic truss shown in fig. 10.19 has the overall shape of an equilateral triangle, and is subjected to a horizontal load of magnitude P at joint **A**. All bars have the same Young's modulus, E, and cross-sectional area, $\mathcal{A}$. Note that this problem features four generalized coordinates: the horizontal and vertical displacement components at joints **A** and **D**. *(1)* Determine the generalized coordinates. *(2)* Find the elongations in each bar. *(3)* Evaluate the forces in each bar. Use the following data: $L = 2$ m, $A = 100$ mm^2, $E = 70$ GPa, $P = 20$ kN. It will be necessary to use a computer to solve this problem.

Problem 10.8. Multi-cable truss structure

A vertical load, P, is supported by seven cables of equal cross-sectional area, $\mathcal{A}$, and elastic modulus, E as shown in fig 10.20. The angles between the cables and the vertical are $60°$, $45°$, $30°$, $0°$, $-30°$, $-45°$, and $-60°$. *(1)* Determine the generalized coordinates. *(2)* Find the elongations in each cable. *(3)* Evaluate the forces in each cable.

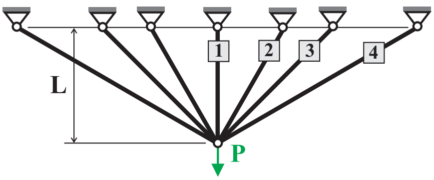

Fig. 10.20. Multiple cables supporting a single load point.

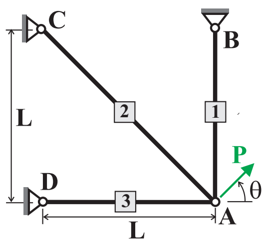

Fig. 10.21. Planar rectangular 3-bar truss with unequal axial stiffnesses.

Problem 10.9. Rectangular 3-bar planar truss

The three-bar, hyperstatic truss shown in fig. 10.21 is subjected to a load of magnitude P with a line of action at an angle $\theta = 45$ degrees with respect to the horizontal. All bars have the same elastic modulus, E; bars 1 and 3 have a cross-sectional area, $\mathcal{A}$, whereas that of bar 3 is $2\mathcal{A}$. *(1)* Determine the generalized coordinates. *(2)* Find the elongations in each bar. *(3)* Evaluate the forces in each bar.

10.6.3 Applications to beams

The principle of minimum total potential energy can also be applied to beam problems. Expressions for the strain energy stored in beams are developed in section 10.4 for beams under axial, transverse, and torsional loads.

Consider, for instance, a beam under a distributed transverse load, $p_2(x_1)$, as shown in fig. 5.14 on page 187. The transverse force applied on a differential element of the beam of length $\mathrm{d}x_1$ is $p_2(x_1)\ \mathrm{d}x_1$, and the work it performs is then $p_2(x_1)\bar{u}_2(x_1)\ \mathrm{d}x_1$, where $\bar{u}_2(x_1)$ is the displacement of the force along its line of action. The work done by this distributed load applied along the beam's span is then found by integration, $W_E = \int_0^L p_2(x_1)\bar{u}_2(x_1)\ \mathrm{d}x_1$. Equation (10.8) then yields the potential of this externally applied load as

$$\Phi = -\int_0^L p_2(x_1)\bar{u}_2(x_1)\ \mathrm{d}x_1. \tag{10.58}$$

The total potential energy of the beam now follows from eq. (10.9) as

$$\Pi = A + \Phi = \frac{1}{2}\int_0^L H_{33}^c \left(\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right)^2 \mathrm{d}x_1 - \int_0^L p_2\bar{u}_2\ \mathrm{d}x_1,$$

where eq. (10.40) is used to express the beam's strain energy in bending in terms of the transverse displacement field, $u_2(x_1)$.

At first glance, this form for the total potential energy is similar to that developed earlier for mechanisms and trusses. A fundamental difference should be pointed out. Whereas in earlier developments the total potential energy is a function of the generalized coordinates, $\Pi = \Pi(\underline{q})$, the potential energy is now a function of another function, $\Pi = \Pi(\bar{u}_2(x_1))$, where $\bar{u}_2(x_1)$ is the beam's transverse displacement field. A "function of a function" is called a *functional.*

Beam problems are *infinite dimensional problems*, or *continuous problems*, because the solution to the problem requires the determination of the transverse displacements field, $\bar{u}_2(x_1)$, at all points $0 \leq x_1 \leq L$, and this is equivalent to an infinite number of unknowns. This contrasts with planar truss problems, for instance, that involve only $2N$ unknowns (*i.e.*, two displacement components at each of the truss' N joints) and are known as *finite dimensional problems* or *discrete problems.*

Minimization of the total potential energy is a standard calculus problem when it is a function of one or a finite number of variables such as the generalized coordinates in eqs. (10.17). When the total potential energy becomes a functional, new mathematical concepts are required to find the configuration of the system that minimizes this functional. The *calculus of variations* [6, 5] is the branch of mathematics that studies functionals, and elements of calculus of variations will be developed in chapter 12.

It is also possible to transform continuous problems into discrete problems by choosing specific functions for $u_2(x_1)$ whose amplitudes can then be determined using the principle of minimum total potential energy. This effectively reduces a problem with an infinite number of degrees of freedom to one with a finite number of degrees of freedom. As will be seen in chapter 11, the principle of minimum total potential energy is a powerful tool for constructing such approximate solutions.

Equation (10.58) gives the potential of the externally applied loads for a beam subjected to transverse distributed transverse loads, $p_2(x_1)$. As illustrated in fig. 6.1 on page 224, three-dimensional beam problems often involve complex loading conditions. In general, the beam can be subjected to distributed loading components, $p_1(x_1)$, $p_2(x_1)$, and $p_3(x_1)$, acting along axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively. Concentrated loads, P_1, P_2, and P_3, can also be applied along the same directions at any point along the span of the beam. Distributed moments, $q_1(x_1)$, $q_2(x_1)$, and $q_3(x_1)$, acting about axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively, can be applied. Finally, concentrated moments, Q_1, Q_2, and Q_3, can also be applied about the same directions at any point along the span of the beam. The potential of these externally applied loads becomes

$$\begin{aligned}\Phi = &-\int_0^L p_1\bar{u}_1 \,\mathrm{d}x_1 - P_1\bar{u}_1(\alpha L) - \int_0^L q_1\Phi_1 \,\mathrm{d}x_1 - Q_1\Phi_1(\alpha L)\\ &-\int_0^L p_2\bar{u}_2 \,\mathrm{d}x_1 - P_2\bar{u}_2(\alpha L) + \int_0^L q_2\frac{\mathrm{d}\bar{u}_3}{\mathrm{d}x_1} \,\mathrm{d}x_1 + Q_2\frac{\mathrm{d}\bar{u}_3}{\mathrm{d}x_1}(\alpha L)\\ &-\int_0^L p_3\bar{u}_3 \,\mathrm{d}x_1 - P_3\bar{u}_3(\alpha L) - \int_0^L q_3\frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1} \,\mathrm{d}x_1 - Q_3\frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1}(\alpha L).\end{aligned} \tag{10.59}$$

The various terms appearing in this lengthy expression can be interpreted individually as follows.

For each concentrated load component, the potential is the negative product of the load by the displacement of its point of application projected along the line of action of the load. For instance, the potential of a concentrated load, P_1, applied at $x_1 = \alpha L$, is $-P_1\bar{u}_1(\alpha L)$. For simplicity, all concentrated loads and moments are assumed to be applied at the same location, $x_1 = \alpha L$. In practical applications, however, each concentrated load must be multiplied by the displacement of its own point of application. For instance, if three concentrated loads, P_1, P_2, and P_3, acting along axes $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively, are applied at location $x_1 = \alpha L$, βL, and γL, respectively, the corresponding potential is $\Phi = -P_1\bar{u}_1(\alpha L) - P_2\bar{u}_2(\beta L) - P_3\bar{u}_3(\gamma L)$.

For each concentrated moment component, the potential is the negative product of the moment by the rotation of its point of application projected along the line of action of the moment. For instance, the potential of a concentrated torque, Q_1, applied at $x_1 = \alpha L$, is $-Q_1\Phi_1(\alpha L)$. Similarly, the potential of a concentrated moment, Q_3, is $-Q_3\Phi_3(\alpha L)$. According to the Euler-Bernoulli assumptions, the rotation of the section equals the slope of the beam, $\Phi_3 = \mathrm{d}\bar{u}_2/\mathrm{d}x_1$, see eq. (5.3). The potential then becomes $-Q_3\mathrm{d}\bar{u}_2(\alpha L)/\mathrm{d}x_1$. For rotation in the orthogonal plane, $\Phi_2 = -\mathrm{d}\bar{u}_3/\mathrm{d}x_1$, see eq. (5.3); the corresponding potential then becomes $-Q_2\Phi_2(\alpha L) = Q_2\mathrm{d}\bar{u}_3(\alpha L)/\mathrm{d}x_1$. Here again, if the concentrated moments are applied at different locations along the beam, the expression for the corresponding potential must be updated accordingly.

When dealing with distributed loads, a similar reasoning applies. The potential of a distributed axial force, $p_1(x_1)$, acting on an infinitesimal slice of the beam of length $\mathrm{d}x_1$ is $-p_1(x_1)\mathrm{d}x_1\ \bar{u}_1(x_1)$. The complete potential of the distributed load is then $-\int_0^L p_1(x_1)\bar{u}_1(x_1)\ \mathrm{d}x_1$. Similar expression are readily derived for the other loading components, as indicated in eq. (10.59).

10.7 Development of a finite element formulation for trusses

The principle of minimum total potential energy provides a powerful tool for the analysis of trusses, as demonstrated in the previous section. While the approach is manageable for simple trusses consisting of only a few bars, it is clear that the algebraic manipulations become increasingly tedious as the number of bars increases. The method is, however, very systematic and reduces the problem to the solution of a set of simultaneous linear equations. While difficult to solve by hand, large sets of simultaneous linear equations are easily solved with the help of computers. In fact, powerful algorithms have been developed that routinely allow the accurate solution of very large systems of linear system, involving millions of degrees of freedom. Since computers take care of the solution phase, *i.e.*, the solution of large sets of linear equations, attention is directed in this section to the development of a systematic approach to generating the equilibrium equations of the problem.

A key to the approach presented here is to first focus on an individual truss member, *i.e.*, an axially loaded bar, rather than on the entire truss. The strain energy and potential of the externally applied loads are generated for each individual bar. Next, the total potential energy of the entire truss is obtained by summing up the contributions from each bar. Equilibrium equations of the problem are then generated by applying the principle of minimum total potential energy to the entire truss. This approach allows the development of *element oriented* methods, which focus on a single element of the structure at a time.

Another key to this approach is the additive property of strain energy: the total strain energy stored in a structure is the sum of the strain energy stored in all of its elastic components. More specifically, the total strain energy in a truss is equal to the sum of the strain energies in each of its bars. For trusses, each bar is an “element” of the system, and the heart of the approach is the evaluation of the strain energy in a generic bar or element of the truss. This simple computation is repeated for each element of the truss. The total strain energy in the system is then found by adding the contributions of the individual elements. This process, known as the *assembly process*, can be performed in an efficient manner through matrix operations that are readily implemented on computers.

From this cursory description of the approach, it is apparent that the process is exceedingly tedious if carried out by hand. It is, however, very systematic: the overall method is broken into a large number of step, each of which is rather simple to complete. The approach is ideally suited for computer implementation, and each step becomes a simple task to be efficiently performed by the computer. First, the strain energy in each bar is computed; next, the contribution of each bar is added to the total strain energy of the truss; finally, the large system of linear equations resulting from the application of the principle of minimum total potential energy is solved. The systematic use of linear algebra and matrix notation greatly simplifies the computer implementation of these procedures. Consequently, a brief summary of key concepts from linear algebra and the matrix and array notation used in this text is provided in appendix A.2. A quick review of the material presented in this appendix may prove useful in understanding the developments that follow.

The approach described in this section is basically an introduction to the *finite element method*, which has become the tool of choice for the solution of complex structural problems. While several key concepts of this method are present in this development, other distinctive features of the method are not required for truss problems. In particular, when applied to more complex structural components, the finite element method involves a discretization procedure that is not required for the problem at hand. This discretization procedure is needed for beams and will be described in section 11.5.

10.7.1 General description of the problem

Figure 10.22 depicts an 11-bar, 7-node planar truss that will be used to illustrate the development of the method. To avoid confusion, each *node number* is circled, and each *bar number* is indicated in a square box; the numbering sequence of both nodes and bars is otherwise arbitrary. The truss is in a plane defined by unit vectors $\bar{\imath}_1$ and $\bar{\imath}_2$ and is pinned to the ground at nodes 1 and 7. The geometry of the truss will be defined in a *global coordinate system* defined by orthonormal basis $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$.

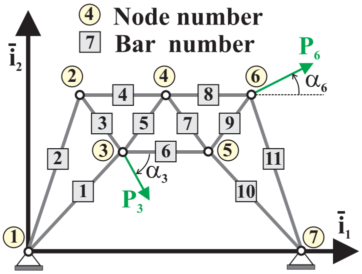

Fig. 10.22. Eleven-bar truss with node and element numbering.

Two concentrated loads are applied to the truss. Loads P_3 and P_6 are applied at nodes 3 and 6, respectively, and are acting at angles α_3 and α_6 with respect to the horizontal, respectively.

The stiffness properties of each bar are denoted $E_{(i)}\mathcal{A}_{(i)}/L_{(i)}$, where $E_{(i)}$, $\mathcal{A}_{(i)}$, and $L_{(i)}$ are the bar's Young's modulus, cross-sectional area, and length, respectively. Throughout this development, subscript $(\cdot)_{(i)}$ will be used to indicate quantities pertaining to the i$^{\text{th}}$ bar or element.

The geometry of the truss is defined by the coordinates of its 7 nodes. For instance, the components of the position vector of node 1 with respect to the origin of the coordinate system are denoted x_1 and y_1, along unit vectors $\bar{\imath}_1$ and $\bar{\imath}_2$, respectively, and stored in array $\underline{p}_1 = \left\{x_1, \ y_1\right\}^T$. Similar arrays[1] can be defined for all the nodes of the truss,

[1] This notation uses symbols x, y, and z, to denote position components, instead of x_1, x_2, and x_3, which are used throughout this book. Notations with multiple subscripts, such as x_{1i} to indicate the position component of node i along axis $\bar{\imath}_1$ are therefore avoided.

$$\underline{p}_1 = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix}, \quad \underline{p}_2 = \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix}, \quad \ldots, \quad \underline{p}_7 = \begin{Bmatrix} x_7 \\ y_7 \end{Bmatrix}. \tag{10.60}$$

The subscript $(\cdot)_i$ will be used to indicate quantities pertaining to the i$^{\text{th}}$ node.

The generalized coordinates of the problem will be selected as the horizontal and vertical displacement components of each of the 7 nodes, denoted u_i and v_i, respectively. The following nodal displacement arrays will be used to contain these generalized coordinates,

$$\underline{q}_1 = \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}, \quad \underline{q}_2 = \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}, \quad \ldots, \quad \underline{q}_7 = \begin{Bmatrix} u_7 \\ v_7 \end{Bmatrix}. \tag{10.61}$$

Array $\underline{q}_1$ stores the two components of displacement at node 1, while array $\underline{q}_i$ stores those at node i. It will also be necessary to define a *global displacement array*, $\underline{q}$, that stores all the nodal displacement arrays in a single column as

$$\underline{q} = \left\{ \underline{q}_1^T, \underline{q}_2^T, \underline{q}_3^T, \underline{q}_4^T, \underline{q}_5^T, \underline{q}_6^T, \underline{q}_7^T \right\}^T. \tag{10.62}$$

As mentioned earlier, the finite element method first focuses on a "generic element" of the system, in this case, a generic bar of the truss, to evaluate the strain energy stored in that specific element. Each bar is connected to two nodes: a root node, denoted *Node 1*, and a tip node, denoted *Node 2*. These nodes are referred to as "local nodes," and are used when focusing on a single bar of the system.

On the other hand, when the complete truss is considered, "global nodes" must be used. For instance, referring to fig. 10.22, bar 4 has two *local nodes*, denoted *Node 1* and *Node 2*, whereas its *global nodes* are nodes 2 and 4. Similarly, bar 9 has two *local nodes*, denoted *Node 1* and *Node 2*, whereas its *global nodes* are nodes 5 and 6. Since the local nodes are denoted *Node 1* and *Node 2* for each and every bar, they are not indicated on the figure as it would lead to confusion. This distinction between local and global nodes is important for the development of the method.

10.7.2 Kinematics of an element

The kinematics of a specific bar in the truss will be studied first. Figure 10.23 depicts a single bar with local nodes denoted *Node 1* and *Node 2*. To simplify the formulation of the problem, a *local coordinate system* is defined: unit vector $\bar{\jmath}_1$ is aligned with the axis of the bar, and $\bar{\jmath}_2$ is normal to the bar. The local coordinate system, $\mathcal{J} = (\bar{\jmath}_1, \bar{\jmath}_2)$, corresponds to a rotation of the global coordinate system, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$, by an angle $\hat{\theta}$, which is the angle between the bar and the horizontal axis, $\bar{\imath}_1$.

The position vectors of the two local nodes of the element are denoted as

$$\hat{\underline{p}}_1 = \begin{Bmatrix} \hat{x}_1 \\ \hat{y}_1 \end{Bmatrix}, \text{ and } \hat{\underline{p}}_2 = \begin{Bmatrix} \hat{x}_2 \\ \hat{y}_2 \end{Bmatrix}, \tag{10.63}$$

For clarity, the quantities pertaining to an element will be indicated with a caret $\hat{(\cdot)}$, to distinguish them from their global counterparts. For example, it is important to distinguish the position vector of node 1, denoted $\underline{p}_1$ as defined by eq. (10.60), from $\underline{\hat{p}}_1$, which indicates the position vector of *Node 1* of a generic bar element.

Similarly, the displacements of the two nodes of the elements, resolved in axis systems $\mathcal{I}$ and $\mathcal{J}$, are denoted

$$\underline{\hat{q}}_1 = \left\{ \begin{matrix} \hat{u}_1 \\ \hat{v}_1 \end{matrix} \right\}, \quad \underline{\hat{q}}_2 = \left\{ \begin{matrix} \hat{u}_2 \\ \hat{v}_2 \end{matrix} \right\}, \quad \text{and} \quad \underline{\hat{q}}_1^* = \left\{ \begin{matrix} \hat{u}_1^* \\ \hat{v}_1^* \end{matrix} \right\}, \quad \underline{\hat{q}}_2^* = \left\{ \begin{matrix} \hat{u}_2^* \\ \hat{v}_2^* \end{matrix} \right\}, \tag{10.64}$$

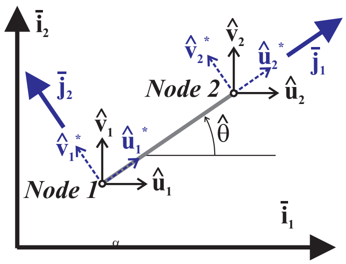

Fig. 10.23. General bar element.

respectively. For each of the two nodes, two sets of displacement components are thus defined. At *Node 1*, the components of the displacement vector resolved in the *global coordinate system* are denoted $\hat{u}_1$ and $\hat{v}_1$, whereas the corresponding components resolved in the *local coordinate system* are denoted $\hat{u}_1^*$ and $\hat{v}_1^*$, respectively. The superscript $(\cdot)^*$ will be used here to indicate the components of quantities resolved in the local coordinate system, $\mathcal{J}$.

The relationships between quantities resolved in two distinct orthonormal bases are discussed in appendix A.3. Since $\underline{\hat{q}}_1$ and $\underline{\hat{q}}_1^*$ are the components of the displacement vectors of *Node 1* resolved in two orthonormal bases, $\mathcal{I}$ and $\mathcal{J}$, eqs. (A.43) apply, and thus

$$\underline{\hat{q}}_1 = \underline{\underline{\hat{R}}}\, \underline{\hat{q}}_1^*, \tag{10.65}$$

where the element rotation matrix, $\underline{\underline{\hat{R}}}$, is similar to that defined in eq. (A.40),

$$\underline{\underline{\hat{R}}} = \begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} \\ \sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix}. \tag{10.66}$$

A similar result can be developed for *Node 2*, $\underline{\hat{q}}_2 = \underline{\underline{\hat{R}}}\, \underline{\hat{q}}_2^*$, where the same rotation matrix is used.

