4

Engineering structural analysis

Solutions of the fifteen governing equations of linear elasticity are not easy to develop for practical problems. Chapter 3 outlined the complexity of the problem and presented solutions to a few of the simpler practical problems that can be easily treated. These equations define what is commonly called the *linearly elastic theory of solid mechanics* or more simply *linear elasticity* theory. Engineers and mathematicians have studied these equations for more than two centuries, and their efforts to develop solutions have led to broad areas of applied mathematics. The range of useful analytical solutions, however, still remains quite limited, and the problems for which solutions are available are usually very simple.

To analyze practical structures that are generally more complicated, it is almost always necessary to make judicious simplifications that reduce the governing equations to a form that can be solved with modest effort. This approach is widely referred to as *engineering structural analysis* or more simply *structural analysis*. These efforts have produced a rich collection of solutions to practical problems, and structural analysis is an important part of many areas of engineering. The chapters that follow treat the subject of structural analysis, but the developments are based on the fundamental theory of solid mechanics presented in the first three chapters. This chapter introduces basic solution processes, and subsequent chapters extend them to a range of useful structural elements.

4.1 Solution approaches

One of the most direct ways to simplify solid mechanics problems is to reduce their dimensionality. The plane stress and plane strain assumptions presented in sections 1.3 and 1.6, respectively, reduce three-dimensional problems to two-dimensional problem. In some cases, a problem can be further simplified to a one-dimensional form. For example, plane stress problems presenting cylindrical symmetry involve stress and strain fields that are functions of only the radial variable when polar coordinates are used to formulate the problem.

In addition to the simplifications mentioned above, various procedures are available to solve the resulting governing equations. Depending on the problem, different solution procedures may require vastly different analytical skills and/or computational efforts.

In general, the objective of structural analysis is to determine the stress and deformation fields that arise from applied loads. Once appropriate simplifications have been made, two approaches to the solution of the problem are possible.

1. In the first approach, a solution for the stress field is developed based on the equilibrium equations of the problem (and possibly using the compatibility equations). Next, the strain field is obtained from the stress field with the help of the constitutive laws. Finally, the strain-displacement equations are integrated to obtain the displacement field. As illustrated in example 3.1, this last step is often very tedious, even for the simplest problem. In addition, because three displacement components must be determined from the six components of the strain field, it is often necessary to invoke the auxiliary compatibility equations, eqs. 1.106. Note that the solution process sequentially moves through the three groups of equations of elasticity.
2. In the second approach, the solution process invokes the three groups of equations of elasticity in the reverse order. First, a set of purely kinematic assumptions are formulated. Typically, the displacement field of the structure under load is assumed. Next, the strain-displacement equations are used to evaluate the strain field, and the constitutive laws then yield the corresponding stress field. Finally, substitution of the stress components in the equilibrium equations leads to a complete solution of the problem.

To illustrate these two solution approaches, this chapter examines the simplest, one-dimensional problems involving a single, direct stress component that can be either tensile or compressive. A slender, homogeneous prismatic bar subjected only to axial loads is a structural component that meets these conditions. The analysis of this type of components and the associated solution procedures are described in the following sections for a variety of such structures. In the process, the two fundamental solution procedures described above are examined in more detail, and solutions are developed for a number of practical cases.

4.2 Bar under constant axial force

Figure 4.1 depicts an idealized problem consisting of an infinitely long, homogeneous bar with constant properties along its span and subjected to end loads P. The first step to the development of an approximate model for this structural component is to describe its kinematic behavior, *i.e.*, to describe how the component deforms under load. Since the axial load and physical properties are constant along the span, the local deformation of the bar must be identical at all points along its span.

Consider now an initially plane cross-section, $\mathcal{S}$, at a point along the span of the bar as shown in fig. 4.1. All the material particles that form cross-section $\mathcal{S}$ before deformation will form a new section, $\mathcal{S}'$, after deformation. The symmetry of the problem requires the two semi-infinite halves of the deformed bar to be identical, and therefore, the deformed section, $\mathcal{S}'$, must remain *planar and normal to the axis of the bar*.

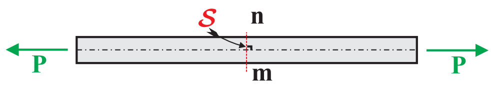

Fig. 4.1. Infinitely long bar under end loads.

For the more realistic problem of a bar of finite length, the above conclusions still hold, except for the portions of the bar near the end points where complex stress distributions may arise. For instance, if the bar is held in the grips of a testing machine, complex stress and displacement fields will develop under the grips. Very different stress and displacement fields will develop in a bar that is loaded by a pin passing through a hole drilled in the bar. In both cases, however, displacements and stresses will eventually become uniform through the cross-section, at large distances from these end zones. The solution developed here is not valid in these end zones, but it does apply in the portions of the bar that are a good distance from these end zones, as implied by Saint-Venant's principle, principle 2 on page 169.

Consider again cross-section nm shown in fig. 4.1. Since the cross-sections of the bar must remain planar, the axial deformation must be identical at all points of the section, and the axial strain, ϵ_1, will also be uniform over the cross-section. Clearly, from the basic definition of extensional strain, it follows that $\epsilon_1 = e/L$, where L is the length of the bar unaffected by the end regions, and e its change in length resulting from the applied load.

If the bar is slender, it is reasonable to assume that the direct stress components in the transverse direction, σ_2 and σ_3, are much smaller than the component, σ_1, aligned with the applied load. This means that $\sigma_2 \approx 0$ and $\sigma_3 \approx 0$. Finally, if the load is not excessive, stress and strain components remain proportional to each other. Hooke's law then applies and eq. (2.1) reduces to $\sigma_1 = E\epsilon_1$.

Since the axial stress component, σ_1, is assumed to be uniformly distributed over the cross-section, equilibrium of the section then requires that

$$\sigma_1 = \frac{P}{\mathcal{A}}, \tag{4.1}$$

where $\mathcal{A}$ is the cross-sectional area of the bar. The elongation of the bar resulting from the application of the load is now easily found as

$$e = \epsilon_1 L = \frac{\sigma_1 L}{E} = \frac{PL}{E\mathcal{A}}. \tag{4.2}$$

The above results are valid for both tensile and compressive load. However, in the case of compressive loads, the equilibrium configuration of the bar might become unstable as the load increases, leading to lateral buckling of the bar; this subject is treated in chapter 14.

Equation (4.2) shows that the elongation of the bar is proportional to the applied load; this can be emphasized by recasting the equation as $e = P/k$, where k is the *axial stiffness of the bar* given by

$$k = \frac{E\mathcal{A}}{L}. \tag{4.3}$$

Under an axial force, the bar behaves like a simple spring of constant stiffness, k, subjected to the same load.

One of the most common structural components that can be modeled as a bar under an axial force is a bar in a truss structure. A truss is a two or three-dimensional structure consisting of slender bars pinned at their ends to joints, which allow only axial forces to be transmitted into each member. In chapter 5, the simple model developed here will be extended to treat a broader class of slender bar problems featuring anisotropic materials, nonuniform cross-sections, and subjected to distributed axial loads varying along the bar's span.

The solution approaches outlined in section 4.1 will now be illustrated for axially loaded uniform bars in several examples.

Example 4.1. Series connection of axially loaded bars
The simplest example of bars subjected to axial forces is a series of bars connected in a straight line and subjected to axial forces applied at the bar ends. Figure 4.2 depicts a configuration featuring two bars connected in series; the left bar is clamped at point **A**, whereas the second bar is loaded by force P at point **C**. An axial load, $3P$, is applied at the junction point **B** between the two bars.

For this problem, the axial force equilibrium conditions can be written for each joint as shown in fig. 4.2. It then follows from equilibrium equations at points **B** and **C** that

$$F_{AB} = 4P, \quad F_{BC} = P, \tag{4.4}$$

where F_{AB} and F_{BC} are the axial forces in bar **AB** and **BC**, respectively. The sign convention used here and consistently throughout this book is that a tensile force in the bar is positive; this is the same convention used for the direct stress components.

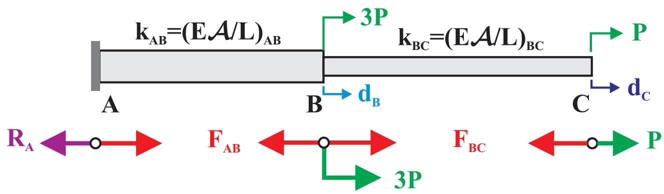

Fig. 4.2. Two bars connected in series and subjected to two loads.

Next, the constitutive law, eq. (4.2), is used to find the extension of each bar as

$$e_{AB} = \frac{4P}{k_{AB}} \quad \text{and} \quad e_{BC} = \frac{P}{k_{BC}}$$

where $k_{AB} = (E\mathcal{A}/L)_{AB}$ and $k_{BC} = (E\mathcal{A}/L)_{BC}$ are the axial stiffnesses of bars **AB** and **BC**, respectively. The notation $(E\mathcal{A}/L)_{AB}$ is used as a shorthand notation for the more cumbersome $E_{AB}\mathcal{A}_{AB}/L_{AB}$, where L_{AB}, $\mathcal{A}_{AB}$ and E_{AB} denote the length, cross-sectional area, and Young's modulus, respectively, for bar **AB**. Similar conventions are used for bar **BC**. Finally, the overall extension of the bar, which is the displacement of point **C**, is found from the compatibility condition, $d_C = e_{AB} + e_{BC}$, to yield

$$d_C = e_{AB} + e_{BC} = \left(\frac{4}{k_{AB}} + \frac{1}{k_{BC}} \right) P.$$

This is a particularly simple example not because two bars only are present, but rather because the forces in the bars and the reaction force at point **A** can be found from equilibrium considerations alone. The deflections then follow immediately from the force-deformation equations.

Example 4.2. Series connection of axially loaded bars (displacement approach)
Consider now the situation shown in fig. 4.3, which is similar to that depicted in fig. 4.2, except that both ends of the system, at points **A** and **C**, are now fixed and only the load applied at point **B** remains. In this case, the problem involves two reactions forces, R_A and R_C, and two bar forces, F_{AB} and F_{BC}, for a total of four unknowns. On the other hand, only three equations of equilibrium can be written, one at each of the three joints: $R_A = F_{AB}$, $F_{BC} - F_{AB} + 3P = 0$, and $R_C = F_{BC}$.

In contrast with the previous example, the equilibrium equations are no longer sufficient to determine the bar forces. Such problems are known as *hyperstatic* systems, or "statically indeterminate," or "statically redundant" systems in contrast with *isostatic* or "statically determinate" systems, such as that presented in example 4.1.

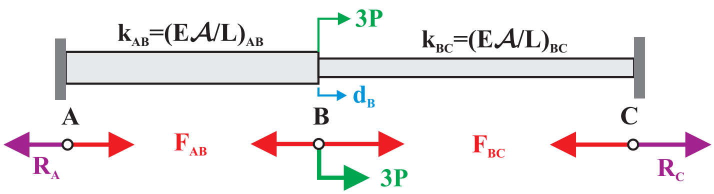

Fig. 4.3. Two bars connected in series with ends fixed.

To find the solution of this problem, deformations must also be considered. The constitutive laws of the system can be expressed as $e_{AB} = F_{AB}/k_{AB}$ and $e_{BC} = F_{BC}/k_{BC}$ for bars **AB** and **BC**, respectively. Introducing these results into the equilibrium equation for point **B** yields

$$k_{AB}\, e_{AB} - k_{BC}\, e_{BC} = 3P. \tag{4.5}$$

Finally, the kinematics of the system are used to express bar extensions in terms of the displacements of points **B** and **C** as $d_B = e_{AB}$ and $d_C = e_{AB} + e_{BC}$, respectively. The displacement at point **C**, however, must vanish because this point is clamped, $d_C = 0$, which implies $e_{AB} = -e_{BC}$ and $d_B = e_{AB} = -e_{BC}$.

