12

Variational and energy principles

Chapter 9 presents the principle of virtual work and its complementary counterpart for particles, systems of particles, and trusses. These principles are introduced by means of simple examples and no attempt is made to formally derive them for three-dimensional solids.

Chapter 10 follows a similar pattern for the derivation of the principle of minimum total potential energy and of its complementary counterpart. Simple applications are presented focusing on mechanical systems, and trusses. Basic concepts of the finite element method applied to truss structures are presented.

Chapter 11 is devoted entirely to the development of approximate solutions for beam problems. The key to this approach is the ability to recast the differential equations of equilibrium into integral forms. The equivalence between the weak statement of equilibrium and the principle of virtual work is demonstrated for simple beam problems. Basic concepts of the finite element method applied to beam structures are presented.

In this chapter, the problem of determining stationary values of functionals (*i.e.*, functions of functions) will be addressed. The basic concepts from the *calculus of variations* [5, 6] that are required for this task will be reviewed first. Next, the principles of virtual and complementary virtual work, the principles of minimum total potential energy and total complementary energy, the Hu-Washizu principle, and the Hellinger-Reissner principle each will be formally presented for three-dimensional solids. Selected structural mechanics problems will then be examined to illustrate the use of these different principles.

12.1 Mathematical preliminaries

The basic equations of elasticity developed in chapter 1 use the formalisms of differential calculus and partial differential equations. Elements of the calculus of variations will be presented in this section.

12.1.1 Stationary point of a function

Consider a function of n variables, $F = F(u_1, u_2, \ldots, u_n)$. The stationary points [7] of this function are defined as those for which

$$\frac{\partial F}{\partial u_i} = 0, \quad i = 1, 2, \ldots, n. \tag{12.1}$$

For a function of a single variable, this condition corresponds to a horizontal tangent to the curve, as illustrated in fig. 12.1. At a stationary point, the function can present a minimum, a maximum, or a saddle point.

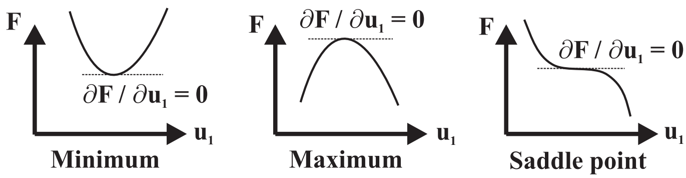

Fig. 12.1. Stationary points of a function.

If a function is stationary at a point, conditions (12.1) hold, and the following statement is then true

$$\frac{\partial F}{\partial u_1}\, w_1 + \frac{\partial F}{\partial u_2}\, w_2 + \ldots + \frac{\partial F}{\partial u_n}\, w_n = 0,$$

where w_1, w_2, ..., w_n are arbitrary quantities. It is convenient to use a special notation for these arbitrary quantities, $w_i = \delta u_i$, where δu_i is called a *virtual change* in u_i. The above statement now becomes

$$\frac{\partial F}{\partial u_1}\, \delta u_1 + \frac{\partial F}{\partial u_2}\, \delta u_2 + \ldots + \frac{\partial F}{\partial u_n}\, \delta u_n = 0.$$

Comparison of this result with a similar expression for the differential, $\mathrm{d}F$, of the same function implies that differentials can be used as virtual changes. Consequently, the virtual change operator, denoted "δ," behaves in a manner similar to the differential operator, denoted "d".

The *variation in* F, denoted δF, is defined as

$$\delta F = \frac{\partial F}{\partial u_1}\, \delta u_1 + \frac{\partial F}{\partial u_2}\, \delta u_2 + \ldots + \frac{\partial F}{\partial u_n}\, \delta u_n, \tag{12.2}$$

and it then follows that

$$\delta F = 0 \tag{12.3}$$

at a stationary point.

The differential condition, eq. (12.1), and the variational condition, eq. (12.3) must both hold at a stationary point. From the above developments, it is clear that eq. (12.1) implies eq. (12.3) and since the above reasoning can be reversed, it is simple to prove that eq. (12.3) implies eq. (12.1). Hence, the two conditions are entirely equivalent.

To determine whether a stationary point is a minimum, a maximum, or a saddle point it is necessary to consider the second derivatives [7] of the functions. If

$$\sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j} \, \mathrm{d}u_i \mathrm{d}u_j > 0 \tag{12.4}$$

at a stationary point for all increments $\mathrm{d}u_i$ and $\mathrm{d}u_j$, the function presents a minimum. If, on the other hand, the same quantity is negative for all $\mathrm{d}u_i$ and $\mathrm{d}u_j$, the function presents a maximum. Finally, if the same quantity can be positive or negative depending on the choice of the increments, the function presents a saddle point.

From the definition of δF, eq. (12.2), the second variation of function F is defined as

$$\delta^2 F = \sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j} \, \delta u_i \delta u_j.$$

It is now clear that a stationary point is a minimum if

$$\delta^2 F > 0, \tag{12.5}$$

for all arbitrary variations δu_i and δu_j. It is a maximum if $\delta^2 F < 0$ for all variations, and a saddle point occurs if the sign of the second variation depends on the choice of the variations of the independent variables.

12.1.2 Lagrange multiplier method

Consider once more the problem of determining a stationary point of a function of several variables, $F = F(u_1, u_2, \ldots, u_n)$, where the variables are not independent. Rather, they are subjected to a constraint of the form

$$f(u_1, u_2, \ldots, u_n) = 0. \tag{12.6}$$

Conceptually the constraint can be used to express one variable, say u_n, in terms of the others. Then, u_n can be eliminated from F to obtain a function of $n-1$ independent variables, $F = F(u_1, u_2, \ldots, u_{n-1})$, which is a problem identical to that treated in the previous section. In many practical situations, however, it might be cumbersome, or even impossible, to completely eliminate one variable of the problem. For example, the constraint equation could be a transcendental equation with no closed-form solution for u_n, or an implicit equation with no simple solution.

This elimination-of-variable process can be avoided by using an alternative approach. At a stationary point, the variation of F vanishes

$$\delta F = \frac{\partial F}{\partial u_1} \, \delta u_1 + \frac{\partial F}{\partial u_2} \, \delta u_2 + \ldots + \frac{\partial F}{\partial u_n} \, \delta u_n = 0. \tag{12.7}$$

This statement, however, does not imply $\partial F/\partial u_i = 0$, for $i = 1, 2, \ldots, n$, because the variations, δu_i, cannot be chosen arbitrarily since they must satisfy the constraint, eq. (12.6).

The relation among the variations can be written explicitly by taking a variation of the constraint to find

$$\delta f = \frac{\partial f}{\partial u_1}\,\delta u_1 + \frac{\partial f}{\partial u_2}\,\delta u_2 + \ldots + \frac{\partial f}{\partial u_n}\,\delta u_n = 0. \tag{12.8}$$

A linear combination of eqs. (12.7) and (12.8) can be constructed to find

$$\frac{\partial F}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial F}{\partial u_n}\,\delta u_n + \lambda\left[\frac{\partial f}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial f}{\partial u_n}\,\delta u_n\right] = 0,$$

where λ is an arbitrary function of variables $u_1, u_2, \ldots, u_n$, called the *Lagrange multiplier*. Regrouping the various terms then leads to

$$\sum_{i=1}^{n}\left[\frac{\partial F}{\partial u_i} + \lambda\,\frac{\partial f}{\partial u_i}\right]\,\delta u_i = 0. \tag{12.9}$$

Conceptually, variation δu_n could now be express in term of the $n-1$ other variations, δu_i, leaving the $n-1$ remaining variations to be independent and arbitrary. To avoid this cumbersome algebraic step, the arbitrary Lagrange multiplier is chosen such that

$$\frac{\partial F}{\partial u_n} + \lambda\,\frac{\partial f}{\partial u_n} = 0.$$

With this choice, the last term of the sum in eq. (12.9) vanishes for all variations δu_n. Hence, it is not necessary to express this variation in terms of the $n-1$ others, which can now be treated as independent, arbitrary quantities. Equation (12.9) then implies

$$\frac{\partial F}{\partial u_i} + \lambda\,\frac{\partial f}{\partial u_i} = 0, \quad i = 1, 2, \ldots, n-1. \tag{12.10}$$

Combining the last two equations then leads to the condition that

$$\delta F + \lambda\delta f = 0,$$

where all variations, δu_i, $i = 1, 2, \ldots, n$, are *independent*.