The bar's length, $\hat{L}$ and orientation angle, $\hat{\theta}$, can be computed from the position vectors of its end nodes. The length is given by

$$\hat{L} = \|\underline{\hat{p}}_2 - \underline{\hat{p}}_1\| = \sqrt{(\hat{x}_2 - \hat{x}_1)^2 + (\hat{y}_2 - \hat{y}_1)^2}. \tag{10.67}$$

Angle $\hat{\theta}$ can be found from the nodal position vectors using the definition of the scalar product, $\bar{\imath}_1 \cdot (\underline{\hat{p}}_2 - \underline{\hat{p}}_1) = \|\bar{\imath}_1\|\, \|(\underline{\hat{p}}_2 - \underline{\hat{p}}_1)\| \cos\hat{\theta}$, and $\bar{\imath}_2 \cdot (\underline{\hat{p}}_2 - \underline{\hat{p}}_1) = \|\bar{\imath}_2\|\, \|(\underline{\hat{p}}_2 - \underline{\hat{p}}_1)\| \sin\hat{\theta}$. It then follows that

$$\cos\hat{\theta} = \frac{\bar{\imath}_1 \cdot (\underline{\hat{p}}_2 - \underline{\hat{p}}_1)}{\hat{L}}, \quad \sin\hat{\theta} = \frac{\bar{\imath}_2 \cdot (\underline{\hat{p}}_2 - \underline{\hat{p}}_1)}{\hat{L}}. \tag{10.68}$$

Finally, it will be convenient to combine the displacements of the element's two nodes into single array, called the *element displacement array*, which can be expressed in either global or local coordinates as

$$\underline{\hat{q}} = \left\{ \begin{matrix} \underline{\hat{q}}_1 \\ \underline{\hat{q}}_2 \end{matrix} \right\}, \quad \underline{\hat{q}}^* = \left\{ \begin{matrix} \underline{\hat{q}}_1^* \\ \underline{\hat{q}}_2^* \end{matrix} \right\}. \tag{10.69}$$

The relationship between these two arrays follows from eq. (10.65) as

$$\underline{\hat{q}} = \left\{ \begin{matrix} \hat{q}_1 \\ \hat{q}_2 \end{matrix} \right\} = \begin{bmatrix} \underline{\underline{\hat{R}}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\hat{R}}} \end{bmatrix} \left\{ \begin{matrix} \hat{q}_1^* \\ \hat{q}_2^* \end{matrix} \right\} = \underline{\underline{\hat{T}}}\, \underline{\hat{q}}^*, \tag{10.70}$$

where $\underline{\underline{0}}$ indicates a 2×2 null matrix, and the element coordinate transformation matrix, $\underline{\underline{\hat{T}}}$, is defined as

$$\underline{\underline{\hat{T}}} = \begin{bmatrix} \underline{\underline{\hat{R}}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\hat{R}}} \end{bmatrix} \tag{10.71}$$

In view eq. (A.41), $\underline{\underline{\hat{R}}}$ is an orthogonal matrix, and therefore, the element coordinate transformation matrix inherits the same property, $\underline{\underline{\hat{T}}}\,\underline{\underline{\hat{T}}}^T = I$. Consequently, it is possible to invert eq. (10.70) to find

$$\underline{\hat{q}}^* = \underline{\underline{\hat{T}}}^{-1}\, \underline{\hat{q}} = \underline{\underline{\hat{T}}}^T\, \underline{\hat{q}}. \tag{10.72}$$

10.7.3 Element elongation and force

Once the kinematics of an element are defined, it becomes possible to evaluate its elongation. From examination of fig. 10.23, the elongation, $\hat{e}$, of the bar is simply $\hat{e} = \hat{u}_2^* - \hat{u}_1^*$. It will be convenient to recast this expression as an array operation by writing $\hat{e} = \hat{u}_2^* - \hat{u}_1^* = \left\{-1, 0, 1, 0\right\}\, \underline{\hat{q}}^* = \underline{\hat{b}}^{*T}\, \underline{\hat{q}}^*$, where $\underline{\hat{b}}^* = \left\{-1, 0, 1, 0\right\}^T$ is an array that relates the element elongation to the nodal displacements, $\underline{\hat{q}}^*$.

Elongations are naturally expressed in terms of the displacement components expressed in the local coordinate system, but it is also possible to express them in terms of displacement components resolved in the global coordinate system as

$$\hat{e} = \underline{\hat{b}}^{*T}\, \underline{\hat{q}}^* = \underline{\hat{b}}^{*T}\, \underline{\underline{\hat{T}}}^T\, \underline{\hat{q}} = \underline{\hat{b}}^T\, \underline{\hat{q}}, \tag{10.73}$$

where eq. (10.72) is used to calculate the nodal displacement components expressed in the local coordinate system in terms of their global coordinate counterparts. Array $\underline{\hat{b}}$ is defined as $\underline{\hat{b}} = \underline{\underline{\hat{T}}}\, \underline{\hat{b}}^*$, where $\underline{\underline{\hat{T}}}$ is given by eq. (10.71).

The bar force, $\hat{F}$, is obtained by multiplying the elongation by the bar's axial stiffness to find

$$\hat{F} = \frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}\, \hat{e} = \frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}\, \underline{\hat{b}}^{*T}\, \underline{\hat{q}}^* = \frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}\, \underline{\hat{b}}^T\, \underline{\hat{q}}. \tag{10.74}$$

10.7.4 Element strain energy and stiffness matrix

Next, the strain energy stored in a typical bar of the truss is evaluated. Because the stiffness of the bar is $\hat{E}\hat{\mathcal{A}}/\hat{L}$, eq. (10.21) yields the element strain energy as

$$\hat{A} = \frac{1}{2}\frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}\,\hat{e}^2 = \frac{1}{2}\frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}\,\hat{e}\cdot\hat{e} = \frac{1}{2}\frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}(\underline{b}^{*T}\underline{\hat{q}}^*)^T(\underline{b}^{*T}\underline{\hat{q}}^*),$$

where the elongation is expressed in terms of the nodal displacement components in local coordinates using the first part of eq. (10.73). Regrouping the terms then leads to

$$\hat{A} = \frac{1}{2}\,\underline{\hat{q}}^{*T}\left[\frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}(\underline{b}^*\underline{b}^{*T})\right]\underline{\hat{q}}^* = \frac{1}{2}\,\underline{\hat{q}}^{*T}\underline{\underline{\hat{k}}}^*\underline{\hat{q}}^*,$$

where $\underline{\underline{\hat{k}}}^*$ is the *element stiffness matrix* expressed in the local coordinate system. Since $\underline{\hat{b}} = \left\{-1, 0, 1, 0\right\}^{*T}$, the entries in this matrix become

$$\underline{\underline{\hat{k}}}^* = \frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}(\underline{b}^*\underline{b}^{*T}) = \frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{10.75}$$

It is also possible to evaluate the components of the same stiffness matrix expressed in the global coordinate system. Equation (10.70) expresses the nodal displacement components resolved in the local coordinate system in terms of their global coordinate counterparts as $\underline{\hat{q}}^* = \underline{\underline{\hat{T}}}^T\underline{\hat{q}}$. It then follows that

$$\hat{A} = \frac{1}{2}\,\underline{\hat{q}}^{*T}\underline{\underline{\hat{k}}}^*\underline{\hat{q}}^* = \frac{1}{2}\,(\underline{\hat{q}}^T\underline{\underline{\hat{T}}})\underline{\underline{\hat{k}}}^*(\underline{\underline{\hat{T}}}^T\underline{\hat{q}}) = \frac{1}{2}\,\underline{\hat{q}}^T(\underline{\underline{\hat{T}}}\,\underline{\underline{\hat{k}}}^*\underline{\underline{\hat{T}}}^T)\underline{\hat{q}} = \frac{1}{2}\,\underline{\hat{q}}^T\underline{\underline{\hat{k}}}\,\underline{\hat{q}}, \tag{10.76}$$

where $\underline{\underline{\hat{k}}} = \underline{\underline{\hat{T}}}\,\underline{\underline{\hat{k}}}^*\underline{\underline{\hat{T}}}^T$ stores the components of the element stiffness matrix expressed in the global coordinate system. Simple algebra reveals that

$$\underline{\underline{\hat{k}}} = \frac{\hat{E}\hat{\mathcal{A}}}{\hat{L}}\left[\begin{array}{cc|cc} \cos^2\hat{\theta} & \sin\hat{\theta}\cos\hat{\theta} & -\cos^2\hat{\theta} & -\sin\hat{\theta}\cos\hat{\theta} \\ \sin\hat{\theta}\cos\hat{\theta} & \sin^2\hat{\theta} & -\sin\hat{\theta}\cos\hat{\theta} & -\sin^2\hat{\theta} \\ \hline -\cos^2\hat{\theta} & -\sin\hat{\theta}\cos\hat{\theta} & \cos^2\hat{\theta} & \sin\hat{\theta}\cos\hat{\theta} \\ -\sin\hat{\theta}\cos\hat{\theta} & -\sin^2\hat{\theta} & \sin\hat{\theta}\cos\hat{\theta} & \sin^2\hat{\theta} \end{array}\right]. \tag{10.77}$$

In eqs. (10.75) and (10.77), the 4×4 element stiffness matrix is partitioned into four, 2×2 sub-matrices. The first two rows and columns of these matrices represent the stiffnesses associated with the two degrees of freedom, *i.e.*, the two displacement components, at *Node 1* of the element, whereas the last two rows and columns represent those associated with the two degrees of freedom at *Node 2* of the element. In eq. (10.75) the degrees of freedom are displacement components resolved in the *local* coordinate system, whereas in eq. (10.77) the degrees of freedom are resolved in the *global* coordinate system.

Not unexpectedly, the expression for element stiffness matrix expressed in the local system, eq. (10.75), is far simpler than its counterpart expressed in the global system, eq. (10.77). Why then is it desirable to derive element stiffness matrices in the global system? This question can be answered by considering fig. 10.22: bars

1, 3, 5, and 6 all connect to a single node, node 3. The local coordinate systems of these four bars are all different, and hence, the four corresponding element stiffness matrices expressed in their individual local systems are associated with local orthogonal displacement components resolved in four different systems. In contrast, when the four element stiffness matrices are expressed in the global system, they are associated with orthogonal displacement components all resolved in the same global system. This latter form of the stiffness matrix will considerable simplify the assembly procedure described below.

It is also important to note that $\hat{A}$ is a positive-definite quantity because the strain energy density for an axially loaded bar, see eq. (10.34), is positive-definite.

10.7.5 Element external potential and load array

When dealing with trusses, it is assumed that the externally applied loads act only at the nodes. Considering the single bar element depicted in fig. 10.23, let $\underline{\hat{F}}_1$ and $\underline{\hat{F}}_2$ be concentrated loads acting at local nodes, *Node 1* and *Node 2*, respectively. These two forces are resolved in the global coordinate system as $\underline{\hat{F}}_1 = \hat{f}_1 \bar{\imath}_1 + \hat{g}_1 \bar{\imath}_2$ and $\underline{\hat{F}}_2 = \hat{f}_2 \bar{\imath}_1 + \hat{g}_2 \bar{\imath}_2$, respectively, and their potential is easily evaluated using eq. (10.13) to find

$$\hat{\Phi} = -\left[\hat{f}_1 \hat{u}_1 + \hat{g}_1 \hat{v}_1\right] - \left[\hat{f}_2 \hat{u}_2 + \hat{g}_2 \hat{v}_2\right] = -\left\{\hat{f}_1, \hat{g}_1, \hat{f}_2, \hat{g}_2\right\} \underline{\hat{q}} = -\underline{\hat{f}}^T \underline{\hat{q}}, \quad (10.78)$$

where the *element load array* is defined as

$$\underline{\hat{f}} = \left\{\hat{f}_1, \hat{g}_1, \hat{f}_2, \hat{g}_2\right\}^T. \quad (10.79)$$

To illustrate this, consider a concentrated load of magnitude P and orientation α with respect to the horizontal, acting at *Node 1* of a bar element. The corresponding element load array is then $\hat{f}_1 = P \cos\alpha$, $\hat{g}_1 = P \sin\alpha$, and $\hat{f}_2 = \hat{g}_2 = 0$. If the same load is applied at *Node 2* instead, the element load array is $\hat{f}_1 = \hat{g}_1 = 0$, $\hat{f}_2 = P \cos\alpha$, and $\hat{g}_2 = P \sin\alpha$.

It is also possible to include the weight of the bar as an externally applied force. Let $\hat{m}$ be the bar's mass and $\hat{m}g$ the corresponding weight acting at its center of mass. For a homogeneous bar, the center of mass will be at its geometric center, and it makes sense to apply half of the weight at each of the element's two nodes. For example, if gravity acts along the negative axis $\bar{\imath}_2$ direction, the corresponding element load array is $\hat{f}_1 = 0$, $\hat{g}_1 = -\hat{m}g/2$, $\hat{f}_2 = 0$, and $\hat{g}_2 = -\hat{m}g/2$.

10.7.6 Assembly procedure

In the previous sections, attention is focused on a single, generic bar to determine its *element* stiffness matrix, see eq. (10.77), and *element* load array, see eq. (10.79). These two quantities are obtained from the element strain energy and external potential, respectively. In this section, attention shifts to the overall truss problem to determine the *global stiffness matrix* and *global load array*. These two quantities will be obtained from the system's total strain energy and total external potential, respectively. Since both strain energy and external potential are scalar quantities, their combined total will be evaluated simply by summing up the contributions from the individual elements.

The total strain energy, A, stored in the truss is the sum of the contributions of all bars. In eq. (10.76), the strain energy of a single, generic bar is $\hat{A}$, and this notation is not ambiguous because only a single element is considered. It now becomes necessary, however, to add the element identification using the subscript $(.)_{(i)}$ introduced earlier. Summing over all elements yields

$$A = \sum_{i=1}^{N_e} \hat{A}_{(i)} = \frac{1}{2} \sum_{i=1}^{N_e} \underline{\hat{q}}^T_{(i)} \underline{\underline{\hat{k}}}_{(i)} \underline{\hat{q}}_{(i)}, \tag{10.80}$$

where N_e is the number of bars in the truss ($N_e = 11$ for the truss illustrated in fig. 10.23). In this case, it is also necessary to add the element identification subscript to both the element stiffness matrix, $\underline{\underline{\hat{k}}}_{(i)}$, and the nodal displacement array, $\underline{\hat{q}}_{(i)}$.

Equation (10.80) gives the total strain energy in the structure, but it is not easy to manipulate because each term in the sum is expressed in terms of a different set of degrees of freedom. For example, with reference to fig. 10.22, element 8 is connected to global nodes 4 and 6 which are local *Node 1* and *Node 2* for the element, respectively. The element stiffness, $\underline{\underline{\hat{k}}}_{(8)}$, is defined in terms of these global nodes, see eq. (10.77), and the corresponding element displacement array is $\underline{\hat{q}}^T_{(8)} = \left\{ \underline{\hat{q}}^T_1, \underline{\hat{q}}^T_2 \right\} = \left\{ \underline{q}^T_4, \underline{q}^T_6 \right\} = \left\{ u_4, v_4, u_6, v_6 \right\}^T$.

To remedy this situation, the *connectivity matrix*, $\underline{\underline{C}}_{(i)}$, for the $i^{\rm th}$ element is introduced. This matrix is designed to extract the element displacement array, $\underline{\hat{q}}_{(i)}$, from the global displacement array, $\underline{q}$, defined by eq. (10.62). This operation can be written as

$$\underline{\hat{q}}_{(i)} = \underline{\underline{C}}_{(i)} \underline{q}. \tag{10.81}$$

To best understand this abstract relationship, consider a specific element of the truss, say bar 6, as shown in fig. 10.22. Its local nodes, *Node 1* and *Node 2*, are associated with the global node numbers 3 and 5, respectively, so that $\underline{\hat{q}}_1 = \underline{q}_3$ and $\underline{\hat{q}}_2 = \underline{q}_5$. The element displacement array, $\underline{\hat{q}}_{(6)}$, can thus be written as

$$\underline{\hat{q}}_{(6)} = \left\{ \begin{matrix} \underline{\hat{q}}_1 \\ \underline{\hat{q}}_2 \end{matrix} \right\}_{(6)} = \left\{ \begin{matrix} \underline{q}_3 \\ \underline{q}_5 \end{matrix} \right\} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{I}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{I}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \left\{ \begin{matrix} \underline{q}_1 \\ \underline{q}_2 \\ \underline{q}_3 \\ \underline{q}_4 \\ \underline{q}_5 \\ \underline{q}_6 \\ \underline{q}_7 \end{matrix} \right\} = \underline{\underline{C}}_{(6)} \underline{q},$$

where $\underline{\underline{0}}$ and $\underline{\underline{I}}$ represent the 2×2 null and identity matrices, respectively. The connectivity matrix, $\underline{\underline{C}}_{(6)}$, is called a *Boolean matrix* because its entries consist solely of 0's and 1's. Matrix $\underline{\underline{C}}_{(6)}$ establishes the connections of bar 6 within the truss by indicating the nodes to which this bar is connected, and this explains its being named a "connectivity matrix."

Expressing the element nodal displacement arrays, $\underline{\hat{q}}_{(i)}$, in terms of the global displacement array, $\underline{q}$, with the help of eq. (10.81), the total strain energy of the truss given by eq. (10.80) now becomes

$$A = \frac{1}{2} \sum_{i=1}^{N_e} \left(\underline{q}^T \underline{\underline{C}}_{(i)}^T \right) \underline{\underline{\hat{k}}}_{(i)} \left(\underline{\underline{C}}_{(i)} \underline{q} \right) = \frac{1}{2} \, \underline{q}^T \left[\sum_{i=1}^{N_e} \underline{\underline{C}}_{(i)}^T \underline{\underline{\hat{k}}}_{(i)} \underline{\underline{C}}_{(i)} \right] \underline{q}.$$

This expression can be simplified to

$$A = \frac{1}{2} \, \underline{q}^T \underline{\underline{K}} \, \underline{q}, \tag{10.82}$$

by defining the *global stiffness matrix*, $\underline{\underline{K}}$, as

$$\underline{\underline{K}} = \sum_{i=1}^{N_e} \underline{\underline{C}}_{(i)}^T \underline{\underline{\hat{k}}}_{(i)} \underline{\underline{C}}_{(i)}. \tag{10.83}$$

The potential of the externally applied loads, Φ, is found by adding the contributions of all bars,

$$\Phi = \sum_{i=1}^{N_e} \hat{\Phi}_{(i)} = - \sum_{i=1}^{N_e} \underline{\hat{q}}_{(i)}^T \underline{\hat{f}}_{(i)}, \tag{10.84}$$

where $\underline{\hat{f}}_{(i)}$ is the load array for the i^{th} element, as defined by eq. (10.79) for a generic bar element. Here again, it is convenient to use the connectivity matrix defined in eq. (10.81) to evaluate the potential,

$$\Phi = - \sum_{i=1}^{N_e} \left(\underline{\underline{C}}_{(i)} \underline{q} \right)^T \underline{\hat{f}}_{(i)} = - \underline{q}^T \left\{ \sum_{i=1}^{N_e} \underline{\underline{C}}_{(i)}^T \underline{\hat{f}}_{(i)} \right\}.$$

This expression can be simplified to

$$\Phi = - \underline{q}^T \underline{Q}, \tag{10.85}$$

by defining the *global load array*, $\underline{Q}$, as

$$\underline{Q} = \sum_{i=1}^{N_e} \underline{\underline{C}}_{(i)}^T \underline{\hat{f}}_{(i)}. \tag{10.86}$$

Finally, the total potential energy, Π, of the truss is obtained by adding the potential of the external loads, eq. (10.85), to the total strain energy, eq. (10.82), to find

$$\Pi = A + \Phi = \frac{1}{2} \, \underline{q}^T \underline{\underline{K}} \, \underline{q} - \underline{q}^T \underline{Q}. \tag{10.87}$$

This compact expression for the total potential energy of the complete system is only possible because the matrix notation encapsulates the nodal and element quantities in arrays and matrices.

The total strain energy is a quadratic form of the generalized coordinates, whereas the potential of the externally applied loads is a linear form of the same variables. It should also be noted that the strain energy of the truss is positive-definite because it is the sum of the positive-definite strain energies for each bar.

10.7.7 Alternative description of the assembly procedure

The assembly procedure described in terms of the connectivity matrix defined in eq. (10.81) is formally correct, but it is not easy to understand nor is it computationally efficient for realistic trusses with many members and nodes. The connectivity matrix, $\underline{\underline{C}}_{(i)}$, has four rows and $2N$ columns, where N is the total number of nodes. For large trusses consisting of many bars and nodes, this matrix becomes very large with a total of $8N$ entries, and yet, only four entries have a unit value while all $(8N - 4)$ others are zero. Furthermore, the evaluation of the global stiffness matrix involves a triple matrix product for each elements, see eq. (10.83). These become increasingly expensive to perform as the problem size increases, and they also are very wasteful because most operations actually are multiplications by zero.