Introducing these results into eq. (4.5) then yields a single equation for the unknown displacement at point **B**, $(k_{AB} + k_{BC})\, d_B = 3P$. This is the equilibrium equation of the problem written in terms of the unknown displacement, d_B. This equation can be solved for the displacement, d_B, and the bar elongations can then be computed as $e_{AB} = -e_{BC} = d_B = 3P/(k_{AB} + k_{BC})$. Back substitution yields the forces in the bars

$$F_{AB} = k_{AB}\, e_{AB} = \frac{3k_{AB}P}{k_{AB} + k_{BC}}, \quad F_{BC} = -k_{BC}\, e_{BC} = -\frac{3k_{BC}P}{k_{AB} + k_{BC}}. \tag{4.6}$$

It is interesting to compare these internal forces with those obtained for the isostatic problem in example 4.2, see eq. (4.4). In the solution of the isostatic problem, the internal forces only depend on the externally applied loads, whereas in the solution of the hyperstatic problem, the internal forces depend on the applied loads, as expected, but also on the stiffness of the structure: indeed, the stiffnesses of the bars, k_{AB} and k_{BC}, appear in the final answer.

Example 4.3. Series connection of axially loaded bars (force approach)
The problem presented in the previous example, see fig. 4.3, will be analyzed again, but a different solution procedure will be followed. As noted previously, the problem involves two reactions forces, R_A and R_C, and two bar forces, F_{AB} and F_{BC}, for a total of four unknowns. Only three equations of equilibrium can be written, one at each of the three joints: $R_A = F_{AB}$, $F_{BC} - F_{AB} + 3P = 0$, and $R_C = F_{BC}$.

If any one of the four internal forces is known, the three others can be directly determined from the equilibrium equations. For instance, if F_{AB} is known, all other internal forces can be readily computed. More formally, the force in bar **AB**, denoted R, is assumed to be known. The three equilibrium equations then yield $F_{BC} = R - 3P$, $R_A = R$, and $R_C = R - 3P$.

The next step is to substitute these forces into the constitutive equations to determine the system deformation, *i.e.*, the bar extensions, as

$$e_{AB} = \frac{F_{AB}}{k_{AB}} = \frac{R}{k_{AB}}, \quad e_{BC} = \frac{F_{BC}}{k_{BC}} = \frac{R - 3P}{k_{BC}}.$$

Next, the strain-displacement equations express the relationship between the system deformations and the displacements of points **A**, **B**, and **C**. Figure 4.3 shows that $d_A = d_C = 0$ and $d_B = e_{AB}$, but the compatibility of deformation between the fixed points **A** and **C** also requires $e_{AB} + e_{BC} = 0$. This compatibility condition provides the necessary equation to solve for R,

$$e_{AB} + e_{BC} = \frac{R}{k_{AB}} + \frac{R - 3P}{k_{BC}} = 0, \quad \text{or} \quad R = \frac{3k_{AB}}{k_{AB} + k_{BC}}\, P. \tag{4.7}$$

Finally, the equilibrium equations yield $F_{AB} = R = 3P\, k_{AB}/(k_{AB} + k_{BC})$ and $F_{BC} = R - 3P = -3P\, k_{BC}/(k_{AB} + k_{BC})$. The displacement of point **B** becomes

$d_B = e_{AB} = F_{AB}/k_{AB}$. The solution is identical to that found in the previous example using the displacement approach.

In the force method, the determination of the unknown force, R, is based on the enforcement of compatibility conditions for system deformations. While this process is carried out here in abstract, mathematical terms, a physical description of the procedure is often helpful in formulating the solution.

In the first step, the system is assumed to be "cut" at a location that reveals the unknown internal force, R. Since this force acts in bar **AB**, the cut is made at an arbitrary point along this bar, for instance at point **A**, as depicted in fig. 4.4.

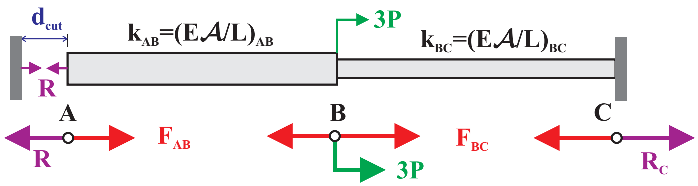

Fig. 4.4. Two bars connected in series with ends fixed and a cut at point **A**.

In the second step, under the action of the externally applied loads, a relative displacement of the two sides of the cut, denoted $d_{\rm cut}$, will develop. Of course, in the real system this cut does not exist, *i.e.*, $d_{\rm cut} = 0$. It is convenient to think of force R as an externally applied load, as illustrated in fig. 4.4. The extensions of the two bars can be written in terms of the forces as

$$e_{AB} = \frac{F_{AB}}{k_{AB}} = \frac{R}{k_{AB}}, \quad e_{BC} = \frac{F_{BC}}{k_{BC}} = \frac{R-3P}{k_{BC}}.$$

In the third step, the compatibility condition is enforced. The displacement at the cut is the sum of the elongations of the two bars, $d_{\rm cut} = e_{AB} + e_{BC}$. In this example, the displacement is positive if the two sides of the cut overlap and negative when a gap forms between the two sides of the cut. In the actual system, the cut is not present and the relative displacement at the cut must vanish: $d_{\rm cut} = 0$. This condition leads to

$$d_{\rm cut} = e_{AB} + e_{BC} = R/k_{AB} + (R-3P)/k_{BC} = 0.$$

This equation expresses the displacement compatibility at the cut, and it is written in terms of forces and flexibilities (*i.e.*, the inverse of stiffnesses). The equation can be solved for the unknown force, R, as

$$R = \frac{3/k_{BC}}{1/k_{AB} + 1/k_{BC}}\, P = \frac{3k_{AB}}{k_{AB} + k_{BC}}\, P.$$

It then follows that $F_{AB} = R = 3P\; k_{AB}/(k_{AB} + k_{BC})$ and $F_{BC} = R - 3P = -3P\; k_{BC}/(k_{AB} + k_{BC})$, and finally, $d_B = e_{AB} = F_{AB}/k_{AB}$.

4.3 Hyperstatic systems

The examples treated in the previous section reveal fundamental differences between two types of systems that are commonly encountered in structural analysis. For some systems, the number of equations of equilibrium is *equal* to the total number of unknown internal forces. Internal forces include reaction forces and forces acting in the members of the system. Such systems are called *statically determinate* or *isostatic* systems. The term "isostatic," where "iso" means "the same," refers to the fact that the number of equilibrium equations is *the same* as the number of force unknowns. For isostatic problems, the unknown forces can be determined from the equations of equilibrium alone, *without* using the strain-displacement equations or constitutive laws. This is a special situation since, in general, the solution of elasticity problems requires the simultaneous solution of the three fundamental groups of equations: the equilibrium, strain-displacement, and constitutive equations. A very simple isostatic system is treated in example 4.1.

For other systems, the total number of unknown internal force and reactions is *larger than* the number of equilibrium equations. Such systems are called *statically indeterminate* or *hyperstatic* systems. The term "hyperstatic," where "hyper" means "larger," refers to the fact the number of force unknowns is *larger than the number of equilibrium equations*. In this case, the equilibrium equations are not sufficient to determine the internal forces in the system. The equilibrium equations by themselves present an infinite number of solutions.

The *degree of redundancy*, N_R, of a system is defined as the number of unknown internal forces minus the number of equations of equilibrium. For instance, the problem presented in example 4.2 features four unknown internal forces and three equations of equilibrium. Hence, its degree of redundancy is $N_R = 4 - 3 = 1$; the system is referred to as having *a single degree of redundancy* or being *hyperstatic of order 1*. The treatment of hyperstatic systems will require the simultaneous solution of the three fundamental groups of equations to evaluate all the unknown quantities of the problem.

The difference between iso- and hyperstatic systems might appear to be rather technical at first, but it is, in fact, very fundamental. A few of the key differences are discussed in the following paragraphs.

First, the solution procedure for the two types of systems is different. For isostatic systems, the equations of equilibrium are written first, then immediately solved for the unknown internal forces. Indeed, no other equations are needed to evaluate these forces. It is only when evaluating deformations and displacements that the constitutive laws and then the strain-displacement equations must be invoked. In contrast, the solution process for hyperstatic problems is somewhat more complex. The equilibrium equations cannot be solved independently of the other two sets of equations of elasticity, the strain-displacement equations and the constitutive laws. Clearly, hyperstatic problems are inherently more difficult to solve because the three sets of equations of elasticity shown in fig. 3.1 are now coupled.

Two main approaches are available for the solution of these coupled equations: the displacement method and the force method, which are presented in examples 4.2

and 4.3, respectively. These two solution procedures will be more formally developed in the next section.

A second difference is observed in the nature of the solution for the unknown internal forces. Compare the expressions given in eqs. (4.4) and (4.6) for the internal forces of an isostatic and a hyperstatic problem, respectively. For isostatic systems, internal forces can be expressed in terms of the externally applied loads, whereas for hyperstatic systems, internal forces depend on the applied loads, as expected, but also on the stiffness of the structure because the bar stiffnesses, k_{AB} and k_{BC}, appear in the final answer. This difference reflects the fact that the solution process for hyperstatic systems requires the use of the material constitutive laws. Consequently, material stiffness characteristics, such as the Young's modulus of the material, explicitly appear in the expressions for the internal forces. In other words, the internal force distribution in hyperstatic systems depends on the stiffness characteristics of the structure, whereas for isostatic systems, this distribution is independent of structural stiffnesses.

The third difference is best explained by considering once again the iso- and hyperstatic systems treated in examples 4.2 and 4.3, respectively. The hyperstatic system features two load paths: one load path, bar **AB**, carries a portion of the applied load to the ground, *i.e.*, to a fixed support while the other load path, bar **BC**, carries the remaining portion of the applied load to the other support. This system is said to present "dual load paths," see fig. 4.3. This contrasts with the isostatic problem that features a single load path: the applied loads are carried back to the single support at point **A** through the serially-connected bars **AB** and **BC**, see fig. 4.2. In the hyperstatic system, the equilibrium equations are not sufficient to determine how much of the load will be carried by load path **AB** and how much will be carried by load path **BC**. In fact, the applied load is split between the two load paths according to their relative stiffnesses, $F_{AB}/F_{BC} = -k_{AB}/k_{BC}$, where the minus sign reflects the sign conventions for the bar internal forces. The stiffer load path will carry more load than the more compliant one.

Systems with multiple load paths are inherently more damage tolerant than systems with a single load path. Indeed, if bar **AB** fails, the single load path system can no longer carry any load, whereas the dual load path system might still be able to carry the applied load, assuming that bar **BC** is designed to safely carry the entire load in the event of a failure of the other bar.

4.3.1 Solution procedures

Two general approaches are available for the solution of hyperstatic systems. The first approach is illustrated in example 4.2 and involves the following steps. First, write the equilibrium equations of the system. Second, use the constitutive laws to express internal forces in terms of member deformations. Third, use the strain-displacement equations to express system deformations in terms of displacements.

At this point, all the equations of elasticity have been written: the rest of the procedure manipulates these equations to obtain the solution of the problem. The deformations written in terms displacements are introduced in the constitutive laws to find the internal forces in terms of displacements, and finally, these internal forces are introduced into the equilibrium equations to yield the equations of equilibrium expressed in terms of displacements. Solution of these equilibrium equations then yield the displacements of the system. Deformations then follow by back substituting the displacements in the strain-displacement equations; finally, the internal forces are obtained from the constitutive laws by back substitution of the deformations. This solution approach is called the *displacement method* or the *stiffness method,* because the governing equations are equilibrium equations written in terms of unknown displacements and component stiffnesses.

The second approach is illustrated in example 4.3 and involves the following steps. First, write the equilibrium equations of the system. Next, determine the system degree of redundancy, N_R, which equals the number of unknown internal forces minus the number of equations of equilibrium. The system is now "cut" at N_R locations. At each of the N_R cuts, a *redundant force* is assumed to act, and a single relative displacement is defined to measure the relative displacement across the cut.