Because of the constraint, eq. (12.6), it is clear that $f\delta\lambda = 0$ for *any arbitrary* $\delta\lambda$. Hence, the stationarity condition can be written as

$$\delta F + \lambda\delta f = \delta F + \lambda\delta f + f\delta\lambda = \delta(F + \lambda f) = 0.$$

A modified function, $F^+ = F + \lambda f$ is introduced and the above statement now implies the vanishing of the variation in F^+ for *all arbitrary variations* δu_i, $i = 1, 2, \ldots, n$, *and* $\delta\lambda$.

In summary, the initial constrained problem can be replaced by an *unconstrained problem*

$$\delta F^{+} = 0, \quad \text{where} \quad F^{+} = F + \lambda f. \tag{12.11}$$

The modified function, F^{+}, involves $n+1$ variables, u_i, $i = 1, 2, \ldots, n$ and λ. The vanishing of the variation in F^{+} then implies

$$\sum_{i=1}^{n} \left[\frac{\partial F}{\partial u_i} + \lambda \, \frac{\partial f}{\partial u_i} \right] \delta u_i + f \, \delta\lambda = 0.$$

Because δu_i, $i = 1, 2, \ldots, n$, and $\delta\lambda$ are all independent, arbitrary variations, it follows that

$$\frac{\partial F}{\partial u_i} + \lambda \, \frac{\partial f}{\partial u_i} = 0, \quad i = 1, 2, \ldots, n; \quad \text{and} \quad f = 0.$$

These form $n+1$ equations to be solved for the $n+1$ unknowns. Note that the Lagrange multiplier method results in an unconstrained problem, but *increases* the number of unknowns from n to $n+1$; the additional unknown is the Lagrange multiplier. On the other hand, if the constraint is used to eliminate one of the unknowns, the resulting problem will be an unconstrained problem for $n-1$ unknowns.

The Lagrange multiplier methods can be readily generalized to problems involving multiple constraints, $f_i = 0$, $i = 1, 2, \ldots, m$. In the presence of m constraints, m Lagrange multipliers, λ_i, $i = 1, 2, \ldots, m$, are introduced. The modified function then becomes

$$F^{+} = F + \sum_{i=1}^{m} \lambda_i f_i. \tag{12.12}$$

12.1.3 Stationary point of a definite integral

Next, the determination of the stationary point of the following definite integral

$$I = \int_a^b F(y, \, y', \, x) \, \mathrm{d}x \tag{12.13}$$

is considered, where the notation $(\cdot)'$ is used to indicate a derivatives with respect to x, and $y(x)$ is an unknown function of x subject to boundary conditions, $y(a) = \alpha$ and $y(b) = \beta$.

This problem seems to be of a completely different nature from those treated in the previous sections. Indeed, I is a *functional* or a "function of a function", *i.e.*, the value of the definite integral I depends on the choice of the unknown function $y(x)$. Since there are an infinite number of values of y between a and b, functional I is equivalent to a function of an infinite number of variables.

This problem will be treated using the variational formalism introduced in section 12.1.1. First, the concept of *variation of a function*, denoted δf, is introduced. Figure 12.2 shows two functions $f(x)$ and $\hat{f}(x)$ such that

$$\delta f = \hat{f}(x) - f(x) = \psi(x),$$

where $\psi(x)$ is a continuous and differentiable, but otherwise arbitrary function such that $\psi(a) = \psi(b) = 0$. In other words, δf is a virtual change that brings the function $f(x)$ to a new, arbitrary function $\hat{f}(x)$. Note that $\delta f(a) = \delta f(b) = 0$ which means that δf does not violate the boundary conditions of the problem.

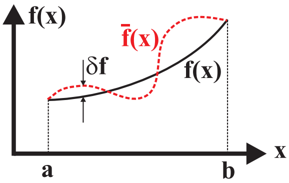

Fig. 12.2. The concept of variation of a function.

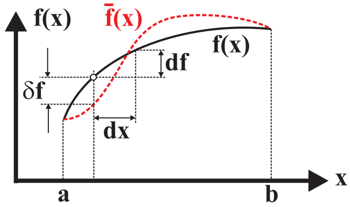

Fig. 12.3. The difference between an increment $\mathrm{d}f$ and a variation δf.

The stationarity of functional I requires

$$\delta I = \delta \int_a^b F(y,\, y',\, x)\, \mathrm{d}x = \int_a^b \delta F(y,\, y',\, x)\, \mathrm{d}x = 0.$$

With the help of eq. (12.2) and treating δ as a differential, this results in

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y}\, \delta y + \frac{\partial F}{\partial y'}\, \delta y' \right] \, \mathrm{d}x = 0.$$

Integration by parts is now applied to the second term in the square bracket

$$\int_a^b \frac{\partial F}{\partial y'} \delta \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}x = \int_a^b \frac{\partial F}{\partial y'} \frac{\mathrm{d}}{\mathrm{d}x} (\delta y)\, \mathrm{d}x = - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \delta y\, \mathrm{d}x + \left[\frac{\partial F}{\partial y'} \delta y \right]_a^b .$$

The boundary term vanishes because $\delta y(a) = \delta y(b) = 0$, and the stationarity condition then becomes

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \right] \delta y \; \mathrm{d}x = 0.$$

The bracketed term must vanish because the integral must go to zero for *all arbitrary variations* δy. This yields

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) = 0 \tag{12.14}$$

which is known as the *Euler-Lagrange equation* for the problem.

Here again, the above reasoning can be reversed. Starting from eq. (12.14), and performing the integration by parts in the reversed order implies $\delta I = 0$. In summary, *the necessary and sufficient condition for the definite integral to be at a stationary point is that eq.* (12.14) be satisfied.

The variational formalism introduced in this section will be systematically applied to elasticity problems in the rest of this chapter. It will be shown that the equations of elasticity can be viewed as the Euler-Lagrange equations associated with the stationarity condition of definite integrals. Various forms of the equations of elasticity can be easily obtained by direct manipulations of these definite integrals. It is therefore important to understand the variational formalism and its implications.

A crucial difference exists between an increment, $\mathrm{d}f$, of a function $f(x)$ and a variation, δf, of the same function, as depicted in fig. 12.3. The differential, $\mathrm{d}f$, is an infinitesimal change in $f(x)$ resulting from an infinitesimal change, $\mathrm{d}x$, in the independent variable, and $\mathrm{d}f/\mathrm{d}x$ represents the rate of change or tangent at the point. On the other hand, δf is an arbitrary virtual change that brings $f(x)$ to $\hat{f}(x)$. The two quantities, $\mathrm{d}f$ and δf, are clearly unrelated, the former is positive in fig. 12.3, and the latter is negative.

Although the concepts associated with the notation $\mathrm{d}f$ and δf are clearly distinct, manipulations of the two symbols are quite similar. For instance, the order of application of the two operations can be interchanged, indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\delta f) = \frac{\mathrm{d}}{\mathrm{d}x}(\hat{f} - f) = \frac{\mathrm{d}\hat{f}}{\mathrm{d}x} - \frac{\mathrm{d}f}{\mathrm{d}x} = \delta\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right). \tag{12.15}$$

Similarly, the order of the integration and variation operations commute

$$\delta \int_a^b F\,\mathrm{d}x = \int_a^b \hat{F}\,\mathrm{d}x - \int_a^b F\,\mathrm{d}x = \int_a^b (\hat{F} - F)\,\mathrm{d}x = \int_a^b \delta F\,\mathrm{d}x. \tag{12.16}$$

12.2 Variational and energy principles

Consider a general elasticity problem consisting of an elastic body of arbitrary shape subjected to surface tractions and body forces as well as geometric boundary conditions such as prescribed displacements at a point or over a portion of its outer surface, as depicted in fig. 12.4.

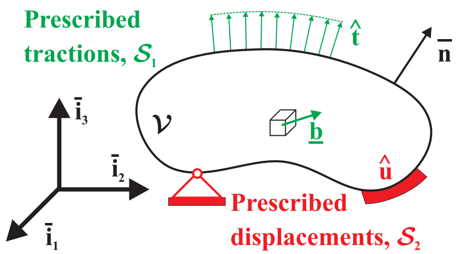

Fig. 12.4. General elasticity problem.