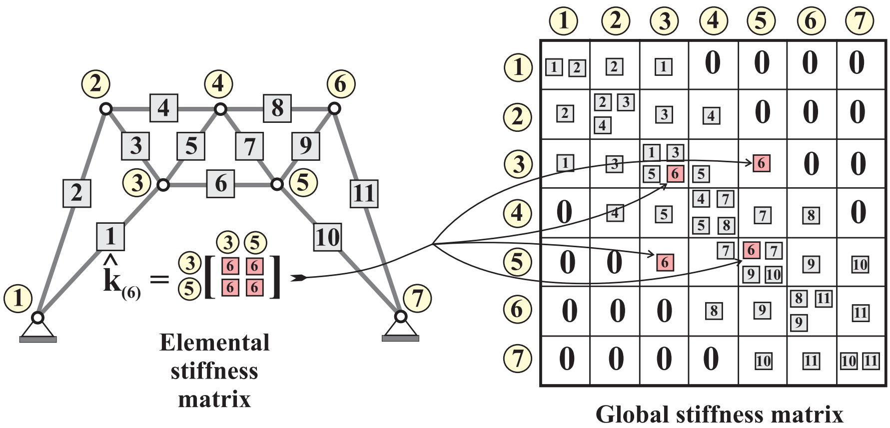

Fig. 10.24. Illustration of the assembly procedure.

It is possible to give a more graphical visualization of the assembly process. Figure 10.24 shows the 11-bar, 7-node truss under consideration. It also depicts the global stiffness matrix; the 7 rows and columns in the matrix are labeled with their corresponding node numbers. Each node has two degrees of freedom (the horizontal and vertical displacement components at that node), so each of the 49 entries is actually a 2×2 matrix and the size of the global stiffness matrix itself is 14×14.

Consider now a typical bar of the truss, say bar 6. Its local nodes, *Node 1* and *Node 2*, are associated with the global node numbers 3 and 5, respectively. The local stiffness matrix for this bar, $\underline{\underline{\hat{k}}}_{(6)}$, can be partitioned into four 2×2 matrices, as shown in eq. (10.77). Bar 6 is connected to global nodes 3 and 5, and therefore, the four sub-matrices of the local stiffness matrix can simply be added to entries $\underline{\underline{K}}(3,3)$, $\underline{\underline{K}}(5,5)$, $\underline{\underline{K}}(3,5)$, and $\underline{\underline{K}}(5,3)$ in the global stiffness matrix, as indicated by the arrows in fig. 10.24. In this discussion, the notation $\underline{\underline{K}}(i,j)$ refers to the 2×2 sub-matrix that appears as the (i,j) entry of the matrix depicted in fig. 10.24.

This procedure is repeated for each bar of the truss to give the final result shown in fig. 10.24. This figure requires careful interpretation. Each of the element numbers shown in square boxes defines a 2×2 sub-matrix extracted from the corresponding element stiffness matrix. These 2×2 sub-matrices are added together to produce the final result in the global stiffness matrix.

Another way to look at the same process is to consider the fully assembled global stiffness matrix in fig. 10.24. Diagonal entry $\underline{\underline{K}}(2,2)$ collects contributions from bars 2, 3, and 4 because these three bars are all physically connected to node 2. Similarly, diagonal entry $\underline{\underline{K}}(5,5)$ collects contributions from bars 6, 7, 9, and 10 because these four bars all connect to node 5.

The off-diagonal entries in the global stiffness matrix can be interpreted in a similar manner. For instance, entries $\underline{\underline{K}}(1,3)$ and $\underline{\underline{K}}(3,1)$ each collect the single contribution stemming from bar 1, because bar 1 connects nodes 1 and 3. Similarly, bar 8 connects nodes 4 and 6, and is the sole contributor to entries $\underline{\underline{K}}(4,6)$ and $\underline{\underline{K}}(6,4)$ in the global stiffness matrix. It is important to note that the symmetry of the local stiffness matrix, see eq. (10.77), and the symmetry of the assembly process, result in the global stiffness matrix also being a symmetric matrix.

At completion of the assembly process, many entries of the global stiffness matrix remain empty or null. For instance, entries $\underline{\underline{K}}(2,6) = \underline{\underline{K}}(6,2) = 0$, because no bar directly connects nodes 2 and 6. Similarly, $\underline{\underline{K}}(1,4) = \underline{\underline{K}}(4,1) = 0$ because nodes 1 and 4 are not directly connected by a bar. For the node numbering sequence selected in fig. 10.24, the non-zero entries in the global stiffness matrix concentrate near the diagonal, and the resulting matrix is called a "banded matrix." It should be obvious that other node and/or element numbering could lead to a more dispersed arrangement of the non-zero entries.

This alternative description of the element assembly process is more graphical than the initial description based on connectivity matrices. When implementing the approach in a computer program, this process of simply adding the entries of the element stiffness matrices to corresponding entries in the global stiffness matrix is the preferred approach, because it is easy to program and efficient to execute.

10.7.8 Derivation of the governing equations

The total potential energy of the truss is given by eq. (10.87), and application of the principle of minimum total potential energy, eq. (10.17), now implies

$$\frac{\partial \Pi}{\partial \underline{q}} = \frac{\partial}{\partial \underline{q}} \left(\frac{1}{2}\, \underline{q}^T \underline{\underline{K}}\, \underline{q} - \underline{q}^T \underline{Q} \right) = \underline{\underline{K}}\, \underline{q} - \underline{Q} = 0. \qquad (10.88)$$

To compute the derivative of the total potential energy, eqs. (A.29) and (A.27) are used to evaluate the derivatives of the strain energy and potential of the externally applied loads, respectively. Appendix A.2.9 also proves that this solution is a minimum.

The governing equation of the system take the form of a linear system of equations,

$$\underline{\underline{K}}\,\underline{q} = \underline{Q}. \tag{10.89}$$

The global stiffness matrix, $\underline{\underline{K}}$, is computed from the given geometry and material properties of the truss, while the global load array, $\underline{Q}$, stems from the external loads applied at the nodes. The unknown quantities are the nodal displacements stored in array $\underline{q}$. The solution of eq. (10.89) yields the displacements at all joints of the truss.

The approach presented here is an element-oriented version of the displacement or stiffness method described in section 4.3.2. Each line of the matrix relationship, eq. (10.89), represents an equilibrium equation of the problem. For instance, the equation obtained by extracting the first line eq. (10.89) represents the equilibrium equation obtained by imposing the vanishing of the sum of the horizontal forces acting at node 1, whereas the second equation corresponds to the vertical equilibrium equation at the same node.

10.7.9 Solution procedure

Efficient algorithms are available for the solution of large sets of linear equations using computers. At this point, however, the linear system given in eq. (10.89) cannot be solved because the global stiffness matrix is singular.

This situation arises because the element stiffness matrices that make up the global stiffness matrix are each singular. Examination of eq. (10.75) reveals that the element stiffness matrix contains two rows of zeros and furthermore, the third row is simply -1 times the first row. Consequently, this 4×4 matrix is three times singular: it is of rank 1, presents a rank deficiency of 3, and has a zero determinant. The element stiffness matrix in global coordinates given by eq. (10.77) has the same rank deficiency because it is the same matrix, expressed in a different coordinate system. Finally, the global stiffness matrix, $\underline{\underline{K}}$, also presents a rank deficiency of 3, and because it is three times singular, the global stiffness matrix cannot be inverted.

Calculation of the eigenvectors and eigenvalues[2] of the element stiffness matrix, $\hat{\underline{\underline{k}}}$, given by eq. (10.77), reveals more information about this rank deficiency. The unit eigenvectors of this matrix are given by the arrays

$$\underline{n}_1 = \frac{1}{\sqrt{2}}\left\{\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix}\right\}, \; \underline{n}_2 = \frac{1}{\sqrt{2}}\left\{\begin{matrix} 0 \\ 1 \\ 0 \\ 1 \end{matrix}\right\}, \; \underline{n}_3 = \frac{1}{\sqrt{2}}\left\{\begin{matrix} \sin\hat{\theta} \\ -\cos\hat{\theta} \\ -\sin\hat{\theta} \\ \cos\hat{\theta} \end{matrix}\right\}, \; \underline{n}_4 = \frac{1}{\sqrt{2}}\left\{\begin{matrix} \cos\hat{\theta} \\ \sin\hat{\theta} \\ -\cos\hat{\theta} \\ -\sin\hat{\theta} \end{matrix}\right\},$$

[2] See appendix A.2.4 for details on the calculation of eigenvalues and eigenvectors of symmetric, positive-definite matrices.

and the corresponding eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and $\lambda_4 = 2\hat{E}\hat{\mathcal{A}}/\hat{L}$, respectively.

The first three eigenvectors, $\underline{n}_1$, $\underline{n}_2$, and $\underline{n}_3$, represent the horizontal rigid body translation of the bar, its vertical rigid body translation, and its rigid body rotation, respectively. The three associated eigenvalues vanish. By definition, rigid body motions create no deformation or straining of the element, and hence, no strain energy is associated with rigid body modes. Using eq. (10.76), the strain energy associated with the first eigenvector is $\hat{A} = 1/2\ \underline{n}_1^T \underline{\underline{\hat{k}}}\, \underline{n}_1$, and as expected, $\hat{A} = 0$, because the definition of eigenvectors implies $\underline{\underline{\hat{k}}}\, \underline{n}_1 = \lambda_1 \underline{n}_1 = 0$. Using a similar reasoning, it is easy to prove the vanishing of the strain energy associated with each of the three rigid body modes of the element. Clearly, the presence of three rigid body modes for the structure implies the rank deficiency of 3 for the element stiffness matrix. The entire truss also presents three rigid body modes, and hence, the global stiffness matrix also features a rank deficiency of 3.

The physical interpretation of this situation is that boundary conditions have not yet been applied to the truss, which is still free to translate and rotate in plane $(\bar{\imath}_1, \bar{\imath}_2)$. Figure 10.22 shows that nodes 1 and 7 are pinned to the ground, preventing any rigid body motion of the truss. These conditions, however, are not reflected in the global stiffness matrix, $\underline{\underline{K}}$, given in eq. (10.83).

The boundary conditions require the vanishing of the displacements at nodes 1 and 7: $\underline{q}_1 = \underline{q}_7 = 0$. Consequently, the reaction forces arising at nodes 1 and 7, denoted $\underline{R}_1$ and $\underline{R}_7$, respectively, should be treated as externally applied forces. The equilibrium equations associated with those two nodes correspond to the first two and last two rows of the global stiffness matrix illustrated in fig. 10.24. These equations can be removed from eq. (10.89) and written separately as

$$\underline{\underline{K}}(1,1)\underline{q}_1 + \underline{\underline{K}}(1,2)\underline{q}_2 + \underline{\underline{K}}(1,3)\underline{q}_3 = \underline{R}_1, \tag{10.90a}$$

$$\underline{\underline{K}}(7,5)\underline{q}_5 + \underline{\underline{K}}(7,6)\underline{q}_6 + \underline{\underline{K}}(7,7)\underline{q}_7 = \underline{R}_7, \tag{10.90b}$$

where the indices on $\underline{\underline{K}}$ denote nodes and not degrees of freedom (therefore these are 2×2 sub-matrices from the global stiffness matrix). This leaves $14 - 4 = 10$ rows remaining in the set of equations. Since the displacements at nodes 1 and 7 vanish, the corresponding terms vanish in eq. (10.90), which can be solved for the unknown reaction forces

$$\underline{R}_1 = \underline{\underline{K}}(1,2)\underline{q}_2 + \underline{\underline{K}}(1,3)\underline{q}_3, \quad \underline{R}_7 = \underline{\underline{K}}(7,5)\underline{q}_5 + \underline{\underline{K}}(7,6)\underline{q}_6. \tag{10.91}$$

Because the displacements at nodes 1 and 7 are zero, the contributions from the terms appearing in the first two and last two columns of the global stiffness matrix vanish. Consequently, the first two and last two columns of $\underline{\underline{K}}$, as well as the first two and last two entries in arrays $\underline{q}$ and $\underline{Q}$ can also be eliminated to create a reduced set of 10 equations that can now be solved for the remaining 10 unknown nodal displacements.

In summary, the boundary conditions can be imposed through the following generalized process. *(1)* Eliminate the rows and columns of the global stiffness matrix corresponding to constrained degrees of freedom to create its reduced counterpart, $\underline{\underline{\bar{K}}}$. *(2)* Eliminate the row of the global displacement array corresponding to constrained degrees of freedom to create its reduced counterpart, $\underline{\bar{q}}$. *(3)* Finally, eliminate the row of the global load array corresponding to constrained degrees of freedom to create its reduced counterpart, $\underline{\bar{Q}}$. The system of equations for the truss then reduces to

$$\underline{\underline{\bar{K}}}\,\underline{\bar{q}} = \underline{\bar{Q}}. \tag{10.92}$$

The reduced stiffness matrix will now be non-singular, and the solution of the problem is found by solving the linear system to find the remaining nodal displacements as $\underline{\bar{q}} = \underline{\underline{\bar{K}}}^{-1}\underline{\bar{Q}}$. Finally, the reactions can be determined from the equations extracted in step 1.

10.7.10 Solution procedure using partitioning

The procedure developed in the previous paragraphs is very descriptive and consists of "eliminating rows and columns" in the global stiffness matrix, global displacement array, and global load array. A more mathematical description of the process is based on partitioning of the same quantities in a manner that allows separate treatment of the constrained and unconstrained nodes.

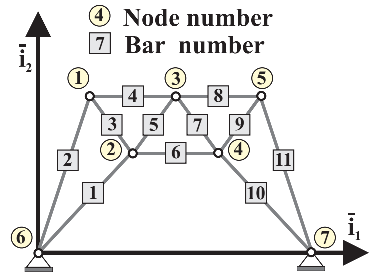

Fig. 10.25. Eleven-bar truss.

First, it should be observed that the node numbering sequence is arbitrary: in fig. 10.22, each node is assigned a number at random. Figure 10.25, shows the same truss, but with a different node numbering for which the two nodes where boundary conditions are to be applied are now numbered as nodes 6 and 7 and are the last in the series.

The global displacement array can now be partitioned into two sub-arrays (using a vertical bar $\{.|.\}$) as follows

$$\underline{q} = \left\{\underline{q}_1^T, \underline{q}_2^T, \underline{q}_3^T, \underline{q}_4^T, \underline{q}_5^T, |\underline{q}_6^T, \underline{q}_7^T\right\}^T = \left\{\underline{q}_u^T, \underline{q}_p^T\right\}^T. \tag{10.93}$$

Array $\underline{q}_u$ is of size N_u and stores the N_u unknown displacements at nodes 1 to 5, while array $\underline{q}_p$ is of size N_p and stores the N_p prescribed displacements at support nodes 6 and 7. For the truss depicted in fig. 10.25, $N_u = 10$ (two displacement components at each of the five nodes numbered 1 to 5), and $N_p = 4$ (two displacement components at both nodes 6 and 7). Figure 10.25 illustrates the case where nodes 6 and 7 have zero prescribed values, *i.e.*, they are pinned to the ground. In some case, the prescribed displacement at a node might be non-zero. For instance, if node 7 are prescribed to move by an amount Δ along axis $\bar{\imath}_1$, then $\underline{q}_7 = \{\Delta, 0\}^T$.

The node numbering sequence is arbitrary, and therefore, the formulation of the problem with the numbering sequence shown in fig. 10.25 is equivalent to that described earlier for the numbering sequence presented in fig. 10.22. It leads to the following partitioned governing equations

$$\begin{bmatrix} \underline{\underline{K}}_{uu} & \underline{\underline{K}}_{up} \\ \underline{\underline{K}}_{up}^T & \underline{\underline{K}}_{pp} \end{bmatrix} \begin{Bmatrix} \underline{q}_u \\ \underline{q}_p \end{Bmatrix} = \begin{Bmatrix} \underline{Q}_u \\ \underline{Q}_p \end{Bmatrix}, \tag{10.94}$$

where subscripts u and p refer to "unconstrained" nodes and nodes with "prescribed" displacements, respectively.

This system of equation corresponds to a partitioned version of the general governing equation for the truss given by eqs. (10.89). In these equations, the global load array is partitioned in the same manner as the global displacement array, *i.e.*,

$$\underline{Q} = \left\{ \underline{Q}_1^T, \underline{Q}_2^T, \underline{Q}_3^T, \underline{Q}_4^T, \underline{Q}_5^T, |\underline{R}_6^T, \underline{R}_7^T \right\}^T = \left\{ \underline{Q}_u^T, \underline{Q}_p^T \right\}^T. \tag{10.95}$$

Array $\underline{Q}_u$ is of size N_u and stores the known forces applied at nodes 1 to 5, while array $\underline{Q}_p$ is of size N_p and stores the reaction forces at nodes 6 and 7. Matrices $\underline{\underline{K}}_{uu}$, $\underline{\underline{K}}_{pp}$, and $\underline{\underline{K}}_{up}$ are of size $(N_u \times N_u)$, $(N_p \times N_p)$, and $(N_u \times N_p)$, respectively.

The first N_u equations of system (10.94) can be rewritten as

$$\underline{\underline{K}}_{uu}\, \underline{q}_u = \underline{Q}_u - \underline{\underline{K}}_{up}\, \underline{q}_p. \tag{10.96}$$

Because the prescribed displacements, $\underline{q}_p$, are known, their contribution is moved to the right-hand side of the equations. The unknown nodal displacements are evaluated as $\underline{q}_u = \underline{\underline{K}}_{uu}^{-1}(\underline{Q}_u - \underline{\underline{K}}_{up}\, \underline{q}_p)$. If the boundary conditions consist solely of nodes rigidly connected to the ground, all prescribed displacement vanish, $\underline{q}_p = 0$, and the system reduces to $\underline{q}_u = \underline{\underline{K}}_{uu}^{-1}\underline{Q}_u$, which is equivalent to eqs. (10.92).

Once the unknown displacements have been evaluated, the last N_p equations of system (10.94) can be rewritten to evaluate the reactions as

$$\underline{Q}_p = \underline{\underline{K}}_{up}^T \underline{q}_u + \underline{\underline{K}}_{pp} \underline{q}_p. \tag{10.97}$$

Here again, all known quantities have been moved to the right-hand side of the equations. If the boundary conditions consist solely of nodes rigidly connected to the ground, all prescribed displacement vanish, $\underline{q}_p = 0$, and $\underline{Q}_p = \underline{\underline{K}}_{up}^T \underline{q}_u$ are the reaction forces at the nodes pinned to the ground.

On the other hand, if some nodal displacements are prescribed to non-vanishing values, eq. (10.96) can still be used to find the unconstrained nodal displacements, $\underline{q}_u$, and eq. (10.97) then yields the reaction forces, $\underline{Q}_p$. These are still reaction forces because they arise from either zero or non-zero prescribed nodal displacements. Those acting at nodes where the displacements are prescribed are sometimes called the "driving forces," *i.e.*, the forces that must be applied at a node to achieve the prescribed displacement. Nodes with prescribed displacements can also be used to represent misalignments in the supports due to non-ideal truss geometry.

In the partitioned system of eq. (10.94), the reduced stiffness matrix, $\underline{\underline{K}}_{uu}$, is obtained by eliminating the last N_p rows and columns of the global stiffness matrix and is equivalent to the reduced stiffness matrix, $\underline{\underline{\bar{K}}}$, in eq. (10.92). The present approach, which is based on partitioning of the reordered system of equations, gives a rigorous justification of the procedure introduced in the previous section.

It should also be noted that both approaches to enforcing the boundary conditions may require re-numbering of the rows and columns in the original set of equations. While tedious to do by hand, such manipulations are easily handled using computer programs.

10.7.11 Post-processing

The last step in the solution process is to determine the bar elongations and forces. The elongation of a bar is given by eq. (10.73) as $\hat{e} = \underline{\hat{b}}^T \underline{\hat{q}}$. For the i^{th} element, this can be written in a formal manner as

$$\hat{e}_{(i)} = \underline{\hat{b}}^T_{(i)} \underline{\hat{q}}_{(i)} = \underline{\hat{b}}^T_{(i)} \underline{\underline{C}}_{(i)} \underline{q}, \tag{10.98}$$

where eq. (10.81) is used to express the element nodal displacement array, $\underline{\hat{q}}_{(i)}$, in terms of the global displacement array, $\underline{q}$. Once the bar's elongation is obtained, the constitutive law is used to evaluate the bar force as

$$\hat{F}_{(i)} = \frac{\hat{E}_{(i)} \hat{\mathcal{A}}_{(i)}}{\hat{L}_{(i)}} \, \hat{e}_{(i)}. \tag{10.99}$$

To illustrate the process, consider bar 6 of the truss shown in fig. 10.22; for this bar, local *Node 1* and *Node 2* correspond global nodes 3 and 5, respectively. Because this bar orientation is parallel to axis $\bar{\imath}_1$, $\hat{\theta}_{(6)} = 0$, the element coordinate transformation matrix, $\underline{\underline{\hat{T}}}_{(6)}$ becomes an identity matrix, and $\underline{\hat{b}} = \underline{\underline{\hat{T}}}\,\underline{\hat{b}}^* = \underline{\hat{b}}^*$. The bar elongation then becomes

$$\hat{e}_{(6)} = \underline{\hat{b}}^T_{(6)} \underline{\underline{C}}_{(6)} \underline{q} = \underline{\hat{b}}^T_{(6)} \begin{Bmatrix} \underline{\hat{q}}_3 \\ \underline{\hat{q}}_5 \end{Bmatrix} = \underline{\hat{b}}^{*T} \begin{Bmatrix} \underline{\hat{q}}_3 \\ \underline{\hat{q}}_5 \end{Bmatrix} = \{-1, 0, 1, 0\} \begin{Bmatrix} \underline{\hat{q}}_3 \\ \underline{\hat{q}}_5 \end{Bmatrix} = -u_3 + u_5.$$

The corresponding bar force is now

$$\hat{F}_{(6)} = \frac{\hat{E}_{(6)} \hat{\mathcal{A}}_{(6)}}{\hat{L}_{(6)}} \, \hat{e}_{(6)} = \frac{\hat{E}_{(6)} \hat{\mathcal{A}}_{(6)}}{\hat{L}_{(6)}} \, (-u_3 + u_5).$$

Example 10.7. Pentagonal truss revisited

The finite element formulation will be applied to the pentagonal truss depicted in fig. 10.26 and analyzed previously in example 10.6. All bars have the same elastic modulus, E, and cross-sectional area, $\mathcal{A}$. This five node, ten-bar, hyperstatic truss involves a total of 10 generalized coordinates, 2 displacement components at each of the five nodes. Three displacement components are prescribed to zero: the vertical displacement component at node 4 and the two displacement components at node 5. These three constraints will eliminate the three rigid body motions of this planar truss. To facilitate partitioning in the solution procedure, the nodes are numbered as indicated in fig. 10.26: the nodes with constraints, *i.e.*, nodes 4 and 5, appear last. The global coordinate system, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2)$, is also shown in the figure.