With the addition of these N_R cuts and the specification of the N_R redundant forces, the *originally hyperstatic system is transformed into an isostatic system* for which the internal forces can be determined in terms of the applied loads and the N_R redundant forces from the equilibrium equations alone, *i.e.*, the redundant forces are treated as externally applied loads. Next, the relative displacements at the N_R cuts are determined by first invoking the constitutive laws to yields system deformations in terms of the applied loads and the N_R redundant forces. Finally, the strain displacement equations can be used to find the relative displacements at the N_R cuts. The original hyperstatic system, however, cannot develop these relative displacements because it has no cuts. These compatibility requirements impose the vanishing of the relative displacements at the cuts, and this leads to a set of N_R equations for the N_R redundant forces. This approach is called the *force method* or the *flexibility method* because the governing equations express compatibility requirements in terms of the redundant forces and component flexibilities.

The displacement and force methods are general solution procedures that can be used to solve a wide range of hyperstatic problems. Hence, it is useful to formally describe these procedures in details. For clarity and simplicity, each step of the procedures is explained in terms of the structural components and variables encountered in the analysis of axially loaded bars. In later chapters, the same methods will be generalized for application to other, more complex structural components and systems.

4.3.2 The displacement or stiffness method

The displacement method focuses on expressing the governing equilibrium equations in terms of displacements, and the resulting equations are solved for these displacements. The forces and moments in the system are then computed from the displacements using the force-deformation relationships. This can be formalized in the following steps.

1. Write the *equilibrium equations of the system*. Equilibrium conditions express the vanishing of the sum of the forces and moments acting on the system. This step typically involves construction of free body diagrams of the various subcomponents of the system, and then formulation of the equilibrium conditions.
2. Use the *constitutive laws* to express internal forces in terms of member deformations or strains.
3. Use the *strain-displacement equations* to express system deformations in terms of displacements. At this point the three groups of equations of elasticity have been utilized. The total number of unknowns of the problem should be equal to the total number of equations.
4. *Introduce the deformations-displacements equations* derived in step 3 *into the constitutive laws* derived in step 2 to find the internal forces in terms of displacements.
5. *Introduce the internal forces* derived in step 4 *into the equilibrium equations* derived in step 1 to yield the equations of equilibrium expressed in terms of displacements.
6. *Solve the equilibrium equations* derived in step 5 to find the displacements of the system.
7. *Find system deformations* by back-substituting the displacements into the strain-displacement equations derived in step 3.
8. *Find system internal forces* by back-substituting the deformations into the constitutive laws derived in step 2.

The displacement method focuses first on determining the displacement of the system, and system deformations are then obtained by back substituting displacements into the strain-displacement equations. Finally, the internal forces follow from back-substitution of deformations into the constitutive laws. The number of displacement variable is exactly equal to the number of equilibrium equations. All equilibrium equations will involve one or more displacement variables, and hence, the solution for the displacements in step 6 typically requires the solution of a set of linear equations. If this system of equations is large, computational tools will ease the solution process.

Example 4.4. Hyperstatic three-bar truss. Displacement method solution
The three-bar truss depicted in fig. 4.5 is a very simple system of axially loaded bars that exhibits all the characteristics of hyperstatic systems. The system is subjected to a vertical load P applied at point **O**, where the three bars are pinned together. The three bars will be identified by the points at which they are pinned to the ground, denoted points **A**, **B**, and **C**. $\mathcal{A}_A$, $\mathcal{A}_B$ and $\mathcal{A}_C$ are the cross-sectional areas of bars **A**, **B**, and **C**, respectively, and E_A, E_B and E_C denote their respective Young's moduli.

The truss features geometric and material symmetry about vertical axis **OB**: the cross-sectional areas of bars **A** and **C** are equal, $\mathcal{A}_A = \mathcal{A}_C$, and so are their Young's moduli, $E_A = E_C$. Consequently, the forces acting in bars **A** and **C**, denoted F_A and F_C, respectively, are also equal, $F_A = F_C$. The vertical displacement of point **O** is denoted Δ, and the displacement method focuses on determining this displacement

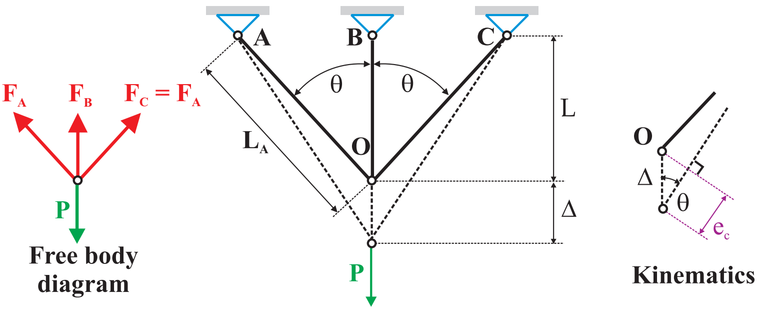

Fig. 4.5. Three bar truss.

first. Due to the symmetry of the problem, the horizontal displacement component vanishes.

Step 1 of the displacement method described in section 4.3.2 is to derive the equation of equilibrium of the problem. The free body diagram drawn in fig. 4.5 yields

$$F_B + 2F_A \cos\theta = P. \tag{4.8}$$

Clearly, the two unknown forces, F_A and F_B, cannot be determined from this single equilibrium equation: this is a hyperstatic system of order 1.

Step 2 invokes the constitutive laws to relate the forces in the bars to the corresponding bar deformations as follows

$$e_A = e_C = \frac{F_A L_A}{(E\mathcal{A})_A} = \frac{F_A L}{(E\mathcal{A})_A \cos\theta}, \quad e_B = \frac{F_B L}{(E\mathcal{A})_B}, \tag{4.9}$$

where e_A, e_B, and e_C are the elongations of the three bars.

Step 3 deals with the last set of equations of elasticity, the strain-displacement equations. Relating the vertical displacement, $\varDelta$, of point **O** to the elongations of the bars is a difficult task if $\varDelta$ is arbitrarily large; for small displacement, however, *i.e.*, when $\varDelta \ll L$, angle θ changes little during deformation, and the kinematics diagram in fig. 4.5 shows that $e_C \approx \varDelta \cos\theta$. It follows that

$$e_A = e_C = \varDelta \cos\theta, \quad e_B = \varDelta. \tag{4.10}$$

All equations of elasticity have now been utilized for this problem. Step 4 is a purely algebraic step combining eqs. (4.9) and (4.10) to express the internal forces in terms of displacements to find

$$\frac{F_A}{(E\mathcal{A})_B} = \frac{F_C}{(E\mathcal{A})_B} = \frac{\varDelta}{L}\bar{k}_A \cos^2\theta, \quad \frac{F_B}{(E\mathcal{A})_B} = \frac{\varDelta}{L}, \tag{4.11}$$

where $\bar{k}_A = (E\mathcal{A})_A/(E\mathcal{A})_B$ is the non-dimensional stiffness of bar **A**.

Step 5 is another purely algebraic step combining eqs. (4.8) and (4.11) to express the single equilibrium condition of the problem in terms of the single displacement component, $\varDelta$, to find

$$\frac{\Delta}{L} + 2\frac{\Delta}{L}\bar{k}_A \cos^3\theta = \frac{P}{(E\mathcal{A})_B}.$$

Step 6 solves this linear equation for the single displacement component, Δ, to find

$$\frac{\Delta}{L} = \frac{1}{1 + 2\bar{k}_A \cos^3\theta} \frac{P}{(E\mathcal{A})_B}. \tag{4.12}$$

This relationship can be written as $\Delta = P/k$, where k is the equivalent vertical stiffness of the three-bar truss, $k = \left[(E\mathcal{A})_B + 2(E\mathcal{A})_A \cos^3\theta\right]/L$.

In step 7, the deformations of the structure are recovered by introducing the displacement given by eq. (4.12) into the strain-displacement equations, eqs. (4.10), to find the elongations as

$$\frac{e_A}{L} = \frac{e_C}{L} = \frac{\cos\theta}{1 + 2\bar{k}_A \cos^3\theta} \frac{P}{(E\mathcal{A})_B}, \quad \frac{e_B}{L} = \frac{1}{1 + 2\bar{k}_A \cos^3\theta} \frac{P}{(E\mathcal{A})_B}. \tag{4.13}$$

The final step of the displacement method, step 8, recovers the forces in the bars by introducing the elongations, given by eq. (4.13) into the constitutive laws, eq. (4.9), to find

$$\frac{F_A}{P} = \frac{F_C}{P} = \frac{\bar{k}_A \cos^2\theta}{1 + 2\bar{k}_A \cos^3\theta}, \quad \frac{F_B}{P} = \frac{1}{1 + 2\bar{k}_A \cos^3\theta}. \tag{4.14}$$

Note that the internal forces in the bars depend on the stiffnesses of the system, $\bar{k}_A = (E\mathcal{A})_A/(E\mathcal{A})_B$. In fact, the ratio of the forces in bars **A** and **B** is $F_A/F_B = \bar{k}_A \cos^2\theta$, *i.e.*, the ratio of the forces in the two bars is in proportion to the ratio of their stiffnesses.

Example 4.5. Rigid plate suspended by four elastic cables: displacement method
The hyperstatic system depicted in fig. 4.6 is more complicated than the previous example, but the same displacement method can be applied. In this example, a rigid square plate of side dimension ℓ is supported by four identical elastic cables of length h, cross-sectional area $\mathcal{A}$, and Young's modulus E. A vertical load P is applied to the rigid plate at point **K** located by coordinates x_{1p} and x_{2p} as indicated in the figure. Find the elongations and forces in the four cables.

The complication in this example arises from the kinematics. Because the plate is assumed to be perfectly rigid, it is easy to understand that the vertical displacements of points **A**, **B**, **C**, and **D**, denoted Δ_A, Δ_B, Δ_C, and Δ_D, respectively, are not independent. Indeed, any three points uniquely define a plane. For example, the displacements of points **A**, **B**, and **C** uniquely defined the configuration of the plate, and the displacement of the fourth point, **D**, follows. For a square plate it is easy to show that $\Delta_A + \Delta_B = \Delta_C + \Delta_D$ is the condition that ensures the infinite rigidity of the plate.

Step 1 of the displacement method described in section 4.3.2 is to derive the equations of equilibrium of the problem from the free body diagram shown in fig. 4.6

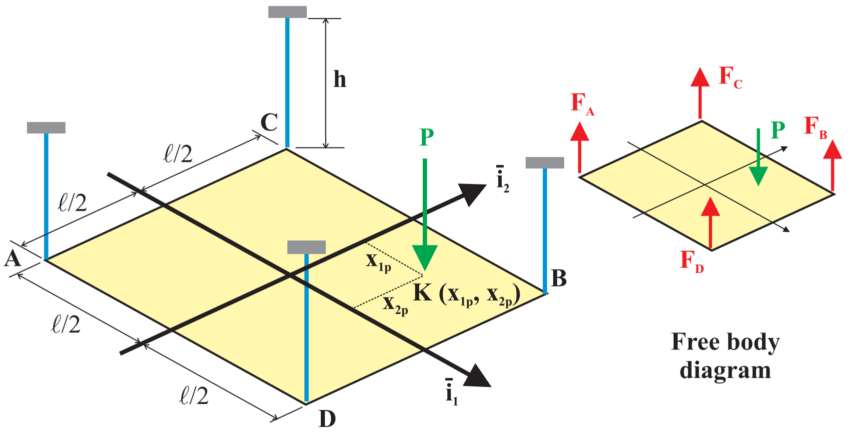

Fig. 4.6. A rigid plate supported by four identical elastic cables.

$$F_A + F_B + F_C + F_D = P, \tag{4.15a}$$

$$-F_A + F_B + F_C - F_D = \frac{2x_{2p}}{\ell} P, \tag{4.15b}$$

$$-F_A + F_B - F_C + F_D = \frac{2x_{1p}}{\ell} P, \tag{4.15c}$$

where the first equation corresponds to the equilibrium of forces in the vertical direction, and the next two equations are moment equilibrium equations about axes $\bar{\imath}_1$ and $\bar{\imath}_2$, respectively. Clearly, the four unknown forces, F_A, F_B, F_C, and F_D, cannot be determined from these three equilibrium equations. This is therefore a hyperstatic system of order 1.