The volume of the body is denoted $\mathcal{V}$ and its outer surface $\mathcal{S}$. The outer normal to $\mathcal{S}$ is the unit vector $\bar{n}$. $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the portions of the outer surface where prescribed tractions $\underline{\hat{t}}$ and prescribed displacements $\underline{\hat{u}}$ are applied, respectively. At a point of the outer surface, either tractions or displacements can be prescribed, but it is impossible to prescribe both. Consequently, $\mathcal{S}_1$ and $\mathcal{S}_2$ share no common points and $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. Note that a point of the outer surface that is traction free belongs to $\mathcal{S}_1$ because vanishing traction conditions, $\underline{\hat{t}} = 0$, are prescribed at that point.

Body forces $\underline{b}$ might also be applied over the entire volume of the body. Gravity forces are a typical example of body forces, but such forces can also arise as a result of electric or magnetic fields. In dynamic problems, inertial forces are also applied as body forces in accordance with D'Alembert's principle.

The basic equations of elasticity developed in chapter 1 form a set of partial differential equations that can be solved to find the displacement, strain, and stress fields at all points in $\mathcal{V}$. These equations will be reviewed in section 12.2.1 where several important definitions are also introduced. In the subsequent sections, a number of variational and energy principles are presented that provide an alternative formalism for the solution of elasticity problems.

12.2.1 Review of the equations of linear elasticity

As depicted in fig. 3.1 on page 101, the equations of elasticity can be broken into three groups. The solution of an elasticity problem involves *(1)* a statically admissible stress field, *(2)* a kinematically admissible displacement field and the corresponding compatible strain field, and *(3)* a constitutive law satisfied at all points in volume $\mathcal{V}$. These concepts are explained below.

Equilibrium equations

The *equations of equilibrium* are the most fundamental equations. They are derived in sections 1.1.2 and 1.1.3 from Newton's law stating that the sum of all the forces acting on a differential element of the structure should vanish.

For reference, the equilibrium equations for a differential element of the body, eqs. (1.4), are rewritten here

$$\begin{aligned}\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 = 0,\\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 = 0,\\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 = 0,\end{aligned} \tag{12.17}$$

and must be satisfied at all points of volume $\mathcal{V}$.

The traction equilibrium equations are

$$t_1 = \hat{t}_1, \quad t_2 = \hat{t}_2, \quad t_3 = \hat{t}_3, \tag{12.18}$$

where the surface tractions are defined in eq. (1.9). The surface equilibrium equations are also called the *force*, or *natural boundary conditions*. The compact stress array, $\underline{\sigma}$, defined be eq. (2.11b), will be used simplify the notation.

Definition 12.1. *A stress field, $\underline{\sigma}$, is said to be statically admissible if it satisfies the equilibrium equations, eqs.* (12.17)*, at all points of volume $\mathcal{V}$ and the surface equilibrium equations, eqs.* (12.18)*, at all points of surface $\mathcal{S}_1$.*

Strain-displacement relationships

The *strain-displacement equations* merely define the strain components that are used for the characterization of the deformation at a point of the body. The strain-displacement relationships are derived in section 1.4.1 from purely geometric considerations.

When the displacements are small, it is convenient to use the engineering strain components to measure the deformation at a point. From section 1.4, axial and shearing strain components are related to the displacements as

$$\begin{gathered}\epsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_3 = \frac{\partial u_3}{\partial x_3}, \\ \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}, \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.\end{gathered} \tag{12.19}$$

To compute strain components, the displacements field must be continuous and differentiable. Furthermore, the displacements must be equal to the prescribed displacements over surface S_2

$$u_1 = \hat{u}_1, \quad u_2 = \hat{u}_2, \quad u_3 = \hat{u}_3; \tag{12.20}$$

these are called the *geometric boundary conditions*.The compact strain array, $\underline{\epsilon}$, defined be eq. (2.11a), will be used simplify the notation.

Definition 12.2. *A displacement field, $\underline{u}$, is said to be kinematically admissible if it is continuous and differentiable at all points in volume $\mathcal{V}$ and satisfies the geometric boundary conditions, eqs.* (12.20)*, at all points on surface $\mathcal{S}_2$.*

Definition 12.3. *A strain field, $\underline{\epsilon}$, is said to be compatible if it is derived from a kinematically admissible displacement field through the strain-displacement relationships, eqs.* (12.19).

Constitutive laws

The *constitutive laws* relate the stress and strain components. They consist of a mathematical idealization of the experimentally observed behavior of materials. The homogeneous, isotropic, linearly elastic material behavior described in section 2.1.1 is a frequently used highly idealized constitutive law. Many materials can present one or more of the following features: anisotropy, plasticity, visco-elasticity, or creep, to name just a few commonly observed material behaviors.

The stress and strain fields are related by the constitutive laws at all points in volume $\mathcal{V}$. For linearly elastic materials, Hooke's law, eq. (2.10), provides a simple linear relationship between the two fields. The positive-definite, symmetric stiffness matrix, $\underline{\underline{C}}$, and the positive-definite, symmetric compliance matrix,$\underline{\underline{S}}$ are given by eqs. (2.12) and (2.14), respectively.

12.2.2 The principle of virtual work

Consider an elastic body that is in equilibrium under applied body forces and surface tractions. This implies that the stress field is statically admissible, *i.e.*, the equilibrium equations, eqs. (12.17), are satisfied at all points in $\mathcal{V}$ and the surface equilibrium equations, eqs. (12.18), at all points on $\mathcal{S}_1$. The following statement is now constructed

$$\int_{\mathcal{V}} \left\{ \left[\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right] \delta u_1 + \left[\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right] \delta u_2 \right.$$
$$\left. + \left[\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right] \delta u_3 \right\} \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \left[\underline{t} - \underline{\hat{t}} \right]^T \delta \underline{u} \, \mathrm{d}\mathcal{S} = 0. \quad (12.21)$$

In this statement each of the three equilibrium equations is multiplied by an arbitrary, virtual change in displacement, then integrated over the range of validity of the equation, volume $\mathcal{V}$. Similarly, each of the three surface equilibrium equations is multiplied by an arbitrary, virtual change in displacement, then integrated over the range of validity of the equation, surface $\mathcal{S}_1$.

Because the stress field is statically admissible, each bracketed term vanishes, and multiplication by an arbitrary quantity results in a vanishing product. Each of the two integral then vanishes, as does their sum.

Next, integration by parts is performed. Using Green's theorem [7], the first term of the volume integral becomes

$$\int_{\mathcal{V}} \frac{\partial \sigma_1}{\partial x_1} \delta u_1 \, \mathrm{d}\mathcal{V} = - \int_{\mathcal{V}} \sigma_1 \frac{\partial \delta u_1}{\partial x_1} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}} n_1 \sigma_1 \, \delta u_1 \, \mathrm{d}\mathcal{S}, \quad (12.22)$$

where n_1 is the component of the outward unit normal along $\bar{\imath}_1$, see fig. 12.4. A similar operation is performed on each stress derivative terms appearing in eq. (12.21).

Finally, the stress and strain arrays are introduced to obtain a compact result,

$$- \int_{\mathcal{V}} \underline{\sigma}^T \delta \underline{\epsilon} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}} \underline{t}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_1} (\underline{t} - \underline{\hat{t}})^T \delta \underline{u} \, \mathrm{d}\mathcal{S} = 0, \quad (12.23)$$

where $\delta \underline{\epsilon}$ denotes a *virtual, compatible strain field* defined as

$$\delta\epsilon_1 = \frac{\partial \delta u_1}{\partial x_1}, \ \delta\epsilon_2 = \frac{\partial \delta u_2}{\partial x_2}, \ \delta\epsilon_3 = \frac{\partial \delta u_3}{\partial x_3},$$
$$\delta\gamma_{23} = \frac{\partial \delta u_2}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_2}, \ \delta\gamma_{13} = \frac{\partial \delta u_1}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_1}, \ \delta\gamma_{12} = \frac{\partial \delta u_1}{\partial x_2} + \frac{\partial \delta u_2}{\partial x_1}. \quad (12.24)$$

The virtual displacements are now chosen to be kinematically admissible, which implies $\delta \underline{u} = 0$ on $\mathcal{S}_2$, and expression (12.23) reduces to

$$- \int_{\mathcal{V}} \underline{\sigma}^T \delta \underline{\epsilon} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} = 0. \quad (12.25)$$

The first term on the left hand side of this expression can be interpreted as the virtual work done by the internal stresses, δW_I, see eq. (9.77a). The remaining two terms correspond to the virtual work done by the externally applied body forces and surface tractions, δW_E. Equation (12.25) therefore becomes $\delta W_I + \delta W_E = 0$.