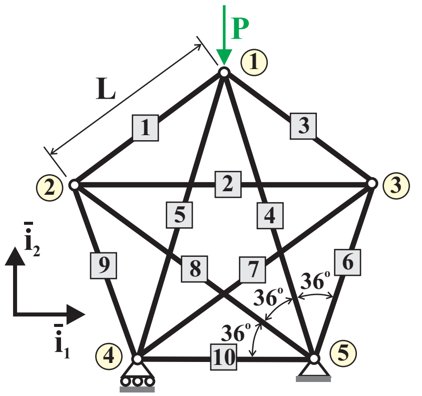

Fig. 10.26. Pentagonal truss with nodes and members defined for analysis using finite element approach.

The local coordinates for each element must be defined with respect to *Node 1* and *Node 2* for the element. Table 10.1 lists these nodes for each element, provides their orientation angle, $\hat{\theta}$, and their length, $\hat{L}$.

Table 10.1. Definition of local nodes and element geometry for pentagonal truss.

Element	*Node 1*	*Node 2*	$\hat{\theta}$	$\hat{L}$
1	2	1	$36°$	L
2	2	3	$0°$	$2L\cos 36°$
3	1	3	$-36°$	L
4	1	5	$-72°$	$2L\cos 36°$
5	1	4	$-108°$	$2L\cos 36°$
6	5	3	$72°$	L
7	4	3	$36°$	$2L\cos 36°$
8	2	5	$-36°$	$2L\cos 36°$
9	4	2	$108°$	L
10	4	5	$0°$	L

Based on the data listed in table 10.1, the stiffness matrices for each element is computed using eq. (10.77). The non-dimensional element stiffness matrices, defined as $\underline{\underline{\bar{k}}}_{(i)} = \underline{\underline{\hat{k}}}_{(i)} L_{(i)}/E\mathcal{A}$, are given here for the four bars connected to node 2,

$$\underline{\underline{\bar{k}}}_{(1)} = \begin{bmatrix} 0.65 & 0.48 & -0.65 & -0.48 \\ 0.48 & 0.35 & -0.48 & -0.35 \\ -0.65 & -0.48 & 0.65 & 0.48 \\ -0.48 & -0.35 & 0.48 & 0.35 \end{bmatrix}, \quad \underline{\underline{\bar{k}}}_{(2)} = \begin{bmatrix} 1.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},$$

$$\underline{\underline{\bar{k}}}_{(8)} = \begin{bmatrix} 0.65 & -0.48 & -0.65 & 0.48 \\ -0.48 & 0.35 & 0.48 & -0.35 \\ -0.65 & 0.48 & 0.65 & -0.48 \\ 0.48 & -0.35 & -0.48 & 0.35 \end{bmatrix}, \quad \underline{\underline{\bar{k}}}_{(9)} = \begin{bmatrix} 0.10 & -0.29 & -0.10 & 0.29 \\ -0.29 & 0.90 & 0.29 & -0.90 \\ -0.10 & 0.29 & 0.10 & -0.29 \\ 0.29 & -0.90 & -0.29 & 0.90 \end{bmatrix}.$$

During the assembly process, these four element matrices will all contribute to the entries in the global stiffness matrix corresponding to the two degrees of freedom at node 2. Given the node numbering shown in fig. 10.26, the top left 2×2 sub-matrix from $\underline{\underline{\hat{k}}}_{(1)}$, $\underline{\underline{\hat{k}}}_{(2)}$, and $\underline{\underline{\hat{k}}}_{(8)}$, and the lower right 2×2 sub-matrix in $\underline{\underline{\hat{k}}}_{(9)}$ will be added together to form the 2×2 sub-matrix in $\underline{\underline{K}}$ for node 2. Adding the various contributions, this sub-matrix becomes

$$\frac{E\mathcal{A}}{L} \begin{bmatrix} 1.7725 & -0.1123 \\ -0.1123 & 1.4635 \end{bmatrix}.$$

The other entries in the global stiffness matrix are constructed in the same manner.

The boundary conditions impose constraints on the degrees of freedom at nodes 4 and 5. At node 5, both horizontal and vertical displacement components must vanish, $u_5 = v_5 = 0$, whereas at node 4, the sole vertical component vanishes, $v_4 = 0$. In view of the node numbering depicted fig. 10.26, those constrained degrees of freedom correspond to the last 3 entries of the global displacement array. The global equations are partitioned into the form given in eq. (10.94) to give

$$\begin{bmatrix} \underline{\underline{K}}_{uu} & \underline{\underline{K}}_{up} \\ \underline{\underline{K}}_{up}^T & \underline{\underline{K}}_{pp} \end{bmatrix} \begin{Bmatrix} \underline{q}_u \\ \underline{q}_p \end{Bmatrix} = \begin{Bmatrix} \underline{Q}_u \\ \underline{Q}_p \end{Bmatrix}, \tag{10.100}$$

where $\underline{\underline{K}}_{uu}$, $\underline{\underline{K}}_{up}$ and $\underline{\underline{K}}_{pp}$, of size (7×7), (7×3), and (3×3), respectively, define a partition of the global stiffness matrix. Array $\underline{q}_u$, of size (7×1), stores the 7 unconstrained degrees of freedom, while array $\underline{q}_p$, of size (3×1), stores degrees of freedom u_5, v_5, and v_4. It follows that the boundary conditions of the problem imply that $\underline{q}_p = 0$. Array $\underline{Q}_u$, of size (7×1), stores the externally loads applied at the unconstrained nodes, and array $\underline{Q}_p$, of size (3×1), stores the reaction forces at the constrained nodes.

Since $\underline{q}_p = 0$, eq. (10.96) can be solved for the unconstrained nodal displacements as

$$\underline{q}_u = \underline{\underline{K}}_{uu}^{-1} \underline{Q}_u = \underline{\underline{K}}_{uu}^{-1} \begin{Bmatrix} 0 \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{PL}{E\mathcal{A}} \begin{Bmatrix} -0.0325 \\ -0.7025 \\ -0.1763 \\ -0.1769 \\ 0.1114 \\ -0.1769 \\ -0.0650 \end{Bmatrix},$$

where the reduced global stiffness matrix, $\underline{\underline{K}}_{uu}$, is

$$\underline{\underline{K}}_{uu} = \frac{E\mathcal{A}}{L}\begin{bmatrix} 1.4271 & 0 & -0.6545 & -0.4755 & -0.6545 & 0.4755 & -0.0590 \\ 0 & 1.8090 & -0.4755 & -0.3455 & 0.4755 & -0.3455 & -0.1816 \\ -0.6545 & -0.4755 & 1.7725 & -0.1123 & -0.6180 & 0 & -0.0955 \\ -0.475 & -0.3455 & -0.1123 & 1.4635 & 0 & 0 & 0.2939 \\ -0.6545 & 0.4755 & -0.6180 & 0 & 1.7725 & 0.1123 & -0.4045 \\ 0.4755 & -0.3455 & 0 & 0 & 0.1123 & 1.4635 & -0.2939 \\ -0.0590 & -0.1816 & -0.0955 & 0.2939 & -0.4045 & -0.2939 & 1.5590 \end{bmatrix}.$$

Now that displacements at all nodes have been computed, the bar elongation can be computed, and finally, eq. (10.99) yields the bar forces as $F_{(1)} = -0.1926\ P$, $F_{(2)} = 0.1778\ P$, $F_{(3)} = -0.1926\ P$, $F_{(4)} = -0.4067\ P$, $F_{(5)} = -0.4067\ P$, $F_{(6)} = -0.1338\ P$, $F_{(7)} = 0.0239\ P$, $F_{(8)} = 0.0239\ P$, $F_{(9)} = -0.1338\ P$, $F_{(10)} = 0.0650\ P$.

Even for this relatively simple problem, the formulation of the individual element stiffness matrices in global coordinates is a tedious numerical exercise. The procedure, however, is systematic and well suited for implementation on computers. In particular, the linear algebra formalism and matrix notation ease the transfer of the different mathematical entities into computer data structures. The various stiffness matrices, displacement and load arrays, are all easily implemented as data arrays in high level computing languages.

The finite element approach described here is particularly well suited for computer implementation because many crucial operations are performed at the element level. When dealing with trusses, this means that many operations, such as the generation of the element stiffness matrix, require only the data associated with a single element. And although developed for planar trusses to simplify the presentation, the approach is readily generalized to three-dimensional truss problems.

In chapter 11, the finite element method introduced here will be extended to deal with beam structures. An additional discretization step will be required to deal with beams, but the assembly process and solution method remain identical to those presented here.

10.7.12 Problems

Problem 10.10. Three-dimensional element stiffness matrix

Section 10.7.4 presents the derivation of the element stiffness matrix for a bar, leading to eq. (10.77). The presentation is limited to planar trusses; the goal of this problem is to generalize the formulation to three-dimensional (3D) problems. *(1)* Generalize the kinematic description of the element give in section 10.7.2. Generalize the element position vector, displacement vectors, and rotation matrix given eqs. (10.63), (10.64), and (10.66) respectively, to 3D. Select the rotation matrix as

$$\underline{\underline{\hat{R}}} = \begin{bmatrix} \ell_1 & -(\ell_2+\ell_3)/\Delta & \ell_1(\ell_2-\ell_3)/\Delta \\ \ell_2 & \ell_1/\Delta & [\ell_2(\ell_2-\ell_3)-1]/\Delta \\ \ell_3 & \ell_1/\Delta & [\ell_3(\ell_2-\ell_3)+1]/\Delta \end{bmatrix},$$

where $\Delta = \sqrt{2\ell_1^2 + (\ell_2 + \ell_3)^2}$. Prove that matrix $\underline{\underline{\hat{R}}}$ is orthogonal. Express the bar's length and direction cosines in terms of the element nodal coordinates. Generalize the element coordinate transformation matrix, $\underline{\underline{\hat{T}}}$, defined by eq. (10.71). *(2)* Generalize the expressions derived in section 10.7.3 for the element elongation and axial force, see eqs. (10.73) and (10.74), respectively. *(3)* Generalize the expressions for the element strain energy and stiffness matrix given in section 10.7.4. Give the expression for the stiffness matrix in the local and global coordinate systems, see eqs. (10.75) and (10.77), respectively. *(4)* The stiffness matrix expressed in the global coordinate system is of size 6×6. Of the six eigenvalues of this matrix, how many are zero? Discuss the nature of the corresponding eigenvectors.

Problem 10.11. Global stiffness matrix assembly process
Figure 10.24 gives a pictorial representation of the assembly process for the truss and node numbering sequence shown in fig. 10.22. Give the corresponding representation of the assembly process for the same truss using the node numbering shown in fig. 10.25.

10.8 Principle of minimum complementary energy

In section 10.2, the principle of minimum total potential energy is derived from the principle of virtual work. Two assumptions are used in this derivation: *(1)* the internal forces are assumed to be conservative and *(2)* the external forces are also assumed to be conservative. This means that the internal forces can be derived from a potential, called the strain energy, and the external forces can also be derived from a potential, called the potential of the externally applied loads.

Figure 10.27 shows the relationship between the principle of minimum total potential energy and the principle of virtual work. The arrow linking the constitutive relationship to the strain energy is unidirectional to indicate that an assumption is made: the internal forces in the solid must be conservative for the strain energy to exist. The figure does not indicate the second assumption that is required to obtain the principle of minimum total potential energy: the externally applied loads must be conservative.

Figure 10.27 also shows how the principle of minimum complementary energy is related to the principle of complementary virtual work developed in section 9.6. Here again, two assumptions are required in the derivation. First, the internal forces are assumed to be conservative. This implies the existence of a strain energy function, and hence, of a complementary strain energy function, see section 10.3. Second, it is assumed that the prescribed displacements can be derived from a potential; this new concept is introduced in the next section. The initial development will focus on a simple three-bar hyperstatic truss considered earlier in chapter 9, but the methodology is general.

10.8.1 The potential of the prescribed displacements

The principle of complementary virtual work is derived in section 9.6, using the three-bar truss depicted in fig. 10.28 to present the concepts associated with this principle. In that development, the vertical displacement of point **B** is prescribed to

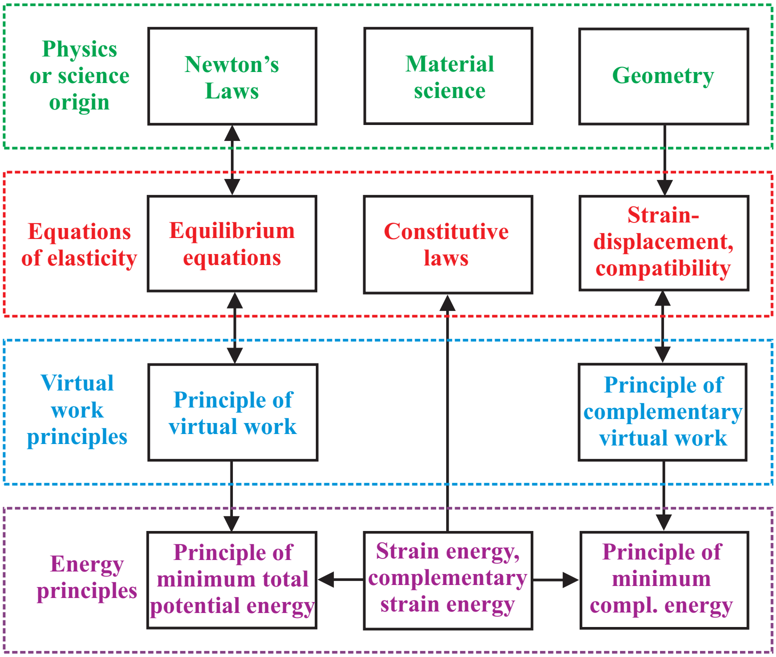

Fig. 10.27. Filiation from elasticity equations to virtual work and energy principles.

be of magnitude Δ. The force required to obtain this desired displacement, denoted D and often called the driving force, is an unknown quantity.

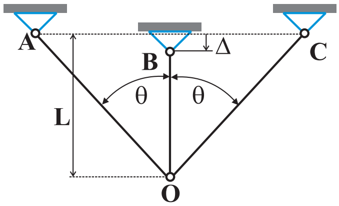

Fig. 10.28. Three-bar truss with prescribed displacement.

The statement of the principle of complementary virtual work, expressed by eq. (9.57), involves the external complementary virtual work, $\delta W'_E = \Delta\ \delta D$, which is directly related to the prescribed displacement. This term is the product of the true prescribed displacement, Δ, and a virtual force, δD. It is now assumed that the prescribed displacement can be derived from a potential, Φ', as

$$\Delta = -\frac{\partial \Phi'(D)}{\partial D}, \tag{10.101}$$

where $\Phi'(D)$ is called the *potential of the prescribed displacement*, or sometimes the *dislocation potential*. With this assumption, the external complementary virtual work becomes

$$\delta W'_E = \Delta\, \delta D = -\frac{\partial \Phi'}{\partial D} \delta D = -\delta \Phi'(D). \tag{10.102}$$

10.8.2 Constitutive laws for elastic materials

Consider a truss consisting of bars made of a linearly elastic material. If a uniform bar of length L, elastic modulus, E, and cross-sectional area, $\mathcal{A}$, is subjected to a force F, an elongation, e, results. Equation (10.29) gives the strain energy in the bar as $A = 1/2\, ke^2$, where $k = E\mathcal{A}/L$. The force applied to the bar can be derived from this strain energy function as $F = \partial A(e)/\partial e = ke$. This result shows that once the strain energy is known, the material's constitutive law follows. Thus, defining the strain energy function for a material is equivalent to defining the constitutive law.

The complementary strain energy for the same bar is given by eq. (10.22) as $A' = 1/2\, F^2/k$, where $1/k$ is the bar's compliance, *i.e.*, the inverse of its stiffness. The bar's elongation can be derived from the complementary strain energy as $e = \partial A'(F)/\partial F = F/k$. Once the complementary strain energy is known, the material's constitutive law follows. The strain energy yields the constitutive law in stiffness form, $F = ke$, whereas the complementary strain energy yields the same relationship in compliance form, $e = F/k$. For a linearly elastic material, the strain energy and its complementary counterpart are equal, $A = A'$, although expressed in terms of different variables: $A(e) = 1/2\, ke^2$, and $A'(F) = 1/2\, F^2/k$.

If the material is elastic but not linear, the strain energy and its complementary counterpart still yield the constitutive laws for the material in stiffness and compliance forms, respectively. The relationship between the two strain energies is given by eq. (10.23) as $A(e) + A'(F) = e\, F$. Taking the differential of this equation leads to $(\partial A/\partial e)\mathrm{d}e + (\partial A'/\partial F)\mathrm{d}F = F\mathrm{d}e + e\mathrm{d}F$. Regrouping the terms in this differential form leads to

$$\left[F - \frac{\partial A}{\partial e}\right] \mathrm{d}e + \left[e - \frac{\partial A'}{\partial F}\right] \mathrm{d}F = 0.$$

Since the differential in elongation and force are arbitrary and independent, the two bracketed terms must vanish, revealing the following relationships

$$F = \frac{\partial A(e)}{\partial e}, \tag{10.103a}$$

$$e = \frac{\partial A'(F)}{\partial F}, \tag{10.103b}$$

which both express the same constitutive law for the material, one in stiffness, the other in compliance form. The existence of the strain energy function guarantees that of its complementary counterpart. Hence, both stiffness and compliance forms of the constitutive law are entirely equivalent.

To illustrate the role of the strain energy and of its complementary counterpart for elastic materials, consider the following strain energy expression for a bar, $A(e) = kL^2(1 - \cos \bar{e})$, where $\bar{e} = e/L$ is the bar's axial strain. The material's constitutive law in stiffness form is readily obtained as $F = \partial A(e)/\partial e = kL \sin \bar{e}$. Due to the periodic nature of the cosine function, this particular strain energy is only meaningful for bar strains of moderate value, *i.e.*, $|\bar{e}| < 1$.

The compliance form of the same constitutive law is obtained by inversion as $\bar{e} = \arcsin \bar{F}$, where $\bar{F} = F/(kL)$ is a non-dimensional force. The complementary strain energy is then obtained from its definition as $A'(F) = \int_0^F e \; \mathrm{d}F = \int_0^{\bar{F}} \bar{e} \; L \; kL \; \mathrm{d}\bar{F} = kL^2(\bar{F} \arcsin \bar{F} + \sqrt{1 - \bar{F}^2} - 1)$. The same result can also be obtained more directly from the relationship the strain energy and its complementary counterpart, eq. (10.23), as $A'(F) = eF - A(e) = kL^2(\bar{F} \arcsin \bar{F} + \sqrt{1 - \bar{F}^2} - 1)$. The material's constitutive law can also be obtained from this expression of the complementary strain energy as

$$e = \frac{\partial A'(F)}{\partial F} = L \arcsin \bar{F},$$

which expresses the same constitutive law as that derived from the strain energy function.

Finally, it must be noted that the existence of the strain energy function or of its complementary counterpart is an *assumption* equivalent to the assumption of a constitutive law. Indeed, as discussed in section 10.1.1, if the material's internal forces are assumed to be conservative, they can be derived from a potential, called the strain energy function. In other words, the existence of a strain energy function, the existence of a complementary energy function, or the fact that the material's internal forces are conservative are three entirely equivalent assumptions.

10.8.3 The principle of minimum complementary energy

The principle of complementary virtual work is introduced for truss structures in section 9.6.2, on page 441. The principle is summarized by eq. (9.55) as $\delta W' = \delta W'_E + \delta W'_I = 0$, and states that a truss undergoes compatible deformations if and only if the sum of the internal and external complementary virtual work vanishes for all statically admissible virtual forces. The internal complementary virtual work for the three-bar truss depicted in fig. 10.28 is given by eq. (9.49) as $\delta W'_I = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C$.