Step 2 invokes the constitutive laws to relate the forces in the cables to the corresponding system deformations as follows

$$e_A = \frac{F_A h}{E\mathcal{A}}, \quad e_B = \frac{F_B h}{E\mathcal{A}}, \quad e_C = \frac{F_C h}{E\mathcal{A}}, \quad e_D = \frac{F_D h}{E\mathcal{A}}, \tag{4.16}$$

where e_A, e_B, e_C, and e_D are the elongations of the four cables. Step 3 deals with the strain-displacement equations which are particularly simple in this case:

$$e_A = \Delta_A, \quad e_B = \Delta_B, \quad e_C = \Delta_C, \quad e_D = \Delta_D. \tag{4.17}$$

All equations of elasticity have now been utilized for this problem.

Step 4 is a purely algebraic step combining eqs. (4.16) and (4.17) to express the internal forces in terms of displacements to find

$$F_A = \frac{E\mathcal{A}}{h}\Delta_A, \quad F_B = \frac{E\mathcal{A}}{h}\Delta_B, \quad F_C = \frac{E\mathcal{A}}{h}\Delta_C, \quad F_D = \frac{E\mathcal{A}}{h}\Delta_D. \tag{4.18}$$

Step 5 is another purely algebraic step combining eqs. (4.15) and (4.18) to express the equilibrium conditions of the problem in terms of the unknown displacements to yield the first three equations below,

$$\Delta_A + \Delta_B + \Delta_C + \Delta_D = \frac{Ph}{E\mathcal{A}}, \tag{4.19a}$$

$$-\Delta_A + \Delta_B + \Delta_C - \Delta_D = \frac{2x_{2p}}{\ell}\frac{Ph}{E\mathcal{A}}, \tag{4.19b}$$

$$-\Delta_A + \Delta_B - \Delta_C + \Delta_D = \frac{2x_{1p}}{\ell}\frac{Ph}{E\mathcal{A}}, \tag{4.19c}$$

$$\Delta_A + \Delta_B - \Delta_C - \Delta_D = 0. \tag{4.19d}$$

The fourth equation expresses the compatibility condition that defines the infinite stiffness of the plate, as discussed earlier.

Step 6 involves the solution of the system of linear equations, eqs. (4.19), to find the displacements of the attachment points of the four cables,

$$\begin{aligned}
\frac{\Delta_A}{h} &= \frac{1}{4}\left(1 - \frac{2x_{1p}}{\ell} - \frac{2x_{2p}}{\ell}\right)\frac{P}{E\mathcal{A}}, \\
\frac{\Delta_B}{h} &= \frac{1}{4}\left(1 + \frac{2x_{1p}}{\ell} + \frac{2x_{2p}}{\ell}\right)\frac{P}{E\mathcal{A}}, \\
\frac{\Delta_C}{h} &= \frac{1}{4}\left(1 - \frac{2x_{1p}}{\ell} + \frac{2x_{2p}}{\ell}\right)\frac{P}{E\mathcal{A}}, \\
\frac{\Delta_D}{h} &= \frac{1}{4}\left(1 + \frac{2x_{1p}}{\ell} - \frac{2x_{2p}}{\ell}\right)\frac{P}{E\mathcal{A}}.
\end{aligned} \tag{4.20}$$

In step 7, the deformations of the structure are recovered by introducing the displacement into the strain-displacement equations, eqs. (4.17), to find the elongations. The final step of the displacement method, step 8, recovers the forces in the cables by introducing the elongations into the constitutive laws, eqs. (4.16), to find

$$\begin{aligned}
\frac{F_A}{P} &= \frac{1}{4}\left(1 - \frac{2x_{1p}}{\ell} - \frac{2x_{2p}}{\ell}\right), \\
\frac{F_B}{P} &= \frac{1}{4}\left(1 + \frac{2x_{1p}}{\ell} + \frac{2x_{2p}}{\ell}\right), \\
\frac{F_C}{P} &= \frac{1}{4}\left(1 - \frac{2x_{1p}}{\ell} + \frac{2x_{2p}}{\ell}\right), \\
\frac{F_D}{P} &= \frac{1}{4}\left(1 + \frac{2x_{1p}}{\ell} - \frac{2x_{2p}}{\ell}\right).
\end{aligned} \tag{4.21}$$

Because the stiffness constants of all four cables are identical, the forces in the cables do not depend on the stiffnesses of the structure. Had the stiffnesses of the cables been different from each other, the final solution for the forces in the cables would depend on the relative stiffnesses of the cables.

4.3.3 The force or flexibility method

The force method focuses on the solution for the system internal forces. Compatibility equations are written in terms of a set of redundant forces. In contrast with the displacement method, the forces are determined first, and strains and displacements are then recovered. The procedure can be formalized in the following steps.

1. Write the *equilibrium equations of the system*. Equilibrium conditions express the vanishing of the sum of the forces and moments acting on the system. This step typically involves the construction of free body diagrams of the various subcomponents of the system and then formulation of the equilibrium conditions.
2. Determine the *system degree of redundancy*, N_R, which equals the number of unknown internal forces minus the number of equilibrium equations.
3. *Cut the system* at N_R locations and define a single relative displacement for each of the cuts. With the N_R cuts, the *originally hyperstatic system is transformed into an isostatic system*.
4. Apply N_R *redundant forces* to the system, each acting along the relative displacement allowed by each of the N_R cuts. Express all internal forces of the system in terms of the applied loads and the N_R redundant forces by means of the equilibrium equations. Note that the choice of where to make the cuts is somewhat arbitrary, and some choices may lead to simpler solution processes. The key requirement in making the cuts is that the resulting system must be an isostatic system, not a mechanism.
5. Use the *constitutive laws* to express system deformations in terms of N_R redundant forces.
6. Use the *strain-displacement equations* to express the relative displacements at the N_R cuts in terms of the N_R redundant forces.
7. Impose the *vanishing of the relative displacements* at the N_R cuts, and use these N_R compatibility equations to solve for the N_R redundant forces.
8. Recover system deformations from the constitutive laws and system displacements from the strain-displacement equations.

The force method directly focuses on the determination of the redundant forces. All internal forces, system deformations and displacements are expressed in terms of redundant forces. The solution process involves the solution of a linear set of equations of size N_R, the degree of redundancy of the system. This contrasts with the displacement method that involves the solution of a system of linear equations of size equal to the number of unknown displacements, N_D. Depending on the relative values of N_R and N_D, the displacement or force methods can be more or less convenient to use.

As a final comment, note that while the force method can be applied quite effectively using good engineering judgement and experience, the displacement method is usually more amenable to automated solution processes using computers.

Example 4.6. Hyperstatic three-bar truss: force method solution

The three-bar truss problem treated in example 4.4 using the displacement method will now be solved using the force method. The truss is depicted in fig. 4.7, and here again, it is subjected to a vertical load P applied at point $\mathbf{O}$, where the three bars are pinned together. The three bars will be identified by the points at which they are pinned to the ground at points **A**, **B**, and **C**. $\mathcal{A}_A$, $\mathcal{A}_B$ and $\mathcal{A}_C$ denote the cross-sectional areas of bars **A**, **B**, and **C**, respectively, and E_A, E_B and E_C denote their respective Young's moduli.

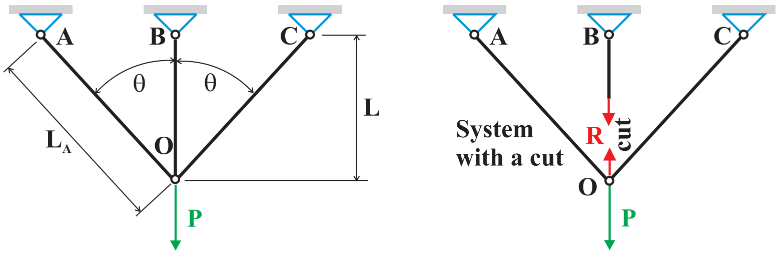

Fig. 4.7. Three bar truss.

The truss features geometric and material symmetry about vertical axis **OB**: the cross-sectional areas of bars **A** and **C** are equal, $\mathcal{A}_A = \mathcal{A}_C$, and so are their Young's moduli, $E_A = E_C$. Consequently, the forces acting in bars **A** and **C**, denoted F_A and F_C, respectively, are also equal, $F_A = F_C$.

Step 1 of the force method described in section 4.3.3 yields a single equation of vertical equilibrium for the problem based on the free body diagram shown in fig. 4.5

$$F_B + 2F_A \cos\theta = P. \tag{4.22}$$

Clearly, the two unknown forces, F_A and F_B, cannot be determined from equilibrium considerations alone. As required by step 2, the system degree of redundancy is determined as $N_R = 2 - 1 = 1$.

Step 3 calls for cutting the system at a single location because $N_R = 1$. As depicted in fig. 4.7, bar **B** is cut for this example, but cutting bars **A** or **C** would lead to a very similar procedure.

Next, in step 4, a single redundant force, R, is applied at the to sides of the cut. With R treated as a known load, it is now possible to solve the equilibrium eq. (4.22) for F_A and F_C as

$$F_A = F_C = \frac{(P-R)}{2\cos\theta}, \quad F_B = R. \tag{4.23}$$

In step 5, bar extensions are expressed in terms of the redundant force, R, using the constitutive laws, eq. (4.9), leading to

$$\frac{e_C}{L} = \frac{e_A}{L} = \frac{F_A}{(E\mathcal{A})_A\cos\theta} = \frac{(P-R)}{2(E\mathcal{A})_A\cos^2\theta}, \quad \frac{e_B}{L} = \frac{F_B}{(E\mathcal{A})_B} = \frac{R}{(E\mathcal{A})_B}. \tag{4.24}$$

Step 6 requires the determination of the relative displacement at the cut, and this is easily obtained from the strain-displacement equations and kinematics as

$$d_{\text{cut}} = \frac{e_A}{\cos\theta} - e_B = \frac{(P-R)L}{2(E\mathcal{A})_A\cos^3\theta} - \frac{RL}{(E\mathcal{A})_B}.$$

Step 7 enforces the vanishing of this relative displacement, $d_{\text{cut}} = 0$. This equation is then solved for the redundant force R to find

$$\frac{R}{P} = \frac{1}{1 + 2\bar{k}_A \cos^3\theta},$$

where $\bar{k}_A = (E\mathcal{A})_A/(E\mathcal{A})_B$ is the non-dimensional stiffness of bar **A**. The internal forces in the bars then follow from eq. (4.23) as $F_B = R$ and $F_A = F_C = (P - R)/(2\cos\theta)$.

In step 8, the bar elongations are recovered from eq. (4.24). As expected, the results obtained using the force method as presented here match those obtained in example 4.4 using the displacement method.

Example 4.7. Rigid plate supported by four cables: force method

The force method can also be used to find the forces and deformations in the rigid plate problem treated in example 4.5. The hyperstatic system is shown again in fig. 4.8. The square rigid square plate with sides of length ℓ is supported by four identical elastic vertical cables of length h, cross-sectional area $\mathcal{A}$, and Young's modulus E. A vertical load, P, is applied to the rigid plate at point **K** located at coordinates x_{1p} and x_{2p} as indicated in the figure.

The kinematics of the rigid plate require that all four corner points remain in a plane. Thus, only three of the vertical displacements of points **A**, **B**, **C**, and **D**, denoted Δ_A, Δ_B, Δ_C, and Δ_D, respectively, are independent, and the fourth can be computed from the other three. Again, it is easy to show that $\Delta_A + \Delta_B = \Delta_C + \Delta_D$ is the condition that expresses the infinite rigidity of the plate.