It can also be shown that if $\delta W_I + \delta W_E = 0$ holds, the stress field must be statically admissible. Indeed, this principle implies eq. (12.23), which in turn implies eq. (12.21) by reversing the integration by parts process. Finally, the volume and surface equilibrium equations are recovered because eq. (12.21) must hold for all arbitrary, kinematically admissible virtual displacements fields. These results imply the principle of virtual work.

Principle 15 (Principle of virtual work) *A body is in equilibrium if and only if the sum of the internal and external virtual work vanishes for all arbitrary kinematically admissible virtual displacements fields and corresponding compatible strain fields.*

In summary, the equations of equilibrium, eqs. (12.17) and (12.18), and the principle of virtual work are two entirely equivalent statements. Because the principle of virtual work is solely a statement of equilibrium, it is always true. For the solution of specific elasticity problems, however, it must be complemented with stress-strain relationships and constitutive laws.

It is interesting to compare the present statement of the principle of virtual work with that derived in chapter 11 for beams under axial and transverse loads and given by eqs.(11.42) and (11.44), respectively. The statements are different because the present formulation deals with general, three-dimensional stress states, whereas the formulation in chapter 11 deals with the stress resultants associated with beam theory. The physical interpretation of these statements, however, is identical in all cases.

12.2.3 The principle of complementary virtual work

Consider an elastic body undergoing kinematically admissible displacements and compatible strains. This implies that the strain-displacement relationships, eqs. (12.19), are satisfied at all points in volume $\mathcal{V}$ and the geometric boundary conditions, eqs. (12.20), at all points on surface $\mathcal{S}_2$. The following statement is now constructed

$$-\int_{\mathcal{V}} \left\{ \left[\epsilon_1 - \frac{\partial u_1}{\partial x_1}\right] \delta\sigma_1 + \left[\epsilon_2 - \frac{\partial u_2}{\partial x_2}\right] \delta\sigma_2 + \left[\epsilon_3 - \frac{\partial u_3}{\partial x_3}\right] \delta\sigma_3 \right.$$
$$+ \left[\gamma_{23} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right] \delta\tau_{23} + \left[\gamma_{13} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right] \delta\tau_{13}$$
$$\left. + \left[\gamma_{12} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right] \delta\tau_{12} \right\} \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_2} \left[\underline{u} - \underline{\hat{u}}\right]^T \delta\underline{t} \, \mathrm{d}\mathcal{S} = 0. \quad (12.26)$$

This statement is constructed in the following manner. Each of the six strain-displacement relationships is multiplied by an arbitrary, virtual change in stress, then integrated over the range of validity of the equations, volume $\mathcal{V}$. Similarly, each of the three geometric boundary conditions is multiplied by an arbitrary, virtual change in surface traction, then integrated over the range of validity of the equation, surface $\mathcal{S}_2$.

Because the strain field is compatible and the displacement field kinematically admissible, each bracketed term vanishes, and multiplication by an arbitrary quantity results in a vanishing product. Each of the two integral then vanishes, as does their sum.

Next, integration by parts is performed. Using Green's theorem [7], the first term of the volume integral becomes

$$\int_{\mathcal{V}} \frac{\partial u_1}{\partial x_1}\, \delta\sigma_1 \,\mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} u_1 \frac{\partial \delta\sigma_1}{\partial x_1}\,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}} u_1 n_1 \delta\sigma_1 \,\mathrm{d}\mathcal{S}, \tag{12.27}$$

where n_1 is the component of the outward unit normal along $\bar{\imath}_1$, see fig. 12.4. A similar operation is performed on each displacement derivative terms appearing in eq. (12.26) to yield

$$\begin{aligned} -\int_{\mathcal{V}} \underline{\epsilon}^T \delta\underline{\sigma}\,\mathrm{d}\mathcal{V} - \int_{\mathcal{V}} \Bigg[\left(\frac{\partial \delta\sigma_1}{\partial x_1} + \frac{\partial \delta\tau_{21}}{\partial x_2} + \frac{\partial \delta\tau_{31}}{\partial x_3} \right) u_1 \\ + \left(\frac{\partial \delta\tau_{12}}{\partial x_1} + \frac{\partial \delta\sigma_2}{\partial x_2} + \frac{\partial \delta\tau_{32}}{\partial x_3} \right) u_2 + \left(\frac{\partial \delta\tau_{13}}{\partial x_1} + \frac{\partial \delta\tau_{23}}{\partial x_2} + \frac{\partial \delta\sigma_3}{\partial x_3} \right) u_3 \Bigg] \,\mathrm{d}\mathcal{V} \\ + \int_{\mathcal{S}} \underline{u}^T \delta\underline{t}\,\mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} (\underline{u} - \underline{\hat{u}})^T \delta\underline{t}\,\mathrm{d}\mathcal{S} = 0. \end{aligned} \tag{12.28}$$

Next, a *statically admissible virtual stress field* is defined as a virtual stress field that satisfies equilibrium equations in volume $\mathcal{V}$,

$$\begin{aligned} \frac{\partial \delta\sigma_1}{\partial x_1} + \frac{\partial \delta\tau_{21}}{\partial x_2} + \frac{\partial \delta\tau_{31}}{\partial x_3} &= 0, \\ \frac{\partial \delta\tau_{12}}{\partial x_1} + \frac{\partial \delta\sigma_2}{\partial x_2} + \frac{\partial \delta\tau_{32}}{\partial x_3} &= 0, \\ \frac{\partial \delta\tau_{13}}{\partial x_1} + \frac{\partial \delta\tau_{23}}{\partial x_2} + \frac{\partial \delta\sigma_3}{\partial x_3} &= 0, \end{aligned} \tag{12.29}$$

and the surface traction equilibrium equation, $\delta\underline{t} = 0$ on surface S_1.

Because the virtual stresses are arbitrary, they can be chosen to be statically admissible and eq. (12.28) reduces to

$$-\int_{\mathcal{V}} \underline{\epsilon}^T \delta\underline{\sigma}\,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta\underline{t}\,\mathrm{d}\mathcal{S} = 0. \tag{12.30}$$

The first term on the left hand side of this expression can be interpreted as the complementary virtual work done by the internal stresses, $\delta W_I'$, see eq. (9.77b). The remaining term corresponds to the complementary virtual work done by the prescribed displacements, $\delta W_E'$. Equation (12.30) therefore becomes $\delta W_I' + \delta W_E' = 0$.

It can also be shown that if $\delta W_I' + \delta W_E' = 0$ holds, the displacement field must be kinematically admissible and the strain field compatible. Indeed, this principle implies eq. (12.28), which in turn implies eq. (12.26) by reversing the integration by parts process. Finally, the strain-displacement relationships and geometric boundary conditions are recovered because eq. (12.26) must hold for all arbitrary stress virtual stress fields. These results imply the principle of complementary virtual work

Principle 16 (Principle of complementary virtual work) *A body is undergoing kinematically admissible displacements and compatible strains if and only if the sum of the internal and external complementary virtual work vanishes for all statically admissible virtual stress fields.*

In summary, the strain-displacement relationships and the geometric boundary conditions, eqs. (12.19) and (12.20), respectively, and the principle of complementary virtual work are two entirely equivalent statements. In addition, comparison of eq. (12.30) with the principle of complementary virtual work, principle 7, developed in chapter 9, shows that principle 16 above is simply a more general statement of principle 7.

12.2.4 Strain and complementary strain energy density functions

In section 10.5, on page 519, the strain and complementary strain energy density functions are developed for a linearly elastic, isotropic material governed by Hooke's law. The strain and complementary strain energy density functions are given by eqs. (10.47) and (10.50), respectively.

If the internal forces in the solid are assumed to be conservative, they can be derived from a potential, as discussed in section 10.1. In this case, the internal forces are the components of stress, and the potential is the strain energy density function. If the stresses in a solid can be derived from a strain energy density function, $a(\underline{\epsilon})$,

$$\underline{\sigma} = \frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}}, \tag{12.31}$$

the material is said to be an *elastic material*. Assuming the material to be elastic or assuming the existence of a strain energy density function are two equivalent assumptions. Linearly elastic materials are elastic materials for which the stress-strain relationship is linear.

If the material is elastic, the work done by the internal stresses when the system is brought from one state of deformation to another depends only on the two states of deformations, but not on the specific path that the system followed from one deformation state to the other. This restricts the types of material constitutive laws that can be expressed in terms of a strain energy density function. For instance, if a material is deformed in the plastic range, the work of deformation will depend on the specific deformation history; hence, there exists no strain energy density function that describes material behavior when plastic deformations are involved.