At this point, it is assumed that the material the bars are made of is elastic, *i.e.*, the existence of a complementary strain energy function is assumed. The material's constitutive law is now expressed in compliance form by eq. (10.103b) and the complementary virtual work becomes

$$\delta W'_I = -\frac{\partial A'_A(F_A)}{\partial F_A} \delta F_A - \frac{\partial A'_B(F_B)}{\partial F_B} \delta F_B - \frac{\partial A'_C(F_C)}{\partial F_C} \delta F_C,$$

where A'_A, A'_B, and A'_C are the complementary strain energies of bars **A**, **B**, and **C**, respectively. Treating δ as a differential, this expression readily simplifies to

$$\delta W'_I = -\delta A'_A - \delta A'_B - \delta A'_C = -\delta A',$$

where $A' = A'_A + A'_B + A'_C$ is the total complementary strain energy for the three-bar truss.

Next, it is assumed that the prescribed displacement at point **B**, see fig. 10.28, can be derived from a potential. As discussed in section 10.8.1, the external complementary virtual work can then be written as $\delta W'_E = -\delta \Phi'(D)$, where Φ' is the potential of the prescribed displacement. The *potential of the prescribed displacements*, Φ', is different from the *potential of the externally applied loads*, Φ, defined in eq. (10.13).

Given these two assumptions, the existence of the material's complementary strain energy function and of the prescribed displacement potential, the principle of complementary virtual work, eq. (9.55), becomes

$$\delta W' = \delta W'_E + \delta W'_I = -\delta A' - \delta \Phi' = -\delta(A' + \Phi') = 0.$$

It is convenient to define the *total complementary energy*, Π' as

$$\Pi' = A' + \Phi', \tag{10.104}$$

and the above statement then further simplifies to

$$\delta \Pi' = 0. \tag{10.105}$$

These developments lead to the principle of stationary complementary energy.

Principle 11 (Principle of stationary complementary energy) *A conservative system undergoes compatible deformations if and only if the total complementary energy vanishes for all statically admissible virtual forces.*

It can be shown in a manner similar to that for the principle of minimum total potential energy that this stationary value is also a minimum value for stable equilibrium. With this, it is now possible to state the principle of minimum complementary energy.

Principle 12 (Principle of minimum complementary energy) *A conservative system undergoes compatible deformations if and only if the total complementary energy is a minimum with respect to arbitrary changes in statically admissible forces.*

Example 10.8. Three-bar truss with prescribed displacement

Consider the hyperstatic three-bar truss treated previously in example 9.15 on page 448 using the principle of complementary virtual work. The configuration is shown again in fig. 10.29. Assume that support joint **B** is given a prescribed displacement, Δ (perhaps due to an initial assembly imperfection). Determine the resulting force in each of the bars.

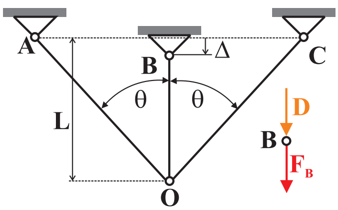

Fig. 10.29. Three-bar truss with prescribed displacement (see fig. 9.37).

For this hyperstatic problem, the three unknown bar forces cannot be evaluated based on the two equilibrium equations for joint **O**. Horizontal equilibrium yields $F_A = F_C$, and vertical equilibrium results in $F_A \cos\theta + F_B + F_C \cos\theta = 0$. The complementary strain energy written in terms of the bar forces, F_A, F_B, and F_C, is

$$A' = \frac{1}{2}\left(\frac{F_A^2}{k_A \cos\theta} + \frac{F_B^2}{k_B} + \frac{F_C^2}{k_C \cos\theta}\right),$$

where $k_A = (E\mathcal{A})_A/L$, $k_B = (E\mathcal{A})_B/L$, and $k_C = (E\mathcal{A})_C/L$ are the bar stiffnesses. With the help of the two equilibrium equations, the three bar forces can be expressed in terms one, say F_C, to find

$$A' = \frac{1}{2}\left(\frac{F_C^2}{k_A \cos\theta} + \frac{(2F_C \cos\theta)^2}{k_B} + \frac{F_C^2}{k_C \cos\theta}\right) = \frac{\bar{k} F_C^2}{2\bar{k}_A k_B \bar{k}_C \cos\theta},$$

where $\bar{k}_A = k_A/k_B$, $\bar{k}_C = k_C/k_B$, and $\bar{k} = \bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C \cos^3\theta$ are non-dimensional stiffness coefficients.

The potential of the prescribed displacement, Δ, at joint **B** is $\Phi' = -D\Delta$. The equilibrium equation at joint **B** states that $D + F_B = 0$ and using the equilibrium equation $F_B = -2F_C \cos\theta$, the potential can now be expressed in terms of bar force F_C as $\Phi' = -2\Delta F_C \cos\theta$. The total complementary potential energy thus takes the following form

$$\Pi' = A' + \Phi' = \frac{\bar{k} F_C^2}{2\bar{k}_A k_B \bar{k}_C \cos\theta} - 2\Delta F_C \cos\theta,$$

and the principle of minimum complementary energy, principle 12, requires that

$$\frac{\partial \Pi'}{\partial F_C} = \frac{\bar{k} F_C}{\bar{k}_A k_B \bar{k}_C \cos\theta} - 2\Delta \cos\theta = 0,$$

which expresses the requirement that the truss must undergo a compatible deformation. This equation yields the bar force, F_C, and the other two bar forces, F_A and F_B, are then obtained from the two equilibrium equations as

$$F_A = F_C = \frac{2\bar{k}_A\bar{k}_C \cos^2\theta}{\bar{k}} k_B \Delta \quad \text{and} \quad F_B = D = \left(1 - \frac{\bar{k}_A + \bar{k}_C}{\bar{k}}\right) k_B \Delta.$$

Finally, the displacement of joint **O** can be found from the extension of bar **B** as $u_1^{(B)} = e_B + \Delta = (\bar{k}_A + \bar{k}_C)\Delta/\bar{k}$.

10.8.4 The principle of least work

The principle of minimum complementary energy developed in the previous section involves the total complementary energy, which is the sum of two scalar quantities: the system's complementary strain energy, and the potential of the prescribed displacements, see eq. (10.104). In the absence of prescribed displacements, the total complementary energy reduces to the complementary strain energy alone. The *principle of least work*, a corollary of the principle of minimum complementary energy, states the following.

Principle 13 (Principle of least work) *In the absence of prescribed displacements, a conservative system undergoes compatible deformations if and only if the complementary strain energy is a minimum with respect to arbitrary changes in statically admissible forces.*

If the system is made of a linearly elastic material, the complementary strain energy is equal the strain energy, see section 10.3. The principle of least work then takes on the following form.

Principle 14 (Principle of least work) *In the absence of prescribed displacements, a linearly elastic system undergoes compatible deformations if and only if the strain energy is a minimum with respect to arbitrary changes in statically admissible forces.*

When using the principle of least work, the system's complementary strain energy or its strain energy must be expressed in terms of the statically admissible forces rather than deformations.

Example 10.9. Three-bar truss with tip load

To illustrate the use of the least work principle, the hyperstatic, three-bar truss treated in example 9.14 on page 447 using the principle of complementary virtual work will be re-examined. As shown in fig. 10.30, a vertical downward load P is applied at joint **O**. The objective is to determine the resulting bar forces.

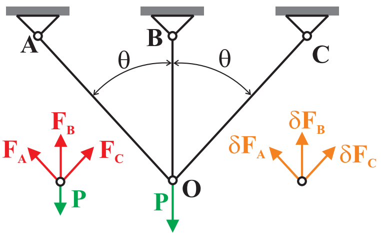

Fig. 10.30. Three-bar truss with applied load (fig. 9.36).

Since it is not necessary to determine the reaction forces at the support joints, the only relevant equilibrium equations are those at joint **O**; the horizontal and vertical equilibrium equations are $F_A = F_C$ and $F_A \cos\theta + F_B + F_C \cos\theta = P$, respectively.

The strain energy is first written in terms of the bar forces F_A, F_B and F_C, as

$$A = \frac{1}{2}\left(\frac{F_A^2}{k_A \cos\theta} + \frac{F_B^2}{k_B} + \frac{F_C^2}{k_C \cos\theta}\right),$$

where $k_A = (E\mathcal{A})_A/L$, $k_B = (E\mathcal{A})_B/L$, and $k_C = (E\mathcal{A})_C/L$ are the bar stiffnesses. Next, using the two equilibrium equations, the three bar forces are expressed in terms of one, say F_C, to find

$$A = \frac{1}{2}\left[\frac{F_C^2}{k_A \cos\theta} + \frac{(P - 2F_C\cos\theta)^2}{k_B} + \frac{F_C^2}{k_C \cos\theta}\right].$$

To impose the condition that the truss must undergo compatible deformations, the least work principle, principle 14, is applied,

$$\frac{\partial A}{\partial F_C} = \left[\frac{F_C}{k_A \cos\theta} - \frac{(P - 2F_C\cos\theta) 2\cos\theta}{k_B} + \frac{F_C}{k_C\cos\theta}\right] = 0.$$

This equation can be solved for the bar force, F_C, and the equilibrium equations then yield the forces in the other two bars as

$$\frac{F_A}{P} = \frac{F_C}{P} = \frac{2\bar{k}_A\bar{k}_C\cos^2\theta}{\bar{k}}, \qquad \frac{F_B}{P} = \frac{\bar{k}_A + \bar{k}_C}{\bar{k}},$$

where $\bar{k}_A = k_A/k_B$, $\bar{k}_C = k_C/k_B$, and $\bar{k} = \bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta$ are the bar non-dimensional stiffness factors.

Application of the principle of minimum complementary energy, or of the principle of least work in the absence of prescribed displacement, leads to a solution process that is very similar to that of the force or flexibility method first developed in section 4.3.3. In this earlier presentation, the compatibility conditions are developed from simple geometric arguments, whereas the principle of minimum complementary energy is used here to derive the same conditions in a more abstract but also more systematic manner.

Example 10.10. Beam on 3 supports

The simply supported beam with an additional mid-span support depicted in fig. 10.31 is subjected to two concentrated loads of equal magnitude, P. Determine the location and magnitude of the maximum bending moment in the beam.

Due to the additional mid-span support, the system is hyperstatic of order 1. Any one of the three support reaction forces, denoted R_1, R_2, and R_3, respectively, can be selected as the redundant force; in this example, R_1 is selected to be the redundant quantity, and therefore will be treated as the unknown.

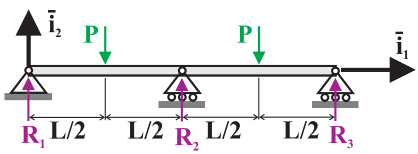

Fig. 10.31. A beam on 3 simple supports.

The overall moment and vertical force equilibrium equations result in $R_3 = R_1$ and $R_2 = 2P - 2R_1$, respectively. The bending moment distribution in the left segment of the beam is found from equilibrium considerations as

$$M_3(\eta) = \begin{cases} R_1 L\eta, & 0 \le \eta \le 1/2, \\ R_1 L\eta - PL(\eta - 1/2), & 1/2 \le \eta \le 1, \end{cases} \tag{10.106}$$

where $\eta = x_1/L$ is the non-dimensional variable along the beam's span. Due to the symmetry of the problem it is only necessary to compute the strain energy in the beam by integrating over the range 0 to L and then doubling the result. Thus

$$A = \int_0^{2L} \frac{M_3^2(x_1)}{2H_{33}^c}\,\mathrm{d}x_1 = 2\int_0^{L} \frac{M_3^2(x_1)}{2H_{33}^c}\,\mathrm{d}x_1 = 2L\int_0^{1} \frac{M_3^2(\eta)}{2H_{33}^c}\,\mathrm{d}\eta$$
$$= \frac{R_1^2L^3}{H_{33}^c}\int_0^{1/2} \eta^2\mathrm{d}\eta + \frac{L}{H_{33}^c}\int_{1/2}^{1} \left[R_1L\eta - PL(\eta-1/2)\right]^2\mathrm{d}\eta.$$

After integration, $A = \left(8R_1^2 - 5PR_1 + P^2\right)L^3/(24H_{33}^c)$, and using the least work principle then yields

$$\frac{\partial A}{\partial R_1} = \frac{L^3}{24H_{33}^c}(16R_1 - 5P) = 0.$$

This equation yields $R_1 = 5P/16$, and the overall equilibrium equations then reveal the remaining reaction forces as $R_1 = R_3 = 5P/16$ and $R_2 = 11P/8$. The beam's bending moment distribution is obtained by substituting these forces in eq. (10.106). The bending moment at the left support vanishes, as expected. At the point of application of the concentrated load, *i.e.*, at $\eta = 1/2$, the bending moment is $M_3 = 5PL/32$; at the mid-span support, *i.e.*, at $\eta = 1$, $M_3 = -6PL/32$. The maximum bending moment is found at the mid-span support, and its magnitude is $|M_3| = 6PL/32$.

Example 10.11. Simply supported beam with a mid-span elastic spring

Consider the simply supported bean of length L with a mid-span spring of stiffness constant k, as depicted in fig. 10.32. The beam carries a uniformly distributed vertical load, p_0. Determine the load in the spring and the reaction forces at the two supports.

Let R_1 and R_2 be the two support reaction forces, and F_s the force that the spring applies on the beam, counted positive downward. The overall equilibrium equations are $R_1+R_2+F_s = p_0L$ and $R_1L+F_sL/2-p_0L^2/2 = 0$. The problem is hyperstatic of order 1, because the two equilibrium equations involve three unknown reaction forces. Two of the reactions can be expressed in terms of the third; for instance, $F_s = p_0L - 2R_1$ and $R_2 = R_1$, where R_1 is treated as the redundant quantity.

Based on equilibrium considerations, the bending moment distribution in the beam is now expressed in terms of the unknown reaction force, R_1, as

$$M_3(\eta) = \begin{cases} p_0L^2\eta^2/2 - R_1L\eta, & 0 \le \eta \le 1/2, \\ p_0L^2(1-\eta)^2/2 - R_1L(1-\eta), & 1/2 \le \eta \le 1, \end{cases}$$

where $\eta = x_1/L$ is the non-dimensional variable along the beam.

The strain energy in the system is

$$A = \frac{1}{2}\int_0^{L} \frac{M_3^2(x_1)}{H_{33}^c}\,\mathrm{d}x_1 + \frac{1}{2}\frac{F_s^2}{k},$$

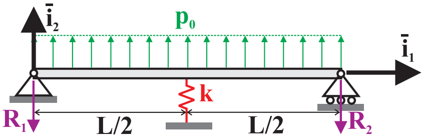

Fig. 10.32. Simply supported beam with a mid-span elastic spring.

where the first term represents the strain energy stored in the beam, and the second represents that stored in the elastic spring.

Rather than substitute for M_3 and F_s in terms of R_1 and then integrate, the principle of least work is applied first,

$$\frac{\partial A}{\partial R_1} = \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial R_1} \, \mathrm{d}x_1 + \frac{F_s}{k} \frac{\partial F_s}{\partial R_1} = 0.$$

Introducing the bending moment distribution and the force in the spring, both expressed in terms of the unknown reaction force, R_1, then leads to

$$2 \int_0^{1/2} \left[\frac{p_0 L^2 \eta^2/2 - R_1 L \eta}{H_{33}^c} \right] (-L\eta) \; L\mathrm{d}\eta + \left[\frac{p_0 L - 2R_1}{k} \right] (-2) = 0.$$

In this expression, the strain energy in the left half of the beam is computed and then doubled, based on symmetry. After integration, this becomes

$$\frac{2L^3}{H_{33}^c} \left(-\frac{p_0 L}{128} + \frac{R_1}{24} \right) - 2\frac{p_0 L - 2R_1}{k} = 0.$$

The solution of this equation yields the non-dimensional reaction forces at the supports and the spring force as

$$\frac{R_1}{p_0 L} = \frac{R_2}{p_0 L} = \frac{1}{2} \frac{384 + 3\bar{k}}{384 + 8\bar{k}}, \quad \frac{F_s}{p_0 L} = \frac{5\bar{k}}{384 + 8\bar{k}},$$

where $\bar{k} = kL^3/H_{33}^c$ is the non-dimensional spring stiffness constant expressing the spring stiffness relative to the beam bending stiffness.

The force in the spring is obtained from the overall equilibrium equation, $F_s = p_0 L - 2R_1$. When $\bar{k} \to 0$, *i.e.*, in the absence of mid-span spring, the reaction forces become $R_1 = R_2 = p_0 L/2$, as expected from symmetry, and $F_s = 0$. For $\bar{k} \to \infty$, *i.e.*, for a mid-span support, $R_1 = R_2 = 3p_0 L/16$ and the mid-span reaction force is $F_s = 5p_0 L/8$.

Example 10.12. Simply supported beam with a misaligned mid-span support

Figure 10.33 shows a simply supported beam with a misaligned mid-span support. In the unloaded configuration, the mid-span support is at a distance d below the beam. As the loading increases, the beam will touch the mid-span support. For the analysis, it is assumed that the beam is touching the support because the applied loads are high

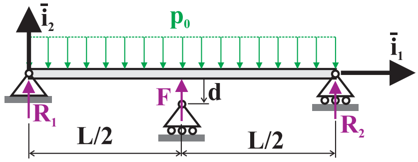

Fig. 10.33. Simply supported beam with a misaligned mid-span support.

enough. Of course, for small loads, the beam does not touch the support, which can then be ignored.

Let R_1 and R_2 be the reaction forces at the two end support and F the force that the mid-span support applies to the beam. The overall equilibrium equations are $R_1 + R_2 + F = p_0 L$ and $R_1 L + FL/2 - p_0 L^2/2 = 0$. The problem is hyperstatic of order 1, because the two equilibrium equations involve three unknown reaction forces. Two of the reactions can be expressed in terms of the third; for instance, $F = p_0 L - 2R_1$ and $R_2 = R_1$. Based on equilibrium considerations, the bending moment distribution in the beam is now expressed in terms of the unknown reaction force, R_1, as

$$M_3(\eta) = \begin{cases} p_0 L^2 \eta^2/2 - R_1 L \eta, & 0 \leq \eta \leq 1/2, \\ p_0 L^2 (1-\eta)^2/2 - R_1 L (1-\eta), & 1/2 \leq \eta \leq 1, \end{cases}$$

where $\eta = x_1/L$ is the non-dimensional variable along the beam.

For this linearly elastic structure, the total complementary energy is $\Pi' = A' + \Phi' = A + \Phi'$. The potential of the prescribed displacements is $\Phi' = -F(-d) = Fd = d(p_0 L - 2R_1)$, where the minus sign stems from the fact that the force and displacements are counted positive in opposite direction.

The principle of minimum complementary energy now states

$$\frac{\partial \Pi'}{\partial R_1} = \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial R_1} \, \mathrm{d}x_1 + \frac{\partial \Phi'}{\partial R_1} = 0.$$

Introducing the above bending moment distribution and potential of the prescribed displacements yields

$$2 \int_0^{1/2} \left[\frac{p_0 L^2 \eta^2/2 - R_1 L \eta}{H_{33}^c} \right] [-L\eta] \; L \mathrm{d}\eta + \frac{\partial}{\partial R_1} d(p_0 L - 2R_1) = 0.$$

In this expression, the strain energy in the left half of the beam is computed and then doubled, based on symmetry. After integration, this condition becomes

$$\frac{2L^3}{H_{33}^c} \left(-\frac{p_0 L}{128} + \frac{R_1}{24} \right) - 2d = 0.$$

The solution of this equation yields the reaction forces at the supports and the mid-span reaction force as

$$R_1 = R_2 = \frac{3p_0 L}{16} + 24\frac{H_{33}^c d}{L^3}, \quad F = \frac{5p_0 L}{8} - 48\frac{H_{33}^c d}{L^3}.$$

The result can be interpreted in different manner. First, if the three supports are on the same level, $d = 0$, and the end point reaction forces are $R_1 = R_2 = 3p_0 L/16$ and the mid-span reaction force is $F = 5p_0 L/8$. If the mid-span support is misaligned and below the beam, *i.e.*, $d > 0$, the loading level for which the beam will just touch the mid-span support is found by setting $F = 0$, *i.e.*, a reaction force is just about to vanish at the support. This leads to $5p_{\rm cr} L/8 - 48H_{33}^c d/L^3 = 0$ and $p_{\rm cr} = 384H_{33}^c d/(5L^4)$. Thus for $p_0 \leq p_{\rm cr}$, the beam does not reach the support and for $p_0 > p_{\rm cr}$, the reaction force give above develops in the misaligned, mid-span support.

The analysis is also valid if $d < 0$. This means that the mid-span support is protruding and pushing the beam upwards. Even in the absence of applied loading, end point reaction forces, $R_1 = R_2 = 24H_{33}^c d/L^3$, and a mid-span reaction force, $F = -48H_{33}^c d/L^3$, will develop. If $d < 0$, it follows that $R_1 < 0$, $R_2 < 0$, and $F > 0$, as expected.