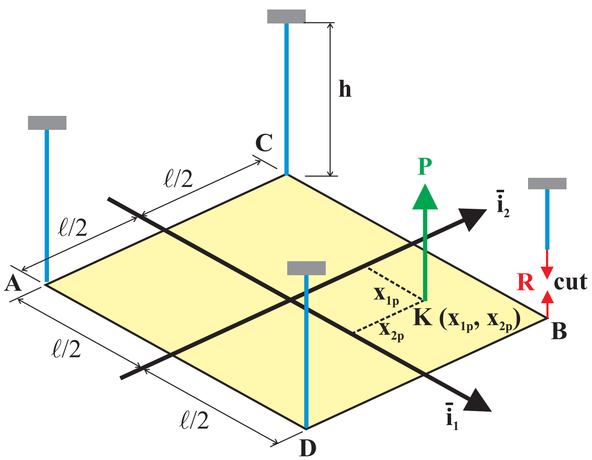

Fig. 4.8. A rigid plate supported by four identical elastic cables.

Step 1 of the force method is to derive the equations of equilibrium of the problem from the free body diagram shown in fig. 4.6. The vanishing of the sum of the forces and moments acting on the rigid plate leads to the equations of equilibrium given by eqs. (4.15). Clearly, the four unknown forces, F_A, F_B, F_C, and F_D, cannot be determined from those three equilibrium equations. As required by step 2, the system degree of redundancy is determined as $N_R = 4 - 3 = 1$.

Step 3 calls for cutting the system at a single location, since $N_R = 1$. As depicted in fig. 4.8, cable **B** is cut in this example, but cutting any one of the four cables would lead to a very similar procedure.

Next, in step 4, a single redundant force, R, is applied at the to sides of the cut. With the help of the equation of equilibrium, eq. (4.15), the internal forces in the cables are expressed in terms of the applied load, P, and the redundant force, R, to find

$$\begin{aligned} F_A &= R - \left(\frac{x_{1p}}{\ell} + \frac{x_{2p}}{\ell}\right) P, \quad F_B = R, \\ F_C &= \left(\frac{1}{2} + \frac{x_{2p}}{\ell}\right) P - R, \quad F_D = \left(\frac{1}{2} + \frac{x_{1p}}{\ell}\right) P - R. \end{aligned} \tag{4.25}$$

In step 5, cable extensions are expressed in terms of the redundant force, R, by introducing the above forces into the constitutive laws, eqs. (4.16), to yield

$$\begin{aligned} \frac{E\mathcal{A}e_A}{h} &= R - \left(\frac{x_{1p}}{\ell} + \frac{x_{2p}}{\ell}\right) P, \quad \frac{E\mathcal{A}e_B}{h} = R, \\ \frac{E\mathcal{A}e_C}{h} &= \left(\frac{1}{2} + \frac{x_{2p}}{\ell}\right) P - R, \quad \frac{E\mathcal{A}e_D}{h} = \left(\frac{1}{2} + \frac{x_{1p}}{\ell}\right) P - R. \end{aligned} \tag{4.26}$$

Step 6 requires determination of the relative displacement at the cut, and step 7 imposes the requirement that it vanish. The condition expressing the infinite rigidity of the plate is $\Delta_A + \Delta_B = \Delta_C + \Delta_D$. If this condition is satisfied, the relative displacement at the cut must vanish. Because the four cables are fixed at their bases, their tip displacements are equal to their elongations and hence, $e_A + e_B = e_C + e_D$. introducing eq. (4.26) into this compatibility equations leads to

$$R - \left(\frac{x_{1p}}{\ell} + \frac{x_{2p}}{\ell}\right) P + R = \left(\frac{1}{2} + \frac{x_{2p}}{\ell}\right) P - R + \left(\frac{1}{2} + \frac{x_{1p}}{\ell}\right) P - R.$$

This equation is now solved for the redundant force R to find

$$R = F_B = \frac{1}{4}\left(1 + \frac{2x_{1p}}{\ell} + \frac{2x_{2p}}{\ell}\right) P.$$

The other internal forces in the cables are then obtained by introducing the redundant force, R, into eqs. (4.25). In step 8, the cable elongations are recovered from eq. (4.26).

As expected, the results obtained using the force method presented here match those obtained in example 4.5 using the displacement method. It is interesting to note that the solution of this problem using the displacement method involves solving a linear system of four equations for the four unknown displacements of the cables, whereas the present force method requires the solution of a single compatibility equation for the unknown redundant force. In other words, the force method requires the solution of a linear system of size equal to the order of redundancy of the hyperstatic system, whereas the displacement method requires the solution of a larger linear system of size equal to the number of unknown displacements.

4.3.4 Problems

Problem 4.1. Simple hyperstatic bars - displacement method solution
Three axially loaded bars, each of length L and all constructed from a material of elasticity modulus E, are arranged as shown in fig. 4.9. Two bars are connected in parallel and one of these has a cross-sectional area that is twice that of the other. A third bar is connected in series at the common point. An axial load, P, is applied at the junction of the three bars. Using the displacement method, determine *(1)* the displacement, d, of the connecting point between the three bars and *(2)* the forces in each of the three bars.

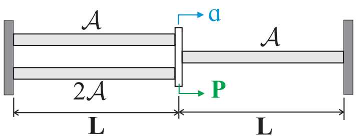

Fig. 4.9. Three bars in a parallel-series configuration.

Problem 4.2. Simple hyperstatic bars - force method solution
Solve problem 4.1 using using the force method.

Problem 4.3. Prestressed steel bar in an aluminum tube
A steel bar of cross-sectional area $\mathcal{A}_s = 800$ mm^2 fits inside an aluminum tube of cross-sectional area $\mathcal{A}_a = 1,500$ mm^2. The assembly is constructed in such a way that initially, the steel bar is prestressed with a compressive force, $-P$, while the aluminum tube is prestressed with a tensile load of equal magnitude, P. Next, the prestressed assembly is subjected to a tensile load F. *(1)* If no prestress is applied, *i.e.*, if $P = 0$, find the maximum external load, F, that can be applied to the assembly without exceeding allowable stress levels in either material. *(2)* Find the optimum prestress level to be applied. This optimum prestress is defined as that for which the allowable stress is reached simultaneously in both steel bar and aluminum tubes when subjected to the externally applied force, F. In other words, when optimally prestressed, both materials are used to their full capacity. *(3)* What improvement, in percent, is achieved by using the optimum prestress level as compared to not prestressing the assembly. Use the following data: $E_s = 210$ and $E_a = 73$ GPa; the yield stresses for steel and aluminum are $\sigma_y^s = 600$ and $\sigma_y^a = 400$ MPa, respectively.

Problem 4.4. Square plate supported by four cables
Consider the rigid square plate of side ℓ supported by four elastic cables each of length h, cross-sectional area $\mathcal{A}$, and Young's modulus E, as depicted in fig. 4.10. A vertical load P is applied at point **K**, located at a distance d from the center of the plate along the line joining points **A** and **B**. *(1)* Determine the degree of redundancy of this system. *(2)* Determine the forces, F_A, F_B, F_C, and F_D, in bars **A**, **B**, **C**, and **D**, respectively. *(3)* On one graph, plot the four non-dimensional forces, F_A/P, F_B/P, F_C/P, and F_D/P, as functions of $\bar{d} = d/\ell$ for $\bar{d} \in [0, 1/\sqrt{2}]$. Hint: See example 4.5. Also note the symmetry of the problem with respect to line **AB**, which simplifies the moment equilibrium equations.

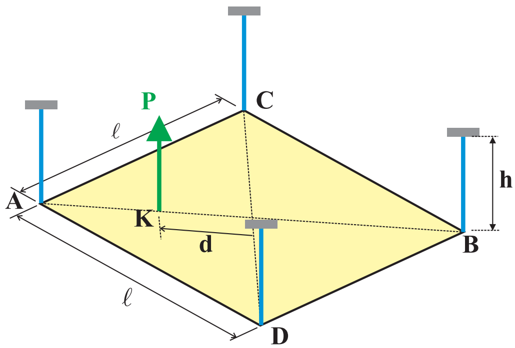

Fig. 4.10. Rigid square plate supported by four elastic cables.

Problem 4.5. Square plate supported by four cables
Consider the rigid square plate of side ℓ supported by four elastic cables each of length h and Young's modulus E, as depicted in fig. 4.10. The cross-sectional areas of the cables are $\mathcal{A}_A$, $\mathcal{A}_B$, $\mathcal{A}_C$, and $\mathcal{A}_D$, for cables **A**, **B**, **C**, and **D**, respectively. A vertical load P is applied at point **K**, located at a distance d from the center of the plate along the line joining points **A** and **B**. It is desired that the plate move straight down under the action of the load, *i.e.*, $\Delta_A = \Delta_B = \Delta_C = \Delta_D = \Delta$, where Δ_A, Δ_B, Δ_C, and Δ_D are the vertical displacements of points **A**, **B**, **C**, and **D**, respectively. *(1)* Determine the degree of redundancy of this system. *(2)* Determine the relationship(s) that must be satisfied by the cross-sectional areas of the cables for the plate to undergo the desired motion. Hints: The relationship between $\mathcal{A}_C$ and $\mathcal{A}_D$ should be obvious from inspection of the problem.

Problem 4.6. Rotor blade hub connection
Figure 4.11 shows a potential design for the attachment of a rotor blade to the rotorcraft hub. The yoke consists of two separate pieces each of which connects the rotor blade to the hub, and the spindle also connects the rotor blade to the hub through an elastomeric bearing. As the rotor blade spins, a large centrifugal force F is applied to the assembly, which can be idealized as three parallel bars of length L, which connect the blade to the hub. The two bars modeling the yoke each have an axial stiffness $(E\mathcal{A})_y$, while the spindle has an axial stiffness $(E\mathcal{A})_s$. The elastomeric bearing is idealized as a very short spring of stiffness k_b in series with the spindle. *(1)* Calculate and plot the non-dimensional forces in the yoke, F_y/F, and in the spindle, F_s/F, as a function of the non-dimensional bearing stiffness, $0 \leq Lk_b/(E\mathcal{A})_s \leq 25$. *(2)* For what value of the stiffness constant k_b is all the centrifugal load carried by the yoke? *(3)* Find the maximum load that can be carried by the spindle. What is the corresponding value of k_b? *(4)* For what value on $Lk_b/(E\mathcal{A})_s$ do the yoke and spindle carry equal loads? Use the following data: $(E\mathcal{A})_y/(E\mathcal{A})_s = 0.8$

4.3.5 Thermal effects in hyperstatic system

It is often the case that hyperstatic systems are more structurally efficient than isostatic systems. They potentially offer the additional advantage of redundant load paths. On the other hand, they present important drawbacks; one of them is sensitivity to *thermal effects*.

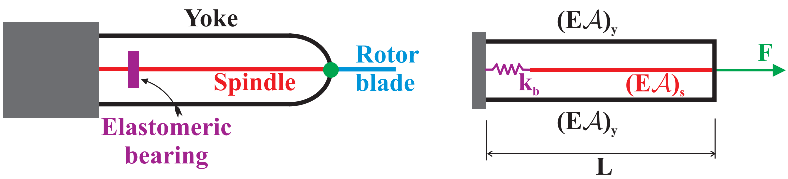

Fig. 4.11. Rotor blade connection to the hub by means of a yoke and spindle.

In isostatic structures, thermal strains simply cause additional deformations of the system, as implied by the modified constitutive laws that account for thermal strains, eqs. (2.22). For hyperstatic structures, however, the presence of thermal strains in the constitutive laws gives rise to additional stresses, called *thermal stresses*. This effect can be significant, even when the entire structure experiences a *uniform temperature change*, although the effect is usually more pronounced in the presence of *temperature gradients*, which result from non-uniform temperature fields, or when different portions of the structure are subjected to different temperatures.