The concept of complementary strain energy is first introduced for springs in section 10.3.1. For nonlinearly elastic materials, the complementary strain energy density function is defined by the following identity

$$a(\underline{\epsilon}) + a'(\underline{\sigma}) = \underline{\epsilon}^T \underline{\sigma}, \tag{12.32}$$

which explains the term "complementary strain energy." Taking a differential of this identity yields

$$\left(\frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}} - \underline{\sigma}\right)^T \mathrm{d}\underline{\epsilon} + \left(\frac{a'(\underline{\sigma})}{\partial \underline{\sigma}} - \underline{\epsilon}\right)^T \mathrm{d}\underline{\sigma} = 0.$$

The term in the first parenthesis vanishes because of eq. (12.31). Because the differentials are arbitrary, the second parenthesis must vanish, leading to

$$\underline{\epsilon} = \frac{a'(\underline{\sigma})}{\partial \underline{\sigma}}. \tag{12.33}$$

In view of eq. (12.32), the existence of the strain energy density function implies the existence of the complementary strain energy density function. The strain energy density function allows the definition of the stresses by eqs. (12.31), which can be viewed as the constitutive laws for the elastic material because they define stresses as a function of strains. Similarly, the complementary strain energy density function allows the definition of the strains by eqs. (12.33), which can be viewed as the constitutive laws for the elastic material because they define strains as a function of stresses. Clearly, the strain and complementary strain energy density functions define the constitutive laws for elastic materials. The stiffness form of the constitutive laws, eqs. (12.31), stems from the strain energy density function, whereas the complementary strain energy density function yields the compliance form of the same constitutive laws, eqs. (12.33).

12.2.5 The principle of minimum total potential energy

Consider a general elastic body that is in equilibrium under applied body forces and surface tractions, and therefore, the principle of virtual work, eq. (12.25), must apply. It is now assumed that the constitutive law for the material can be expressed in terms of a strain energy density function, eq. (12.31). The virtual work done by the internal stresses appears in the first term of eq. (12.25), and it is readily evaluated as

$$-\int_{\mathcal{V}} \delta\underline{\epsilon}^T \underline{\sigma} \,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta\underline{\epsilon}^T \frac{\partial a(\underline{\epsilon})}{\partial \underline{\epsilon}} \,\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta a(\underline{u}) \,\mathrm{d}\mathcal{V} = \delta \int_{\mathcal{V}} a(\underline{u}) \,\mathrm{d}\mathcal{V} = \delta A(\underline{u}),$$

where the chain rule for derivatives is used at the second equality.

The strain energy density and the *total strain energy* of the body, $A = \int_{\mathcal{V}} a \,\mathrm{d}\mathcal{V}$, must be expressed in terms of the displacement field $\underline{u}$ using the strain displacement relationships because the principle of virtual work requires a compatible strain field. The principle of virtual work, eq. (12.25), now becomes

$$-\delta A(\underline{u}) + \int_{\mathcal{V}} \underline{b}^T \delta\underline{u} \,\mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta\underline{u} \,\mathrm{d}\mathcal{S} = 0. \tag{12.34}$$

Next, the body forces and surface tractions are assumed to be derivable from potential functions

$$\underline{b} = -\frac{\partial \phi}{\partial \underline{u}}; \quad \underline{\hat{t}} = -\frac{\partial \psi}{\partial \underline{u}},$$

where ϕ is the *potential of the body forces*, and ψ the *potential of the surface tractions*.

With these definitions, second and third terms (*i.e.*, the external work terms) in eq. (12.34) become

$$\begin{aligned}
&\int_{\mathcal{V}} \underline{b}^T \delta\underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta\underline{u} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{V}} \frac{\partial\phi}{\partial\underline{u}}^T \delta\underline{u} \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \frac{\partial\psi}{\partial\underline{u}}^T \delta\underline{u} \, \mathrm{d}\mathcal{S} \\
&= -\int_{\mathcal{V}} \delta\phi(\underline{u}) \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \delta\psi(\underline{u}) \, \mathrm{d}\mathcal{S} = -\delta\int_{\mathcal{V}} \phi(\underline{u}) \, \mathrm{d}\mathcal{V} - \delta\int_{\mathcal{S}_1} \psi(\underline{u}) \, \mathrm{d}\mathcal{S} \\
&= -\delta\Phi(\underline{u}),
\end{aligned}$$

where $\Phi(\underline{u}) = \int_{\mathcal{V}} \phi(\underline{u}) \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \psi(\underline{u}) \, \mathrm{d}\mathcal{S}$ is the *total potential the externally applied loads*.

Introducing this result into the principle of virtual work expressed in eq. (12.34) leads to

$$-\delta A(\underline{u}) - \delta\Phi(\underline{u}) = 0, \text{ or } \quad \delta\left(A(\underline{u}) + \Phi(\underline{u})\right) = 0. \tag{12.35}$$

The *total potential energy* of the body is now defined as

$$\Pi(\underline{u}) = A(\underline{u}) + \Phi(\underline{u}), \tag{12.36}$$

and it follows that

$$\delta\Pi(\underline{u}) = 0. \tag{12.37}$$

This statement expresses the requirement that the total potential energy must assume a stationary value with respect to compatible deformations when the body is in equilibrium. As discussed in section 12.1.1, the sign of the second variation, $\delta^2\Pi$, will determine whether the stationary point is actually a minimum. The first variation in Π is

$$\delta\Pi(\underline{u}) = \int_{\mathcal{V}} \left(\frac{\partial a}{\partial\underline{\epsilon}}\right)^T \delta\underline{\epsilon} \, \mathrm{d}\mathcal{V} - \int_{\mathcal{V}} \underline{b}^T \delta\underline{u} \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta\underline{u} \, \mathrm{d}\mathcal{S},$$

and its second variation is then

$$\delta^2\Pi(\underline{u}) = \int_{\mathcal{V}} \delta\underline{\epsilon}^T \frac{\partial^2 a}{\partial\underline{\epsilon}\partial\underline{\epsilon}} \delta\underline{\epsilon} \, \mathrm{d}\mathcal{V}.$$

Based on physical reasoning, the strain energy density function must be a positive-definite function of the strain components, which implies $\delta\underline{\epsilon}^T \partial^2 a/(\partial\underline{\epsilon}\partial\underline{\epsilon}) \, \delta\underline{\epsilon} \geq 0$ for all $\delta\underline{\epsilon}$. Indeed, if the strain energy function is not positive-definite, strain states will exist that generate a negative strain energy, *i.e.* the elastic body will generate energy under deformation, a situation that is physically impossible. It follows that $\delta^2\Pi \geq 0$, and hence, Π presents an absolute minimum at its stationary point. These results can be interpreted as follows.

Principle 17 (Principle of minimum total potential energy) *Among all kinematically admissible displacements fields, the actual displacement field that corresponds to the equilibrium configuration of the body makes the total potential energy an absolute minimum.*

The reverse is also true: if the principle of minimum total potential energy holds, the total potential energy must present a stationary point, implying eq. (12.35). In turn, this equation implies the principle of virtual work in which the stresses are expressed in terms of the strains using constitutive laws of the form of eq. (12.31), and strains are themselves expressed in terms of displacements using the strain-displacement relationships. The principle of minimum total potential energy implies the equations of equilibrium of the problem expressed in terms of the displacement field. From section 12.1.3, it is clear that these equations are the Euler-Lagrange equations arising from the stationarity condition for the total potential energy.

The principle of minimum total potential energy implies the principle of virtual work, but the principle of virtual work only implies the principle of minimum total potential energy under restrictive assumptions on existence of a strain energy density function and of potentials of the body forces and surface tractions. In other words, the principle of virtual work is a more general but possibly less useful statement.

12.2.6 The principle of minimum complementary energy

Consider an elastic body undergoing kinematically admissible displacements and compatible strains. In this case, the principle of complementary virtual work, eq. (12.30), must apply. It is now assumed that the constitutive law for the material can be expressed in terms of a stress energy density function, eq.(12.33). The virtual work done by the internal strains appears in the first term of eq. (12.30) and is now readily evaluated as

$$\int_{\mathcal{V}} \delta\underline{\sigma}^T \underline{\epsilon} \, \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta\underline{\sigma}^T \, \frac{\partial b(\underline{\sigma})}{\partial \underline{\sigma}} \, \mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \delta b(\underline{\sigma}) \, \mathrm{d}\mathcal{V} = \delta \int_{\mathcal{V}} b(\underline{\sigma}) \, \mathrm{d}\mathcal{V} = \delta A'(\underline{\sigma}),$$

where the chain rule for derivatives is used at the second equality. The quantity $A'(\underline{\sigma})$ is the *total stress energy* in the body.