10.8.5 Problems

Problem 10.12. Cantilevered beam with intermediate spring support
The cantilevered beam shown in fig. 10.34 carries a uniform loading distribution, p_0, and is supported by a spring of stiffness constant k, located at a distance a from the root of the beam. *(1)* Use the least work principle to determine the force in the spring. *(2)* Find the bending moment distribution in the beam.

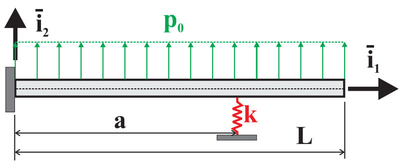

Fig. 10.34. Cantilevered beam with intermediate spring support.

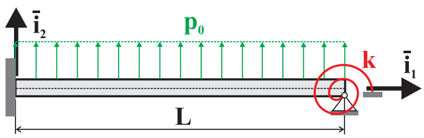

Fig. 10.35. Cantilevered beam with tip rotational spring.

Problem 10.13. Cantilevered beam with tip rotational spring
A cantilevered beam of span L is subjected to a uniform loading distribution, p_0, as depicted in fig. 10.35. An additional support is located at the beam's tip, and a rotational spring of stiffness constant k acts at the same point. *(1)* Use the least work principle to determine the tip support reaction force. *(2)* Find the bending moment distribution in the beam.

Problem 10.14. 3-bar truss with unequal bar stiffness properties
The three-bar, hyperstatic truss shown in fig. 10.36 is subjected to a tip vertical load P. The three bars have a Young's modulus E, bar 1 is of cross-sectional area $\mathcal{A}$, while that bars 2 and 3 is $2\mathcal{A}$. *(1)* Use the principle of complementary energy to find the bar forces.

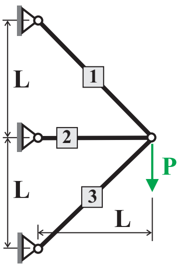

Fig. 10.36. 3-bar truss with unequal bar stiffness properties.

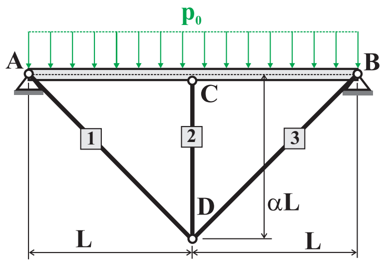

Fig. 10.37. Simply supported beam with uniform load and truss bracing.

Problem 10.15. Combined beam and truss problem
The structure shown in fig. 10.37 consists of a simply supported, continuous beam, **AB**, subjected to a uniformly distributed load, p_0. Additional support is provided by a truss consisting of bars **AD**, **CD**, and **BD**, which are pinned together to provide a mid-span support for the beam. Bar **CD** is of length αL. *(1)* Using the principle of least work, find the forces in the three bars. *(2)* Determine the bending moment distribution in the beam. Use the following data: $\alpha = 1$. Ignore the axial forces in the beam.

Problem 10.16. Cantilevered beam with simple support and concentrated load
A cantilevered beam with a mid-span support carries a tip concentrated load, P, as depicted in fig. 10.38. *(1)* Using the principle of least work, determine the reaction forces. *(2)* Find the bending moment distribution in the beam.

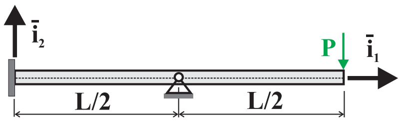

Fig. 10.38. Cantilevered beam with simple support and concentrated load.

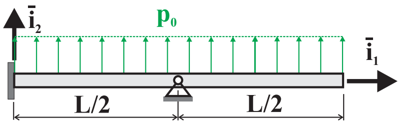

Fig. 10.39. Cantilevered beam with simple support under uniform loading.

Problem 10.17. Cantilevered beam with simple support and uniform load
A cantilevered beam with a mid-span support carries a uniformly distributed load, p_0, as depicted in fig. 10.39. *(1)* Using the principle of least work, determine the reaction forces. *(2)* Find the bending moment distribution in the beam.

10.9 Energy theorems

A number of important energy theorems will be developed in this section. These theorems are corollaries of the fundamental energy principles developed earlier. Consequently, all theorems are valid for elastic structures only. The application of two of these theorems, Clapeyron's theorem and Castigliano's second theorem, is further limited to linearly elastic materials.

A properly constrained[3] elastic body subjected to various concentrated loads and couples is shown in fig. 10.40. The first loading type consists of N concentrated loads, P_i, $i = 1, 2, \ldots, N$, and the displacements of their points of application projected in the direction of the loads are denoted Δ_i, $i = 1, 2, \ldots, N$, respectively. The second type of loading consists of M couples, Q_j, $j = 1, 2, \ldots, M$, and the rotations at their points of application about the axis of the couple are denoted Φ_j, $i = 1, 2, \ldots, M$, respectively.

A special case of these loading conditions consists of two forces sharing the same line action, such as forces P_3 and P_4 in fig. 10.40. In some cases, the two forces are of equal magnitude and opposite direction, $P_3 = P_4 = P$, and the relative displacement of their points of application is then denoted $\Delta_0 = \Delta_3 + \Delta_4$. A similar situation could occur with two couples of equal magnitude and opposite direction sharing a common axis, say $Q_3 = Q_4 = Q$, and the relative rotation at their points of application is $\Phi_0 = \Phi_3 + \Phi_4$.

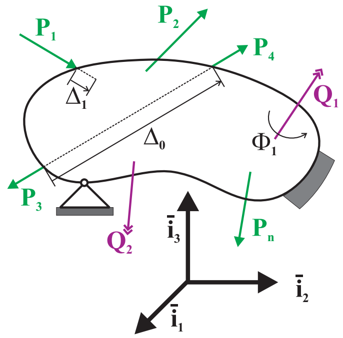

Fig. 10.40. Elastic body subjected to various loads.

10.9.1 Clapeyron's theorem

For an elastic system, eq. (10.12) implies that the strain energy stored in the body equals the work done by the external forces as they are increased quasi-statically from zero to their final values. Consider a suitably restrained body that is subjected to N external loads, P_i, and M external couples, Q_j. Equation (10.12) now implies

$$A = W_E = \sum_{i=1}^{N} \int_0^{\Delta_i} P_i \, \mathrm{d}u_i + \sum_{j=1}^{N} \int_0^{\Phi_j} Q_j \, \mathrm{d}\theta_j,$$

where u_i are the displacements of the external load projected along their line of action and Δ_i are the final displacements, θ_j are the rotations of the external moments about their axes, and Φ_j are the final rotations.

Next, the body is assumed to be linearly elastic and hence, the applied loads are proportional to the displacements of their point of application, $P_i \propto u_i$, and the applied couples are proportional to the rotation of their point of application, $Q_j \propto \theta_j$. It now becomes possible to evaluate the two integrals to find

$$A = W_E = \sum_{i=1}^{N} \frac{P_i \Delta_i}{2} + \sum_{j=1}^{N} \frac{Q_j \Phi_j}{2}. \tag{10.107}$$

This result is known as Clapeyron's theorem.

[3] Properly constrained means that the body cannot rotate or translate freely but may be subjected to a hyperstatic set of reactions.

Theorem 10.1 (Clapeyron's theorem). *The strain energy stored in a linearly elastic structure equals the sum of the half product of the applied loads by the displacements of their respective points of applications projected along their lines of action.*

While Clapeyron's theorem is useful for evaluating the strain energy, it can also be used to compute the deflection, Δ, at the point of application of a load, P, when this single load is the only load applied. In such a case, eq. (10.107) becomes $\Delta = 2A/P$. A similar result is obtain for the rotation at the point of application of a single moment.

It is interesting to compare eqs. (10.107) and (10.13), which differ by a factor of two. In the derivation of eq. (10.107), load P is assumed to grow in proportion to the displacement, whereas for eq. (10.13), load P is assumed to remain constant. As discussed in section 10.4.4, this difference in the nature of the applied loading explains the factor of two in the work they perform.

Clapeyron's theorem can also be proved by the following alternative reasoning. First, the total potential energy of the system is written as

$$\Pi = A - \sum_{i=1}^{N} P_i \Delta_i,$$

where $A = 1/2 \int_V \underline{\epsilon}^T \underline{\underline{C}}\, \underline{\epsilon} \; \mathrm{d}V$ is the strain energy for the general three-dimensional linearly elastic body, see eq. (10.49). The principle of minimum total potential energy implies the stationarity of the total potential energy, which can be expressed as[4] $\delta\Pi = \int_V \underline{\epsilon}^T \underline{\underline{C}}\, \delta\underline{\epsilon} \; \mathrm{d}V - \sum_{i=1}^{N} P_i \delta\Delta_i = 0$. Since this relationship must hold for all arbitrary virtual displacement and associated compatible strain fields, a valid choice is $\delta\Delta_i = \Delta_i$ and $\delta\underline{\epsilon} = \underline{\epsilon}$, where Δ_i and $\underline{\epsilon}$ correspond to the displacement and strain fields for the equilibrium configuration of the structure, respectively. It follows that $\int_V \underline{\epsilon}^T \underline{\underline{C}}\, \underline{\epsilon} \; \mathrm{d}V = \sum_{i=1}^{N} P_i \Delta_i$ and finally, $A = \sum_{i=1}^{N} P_i \Delta_i/2$, which is the statement of Clapeyron's theorem, see eq. (10.107). Note that the order in which the forces are applied is immaterial, and the strain energy stored in the structure depends only on the magnitude of the forces and the resulting projected displacements.

Consider now the case of two forces, P_3 and P_4, of equal magnitude and opposite sign sharing a common line of action, as shown in fig. 10.40. Clapeyron's theorem yields $A = (P_3\Delta_3 + P_4\Delta_4)/2$; let P denote the intensity of the forces, $P_3 = P_4 = P$, and hence, $A = P(\Delta_3 + \Delta_4)/2 = P\Delta_0/2$, where $\Delta_0 = \Delta_3 + \Delta_4$ is the relative distance between the points of application of the two forces. This relative distance is a positive quantity if the forces pull away from each other, and is negative in the opposite case.

A similar reasoning yields $A = Q\Phi_0/2$, where Q is the common magnitude of two couples of equal magnitude and opposite direction sharing a common axis and Φ_0 is the relative rotation at their points of application. In summary, each of the loading terms appearing in Clapeyron's theorem, eq. (10.107), could be of either of the following four types: $P_i\Delta_i/2$, $Q_j\Phi_j/2$, $P\Delta_0/2$, or $Q\Phi_0/2$.

[4] See appendix A.2.7 for details on taking the differential of a quadratic form.

Example 10.13. Simply supported beam with concentrated load
Consider a simply supported, uniform beam of length L subjected to a concentrated load P acting at a distance αL from the left support, as depicted in fig. 10.41. This problem is treated using classical approaches in examples 5.5 and 5.6 on pages 197 and 199, respectively. The bending moment distribution is readily obtained from equilibrium considerations and is given by eq. (5.52).

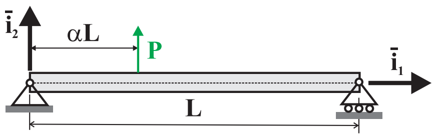

Fig. 10.41. Simply supported beam with concentrated load.

The strain energy stored in the beam is now obtained from eq. (10.41) as

$$A = \frac{1}{2}\int_0^L \frac{M_3^2}{H_{33}^c}\,\mathrm{d}x_1 = \frac{P^2L^2}{2H_{33}^c}\left[\int_0^\alpha (1-\alpha)^2\eta^2\, L\mathrm{d}\eta + \int_\alpha^1 \alpha^2(1-\eta)^2\, L\mathrm{d}\eta\right],$$

where $\eta = x_1/L$ is a non-dimensional variable along the beam's span. Performing the integrations leads to

$$A = \frac{P^2L^3}{2H_{33}^c}\left[(1-\alpha)^2\frac{\alpha^3}{3} + \alpha^2\frac{(1-\alpha)^3}{3}\right] = \frac{P^2L^3}{6H_{33}^c}\alpha^2(1-\alpha)^2.$$

Clapeyron's theorem, eq. (10.1), yields $A = P\Delta/2 = P^2L^3\alpha^2(1-\alpha)^2/(6H_{33}^c)$, and solving for the displacement, Δ, at the point of application of the load leads to $\Delta = PL^3\alpha^2(1-\alpha)^2/(3H_{33}^c)$. This result is identical to that obtained earlier with the classical approach. Indeed, $\Delta = \bar{u}_2(\alpha)$, where $\bar{u}_2(\eta)$ is the beam's transverse displacement field given by eq. (5.51) and $\bar{u}_2(\alpha)$ its value at $\eta = \alpha$.

Clapeyron's theorem yields the deflection under the load in a very expeditious manner. A disadvantage of this theorem, however, is that it is useful only when a single concentrated load is applied. In contrast, the principle of virtual work developed in section 9.7.6 is nearly as simple to formulate and can be used for any type or combination of loading. Finally, the classical approaches presented in section 5.5 and demonstrated in examples 5.6 and 5.6 require the solution of the governing differential equation of the problem but yield the distributions of transverse displacements, bending moments, and shear forces over the beam's entire span. More detailed information is obtained, but at a higher cost.

10.9.2 Castigliano's first theorem

Consider, again, a properly constrained elastic body subjected to various concentrated loads and couples as shown in fig. 10.40. The total potential energy, see eq. (10.10), can be written as $\Pi = A + \Phi = A - \sum_{i=1}^{N} P_i \Delta_i$, where P_i is an externally applied load and Δ_i the displacements of its point of application projected along its line of action.

The principle of minimum total potential energy now implies the stationarity of the total energy, eq. (10.17), and hence,

$$\frac{\partial \Pi}{\partial \Delta_j} = \frac{\partial A}{\partial \Delta_j} - \frac{\partial}{\partial \Delta_j} \sum_{i=1}^{N} P_i \Delta_i = \frac{\partial A}{\partial \Delta_j} - P_j = 0.$$

This equation leads to *Castigliano's first theorem*,

$$P_i = \frac{\partial A}{\partial \Delta_i}. \tag{10.108}$$

Theorem 10.2 (Castigliano's first theorem). *For an elastic system, the magnitude of the load applied at a point is equal to the partial derivative of the strain energy with respect to the projected load's displacement.*

To make use of this theorem the strain energy in the structure must be expressed in term of the projected displacements, Δ_i. Because this theorem is derived directly from the principle of minimum total potential energy, eq. (10.108) is simply an equilibrium statement for the problem.

Castigliano's first theorem is easily extended to other loading conditions such as applied couples, loads of equal magnitude and opposite directions sharing a common line of action, or couples of equal magnitude and opposite directions sharing a common axis.

10.9.3 Crotti-Engesser theorem

Clapeyron's and Castigliano's first theorems are corollaries of the principle of minimum total potential energy. Not unexpectedly, parallel developments based on the principle of minimum complementary energy will lead to similar results.

The total complementary energy, Π', is defined in eq. (10.104) as the sum of the complementary strain energy, A', and potential of the prescribed displacements, Φ'. If the system is subjected to N prescribed displacements, Δ_i, $i = 1, 2, \ldots, N$, the potential of the prescribed displacements is $\Phi' = -\sum_{i=1}^{N} P_i \Delta_i$, where P_i, $i = 1, 2, \ldots, N$, are the driving forces required to obtain the prescribed displacements. The total complementary energy now becomes

$$\Pi' = A' + \Phi' = A' - \sum_{i=1}^{N} P_i \Delta_i.$$

Next, the statically admissible stress field in the elastic body is expressed in terms of the driving forces, *i.e.*, $A' = A'(P_i)$. The principle of minimum complementary energy, principle 12, then implies

$$\frac{\partial \Pi'}{\partial P_j} = \frac{\partial A'}{\partial P_j} - \frac{\partial}{\partial P_j} \sum_{i=1}^{N} P_i \Delta_i = \frac{\partial A'}{\partial P_j} - \Delta_j = 0.$$

It now follows that

$$\Delta_i = \frac{\partial A'}{\partial P_i}, \tag{10.109}$$

where, clearly, the complementary energy in the structure must be expressed in terms of the driving forces, P_i. This result is known as the *Crotti-Engesser theorem* which can be stated as follows.

Theorem 10.3 (Crotti-Engesser theorem). *For an elastic structure, the prescribed deflection at a point is given by the partial derivative of the complementary energy with respect to the driving force.*

Unlike Clapeyron's theorem, 10.1, the Crotti-Engesser theorem can be applied to problems with multiple applied loads.

10.9.4 Castigliano's second theorem

In the derivation of the Crotti-Engesser theorem, the existence of the complementary energy is assumed for the elastic material. If the material is assumed to be linearly elastic, the strain energy and its complementary counterpart become equal, $A = A'$, and the Crotti-Engesser theorem, eq. (10.109), then leads to *Castigliano's second theorem*,

$$\Delta_i = \frac{\partial A}{\partial P_i}. \tag{10.110}$$

Theorem 10.4 (Castigliano's second theorem). *For a linearly elastic structure, the prescribed deflection at a point is given by the partial derivative of the strain energy with respect to the driving force.*

Note the obvious symmetry between eq. (10.110) and eq. (10.108). It should be noted, however, that Castigliano's first theorem applies to any elastic system, whereas Castigliano's second theorem only applies to linearly elastic structures.

10.9.5 Applications of energy theorems

The energy theorems are useful for determining deflections at specific points of a structure. In particular, Castigliano's second theorem yields structural deflections under applied loads.

Castigliano's second theorem is also useful when dealing with hyperstatic systems. Imagine a cantilevered beam with a tip support. One way to look at this problem is to consider a cantilevered beam with a prescribed tip displacement, which is required to vanish. The driving force, in this case, is the reaction force at the support. If P_i denotes this reaction force and $\Delta_i = 0$ the prescribed tip displacement, Castigliano's second theorem, eq. (10.110), requires $\partial A / \partial P_i = 0$. This equation is the compatibility equation at the tip support. It is interesting to note that in this case, Castigliano's second theorem reduces to the principle of least work, principle 13 on page 554.

Example 10.14. Deflection of a cantilever under transverse load
Consider a cantilevered beam of length L subjected to a tip transverse load P acting at a distance αL from the left support, as depicted in fig. 5.26 on page 201. This problem is treated using the classical approach in example 5.8 on page 201.

Simple equilibrium arguments yield the following bending moment distribution: $M_3 = PL(\alpha - \eta)$, for $\eta \leq \alpha$, and $M_3 = 0$, for $\alpha \leq \eta \leq 1$, where $\eta = x_1/L$ is the non-dimensional variable along the beam's span. The strain energy can be expressed in terms of the bending moment distribution as

$$A = \frac{1}{2H_{33}^c} \int_0^L M_3^2 \, \mathrm{d}x_1 = \frac{1}{2H_{33}^c} \int_0^\alpha (PL)^2 (\alpha - \eta)^2 \, L\mathrm{d}\eta = \frac{P^2(\alpha L)^3}{6H_{33}^c}.$$

Castigliano's second theorem then yields the deflection under the load as $\Delta = \partial A/\partial P = P(\alpha L)^3/(3H_{33}^c)$. This result is identical to that found with the classical approach, see eq. (5.55).

Example 10.15. Rotation of a cantilever under couple
Consider a cantilevered beam of length L subjected to a concentrated couple Q acting at a distance αL from the left support. Find the rotation at the point where Q acts.

Simple equilibrium arguments yield the following bending moment distribution: $M_3 = Q$, for $\eta \leq \alpha$, and $M_3 = 0$, for $\alpha \leq \eta \leq 1$, where $\eta = x_1/L$ is the non-dimensional variable along the beam's span. The strain energy can be expressed in terms of the bending moment distribution as

$$A = \frac{1}{2H_{33}^c} \int_0^L M_3^2 \, \mathrm{d}x_1 = \frac{1}{2H_{33}^c} \int_0^\alpha Q^2 \, L\mathrm{d}\eta = \frac{Q^2(\alpha L)}{2H_{33}^c}.$$

Castigliano's second theorem then yields the rotation at the point of application of the couple as $\Phi = \partial A/\partial Q = Q\alpha L/H_{33}^c$.

Example 10.16. Simply supported beam with concentrated load
Consider a simply supported, uniform beam of length L subjected to a concentrated load P acting at a distance αL from the end supports, as depicted in fig. 10.41. This problem is treated using classical approaches in examples 5.6 on page 199, and in example 10.13 using Clapeyron's theorem. The use of Clapeyron's theorem will be contrasted here with Castigliano's second theorem.