Example 4.8. Series connected bars subjected to temperature change
Consider the system depicted in fig. 4.12 featuring two bars connected in series and constrained by rigid walls at points **A** and **C**. Load P is applied at point **B**, and in addition, both bars are subjected to a temperature change ΔT. Except for this thermal effect, the problem is identical to that treated in example 4.2.

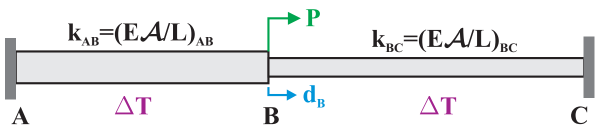

Fig. 4.12. Two bars connected in series with ends fixed.

The equilibrium equation of the system remains unchanged, $F_{AB} - F_{BC} = P$, and the displacement of point **B**, d_B, is still related to the elongations of the bars $d_B = e_{AB} = -e_{BC}$. The constitutive equations, however, must now be modified to account for the thermal strains. In view of eq. (2.22), the total strain in each bar is the sum of the mechanical and thermal strains, $\epsilon = \epsilon^m + \epsilon^t$, where the mechanical strain is related to the stress in the bar, $\epsilon^m = \sigma/E$, and the thermal strain depends on the temperature change, $\epsilon^t = \alpha \Delta T$. The extension in the bar now becomes

$$e_{AB} = \epsilon L_{AB} = \frac{\sigma_{AB}}{E_{AB}} L_{AB} + \alpha \Delta T L_{AB} = \frac{F_{AB}}{k_{AB}} + \alpha \Delta T L_{AB}.$$

A similar equation can also be developed for the elongation of the other bar, e_{BC}.

Following the steps of the displacement method, the internal forces are expressed in terms of deformations, then in terms of displacements, leading to the equilibrium equation expressed in terms of the displacement as

$$(k_{AB} + k_{BC})d_B = P + \alpha \Delta T\left[(E\mathcal{A})_{AB} - (E\mathcal{A})_{BC}\right].$$

The displacement of point **B** is then

$$d_B = \frac{P}{k_{AB} + k_{BC}} + \frac{\alpha \Delta T\left[(E\mathcal{A})_{AB} - (E\mathcal{A})_{BC}\right]}{k_{AB} + k_{BC}} = d_B^m + d_B^t.$$

This rather complex result shows that total displacement of point **B** is the superposition of the displacement d_B^m due to applied mechanical loads, and the displacement d_B^t due to thermal effects. This should not be unexpected, because mechanical and thermal effects are superposed in the constitutive law.

The internal forces are obtained by substituting the displacement back into the constitutive laws, to find

$$F_{AB} = \frac{k_{AB}}{k_{AB} + k_{BC}}\left[P - \alpha \Delta T\left(L_{AB} + L_{BC}\right) k_{BC}\right],$$

and

$$F_{BC} = \frac{k_{BC}}{k_{AB} + k_{BC}}\left[-P - \alpha \Delta T\left(L_{AB} + L_{BC}\right) k_{AB}\right].$$

It is interesting to consider the case when the two bars are identical, $k_{AB} = k_{BC} = k$. The displacement of point **B** simply becomes $d_B = P/(2k)$. In this case, the thermal displacement vanishes due to the symmetry of the problem, and the total displacement is due solely to the mechanical loads. The axial forces in the bars become $F_{AB} = P/2 - E\mathcal{A}\alpha\Delta T$ and $F_{BC} = -P/2 - E\mathcal{A}\alpha\Delta T$. Due to the symmetry of the problem, both bars share an equal burden in carrying the mechanical loads, $\pm P/2$, and are both subjected to the same compressive thermal stress, $E\mathcal{A}\alpha\Delta T$.

It is also interesting to consider the impact of thermal stresses on the load carrying capability of this system. The bars will yield when the yield stress is reached, that is when $F_{AB} = \pm\sigma_y\mathcal{A}_{AB}$ and when $F_{BC} = \pm\sigma_y\mathcal{A}_{BC}$. In the absence of thermal effects, the load carrying capacity of the system is then $P_{\max} = 2\mathcal{A}\sigma_y$, whereas in the presence of thermal effects, the load carrying capacity becomes $P_{\max} = 2\mathcal{A}(\sigma_y - E\alpha\Delta T) = 2\mathcal{A}\bar{\sigma}_y$. In other words, in the presence of thermal effects, the effective yield stress, $\bar{\sigma}_y$, is the yield stress of the material, σ_y, *reduced by the thermal stress*, $E\alpha\Delta T$.

Example 4.9. Hyperstatic three-bar truss subject to temperature change
The three-bar truss depicted in fig. 4.13 is assembled when all components are at common temperature T_0 and no initial stresses are present in the bars. The three bars will be identified by the points at which they are pinned to the ground at **A**, **B**, and **C**. $\mathcal{A}_A$, $\mathcal{A}_B$ and $\mathcal{A}_C$ denote the cross-sectional areas of bars **A**, **B**, and **C**, respectively, while E_A, E_B and E_C denote their respective Young's moduli. The truss features geometric and material symmetry about the vertical axis **OB**: the cross-sectional areas of bars **A** and **C** are equal, $\mathcal{A}_A = \mathcal{A}_C$, and so are their Young's moduli, $E_A = E_C$. Consequently, the forces acting in bars **A** and **C**, denoted F_A and

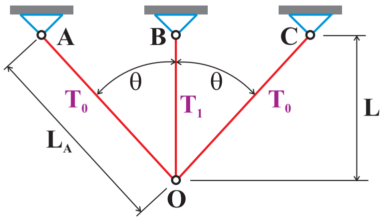

Fig. 4.13. Three bar truss subjected to temperature differentials.

F_C, respectively, are also equal, $F_A = F_C$. Assume that only bar **B** is now heated to a temperature $T_1 = T_0 + \Delta T$, thus preserving the symmetry of the problem.

Due to the heating, the center bar tries to expand by an amount $\alpha(T_1 - T_0)L = \alpha \Delta T\ L$ but is prevented from doing so by the other two bars. An equilibrium point will be reached where the truss expands, although less than the unconstrained bar would, and internal stresses will appear. Intuitively, bar **B** will be in compression, whereas bars **A** and **C** will be in tension.

This thermal problem will be treated using the force method. A a similar problem featuring the same three-bar truss subjected to external loading is treated using the same approach in example 4.6. The equation of equilibrium for this example is given by eq. (4.22) and remains valid for the present example: $F_B + 2F_A \cos\theta = 0$. Since the problem features a single degree of redundancy, a single cut is required. Here again, bar **B** is cut and an unknown redundant force, R, is assumed to act at the cut. The internal forces in the bars are expressed in terms of R, and eqs. (4.23) become $F_A = F_C = -R/(2\cos\theta)$ and $F_B = R$.

The constitutive laws are now used to express the non-dimensional bar elongation in terms of the unknown redundant force to find

$$\frac{e_C}{L} = \frac{e_A}{L} = -\frac{1}{2\bar{k}_A \cos^2\theta} \frac{R}{(E\mathcal{A})_B}, \quad \frac{e_B}{L} = \frac{R}{(E\mathcal{A})_B} + \alpha \Delta T,$$

where $\bar{k}_A = (E\mathcal{A})_A/(E\mathcal{A})_B$ is the non-dimensional stiffness of bar **A**.These expressions are almost identical to those of eqs.(4.24), except for the thermal strain terms now contributing to the elongation of bar **B**. The relative displacement at the cut is now easily obtained

$$d_{\text{cut}} = \frac{e_A}{\cos\theta} - e_B = \frac{-L}{2\bar{k}_A \cos^3\theta} \frac{R}{(E\mathcal{A})_B} - L\frac{R}{(E\mathcal{A})_B} - L\alpha\Delta T.$$

The vanishing of this relative displacement implies $d_{\text{cut}} = 0$ and yields the unknown non-dimensional redundant force as

$$\frac{R}{(E\mathcal{A})_B} = \frac{F_B}{(E\mathcal{A})_B} = -\frac{2\bar{k}_A \cos^3\theta}{1 + 2\bar{k}_A \cos^3\theta}\, \alpha\Delta T.$$

The non-dimensional forces in bars **A** and **B** follow from the equilibrium equation as

$$\frac{F_A}{(E\mathcal{A})_B} = \frac{F_C}{(E\mathcal{A})_B} = \frac{\bar{k}_A \cos^2\theta}{1 + 2\bar{k}_A \cos^3\theta}\, \alpha\Delta T.$$

These internal forces, called *thermal forces*, are proportional to the thermal strain, $\alpha\Delta T$. As expected, the force in bar **B** is compressive, in contrast with the tensile forces present in bars **A** and **C**. Finally, the vertical displacement of the truss at point **O** is given by the elongation of bar **B**, and this is easily recovered as

$$\frac{d_B}{L} = \frac{e_B}{L} = \frac{\left[1 + 2(\bar{k}_A - 1)\cos^3\theta\right]}{1 + 2\bar{k}_A \cos^3\theta}\,\alpha\Delta T.$$

4.3.6 Manufacturing imperfection effects in hyperstatic system

An additional drawback of hyperstatic systems is their sensitivity to *dimensional* or *manufacturing imperfections*. Consider, here again, the three-bar truss depicted in fig. 4.13. Assume all bars to be at the same temperature, but due to manufacturing imperfections, bar **B** was made too long. It is impossible to assemble the system: if bars **A** an **C** are first connected together at point **O**, bar **B** is longer than the distance from point **B** to **O**. The only way to assemble the system is to compress bar **B** to the right length, pin the three bars together at point **O**, then release the compression in bar **B**. In the final assembly, *residual forces* will be present; intuitively, it follows that bar **B** is left under compression, whereas bars **A** and **C** have a residual tensile stress.

It is worth noting the close connection between thermal strain and manufacturing imperfections. In example 4.9, bar **B** is subjected to a temperature differential resulting in a thermal elongation $L\alpha\Delta T$. In other words, bar **B** is now too long by an amount $L\alpha\Delta T$. This is identical to a manufacturing imperfection where bar **B** is too long by an amount $\mu = L\alpha\Delta T$. This means that the *residual stress* due to thermal effects computed in example 4.9 are identical to the residual stress due to manufacturing imperfections in the same system, provided that $\alpha\Delta T$ is replaced by μ/L in all results of example 4.9.

Example 4.10. Rigid plate supported by four elastic bars

Consider the hyperstatic system depicted in fig. 4.14 in which a rigid square plate of side ℓ is supported by four identical elastic bars of length h, cross-sectional area $\mathcal{A}$, and Young's modulus E. This example is similar to the previous examples in which a rigid plate is suspended from four cables, but in this case, the support is provided by the four bars or legs. Assume that one of the bars is too short by an amount μ due, for example, to manufacturing imperfections.

Since the plate is assumed to be infinitely rigid, the vertical displacements of points **A**, **B**, **C**, and **D**, denoted Δ_A, Δ_B, Δ_C, and Δ_D, respectively, are not independent. Indeed, three points uniquely define a plane, hence the displacements of points **A**, **B**, and **C** uniquely define the configuration of the plate, and the displacement of the fourth point, **D**, follows. As in the previous examples, this constraint can be expressed for a square plate as, $\Delta_A + \Delta_B = \Delta_C + \Delta_D$.

In example 4.5, a similar configuration is considered, but a vertical load is applied at an arbitrary point on the plate as shown in fig. 4.6. The displacement method is used to solve the problem, and a similar procedure is used here. In the first step, the equations of equilibrium of the system are derived from the free body diagram

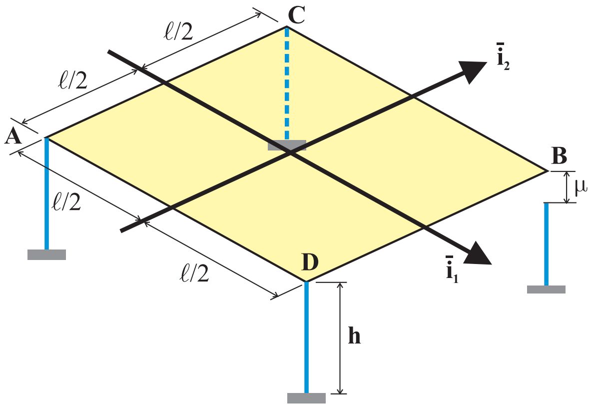

Fig. 4.14. A rigid plate supported by four identical elastic bars with a manufacturing imperfection.

shown in fig. 4.6, to find eqs. (4.15). Next, the constitutive laws relating the bar forces to the corresponding deformations are still given by eqs. (4.16). Finally, the strain displacement equations are still given by eqs. (4.17), except for bar **B** where, due to the manufacturing imperfection, $\Delta_B = e_B - \mu$.