The principle of complementary virtual work, eq. (12.30), can now be written as

$$-\delta A'(\underline{\sigma}) + \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = 0 \qquad (12.38)$$

Next, the prescribed displacements are *assumed* to be derivable from a potential function

$$\underline{\hat{u}} = -\frac{\partial \chi(\underline{t})}{\partial \underline{t}},$$

where $\chi(\underline{t})$ is the *potential of the prescribed displacements*. For instance, the potential of prescribed displacements is simply $\chi = -\underline{\hat{u}}^T \underline{t}$. It is important to note that potential functions do not exist for all types of prescribed displacements. For example, potential functions do not always exist for displacements that depend on the surface tractions, although such cases are not common in practice.

The second term in eq. (12.38) now becomes

$$\int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}_2} \frac{\partial \chi}{\partial \underline{t}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}_2} \delta \chi(\underline{t}) \, \mathrm{d}\mathcal{S} = -\delta \int_{\mathcal{S}_2} \chi(\underline{t}) \, \mathrm{d}\mathcal{S} = -\delta \Phi'.$$

where $\varPhi'(\underline{t}) = \int_{\mathcal{S}_2} \chi(\underline{t}) \, \mathrm{d}\mathcal{S}$ is the total potential the prescribed displacements. Introducing this result into eq. (12.38) leads to

$$-\delta A'(\underline{\sigma}) - \delta\varPhi'(\underline{t}), \text{ or } \quad \delta\left[A'(\underline{\sigma}) + \varPhi'(\underline{t})\right] = 0. \tag{12.39}$$

The *total complementary energy* of the body is now defined as

$$\varPi'(\underline{\sigma}) = A'(\underline{\sigma}) + \varPhi'(\underline{t}), \tag{12.40}$$

and it follows that

$$\delta\varPi'(\underline{\sigma}) = 0. \tag{12.41}$$

This statement can be interpreted as follows

Principle 18 (Principle of minimum complementary energy) *Among all statically admissible stress fields, the actual stress field that corresponds to the compatible deformations of the body makes the total complementary energy an absolute minimum.*

Equation (12.41) only proves that for compatible deformations, the total complementary energy presents a stationary point. As discussed in section 12.1.1, the sign of the second variation $\delta^2\varPi'$ will determine whether the stationary point actually is a minimum. The first variation in $\varPi'$ is

$$\delta\varPi'(\underline{\sigma}) = \int_{\mathcal{V}} \sum_{i=1}^{6} \frac{\partial a'}{\partial \sigma_i} \delta\sigma_i \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta\underline{t} \, \mathrm{d}\mathcal{S}, \tag{12.42}$$

and its second variation is then

$$\delta^2\varPi'(\underline{\sigma}) = \int_{\mathcal{V}} \sum_{i,j=1}^{6} \frac{\partial^2 a'}{\partial\sigma_i \partial\sigma_j} \, \delta\sigma_i \, \delta\sigma_j \, \mathrm{d}\mathcal{V}. \tag{12.43}$$

Just as for the strain energy density function, the stress energy density function must be a positive-definite function of the stress components, which implies $\sum_{i,j=1}^{6} \partial^2 a'/(\partial\sigma_i\partial\sigma_j) \, \delta\sigma_i \, \delta\sigma_j \geq 0$ for all $\delta\sigma_i$. Indeed, if the stress energy function is not positive-definite, stress states will exist that generate a negative stress energy, *i.e.* the elastic body will generate energy under stress, a situation that is physically impossible. It follows that $\delta^2\varPi' \geq 0$, and hence, $\varPi'$ presents an absolute minimum at its stationary point.

If the principle of minimum complementary energy holds, the complementary energy must present a stationary point, implying eq. (12.39). In turn, this equation implies the principle of complementary virtual work, in which the strains are expressed in terms of the stresses using constitutive laws of the form of eq. (12.33). As a result, the principle of minimum complementary energy implies the strain-displacement relationships of the problem expressed in terms of the stress field, which must satisfy equilibrium equations. From section 12.1.3,it follows that these equations are the Euler-Lagrange equations arising from the stationarity condition for the complementary energy.

The principle of minimum complementary energy implies the principle of complementary virtual work, but the principle of complementary virtual work only implies the principle of minimum complementary energy under restrictive assumptions on existence of a stress energy density function and of a potential for the prescribed displacements.

12.2.7 Energy theorems

In section 10.9, a number of energy theorems are presented that all are corollaries of the fundamental energy principles developed above. Clapeyron's theorem, theorem 10.1, and Castigliano's first theorem, theorem 10.2, are corollaries of the principle of minimum total potential energy. The principle of least work, principle 14, Crotti-Engesser theorem, theorem 10.3, and Castigliano's second theorem, theorem 10.4, are corollaries of the principle of complementary total potential energy. Finally, the reciprocity theorems of Betti and Maxwell, theorems 10.5 and 10.6, respectively, are direct consequences of these theorem. Because the principle of minimum total potential energy and its complementary counterpart have now been established for general, three-dimensional structures, the theorems listed above are also valid for the same three-dimensional structures.

12.2.8 Hu-Washizu's principle

The principle of virtual work developed in section 9.3 is shown to be entirely equivalent to the equations of equilibrium of a three-dimensional solid, eqs. (12.17) and (12.18). Because this principle is solely a statement of equilibrium, it must be complemented with stress-strain relationships and constitutive laws in order to obtain a complete set of equations for the solution of specific elasticity problems.

On the other hand, the principle of complementary virtual work developed in section 12.30 is equivalent to the strain-displacement relationships and the geometric boundary conditions, eqs. (12.19) and (12.20), respectively. This principle must be complemented with equilibrium equations and constitutive laws in order to obtain a complete set of equations for the solution of specific elasticity problems.

In summary, the principle of virtual work is a statement of equilibrium whereas the principle of complementary virtual work is a statement of compatibility. Clearly, these principles are equivalent to a subset of all the equations required for the solution of elasticity problems. Hu-Washizu's principle remedies this shortcoming, and it is equivalent to the complete set of equations required to solve elasticity problems.

Consider an elastic body that is in equilibrium under the applied body forces and surface tractions, that is undergoing compatible strains whose displacement field is kinematically admissible, and for which the stress and strain fields satisfy the material constitutive laws. This implies that the stress field is statically admissible, *i.e.*, the equilibrium equations, eqs. (12.17), are satisfied at all points in $\mathcal{V}$ and the surface equilibrium equations, eqs. (12.18), at all points on $\mathcal{S}_1$. This further implies that the strain-displacement relationships, eqs. (12.19), are satisfied at all points in $\mathcal{V}$ and the geometric boundary conditions, eqs. (12.20), at all points on $\mathcal{S}_2$. Finally, the constitutive equations, assumed to be expressed in terms of a strain energy density function, eq. (12.31), must hold at all points in $\mathcal{V}$.