The algebra associated with the use of Castigliano's second theorem is somewhat simplified if the following manipulation is performed first

$$\Delta = \frac{\partial A}{\partial P} = \frac{\partial}{\partial P} \int_0^L \frac{M_3^2}{2H_{33}^c} \, \mathrm{d}x_1 = \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial P} \, \mathrm{d}x_1.$$

The bending moment distribution is given by eq. (5.52), or it can be obtained directly from equilibrium considerations as

$$M_3(\eta) = PL \begin{cases} -(1-\alpha)\eta, & 0 \le \eta \le \alpha, \\ -\alpha(1-\eta), & \alpha < \eta \le 1, \end{cases}$$

where $\eta = x_1/L$ is the non-dimensional variable along the beam's span. Substituting this in the previous equation results in

$$\Delta = \frac{PL^2}{H_{33}^c} \left[\int_0^{\alpha} (1-\alpha)^2 \eta^2 \, L\mathrm{d}\eta + \int_{\alpha}^{1} \alpha^2 (1-\eta)^2 \, L\mathrm{d}\eta \right],$$

and performing the integrations then yields

$$\Delta = \frac{PL^3}{H_{33}^c} \left[(1-\alpha)^2 \frac{\alpha^3}{3} + \alpha^2 \frac{(1-\alpha)^3}{3} \right] = \frac{PL^3}{3H_{33}^c} \alpha^2 (1-\alpha)^2.$$

This result is identical to that obtained in example 10.13 using Clapeyron's theorem.

Example 10.17. Ring under internal forces

Consider the open circular ring of radius R shown in fig. 10.42. The ring is cut at one location and two opposite tangential forces of equal magnitude P are applied along the same tangential line of action in the plane of the ring. Evaluate the relative displacement, Δ, of the points of application of the two forces using Castigliano's second theorem.

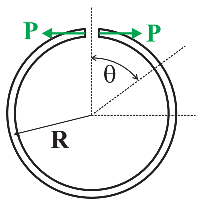

Fig. 10.42. Ring subjected to internal forces acting *in the plane of the ring*.

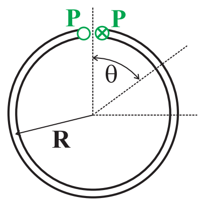

Fig. 10.43. Ring subjected to internal forces acting *out of the plane of the ring*.

For this configuration, the ring is subjected to both bending and axial loading, and hence, the total strain energy in the system is the sum of the strain energies due to bending and extension, given eqs. (10.41) and (10.36), respectively. Castigliano's second theorem now becomes

$$\Delta = \frac{\partial A}{\partial P} = \int_0^{2\pi} \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial P} \, R\mathrm{d}\theta + \int_0^{2\pi} \frac{N_1}{S} \frac{\partial N_1}{\partial P} \, R\mathrm{d}\theta,$$

where $R\mathrm{d}\theta$ is the infinitesimal axial distance around the ring.

The ring's bending moment and axial force distributions are evaluated by considering the equilibrium conditions of a segment of the beam of length $R\theta$. They are found to be $M_3(\theta) = -PR(1-\cos\theta)$ and $N_1 = -P\cos\theta$, respectively. Substituting these expression into the statement of Castigliano's theorem then yields

$$\Delta = \frac{PR^3}{H_{33}^c}\int_0^{2\pi}(1-\cos\theta)^2\,\mathrm{d}\theta + \frac{PR}{S}\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \frac{3\pi PR^3}{H_{33}^c} + \frac{\pi PR}{S}.$$

If the ring has a rectangular cross-section of radial thickness h and width b, the bending stiffness becomes $H_{33}^c = Ebh^3/12$, and it then follows that

$$\Delta = \frac{36\pi PR^3}{Ebh^3} + \frac{\pi PR}{Ebh} = \frac{\pi PR}{Ebh}\left[36\left(\frac{R}{h}\right)^2 + 1\right].$$

For a thin ring, $R/h \gg 1$, and the first term becomes dominant, implying that bending rather than extension of the ring is the principal contributor to the relative displacement of the points of application of the two forces.

The problem can be modified to introduce torsional deformation into the ring by changing the orientation of the applied forces, P, to now act along the same line of action but *normal to the plane of the ring.* This configuration is shown in fig. 10.43, where the two forces of magnitude P are acting normal to the plane of the figure. The total strain energy in the system is now the sum of the strain energies due to bending and torsion, given by eqs. (10.41) and (10.45), respectively. Castigliano's second theorem can now be written as

$$\Delta = \frac{\partial A}{\partial P} = \int_0^{2\pi}\frac{M_3}{H_{33}^c}\frac{\partial M_3}{\partial P}\,R\mathrm{d}\theta + \int_0^{2\pi}\frac{M_1}{H_{11}}\frac{\partial M_1}{\partial P}\,R\mathrm{d}\theta,$$

where $M_3 = PR\sin\theta$ is the bending moment distribution in the ring and $M_1 = PR(1-\cos\theta)$ the torsion moment distribution. Note that Δ is the relative displacement of the points of application of the forces projected along their line of action, *i.e.*, measured in the direction perpendicular to the plane of the ring. This relative displacement is then given by

$$\Delta = \frac{PR^3}{H_{33}^c}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta + \frac{PR^3}{H_{11}}\int_0^{2\pi}(1-\cos\theta)^2\,R\mathrm{d}\theta = \frac{\pi PR^3}{H_{33}} + \frac{3\pi PR^3}{H_{11}}.$$

If the ring has a circular cross-section of radius a, and is made of a linearly elastic, homogeneous material so that $G = E/[2(1+\nu)]$, the relative displacement becomes $\Delta = 4PR^3[1+3(1+\nu)]/(Ea^4)$. For this configuration, torsional deformation in the ring contributes $[3(1+\nu)]/[1+3(1+\nu)] \approx 80\%$ of the total relative displacement, for $\nu = 0.3$.

Example 10.18. Cantilevered beam with intermediate support

The cantilevered beam subjected to a uniform loading, p_0 and with an intermediate support located a distance $x_1 = \alpha L$ from the left clamp is depicted in fig. 10.44. This is a hyperstatic problem of order 1, and the reaction force at the support will be determined using Castigliano's second theorem.

Within the framework of Castigliano's second theorem, the reaction force, R, at the support is the driving force that prescribes a vanishing displacement at the intermediate support. This theorem, eq. (10.110), now implies $\Delta = \partial A/\partial R = 0$

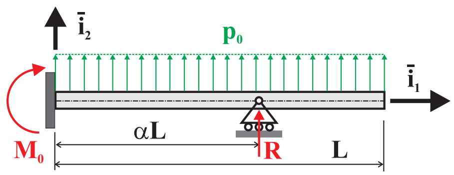

Fig. 10.44. Cantilevered beam with an intermediate support at location αL.

and this equation will be used to compute the unknown driving force, which is the desired reaction force.

The bending moment distribution in the beam can be found from equilibrium considerations as

$$M_3 = \begin{cases} p_0 L^2 (1-\eta)^2/2 - RL(\alpha - \eta), & 0 \leq \eta \leq \alpha, \\ p_0 L^2 (1-\eta)^2/2, & \alpha < \eta \leq 1, \end{cases}$$

where $\eta = x_1/L$ is the non-dimensional span-wise variable.

The compatibility condition at the support now follows from the application of Castigliano's second theorem, to find

$$\Delta = \frac{\partial A}{\partial R} = \int_0^\alpha \left[\frac{p_0 L^2}{2}(1-\eta)^2 - RL(\alpha - \eta) \right] \left[-L(\alpha - \eta) \right] L \mathrm{d}\eta = 0.$$

The second part of this integral, extending from α to 1, vanishes because $\partial M_3/\partial R$ vanishes over that portion of the beam. This equation can be integrated and solved to determine the unknown reaction force

$$R = \frac{p_0 L}{8} \, \frac{6 - 4\alpha + \alpha^2}{\alpha}.$$

A free body diagram of the entire beam reveals that the reaction moment at the clamped end of the beam is $M_0 = p_0 L^2/2 - \alpha RL$, and substituting for R, it follows that $M_0 = -(p_0 L^2/8)(\alpha^2 - 4\alpha + 2)$.

The same hyperstatic problem can be handled in a different manner within the framework of Castigliano's second theorem by choosing another reaction as the driving force. In this case, the reaction moment, M_0, at the root clamp is the driving moment that prescribes a vanishing rotation at the clamp. Castigliano's second theorem now implies $\Phi = \partial A/\partial M_0 = 0$ and this equation will be used to compute the unknown driving moment, which is the desired reaction moment.

The bending moment distribution in what is now a simply supported beam is found from equilibrium considerations as

$$M_3 = \begin{cases} p_0 L^2 \left[\eta^2 + (1/\alpha - 2)\eta \right]/2 + M_0(1 - \eta/\alpha), & 0 \leq \eta \leq \alpha, \\ p_0 L^2 (1-\eta)^2/2, & \alpha < \eta \leq 1. \end{cases}$$

The compatibility condition at the clamp now follows from the application of Castigliano's second theorem, to find

$$\Phi = \int_0^\alpha \frac{1}{H_{33}^c} \left\{ \frac{p_0 L^2}{2} \left[\eta^2 + (\frac{1}{\alpha} - 2)\eta \right] + M_0 (1 - \frac{\eta}{\alpha}) \right\} \left(1 - \frac{\eta}{\alpha}\right) L \mathrm{d}\eta = 0.$$

This equation can be solved to determine the reaction moment $M_0 = -p_0 L^2(\alpha^2 - 4\alpha + 2)/8$, which is identical to that found earlier.

Clearly, the two approaches are identical, and the choice between the two is dictated by simplicity of the required algebra. Again, it should be noted that this example can also be solved in almost the same way using the principle of least work, principle 13 on page 554

10.9.6 The dummy load method

Castigliano's second theorem as expressed in eq. (10.110) gives the deflection at the point of application of a concentrated load. This prompts the following question: is it possible to use Castigliano's second theorem to compute the deflection of a structure at a point where no load is applied? The *dummy load method* is a procedure that enables the use of Castigliano's second theorem to compute the deflection at any point of a structure whether or not a concentrated load is applied at that point.

In the first step of the procedure, a fictitious or "dummy load," $\mathcal{P}$, is applied to the structure at the point where the displacement is to be computed. Furthermore, the line of action of this dummy load is aligned with the direction of the desired displacement component. In the second step of the procedure, the displacement component, $\hat{\Delta}$, is computed using Castigliano's second theorem as $\hat{\Delta} = \partial A/\partial \mathcal{P}$. In the last step, the dummy load is removed by setting it equal to zero to find the desired displacement, $\Delta = \lim_{\mathcal{P}\to 0} \hat{\Delta}$. Load $\mathcal{P}$ is just an artifact that enables the use of Castigliano's second theorem, and this observation explains why load $\mathcal{P}$ is called a dummy load, and why the method is called the dummy load method.

The dummy load method can be summarized by the following equation

$$\Delta = \lim_{\mathcal{P}\to 0} \frac{\partial A}{\partial \mathcal{P}}. \tag{10.111}$$

The strain energy, A, must be determined as a function of the applied loads, including the dummy load $\mathcal{P}$, and any redundant quantities. If the material the structure is made of is elastic, but nonlinear, the complementary strain energy, A', must be used instead of the strain energy.

Example 10.19. Tip deflection of a cantilevered beam

Consider a cantilevered beam of length L subjected to a uniform loading p_0, as shown in fig. 10.45. Determine the beam's tip deflection, Δ, using the dummy load method.

Since no concentrated load is applied at the beam's tip, the dummy load method described in section 10.9.6 will be used. In the first step of the procedure, a dummy load, $\mathcal{P}$, is applied at the beam's tip, as illustrated in fig. 10.45. The bending moment distribution in the beam is readily obtained from equilibrium considerations as $M_3(x_1) = p_0(L - x_1)^2/2 + \mathcal{P}(L - x_1)$.

The strain energy in the structure can then be written as

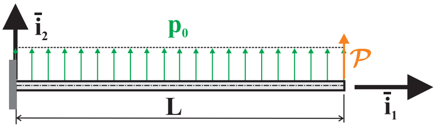

Fig. 10.45. Cantilevered beam under uniform loading.

$$A = \frac{1}{2H_{33}^c} \int_0^1 \left[\frac{p_0 L^2}{2} (1-\eta)^2 + \mathcal{P} L (1-\eta) \right]^2 L \mathrm{d}\eta$$
$$= \frac{1}{2H_{33}^c} \left(\frac{p_0^2 L^5}{20} + \frac{\mathcal{P} p_0 L^4}{4} + \frac{\mathcal{P}^2 L^3}{3} \right).$$

As expected, the strain energy explicitly depends on the dummy load. The second step of the procedure uses Castigliano's second theorem to compute the tip deflection due to all loads as

$$\hat{\Delta} = \frac{\partial A}{\partial \mathcal{P}} = \frac{1}{2H_{33}^c} \left(\frac{p_0 L^4}{4} + \frac{2\mathcal{P} L^3}{3} \right).$$

Finally, the last step of the procedure reveals the desired tip displacement as

$$\Delta = \lim_{\mathcal{P} \to 0} \hat{\Delta} = \frac{p_0 L^4}{8 H_{33}^c}.$$

This result is identical to that obtained using the classical approach; indeed, $\Delta = \bar{u}_2(\eta = 1)$, where the transverse displacement field is given by eq. (5.54). The same result is also obtained using the unit load method described in section 9.7.6.

The solution just presented has scrupulously followed the dummy load procedure described in section 10.9.6. While easily understood, this procedure involves unnecessarily complicated integrations. The desired displacement can be obtained more directly by carrying out the derivative with respect to the dummy load and taking the limit before carrying out the integrations. This is illustrated as follows,

$$\Delta = \left[\frac{\partial}{\partial \mathcal{P}} \int_0^L \frac{M_3^2}{2H_{33}^c} \mathrm{d}x_1 \right]_{\mathcal{P}=0} = \int_0^L \frac{M_3}{H_{33}^c} \left[\frac{\partial M_3}{\partial \mathcal{P}} \right]_{\mathcal{P}=0} \mathrm{d}x_1. \tag{10.112}$$

For the present problem, this yields

$$\Delta = \int_0^1 \left[\frac{p_0 L^2}{2H_{33}^c} (1-\eta)^2 \right] L(1-\eta)\, L\mathrm{d}\eta = \frac{p_0 L^4}{2H_{33}^c} \int_0^1 (1-\eta)^3 \,\mathrm{d}\eta = \frac{p_0 L^4}{8H_{33}^c},$$

which is the same result as before.

Example 10.20. Deflection of a simply supported beam

Consider the simply supported beam of length L subjected to a uniform transverse loading, p_0, as depicted in fig. 10.46. Determine the transverse deflection of the beam at location αL.

Using the dummy load method, a dummy load, $\mathcal{P}$, is added at location αL. To evaluate the strain energy in the structure, the bending moment distribution is computed first as

$$M_3(\eta) = \begin{cases} -p_0 L^2 \eta(1-\eta)/2 - \mathcal{P}L(1-\alpha)\eta, & 0 \leq \eta \leq \alpha, \\ -p_0 L^2 \eta(1-\eta)/2 - \mathcal{P}L\alpha(1-\eta), & \alpha < \eta \leq 1, \end{cases}$$

where $\eta = x_1/L$ is a non-dimensional span-wise variable. The first term represents the contribution of the distributed load, p_0, and the second term represents that due to the dummy load, $\mathcal{P}$.

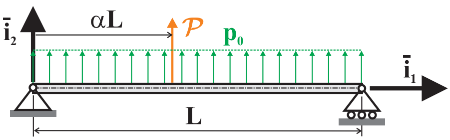

Fig. 10.46. Simply supported beam under uniform loading.

Using the simplified approach given by eq. (10.112), the deflection under the dummy load at $x_1 = \alpha L$ becomes

$$\Delta(\alpha) = \frac{p_0 L^4}{2H_{33}^c} \left[\int_0^\alpha \eta(1-\eta)(1-\alpha)\eta \, d\eta + \int_\alpha^1 \eta(1-\eta)\alpha(1-\eta) \, d\eta \right].$$

Performing the integrations and simplifying leads to

$$\begin{aligned} \Delta(\alpha) &= \frac{p_0 L^4}{2H_{33}^c} \left[(1-\alpha)(\frac{\alpha^3}{3} - \frac{\alpha^4}{4}) + \alpha[\frac{(1-\alpha)^3}{3} - \frac{(1-\alpha)^4}{4}] \right] \\ &= \frac{p_0 L^4}{24H_{33}^c} \left(\alpha^4 - 2\alpha^3 + \alpha \right). \end{aligned}$$

This result gives the transverse displacement at an arbitrary point along the beam and is, in fact, the transverse displacement field for the beam, $\bar{u}_2(\alpha)$, $0 \leq \alpha \leq 1$. This result is identical to that obtained with the classical approach in eq. (5.48), but note that with the present procedure, the transverse displacement field is obtained without having to integrate the governing differential equation of the problem.

10.9.7 Unit load method revisited

The unit load method is developed in section 9.7.6 based on the principle of complementary virtual work. In this section, the same method will be derived from the dummy load method presented in section 10.9.6. The close relationship between the two methods should not come as a surprise: the unit load method is a direct consequence of the principle of complementary virtual work, whereas the dummy load method is derived from Castigliano's second theorem, which itself, stems from the principle of minimum complementary energy. Clearly, both unit and dummy load methods have their roots in the principle of complementary virtual work, and hence, are statements of compatibility conditions.

When using the dummy load method, the expression for the strain energy in an isostatic beam is

$$A = \int_0^L \frac{\mathcal{M}_3^2}{2H_{33}^c} \, \mathrm{d}x_1,$$

where $\mathcal{M}_3(x_1)$ is the bending moment distribution generated by the externally applied loads and the dummy load. Castigliano's second theorem now implies that the deflection at the point of application of the dummy load is

$$\Delta = \lim_{\mathcal{P}\to 0} \frac{\partial A}{\partial \mathcal{P}} = \lim_{\mathcal{P}\to 0} \int_0^L \frac{\mathcal{M}_3}{H_{33}^c} \frac{\partial \mathcal{M}_3}{\partial \mathcal{P}} \mathrm{d}x_1. \tag{10.113}$$

As the dummy load tends to zero, the quantities appearing in this equation can be interpreted as follows.

$$\lim_{\mathcal{P}\to 0} \mathcal{M}_3 = M_3 = \text{bending moment due to externally applied loads only,}$$

$$\lim_{\mathcal{P}\to 0} \frac{\partial \mathcal{M}_3}{\partial \mathcal{P}} = \hat{M}_3 = \text{bending moment due to a unit load only.}$$

With this notation, eq. (10.113) becomes the familiar statement of the unit load method, see eq. (9.83),

$$\Delta = \int_0^L \frac{\hat{M}_3 M_3}{H_{33}^c} \mathrm{d}x_1. \tag{10.114}$$

Although this expression seems to be identical to that derived for the unit load method, important differences exist. The bending moment distribution due to externally applied loads, denoted M_3, is identical for both unit and dummy load methods, and is the bending moment distribution acting in the actual structure under the action of the externally applied loads.

A subtle difference exists, however, between the bending moment distribution, $\hat{M}_3$, defined in the two methods. For the dummy load method, $\hat{M}_3$ is the bending moment acting in the structure subjected to a unit dummy load. For the unit load method, $\hat{M}_3$ is *any statically admissible* bending moment distribution in equilibrium with the unit load. In this latter case, $\hat{M}_3$ is not necessarily the actual bending moment distribution acting in the structure subjected to the unit load, but rather, any statically admissible bending moment distribution in equilibrium with the unit load. Consequently, the unit load method is more versatile, and the fact that any statically admissible bending moment distribution can result in a significant simplification of the procedure.

Example 10.21. Deflection of a hyperstatic beam (Dummy load method)
Consider the cantilevered beam with a mid-span support subjected to a uniformly distributed loading, p_0, as depicted in fig. 10.47. Determine the beam's tip deflection using the dummy load method.

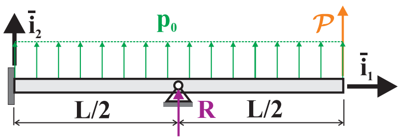

Fig. 10.47. Application of Castigliano's second theorem to calculate deflections in a hyperstatic beam.

Because of the presence of the mid-span support, the system is hyperstatic of order 1. The mid-span reaction force, R, is selected as a redundant quantity, and the bending moment distribution in the bean is then obtained from equilibrium considerations as

$$M_3(\eta) = \begin{cases} \mathcal{P}L(1-\eta) + p_0 L^2(1-\eta)^2/2 + RL(1/2-\eta), & 0 \leq \eta \leq 1/2, \\ \mathcal{P}L(1-\eta) + p_0 L^2(1-\eta)^2/2, & 1/2 \leq \eta \leq 1. \end{cases}$$

First, Castigliano's second theorem (or the principle of least work) is used to find the unknown reaction force, R, by imposing the vanishing of the displacement at the mid-span support, leading to

$$\frac{\partial A}{\partial R} = \frac{\partial}{\partial R} \int_0^L \frac{1}{2} \frac{M_3^2}{H_{33}^c} \mathrm{d}x_1 = \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial R} \, \mathrm{d}x_1 = 0.$$

Introducing the above bending moment distribution, this compatibility condition leads to

$$\int_0^{1/2} \left[\frac{\mathcal{P}L(1-\eta) + p_0 L^2(1-\eta)^2/2 + RL(1/2-\eta)}{H_{33}^c} \right] \left[L\left(\frac{1}{2} - \eta\right) \right] \, L\mathrm{d}\eta = 0.$$

Evaluating the integrals then yields the reaction force at the mid-span support as

$$R = -\frac{5\mathcal{P}}{2} - \frac{17 p_0 L}{16}.$$

Next, the beam's tip deflection, Δ, is computed using the dummy load method as

$$\Delta = \frac{\partial A}{\partial \mathcal{P}} = \int_0^L \frac{M_3}{H_{33}^c} \frac{\partial M_3}{\partial \mathcal{P}} \, \mathrm{d}x_1.$$

Introducing the above bending moment distribution then yields

$$\Delta = \int_0^{1/2} \left[\frac{p_0 L^2(1-\eta)^2/2 + RL(1/2-\eta)}{H_{33}^c} \right] [L(1-\eta)] \; L\mathrm{d}\eta + \int_{1/2}^{1} \left[\frac{p_0 L^2(1-\eta)^2/2}{H_{33}^c} \right] [L(1-\eta)] \; L\mathrm{d}\eta.$$

Regrouping and evaluating these integrals gives the beam's tip deflection as $\Delta = p_0 L^4/(8H_{33}^c) + 5RL^3/(48H_{33}^c)$, and introducing the value of the mid-span reaction force leads to

$$\Delta = \frac{11 p_0 L^4}{768 H_{33}^c}.$$

Example 10.22. Deflection of a hyperstatic beam (Unit load method)
Consider the same cantilevered beam with a mid-span support subjected to a uniformly distributed loading, p_0, treated in example 10.21 and shown in fig. 10.47. Determine the beam's tip deflection, but now using the unit load method.