The remaining steps of the displacement method closely follow the development presented in example 4.5 and lead to the following equations of equilibrium written in terms unknown displacements

$$\Delta_A + \Delta_B + \Delta_C + \Delta_D = -\mu, \tag{4.27a}$$
$$-\Delta_A + \Delta_B + \Delta_C - \Delta_D = -\mu, \tag{4.27b}$$
$$-\Delta_A + \Delta_B - \Delta_C + \Delta_D = -\mu, \tag{4.27c}$$
$$\Delta_A + \Delta_B - \Delta_C - \Delta_D = 0, \tag{4.27d}$$

where the last equation expresses the infinite stiffness of the plate as discussed earlier. The solution of this linear system yields the displacements of the corner points as

$$\Delta_A = \frac{\mu}{4}, \quad \Delta_B = -\frac{3\mu}{4}, \quad \Delta_C = \Delta_D = -\frac{\mu}{4}. \tag{4.28}$$

Finally, the bar forces are recovered as

$$F_A = F_B = \frac{1}{4}\frac{\mu}{h}E\mathcal{A}, \quad F_C = F_D = -\frac{1}{4}\frac{\mu}{h}E\mathcal{A}. \tag{4.29}$$

These are the *residual forces* due to manufacturing imperfections. The two opposite bars **A** and **B** are subjected to tension, whereas the two opposite bars **C** and **D** are under compression. The magnitudes of the forces in the four bars are equal and proportional to the manufacturing imperfection, μ.

Assume now that a vertical load, P, is applied at the center of the plate. The total forces in the bars are now the superpositions of the forces due to the applied loads, as given by eqs. (4.21), and the forces due to the manufacturing imperfection, as given by eqs. (4.29), to find

$$F_A = F_B = \frac{P}{4} + \frac{1}{4}\frac{\mu}{h}E\mathcal{A}, \quad F_C = F_D = \frac{P}{4} - \frac{1}{4}\frac{\mu}{h}E\mathcal{A}.$$

In view of the symmetry of the problem, the applied load is carried equally by the four bars, whereas the manufacturing imperfection put additional loads into bars **A** and **B**, but unloads bars **C** and **D**.

The maximum load the structure can carry is $P_{\max}/4 + E\mathcal{A}\mu/(4h) = \mathcal{A}\sigma_y$, where σ_y is the material yield stress. Hence, $P_{\max} = 4\mathcal{A}\left[\sigma_y - 1/4\, E\, \mu/h\right] = 4\mathcal{A}\bar{\sigma}_y$. Due to manufacturing imperfections, the effective yield stress, $\bar{\sigma}_y$, is the actual yield stress for the material, σ_y, reduced by $1/4\, E\, \mu/h$.

The residual forces are proportional to the magnitude of the manufacturing imperfections, as expected, but also to the Young's modulus of the material, see eqs. (4.29). Hence, the stiffer the system, the more sensitive it will be to manufacturing imperfections.

Example 4.11. Prestress in a bolt

Geometric incompatibility may also be created intentionally; indeed, it is sometimes desirable to introduce a prestress into a structural member. Consider, for instance, the prestress created in a bolt when tightened. Typically, a tensile force is created in the bolt to develop a compressive force acting on the bolted assembly. This situation is illustrated in fig. 4.15, which depicts a prestressed bolt-sleeve assembly.

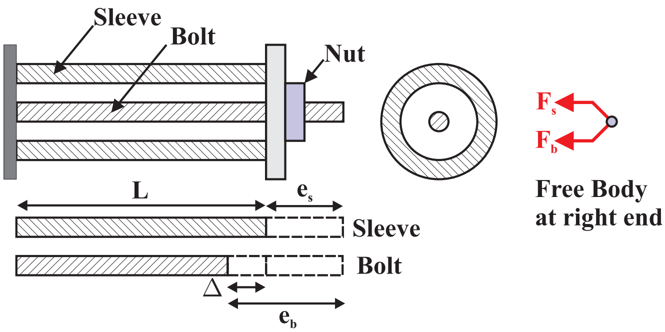

Fig. 4.15. Prestressed bolt-sleeve assembly.

The sleeve is assumed to be a hollow circular cylinder of cross-sectional area $\mathcal{A}_s$ and the bolt has a cross-sectional area $\mathcal{A}_b$; both are of initial length L. The Young's moduli of the sleeve and bolt are E_s and E_b, respectively. Assume that the nut on the bolt is turned until the entire assembly is snug, and then, the nut is rotated by an additional N turns. This will shorten the portion of the bolt between the end plates by an amount $\Delta = pN$, where p is the bolt's thread pitch or distance between successive threads.

The analysis follows a procedure similar to that used in example 4.10. The single equilibrium equation of the system is $F_s + F_b = 0$, where F_s and F_b are the forces in the sleeve and bolt, respectively, both assumed positive in tension. The constitutive equations for the sleeve and the bolt are simply, $F_s = k_s e_s$ and $F_b = k_b e_b$, where $k_s = (E\mathcal{A})_s/L$ and $k_b = (E\mathcal{A})_b/L$ are the equivalent sleeve and bolt stiffnesses, respectively.

Let the elongations of the sleeve and bolt be denoted e_s and e_b, respectively, as illustrated in the lower part of fig. 4.15. Due to the initial tightening of the nut, the bolt is shortened by an amount $\Delta = pN$, and hence, displacement compatibility requires $e_s = e_b - \Delta$.

Since only the prestress forces in the bolt and sleeve are to be determined, the force method provides the most direct solution procedure. Let the sleeve force be the redundant force in the system, and hence, $F_s = R$. The equilibrium equation then implies $F_b = -F_s = -R$, and substitution into the constitutive equations yields the sleeve and bolt extensions as $e_s = F_s/k_s = R/k_s$ and $e_b = F_b/k_b = -R/k_b$, respectively. Finally, introducing these results into the compatibility equation yields $R/k_s = -R/k_b - \Delta$. Solving this equation yields $R = -k_s k_b/(k_s + k_b)\Delta$. The forces in the bolt and sleeve are then

$$F_s = R = -\frac{k_s k_b}{k_s + k_b}\Delta, \quad \text{and} \quad F_b = -R = \frac{k_s k_b}{k_s + k_b}\Delta,$$

respectively. As expected, the bolt is in tension while the sleeve is in compression. From a practical point of view, the desired prestress level, F_s or F_b, would be specified first, and the required number of turns, N, would then be computed. For instance, for a prescribed compressive F_s, $N = (k_s + k_b)|F_s|/(pk_s k_b)$.

4.3.7 Problems

Problem 4.7. Constrained bar at uniform temperature

A uniform aluminum bar is constrained at its two end. If the bar is stress free for a temperature $T_0 = 20°$ C, find the compressive stress in the bar if the temperature is raised to value $T = 140°$ C. Note: $E_{\text{al}} = 73$ GPa, $\alpha_{\text{al}} = 16.5$ μ/C.

Problem 4.8. Steel bar inside a copper tube

A steel bar with a 750 mm^2 section is placed inside a copper tube with a section of 1250 mm^2. The bar and tube have a common length of 0.5 m and are connected at their ends. At the reference temperature, both elements are stress free. *(1)* If the assembly is heated up to 80° C, find the axial stresses in both elements. Note: $E_{\text{steel}} = 210$ GPa, $\alpha_{\text{steel}} = 12$ μ/C; $E_{\text{copper}} = 120$ GPa, $\alpha_{\text{copper}} = 17$ μ/C.

Problem 4.9. Bolt-sleeve assembly subjected to temperature rise

Consider the sleeve and bolt assembly shown in fig. 4.15, where the bolt is made of stainless steel, which presents a larger coefficient of thermal expansion than the titanium sleeve. Consequently, under a temperature rise $\Delta T = 100$ C, the bolt will extend more than the sleeve and will become loose, *i.e.*, a gap will develop between the nut and washer plate. To prevent this, a pre-stress is applied to the assembly by turning the nut N turns before the temperature rise. Determine the number of turns N that must be used to create the required pre-stress for the following conditions: $p = 0.5$ mm (bolt thread pitch), $L = 100$ mm, $\mathcal{A}_b = 100$ mm^2, $\mathcal{A}_s = 800$ mm^2, $E_b = 210$ GPa, $E_s = 120$ GPa, $\alpha_b = 18$ μ/C, and $\alpha_s = 8$ μ/C.

Problem 4.10. Three-bar truss

Consider the three-bar truss shown in fig. 4.7. The truss is not subjected to any external load, but due to a manufacturing imperfection, the middle bar is of length $L + \mu$ in its unstressed configuration. *(1)* Find the forces in bars **A**, **B**, and **C** as a function of the magnitude of the manufacturing imperfection, μ. *(2)* Find the displacement of point **O** as a function of μ.

4.4 Pressure vessels

This section briefly describes the behavior of structures operating under internal pressure such as rings, and cylindrical or spherical pressure vessels. Typically, these thin-walled structures are designed to contain fluids or gases under pressure. Two particular geometric shapes, the sphere and the cylinder with hemispherical end caps, are widely utilized, and for these shapes, a two-dimensional stress state develops in the thin walls.

4.4.1 Rings under internal pressure

Consider the thin-walled ring or tube of mean radius R and thickness t subjected to an internal pressure p_i, as depicted in fig. 4.16. Due of the internal pressure, a hoop stress, σ_h, will develop in the wall. This hoop stress is readily found by equilibrium consideration: fig. 4.16 shows a free body diagram for the half portion of the ring cut by a plane passing through the axis of the cylinder, revealing the hoop stress acting in the wall. The total vertical force per unit length of the ring due to the pressure acting on its upper half is $p2R$; this force is equilibrated by the hoop stress. Assuming that the hoop stress is uniformly distributed through the wall thickness, it follows that

$$\sigma_h = \frac{p2R}{2t} = \frac{pR}{t}. \tag{4.30}$$

The hoop stress is sometimes called the circumferential stress.

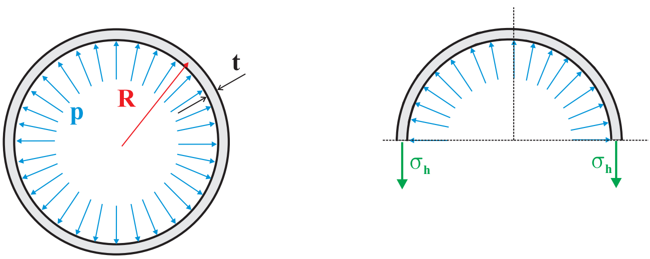

Fig. 4.16. Thin ring under internal pressure.

If the material is homogeneous and linearly elastic, the hoop strain, ϵ_h, is obtained from Hooke's law as $\epsilon_h = \sigma_h/E = pR/(tE)$ and the radius of the ring increases by an amount $\varDelta R = R^2 p/(Et)$.