The following statement is now constructed by combining eqs. (12.21), (12.26) and (12.31) into a single integral equation

$$\begin{aligned}
&\int_{\mathcal{V}} \left\{ \left[\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right] \delta u_1 + \left[\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right] \delta u_2 \right. \\
&+ \left. \left[\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right] \delta u_3 \right\} \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \left[\underline{t} - \underline{\hat{t}} \right]^T \delta \underline{u} \, \mathrm{d}\mathcal{S} \\
&- \int_{\mathcal{V}} \left\{ \left[\epsilon_1 - \frac{\partial u_1}{\partial x_1} \right] \delta \sigma_1 + \left[\epsilon_2 - \frac{\partial u_2}{\partial x_2} \right] \delta \sigma_2 + \left[\epsilon_3 - \frac{\partial u_3}{\partial x_3} \right] \delta \sigma_3 \right. \\
&+ \left[\gamma_{23} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right] \delta \tau_{23} + \left[\gamma_{13} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right] \delta \tau_{13} \\
&+ \left. \left[\gamma_{12} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right] \delta \tau_{12} \right\} \mathrm{d}\mathcal{V} - \int_{S_2} \left[\underline{u} - \underline{\hat{u}} \right]^T \delta \underline{t} \, \mathrm{d}\mathcal{S} \\
&+ \int_{\mathcal{V}} \left\{ \left[\frac{\partial a}{\partial \epsilon_1} - \sigma_1 \right] \delta \epsilon_1 + \left[\frac{\partial a}{\partial \epsilon_2} - \sigma_2 \right] \delta \epsilon_2 + \left[\frac{\partial a}{\partial \epsilon_3} - \sigma_3 \right] \delta \epsilon_3 \right. \\
&+ \left. \left[\frac{\partial a}{\partial \gamma_{23}} - \tau_{23} \right] \delta \gamma_{23} + \left[\frac{\partial a}{\partial \gamma_{13}} - \tau_{13} \right] \delta \gamma_{13} + \left[\frac{\partial a}{\partial \gamma_{12}} - \tau_{12} \right] \delta \gamma_{12} \right\} \mathrm{d}\mathcal{V} = 0.
\end{aligned} \tag{12.44}$$

This lengthy statement can be manipulated in several different ways. *(1)* The terms appearing in the equilibrium equations could be integrated by parts (as is done for the derivation of the principle of virtual work, section 12.2.2), *(2)* the terms appearing in the strain-displacement relationships could be integrated by parts (as is done for the derivation of the principle of complementary virtual work, section 12.2.3), or *(3)* both integrations by parts could be carried out. These three approaches will give rise to three different statements of Hu-Washizu's principle.

First statement of Hu-Washizu's principle

In the first approach, the terms appearing in the equations of equilibrium are integrated by parts using eq. (12.22). After regrouping all terms, this yields the *first statement of Hu-Washizu's principle*

$$\begin{aligned}
&\delta \int_{\mathcal{V}} \left[a(\underline{\epsilon}) - \left(\epsilon_1 - \frac{\partial u_1}{\partial x_1} \right) \sigma_1 - \left(\epsilon_2 - \frac{\partial u_2}{\partial x_2} \right) \sigma_2 - \left(\epsilon_3 - \frac{\partial u_3}{\partial x_3} \right) \sigma_3 \right. \\
&- \left(\gamma_{23} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right) \tau_{23} - \left(\gamma_{13} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \tau_{13} \\
&- \left. \left(\gamma_{12} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \tau_{12} \right] \mathrm{d}\mathcal{V} \\
&- \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} (\underline{u} - \underline{\hat{u}})^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = 0.
\end{aligned} \tag{12.45}$$

This principle involves three independent fields: the strain, stress, and displacement fields. Hence, Hu-Washizu's principle is a *three field principle*, whereas the principle of minimum total potential energy and the principle of minimum complementary energy both are single field principles, involving only the displacement and stress fields, respectively.

This principle is closely related to the principle of minimum total potential energy, eq. (12.34). Indeed, starting from eq. (12.45), the strain field is assumed to be compatible and the displacement field kinematically admissible. Hence, the strain displacement relationships are satisfied and the last six terms in the left hand side integrand vanish; furthermore, the displacement field satisfies the geometric boundary conditions on S_2 and the last integral on the right hand side vanishes as well. The remaining terms then express the principle of minimum total potential energy, eq. (12.34).

The first statement of Hu-Washizu's principle can also be obtained by starting from the principle of minimum total potential energy, eq. (12.34). This principle is a statement of equilibrium, because it is derived from the principle of virtual work, and the constitutive laws of the material are also included in the principle by means of the strain energy density function. However, this principle provides no information about the strain displacement relationships of the problem. Consequently, the principle of minimum total potential energy can be viewed as a constrained minimization problem that yields all the equations of elasticity: minimization of the total potential energy yields the equations of equilibrium and the constitutive laws, while the external constraints, the strain displacement equations, then yield the last set of equations.

This *constrained* minimization problem is then transformed into an *unconstrained* minimization problem using the Lagrange multiplier technique described in section 12.1.2. The modified function to be minimized is now in the form of eq. 12.12, where the f_i, $i = 1, 2, \ldots 6$, are the strain displacement relationships and the λ_i are six Lagrange multipliers. This is exactly the form of Hu-Washizu's principle, eq. (12.45), where the Lagrange multipliers are identified to be the stress components. This leads to an interesting interpretation of the stress field: the six stress components are the Lagrange multipliers used to enforce the corresponding compatibility equations.

Second statement of Hu-Washizu's principle

In the second approach, the terms appearing in the strain-displacement relationships of eq. 12.44 are integrated by parts using eq. 12.27. After regrouping all terms, this yields the *second statement of Hu-Washizu's principle*

$$\delta \int_{\mathcal{V}} \left[\left(a(\underline{\epsilon}) - \underline{\epsilon}^T \underline{\sigma}\right) + \left(\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right) u_1 \right.$$
$$\left. + \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right) u_2 + \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right) u_3 \right] \mathrm{d}\mathcal{V}$$
$$- \int_{\mathcal{S}_1} (\underline{t} - \hat{\underline{t}})^T \delta \underline{u} \; \mathrm{d}S - \int_{\mathcal{S}_2} \hat{\underline{u}}^T \delta \underline{t} \; \mathrm{d}\mathcal{S} = 0. \qquad (12.46)$$

This principle is closely related to the principle of minimum complementary energy, eq. (12.38). Indeed, starting from the above statement, the stress field is assumed to be statically admissible. Hence, the equations of equilibrium relationships are satisfied and the last three terms in the volume integral vanish; furthermore, equilibrium of the surface tractions is satisfied on $\mathcal{S}_1$ and the corresponding surface integral vanishes as well. The remaining terms then express the principle of minimum complementary energy, eq. (12.38).

The second statement of Hu-Washizu's principle can also be obtained by starting from the principle of minimum complementary energy, eq. 12.38. This principle is a statement of compatibility because it is derived from the principle of complementary virtual work, and the constitutive laws of the material are also included in the principle by means of the stress energy density function. This principle provides no information about the equilibrium equations of the problem. Consequently, the principle of minimum complementary energy can be viewed as a constrained minimization problem that yields all the equations of elasticity: minimization of the complementary energy yields the compatibility equations and the constitutive laws while the external constraints, the equilibrium equations, then yield the last set of equations.

This *constrained* minimization problem is then transformed into an *unconstrained* minimization problem using the Lagrange multiplier technique described in section 12.1.2. The modified function to be minimized is now in the form of eq. 12.12, where the f_i, $i = 1, 2, 3$, are the equilibrium equations and the λ_i are three Lagrange multipliers. This is exactly the form of Hu-Washizu's principle, eq. (12.46), where the Lagrange multipliers are identified to be the displacement components. This leads to an interesting interpretation of the displacement field: the three displacement components are the Lagrange multipliers used to enforce the corresponding equilibrium equations.

Third statement of Hu-Washizu's principle

Finally, in the last approach the terms appearing in both the equations of equilibrium and strain-displacement relationships of eq. (12.44) are integrated by parts using eqs. (12.22 and eq. (12.27), respectively. After regrouping all terms, this yields the *third statement of Hu-Washizu's principle*

$$\int_{\mathcal{V}} \left\{ \delta \left[a(\underline{\epsilon}) - \underline{\epsilon}^T \underline{\sigma} \right] + \sigma_1 \frac{\partial \delta u_1}{\partial x_1} + \sigma_2 \frac{\partial \delta u_2}{\partial x_2} + \sigma_3 \frac{\partial \delta u_3}{\partial x_3} + \tau_{23} \left[\frac{\partial \delta u_2}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_2} \right] \right.$$
$$+ \tau_{13} \left[\frac{\partial \delta u_1}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_1} \right] + \tau_{12} \left[\frac{\partial \delta u_1}{\partial x_2} + \frac{\partial \delta u_3}{\partial x_2} \right] - u_1 \left[\frac{\partial \delta \sigma_1}{\partial x_1} + \frac{\partial \delta \tau_{12}}{\partial x_2} + \frac{\partial \delta \tau_{13}}{\partial x_3} \right]$$
$$\left. - u_2 \left[\frac{\partial \delta \tau_{12}}{\partial x_1} + \frac{\partial \delta \sigma_2}{\partial x_2} + \frac{\partial \delta \tau_{23}}{\partial x_3} \right] - u_3 \left[\frac{\partial \delta \tau_{13}}{\partial x_1} + \frac{\partial \delta \tau_{23}}{\partial x_2} + \frac{\partial \delta \sigma_3}{\partial x_3} \right] \right\} \mathrm{d}\mathcal{V}$$
$$- \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} - \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} + \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = 0. \tag{12.47}$$

The main advantage of this third statement of Hu-Washizu's principle is that no derivatives of the three unknown fields are present; derivatives only show up in the variations. In numerical applications, this observation has important implications on the way in which the unknown fields can be approximated, because minimal continuity requirements are imposed.