A cut is made at the mid-span support and the mid-span reaction force, R, is selected as the redundant quantity. The bending moment distribution in the beam due to the externally applied loads is obtained from equilibrium considerations as $M_3(\eta) = p_0 L^2 (1-\eta)^2/2$. Next, a statically admissible bending moment distribution is evaluated that is in equilibrium with a unit load applied upwards to the beam at the cut support point: $\hat{M}_3 = L(1/2-\eta)$ for $0 \le \eta \le 1/2$ and $\hat{M}_3 = 0$ for $1/2 \le \eta \le 1$. The deflection at the support under the externally applied loads is

$$\Delta_c = \int_0^L \frac{M_3 \hat{M}_3}{H_{33}^c} \, \mathrm{d}x_1 = \int_0^{1/2} \left[\frac{p_0 L^2 (1-\eta)^2/2}{H_{33}^c} \right] \left[L(1/2-\eta) \right] \, L \mathrm{d}\eta = \frac{17 p_0 L^4}{384 H_{33}^c}.$$

The deflection at mid-span support due to the unit load alone is

$$\Delta_1 = \int_0^L \frac{\hat{M}_3^2}{H_{33}^c} \, \mathrm{d}x_1 = \int_0^{1/2} \frac{L^2 (1/2-\eta)^2}{H_{33}^c} \, L \mathrm{d}\eta = \frac{L^3}{24 H_{33}^c}.$$

The reaction force at the support is now $R = -\Delta_c/\Delta_1 = -17 p_0 L/16$, from which

$$M_3(\eta) = \begin{cases} p_0 L^2 (1-\eta)^2/2 + RL(1/2-\eta), & 0 \le \eta \le 1/2, \\ p_0 L^2 (1-\eta)^2/2, & 1/2 \le \eta \le 1, \end{cases}$$

which is identical to the result found with the dummy load method, provided that the dummy load is set to zero.

To determine the beam's tip deflection, a unit load is applied at its tip. A statically admissible bending moment distribution that is in equilibrium with this tip unit load is found from equilibrium consideration as $\hat{M}_3 = L(1-\eta)$. When evaluating this bending moment distribution, the mid-span reaction force is set to zero, because all that is required of this distribution is that it be statically admissible for the tip unit load. The beam's tip deflection, Δ, now becomes

$$\Delta = \int_0^L \frac{M_3 \hat{M}_3}{H_{33}^c} \mathrm{d}x_1 = \int_0^{1/2} \left[\frac{p_0 L^2 (1-\eta)^2/2 + RL(1/2-\eta)}{H_{33}^c} \right] \left[L(1-\eta) \right] L \mathrm{d}\eta$$
$$+ \int_{1/2}^1 \left[\frac{p_0 L^2 (1-\eta)^2/2}{H_{33}^c} \right] \left[L(1-\eta) \right] L \mathrm{d}\eta = \frac{11 p_0 L^4}{768 H_{33}^c}.$$

The solution is identical to that found in the previous example.

The choice between the unit and the dummy load methods is largely a matter of convenience, although a hybrid approach using Castigliano's second theorem (or the principle of least work) to find R and the unit load method to find Δ will often lead to simpler integrals.

10.9.8 Problems

Problem 10.18. Cantilevered beam subjected to distributed load
Consider the cantilevered beam subjected to a uniformly distributed load over half its span, as depicted in fig. 10.48. *(1)* Use the dummy load method to compute the deflection of the beam at point **M**.

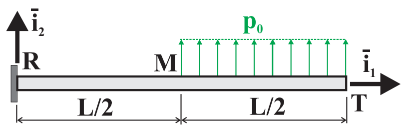

Fig. 10.48. Cantilevered beam subjected to half-span loading.

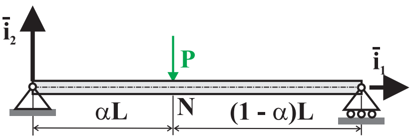

Fig. 10.49. Simply-supported beam under concentrated load

Problem 10.19. Simply supported beam with concentrated load
Consider the simply supported beam subjected to a concentrated load applied at a distance αL from the left support, as depicted in fig. 10.49. *(1)* Use the dummy load method to compute the deflection of the beam at point **N**.

Problem 10.20. Cantilevered beam subjected to triangular loading
Consider the cantilevered beam subjected to a triangular loading, as depicted in fig. 10.50. *(1)* Use the dummy load method to compute the deflection of the beam at point **T**.

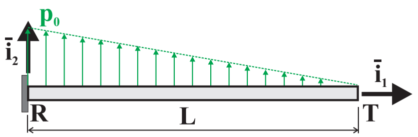

Fig. 10.50. Cantilevered beam subjected to half-span loading.

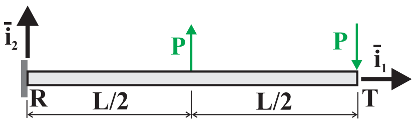

Fig. 10.51. Cantilevered beam subjected to two concentrated loads.

Problem 10.21. Simply supported beam with concentrated load
Consider the cantilevered beam subjected to two concentrated loads, each of magnitude P, as depicted in fig. 10.51. *(1)* Use the dummy load method to compute the deflection of the beam at point **T**.

Problem 10.22. Semi-circular beam with rigid arm
Consider the uniform, semi-circular beam with a rigid arm attached at its tip, as shown in fig. 10.52. The beam is made of a linearly elastic material and the radius of its centerline is R. A load of magnitude P acts at the tip of the rigid arm in the plane of the beam, but its orientation in this plane is otherwise arbitrary. Prove that: *(1)* The displacement, Δ, of point **O** is in the direction of the applied load *for any arbitrary orientation* of P, and *(2)* the spring constant $k = P/\Delta$ is independent of the orientation of the load P. *Hint*: At first, study the behavior of the beam under a horizontal force, H. Next, turn to a vertical force, V. The

behavior of the system under a general loading is then obtained by invoking the principle of superposition for a linear system. You should assume that only bending deformations will contribute to the strain energy in the beam, *i.e.*, ignore axial deformation.

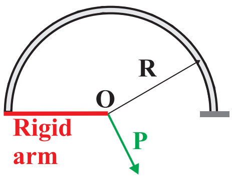

Fig. 10.52. Semi-circular beam with a rigid arm.

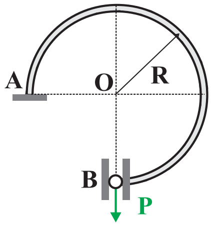

Fig. 10.53. Uniform spring under a vertical load P.

Problem 10.23. Circular beam with vertical tip load
The uniform circular beam with centerline radius R shown in fig. 10.53 is clamped at point **A** and constrained to move in the only in the vertical direction at point **B**, where it is also subjected to an applied vertical load, P. *(1)* Find the displacement, Δ, in the direction of the applied load. *(2)* Find the horizontal reaction Q at point **B**. *(3)* Find the equivalent spring constant $k = P/\Delta$. Assume that only bending deformations are significant, *i.e.*, ignore axial deformation.

Problem 10.24. Deflection of 3-bar truss with different member properties
The three-bar, hyperstatic truss shown in fig. 10.36 is subjected to a tip vertical load P. The three bars have a Young's modulus E, bar 1 is of cross-sectional area $\mathcal{A}$, while that bars 2 and 3 is $2\mathcal{A}$. *(1)* Determine the vertical deflection of the loaded joint. *(1)* Determine the horizontal deflection of the loaded joint.

Problem 10.25. Deflection of cantilevered beam with simple support and concentrated load
A cantilevered beam with a mid-span support carries a tip concentrated load, P, as depicted in fig. 10.38. *(1)* Compute the deflection at the beam's tip.

10.10 Reciprocity theorems

For linearly elastic structures, a useful reciprocity exists between loads applied at one set of locations and deflections produced at another set of locations. This reciprocity can be stated in the form of two theorems that are developed in the following sections.

10.10.1 Betti's theorem

Consider a properly constrained elastic body subjected to various concentrated loads and couples, as shown in fig. 10.40. If the displacements of the points of application of these loads projected along their lines of action are denoted $\Delta_i^{[1]}$, Clapeyron's theorem, eq. 10.107, implies $A^{[1]} = \sum_{i=1}^N P_i^{[1]}\Delta_i^{[1]}/2$, where $A^{[1]}$ is the total strain energy of the system under this loading condition, denoted *state 1*. Let the magnitude of the applied loads be changed to $P_i^{[2]}$ and the corresponding projected displacements will then be $\Delta_i^{[2]}$. In this new state, denoted *state 2*, the points of application of the loads and their lines of action are identical to those in *state 1*. The strain energy in *state 2* is $A^{[2]} = \sum_{i=1}^N P_i^{[2]}\Delta_i^{[2]}/2$. When going from *state 1* to *state 2*, the added work done by the applied loads equals the change in total strain energy

$$A^{[2]} - A^{[1]} = \sum_{i=1}^{N} \frac{P_i^{[2]}\Delta_i^{[2]}}{2} - \sum_{i=1}^{N} \frac{P_i^{[1]}\Delta_i^{[1]}}{2}. \tag{10.115}$$

This difference can be computed in an alternative manner. It is possible to go from *state 1* to *state 2* by gradually adding to the forces $P_i^{[1]}$ of *state 1* the forces $P_i^{[2]} - P_i^{[1]}$ with unchanged lines of action. During this transition, additional work will be done by the forces $P_i^{[1]}$ which remain constant, and the forces, $P_i^{[2]} - P_i^{[1]}$, which are allowed to increase gradually to *state 2*. The work done by $P_i^{[1]}$ is $\sum_{i=1}^N P_i^{[1]}(\Delta_i^{[2]} - \Delta_i^{[1]})$, and the work done by the gradually increasing forces, $P_i^{[2]} - P_i^{[1]}$, is $\sum_{i=1}^N (P_i^{[2]} - P_i^{[1]})(\Delta_i^{[2]} - \Delta_i^{[1]})/2$. Thus, the change in strain energy between the two states is equal to the work done by these forces, and this can be written as

$$\begin{aligned} A^{[2]} - A^{[1]} &= \sum_{i=1}^{N} P_i^{[1]}(\Delta_i^{[2]} - \Delta_i^{[1]}) + \sum_{i=1}^{N} \frac{(P_i^{[2]} - P_i^{[1]})(\Delta_i^{[2]} - \Delta_i^{[1]})}{2} \\ &= \frac{1}{2}\sum_{i=1}^{N} (P_i^{[2]} + P_i^{[1]})(\Delta_i^{[2]} - \Delta_i^{[1]}). \end{aligned} \tag{10.116}$$

Comparing eqs. (10.115) and (10.116) then yields

$$\sum_{i=1}^{N} P_i^{[1]}\Delta_i^{[2]} = \sum_{i=1}^{N} P_i^{[2]}\Delta_i^{[1]}. \tag{10.117}$$

This result be interpreted as follows

Theorem 10.5 (Reciprocity theorem or Betti's theorem). *A linearly elastic body is subjected to two loading states characterized by loads of different magnitudes but identical points of applications and lines of action. The sum of the product of the loads in one state by the projected displacements of the other is identical to that obtained when the two states are interchanged.*

Because Betti's theorem is a direct consequence of Clapeyron's theorem, theorem 10.1, both theorems are valid for the same loading cases. The loadings defining *states 1* and *2* can involve one or more of the following *(1)* a concentrated load and the projected displacement of its point of application, *(2)* a couple and the projected rotation at its point of application, *(3)* two opposite forces of identical magnitude sharing a common line of action and the projected relative displacement of their points of application, and *(4)* two opposite couples of identical magnitude sharing a common line of action and the projected relative rotation at their points of application.

10.10.2 Maxwell's theorem

Let the simply supported beam depicted in fig. 10.54 be subjected to two loading states. *State 1* consists of load $P^{[1]}$ applied at point **1**, whereas *state 2* consists of load $P^{[2]}$ applied at point **2**. For *state 1*, the displacements at points **1** and **2** will be denoted $\Delta_1^{[1]}$ and $\Delta_2^{[1]}$, respectively. Similarly, the corresponding displacements for *state 2* are $\Delta_1^{[2]}$ and $\Delta_2^{[2]}$, respectively.

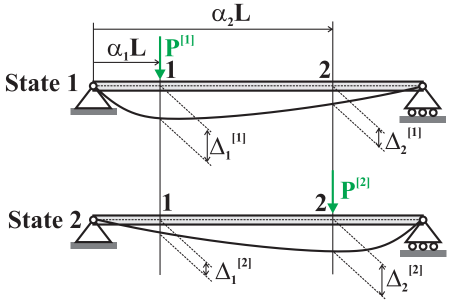

Fig. 10.54. Simply supported beam under two loading states.

If the structure is made of a linearly elastic material, Betti's theorem, theorem 10.5, is applicable and implies

$$P^{[1]}\,\Delta_1^{[2]} = P^{[2]}\,\Delta_2^{[1]}, \text{ or } \quad \frac{\Delta_1^{[2]}}{P^{[2]}} = \frac{\Delta_2^{[1]}}{P^{[1]}}. \tag{10.118}$$

The *influence coefficient* is defined as the displacement at a point due to the application of a unit load at another point. For instance, influence coefficient η_{12} gives the displacement at point **1** due to the application of a unit load at point **2**. Clearly, $\eta_{12} = \Delta_1^{[2]}/P^{[2]}$ and η_{21}, the displacement at point **2** due to a unit load applied at point **1**, is $\eta_{21} = \Delta_2^{[1]}/P^{[1]}$. Eq. (10.118) becomes

$$\eta_{12} = \eta_{21}. \tag{10.119}$$

This result be interpreted as follows.

Theorem 10.6 (Maxwell's theorem). *For a linearly elastic structure, the influence coefficient of point **1** on point **2** equals that of point **2** on point **1**, for any choice of points **1** and **2**.*

Maxwell's theorem is a simple corollary of Betti's theorem, and applies to all the loading conditions for which Betti's theorem is valid. Hence, the concept of influence coefficient should be understood as the projected displacement or rotation at a point, relative displacement of two points, or relative rotation at two points due to any of the following four loading types applied at another location: a concentrated load or couple, or two opposite forces of identical magnitude sharing a common line, or two opposite couples of identical magnitude sharing a common line of action.

Example 10.23. Simply supported beam

Consider a simply supported, uniform beam of length L as shown in fig. 10.54. Let points **1** and **2** be located at distances $\alpha_1 L$ and $\alpha_2 L$ from the left end support.

The influence coefficient η_{12} can be evaluated from the exact solution of the problem given in example 5.5. In this case, η_{12} is the displacement at point **1** when the beam is subjected to a unit load at point **2**. Equation (5.51) gives this displacement as

$$\begin{aligned}\eta_{12} &= \frac{L^3}{6H_{33}^c}\left[-(1-\alpha_2)\alpha_1^3 + \alpha_2(2-\alpha_2)(1-\alpha_2)\alpha_1\right] \\ &= \frac{L^3}{6H_{33}^c}\left[-\alpha_1^3 + \alpha_1^3\alpha_2 + 2\alpha_1\alpha_2 - 3\alpha_1\alpha_2^2 + \alpha_1\alpha_2^3\right].\end{aligned}$$

The following values are used in eq. (5.51): $P = 1$ because a unit load is applied, $\alpha = \alpha_2$ because this load is applied at $\eta = \alpha_2$, and $\eta = \alpha_1$ to find the displacement $\bar{u}_2(\alpha_1)$.

The influence coefficient η_{21}, corresponding to the displacement at point **2** when the beam is subjected to a unit load at point **1** is evaluated in a similar manner, to find

$$\begin{aligned}\eta_{21} &= \frac{L^3}{6H_{33}^c}\left[\alpha_1(\alpha_2^3 - 3\alpha_2^2) + \alpha_1(2+\alpha_1^2)\alpha_2 - \alpha_1^3\right] \\ &= \frac{L^3}{6H_{33}^c}\left[\alpha_1\alpha_2^3 - 3\alpha_1\alpha_2^2 + 2\alpha_1\alpha_2 + \alpha_1^3\alpha_2 - \alpha_1^3\right].\end{aligned}$$

The following values are used in eq. (5.51): $P = 1$ because a unit load is applied, $\alpha = \alpha_1$ because this load is applied at $\eta = \alpha_1$, and $\eta = \alpha_2$ to find the displacement $\bar{u}_2(\alpha_2)$. As expected, $\eta_{12} = \eta_{21}$, in accordance with Maxwell's theorem.

Example 10.24. Symmetry of the flexibility matrix

The concept of flexibility matrix is introduced in example 5.9 on page 203. Since the flexibility matrix simply stores the influence coefficients, see eq. (5.58), Maxwell's theorem implies the *symmetry of the flexibility matrix*. In example 5.10 on page 204, the flexibility matrix of a cantilevered beam is determined analytically, see eq. (5.60), and is found to be symmetric, as expected.

Of course, if the flexibility matrix is determined experimentally according to the procedure described in example 5.9, the symmetry condition will only be satisfied within the bounds of experimental errors. Consider the case of the cantilevered beam depicted in fig. 5.28 on page 203. Table 10.2 lists the displacements measured on a cantilevered beam subjected to three loading cases. The first column of

Table 10.2. Measured displacements for the three loading cases. Load are measured in kN, displacements in mm.

	$P_1 = 1.5$	$P_2 = 1.0$	$P_3 = 0.5$
Δ_1	10.9	18.3	14.6
Δ_2	27.7	59.1	51.1
Δ_3	43.1	104.	98.5

this table lists the displacements at locations $\alpha_1 L$, $\alpha_2 L$, and $\alpha_3 L$ for $P_1 = 1.5$ kN, $P_2 = P_3 = 0$. The next two columns list the corresponding data for $P_2 = 1.0$ kN, $P_1 = P_3 = 0$ and $P_3 = 0.5$ kN, $P_1 = P_2 = 0$, respectively. The influence coefficients are easily obtained from the experimental data: for instance, $\eta_{11} = \Delta_{11}/P_1 = 10.9\ 10^{-3}/1.5\ 10^3 = 7.27\ 10^{-6}$ m/N. Proceeding similarly with all other influence coefficients, the flexibility matrix is found as

$$\underline{\underline{F}} = \begin{bmatrix} 0.0073 & 0.0183 & 0.0292 \\ 0.0185 & 0.0591 & 0.1022 \\ 0.0287 & 0.1040 & 0.1970 \end{bmatrix} 10^{-3}\ \mathrm{m/N},$$

which is not exactly symmetric. From the measurements, the influence coefficient, η_{23}, can be estimated by averaging the corresponding entries of the measured flexibility matrix as $\bar{\eta}_{23} = \bar{\eta}_{32} \approx (0.1022\ 10^{-3} + 0.1040\ 10^{-3})/2 = 0.1031\ 10^{-3}$ m/N. The estimated displacements now becomes $\bar{\Delta}_{23} = \bar{\eta}_{23} P_3 = 51.6$ mm and $\bar{\Delta}_{32} = \bar{\eta}_{23} P_2 = 103.1$ mm. The relative experimental error is now $e = |\bar{\Delta}_{23} - \Delta_{23}|/\Delta_{23} \approx 0.9\%$ or $e = |\bar{\Delta}_{32} - \Delta_{32}|/\Delta_{32} \approx 0.9\%$. An error estimation using the other off-diagonal terms of the flexibility matrix reveals a relative error of approximatively the same magnitude. Based on Maxwell's theorem, the experimental error is of the order of one percent.

10.10.3 Problems

Problem 10.26. Direct proof of Maxwell's theorem

Prove Maxwell's theorem directly from the unit load method. Use the simply supported beam depicted in fig. 10.54to support your reasoning.

Problem 10.27. Direct proof of Maxwell's theorem

Prove Maxwell's theorem directly from the dummy load method. Use the simply supported beam depicted in fig. 10.54to support your reasoning.