4.4.2 Cylindrical pressure vessels

Consider now a thin-walled pressure vessel consisting of a cylindrical tube of radius R, length L and thickness t closed by spherical end caps, as depicted in fig. 4.17. Pressure vessels operate under a multi-axial state of stress that includes a hoop, axial and radial stress components. The hoop stress is readily found from the same equilibrium arguments used for the ring; assuming the hoop stress to be uniformly distributed through the wall thickness, its magnitude then becomes

$$\sigma_h = \frac{pR}{t}. \tag{4.31}$$

The resultant axial force of the pressure loading on the end caps is independent of their shape and is equal to $p\pi R^2$. For a thin-walled pressure vessel, the stress along the axis of the vessel, σ_a, is assumed to be uniformly distributed through the wall thickness, and axial equilibrium reveals its magnitude to be

$$\sigma_a = \frac{p\pi R^2}{2\pi R t} = \frac{pR}{2t} = \frac{\sigma_h}{2}. \tag{4.32}$$

This gives rise to a biaxial stress state where the hoop stress twice as large as the axial stress.

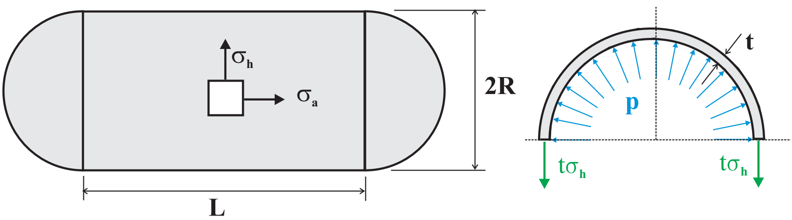

Fig. 4.17. Pressure vessel under internal pressure.

In addition, it should be noted that a radial stress component σ_r also exists. This stress acts along the radius of the cylindrical part of the vessel, and it varies from $\sigma_r = -p$ on the internal surface of the vessel to $\sigma_r = 0$ on the external surface. In most practical designs the ratio R/t is a large quantity and hence $\sigma_r \ll \sigma_h = 2\sigma_a$. Consequently, the radial stress is generally ignored.

If the material can be assumed to behave as a linearly elastic material, Hooke's law, eq. (2.4), implies

$$\epsilon_h = \frac{\sigma_h - \nu\sigma_a}{E} = \frac{\sigma_h}{E}\left(1 - \frac{\nu}{2}\right), \quad \epsilon_a = \frac{\sigma_a - \nu\sigma_c}{E} = \frac{\sigma_h}{E}\left(\frac{1}{2} - \nu\right).$$

Finally, the changes in vessel radial and longitudinal dimensions are

$$\Delta R = R\epsilon_h = \frac{R\sigma_h}{E}\left(1 - \frac{\nu}{2}\right), \quad \Delta L = L\epsilon_a = \frac{L\sigma_h}{E}\left(\frac{1}{2} - \nu\right),$$

respectively.

Since the hoop and axial stresses are the only stress components acting on the vessel, they are the principal stresses, $\sigma_{p1} = \sigma_h = pR/t$ and $\sigma_{p2} = \sigma_a = pR/2t$. According to Tresca's criterion, see eq. (2.29), the yield criterion reduces to $p_y R/t \leq \sigma_y$. This means that the internal pressure for which the yield stress is reached in the material is $p_y = t\sigma_y/R$. On the other hand, if von Mises' criterion is used, see eq. (2.32), the yield criterion becomes $\sigma_{\rm eq} = \sqrt{3}/2\ p_y R/t \leq \sigma_y$. The internal pressure for which the yield stress is reached in the material is $p_y = 2/\sqrt{3}\ t\sigma_y/R$.

4.4.3 Spherical pressure vessels

Consider now a thin-walled sphere of radius R and thickness t subjected to an internal pressure p, as shown in fig. 4.18. This type of configuration is representative of spherical pressure vessels. To begin, the sphere is cut by a horizontal plane passing through its center, to reveal the free body diagram shown in the figure. Due to the symmetry of the problem, the pressure acting on the upper half of the sphere will be equilibrated by a hoop stress, σ_h, which is uniformly distributed around the circle at the intersection of the sphere with the plane of the cut. The total upward force generated by the pressure, $\pi R^2 p$, is equilibrated by the downward force generated by the distributed hoop stress, $2\pi R t\sigma_h$, where the hoop stress is assumed to be uniformly distributed through the wall thickness. This yields the following result

$$\sigma_h = \frac{pR}{2t}. \tag{4.33}$$

The hoop stress is half of that in a pressurized tube of equal radius and thickness, see eq. (4.31).

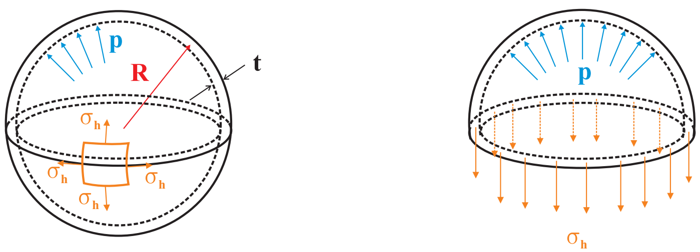

Fig. 4.18. Thin sphere under internal pressure.

Of course, in view of the spherical symmetry of the problem, the orientation of the plane of the cut is arbitrary. Hence, the hoop stress derived above is acting on a face with an arbitrary orientation. As shown in fig. 4.18, the stresses acting on an arbitrary differential element cut from the thin-walled sphere are σ_h in two orthogonal directions. Since the shear stress component vanish, these are the principal stresses, and hence, $\sigma_{p1} = \sigma_{p2} = \sigma_h$. Mohr's circle reduces to a single point at ordinate σ_h.

For a linearly elastic material, the hoop strain, ϵ_h, is obtained from Hooke's law, eq. (2.4), as

$$\epsilon_1 = \epsilon_2 = \epsilon_h = \frac{1-\nu}{2} \frac{R}{t} \frac{p}{E}. \tag{4.34}$$

The deformation is identical in all directions, due to the spherical symmetry of the problem. Since the shear strain components vanish, the principal strains are $\epsilon_{p1} = \epsilon_{p2} = \epsilon_h$. The radius of the sphere increases by an amount $\Delta R = (1-\nu)(pR^2)/(2Et)$.

4.4.4 Problems

Problem 4.11. Copper ring on a steel shaft

A copper ring is heated to a temperature of 150° C and then exactly fits onto a steel shaft at a uniform temperature of 25° C. *(1)* Find the hoop stress in the ring when the assembly has cooled down to a uniform temperature of 25° C. *(2)* Find the common temperature at which both ring and shaft must be brought to if the ring is to slip out of the shaft. Hint: since the steel cylinder is very stiff, it is reasonable to assume that it is remains rigid as the copper ring cools down. Of course, under heating, the steel cylinder will expand. Note: $\alpha_{\rm steel} = 12.5\mu$/C; $E_{\rm copper} = 110$ GPa, $\alpha_{\rm copper} = 16.5\mu$/C.

Problem 4.12. Bi-material fly wheel

A fly wheel shown in fig. 4.19 is made of two concentric ring of metal: the inside ring, of thickness t_ℓ, is made of lead and the outside ring, of thickness t_s, is made of steel. The fly wheel has a radius R_m and $t_\ell \ll R_m$, $t_s \ll R_m$. It will be assumed that the lead ring provides little strength and stiffness to the assembly and hence, all stresses are carried in the steel ring. *(1)* Find the maximum angular velocity, $\Omega_{\max}$, the fly wheel can rotate at if the yield stress in the steel is σ_y. *(2)* Find the maximum kinetic energy that can be stored in the fly wheel. *(3)* Is this bi-material design a good concept for a high performance fly wheel? Use the following data: density of lead, $\rho_\ell = 11,300$ and of steel, $\rho_s = 7,700$ kg/m^3; thickness of lead, $t_\ell = 5$ and of steel $t_s = 3$ mm; radius of the fly wheel $R_m = 250$ mm, its width $b = 20$ mm; yield stress for steel $\sigma_y = 800$ MPa.

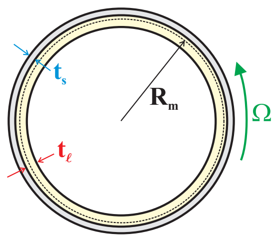

Fig. 4.19. Configuration of the bi-material fly wheel.

Problem 4.13. Cylindrical versus spherical pressure vessels

Spacecrafts often require pressure vessels to carry fuel under pressure. The question investigated here is the relative structural performance of cylindrical and spherical pressure vessels. Consider a cylindrical pressure vessel of radius R_c, length L_c and wall thickness t_c; $L_c = 2R_c$. On the other hand, consider a spherical pressure vessel of radius R_s and wall thickness t_s. The two vessels must carry the same amount of fluid, *i.e.*, must have the same volume; the two vessels are made of the same material with the yield stress σ_y, and must be able to withstand the same internal pressure. *(1)* Find the ratio of the structural masses of the two vessels. *(2)* For weight sensitive applications such as spacecrafts, is it better to use cylindrical or spherical pressure vessels?

4.5 Saint-Venant's principle

An important concept in structural engineering concerns the effects of local loading and constraint conditions on the stresses and deformations that develop throughout a structure. An obvious example is a concentrated force, which is assumed to act at a point on the surface of a structure. Clearly, this will result in an infinite value for the stresses at the point of application, but yet the reactions and stresses at other parts of the structure are finite.

Consider a body subjected to a set of self-equilibrating loads, as depicted in fig. 4.20. In the vicinity of the applied loads, internal stresses will arise, as expected. However, since the net resultant of the applied load vanishes, it seems reasonable to expect their net effect to decrease away from their point of application. In other words, the effect of a set of self-equilibrating loads is expected to be localized. Typically, if the loads are applied over an area of characteristic dimension δ, the affected zone approximately extends a distance δ in all directions from the point of application.

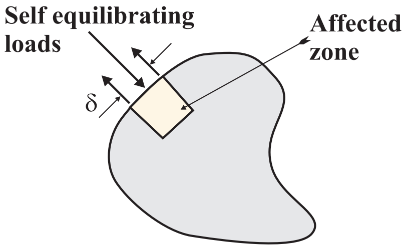

Fig. 4.20. Body subjected to a set of self-equilibrating loads

This behavior has been observed experimentally, and is known as Saint-Venant's principle.

Principle 2 (Saint-Venant's principle) *If self-equilibrating loads are applied to a body over an area of characteristic dimension δ, the internal stresses resulting from these loads are only significant over a portion of the body of approximate characteristic dimension δ.*

Note that this principle is rather vague, as it deals with "approximate characteristic dimensions." It allows qualitative rather that quantitative conclusions to be drawn.

An important application of Saint-Venant's principle deals with end effects in bars and beams. In section 4.2, the stress distribution in bars subjected to end loads is studied. Clearly, the assumed uniform stress distribution over the cross-section of the bar is only valid far away from the end section of the bar. Consider fig. 4.21 where the end section of a bar of height h is subjected to a concentrated load P. This concentrated load is statically equivalent to a distributed load $p_0 = P/h$ plus a set of self-equilibrating loads, as depicted on the figure. Saint-Venant's principle implies that the self-equilibrating set of loads only affect a small zone of length h near the end of the bar.

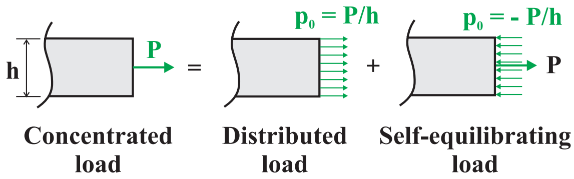

Fig. 4.21. Bar subjected to an end concentrated load.

According to Saint-Venant's principle, the stress distribution in the bar, namely the uniform axial stress distribution of eq. (4.1), is identical whether the bar is subjected to end distributed or concentrated loads, except in the two end zones of length h. If the end loads are applied as a uniform distribution, the axial stresses in the bar are uniformly distributed over the cross-section at all sections. On the other hand, if the end loads are concentrated loads, the axial stresses in the bar are uniformly distributed over the cross-section only in the central portion of the beam. Near the end points, a complex state of stress will arise; indeed, the axial stress should grow to infinity right at the point of application of the concentrated load. These end zones approximately extend a distance h at either end of the beam. The solution discussed in section 4.2 is sometimes called the *central solution*, *i.e.*. the solution valid in the central portion of the bar, away from the end zones.