12.2.9 Hellinger-Reissner's principle

Due to the complexity of the three-field Hu-Washizu's principle, a simpler, two-field principle is preferred for some applications. Hellinger-Reissner's principle is such a principle, and it is easily derived from Hu-Washizu's principle by eliminating the strain field.

Starting from the first statement of Hu-Washizu's principle, eq. (12.45), eq. (12.32) is used to eliminate the strain field: $\delta[a(\underline{\epsilon}) - \underline{\epsilon}^T \underline{\sigma}] = -\delta a'(\underline{\sigma})$. This simple operation yields the *first statement of Hellinger-Reissner's principle*

$$\delta \int_{\mathcal{V}} \left[\frac{\partial u_1}{\partial x_1} \sigma_1 + \frac{\partial u_2}{\partial x_2} \sigma_2 + \frac{\partial u_3}{\partial x_3} \sigma_3 + \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \tau_{23} \right.$$
$$\left. + \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \tau_{13} + \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \tau_{12} - a'(\underline{\sigma}) \right] \mathrm{d}\mathcal{V} \tag{12.48}$$
$$- \int_{\mathcal{V}} \underline{b}^T \delta \underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta \underline{u} \, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} (\underline{u} - \underline{\hat{u}})^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = 0.$$

Next, starting from the second statement of Hu-Washizu's principle, eq. (12.46), the strain field is eliminated in a similar manner to obtain the *second statement of Hellinger-Reissner's principle*

$$\delta \int_{\mathcal{V}} \left[\left(\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right) u_1 + \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right) u_2 \right.$$
$$\left. + \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right) u_3 - a'(\underline{\sigma}) \right] \mathrm{d}\mathcal{V}$$
$$- \int_{\mathcal{S}_1} (\underline{t} - \underline{\hat{t}})^T \delta \underline{u} \, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta \underline{t} \, \mathrm{d}\mathcal{S} = 0. \tag{12.49}$$

Finally, a similar procedure starting from the third statement of Hu-Washizu's principle, eq. 12.47, leads to the the *third statement of Hellinger-Reissner's principle*

$$\int_{\mathcal{V}} \left[\delta a'(\underline{\sigma}) + u_1 \left(\frac{\partial \delta\sigma_1}{\partial x_1} + \frac{\partial \delta\tau_{12}}{\partial x_2} + \frac{\partial \delta\tau_{13}}{\partial x_3} \right) + u_2 \left(\frac{\partial \delta\tau_{12}}{\partial x_1} + \frac{\partial \delta\sigma_2}{\partial x_2} + \frac{\partial \delta\tau_{23}}{\partial x_3} \right) \right.$$
$$+ u_3 \left(\frac{\partial \delta\tau_{13}}{\partial x_1} + \frac{\partial \delta\tau_{23}}{\partial x_2} + \frac{\partial \delta\sigma_3}{\partial x_3} \right) - \sigma_1 \frac{\partial \delta u_1}{\partial x_1} - \sigma_2 \frac{\partial \delta u_2}{\partial x_2} - \sigma_3 \frac{\partial \delta u_3}{\partial x_3}$$
$$\left. - \tau_{23} \left(\frac{\partial \delta u_2}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_2} \right) - \tau_{13} \left(\frac{\partial \delta u_1}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_1} \right) - \tau_{12} \left(\frac{\partial \delta u_1}{\partial x_2} + \frac{\partial \delta u_3}{\partial x_2} \right) \right] \mathrm{d}\mathcal{V}$$
$$+ \int_{\mathcal{V}} \underline{b}^T \delta\underline{u} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{S}_1} \underline{\hat{t}}^T \delta\underline{u} \, \mathrm{d}\mathcal{S} - \int_{\mathcal{S}_2} \underline{\hat{u}}^T \delta\underline{t} \, \mathrm{d}\mathcal{S} = 0. \tag{12.50}$$

12.3 Applications of variational and energy principles

The general formulation of the equations of linear elasticity as a series of variational problem provides powerful capabilities to analyze more complex structural problems than those considered in the previous chapters. As illustrated in chapter 11, this is particularly important when approximate solutions are sought because the integral forms appearing the variational equations involve lower order derivatives than those appearing in the corresponding differential equations, thereby decreasing the continuity requirements.

The equivalence of the differential equation and variational approaches is proven at several points in the above developments. This equivalence can also be demonstrated by deriving the governing differential equations from the variational formulation. This approach provides the additional benefit of yielding the associated boundary conditions.

The beam bending problem

Beam bending under transverse loading is discussed extensively in chapters 5, 6 and 8. In this section, the governing differential equations for beam bending will be derived from the principle of minimum total potential energy.

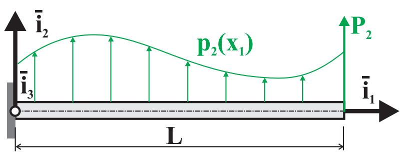

Fig. 12.5. Cantilevered beam with distributed transverse load.

Consider a general beam bending problem consisting of a uniform cantilevered beam with a symmetric cross-section subjected to a distributed transverse load, $p_2(x_1)$, and a concentrated tip load, P_2, as illustrated in fig. 12.5. The total potential energy of the system is given by eq. (10.40) as

$$\Pi = \frac{1}{2} \int_0^L H_{33}^c \left(\frac{\mathrm{d}^2 \bar{u}_2}{\mathrm{d}x_1^2} \right)^2 \mathrm{d}x_1 - \int_0^L p_2(x_1) \bar{u}_2(x_1) \, \mathrm{d}x_1 - P_2 \bar{u}_2(L), \tag{12.51}$$

where it is clear that $\Pi = \Pi(\bar{u}_2(x_1))$ is a functional.

The principle of minimum total potential energy, principle 17, implies that the total potential energy must be stationary, $\delta\Pi = 0$. Section 12.1.3 outlines the procedure to determine the stationary point of a functional in the form given by eq. (12.13), and the stationarity condition leads the the Euler-Lagrange equation, eq. (12.14). The same procedure will be followed here to determine the stationary point of the total potential energy, as required by the principle of minimum total potential energy.

Variation of of the total potential energy, eq. (12.51), can be written as

$$\delta\Pi = \frac{1}{2}\int_0^L H_{33}^c \, 2\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} \, \delta\left(\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right) \mathrm{d}x_1 - \int_0^L p_2 \, \delta\bar{u}_2 \, \mathrm{d}x_1 - P_2\delta\bar{u}_2(L) = 0.$$

Using eq. (12.15), it is possible to interchange the order of the variational and partial differential operators in the third term in the first integral to find

$$\delta\Pi = \frac{1}{2}\int_0^L H_{33}^c \, 2\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} \, \frac{\mathrm{d}^2}{\mathrm{d}x_1^2}(\delta\bar{u}_2)\mathrm{d}x_1 - \int_0^L p_2 \, \delta\bar{u}_2 \, \mathrm{d}x_1 - P_2\delta\bar{u}_2(L) = 0.$$

To eliminate the higher differential order of the virtual displacement field appearing in the first integral, two integration by parts are carried out, leading to

$$\begin{aligned} \delta\Pi = &\int_0^L \left[\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left(H_{33}^c \, \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right) - p_2\right] \delta\bar{u}_2\mathrm{d}x_1 + \left[H_{33}^c \frac{\mathrm{d}\bar{u}_2^2}{\mathrm{d}x_1^2} \, \delta\left(\frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1}\right)\right]_0^L \\ &- \left[\frac{\mathrm{d}}{\mathrm{d}x_1}\left(H_{33}^c \frac{\mathrm{d}\bar{u}_2^2}{\mathrm{d}x_1^2}\right) \delta\bar{u}_2\right]_0^L - P_2\delta\bar{u}_2(L) = 0. \end{aligned} \tag{12.52}$$

This equation must be satisfied for all arbitrary displacements, $\delta\bar{u}_2(x_1)$ for $0 \leq x_1 \leq L$. This can only happen if the first bracketed term vanishes, leading to the following differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left(H_{33}^c \, \frac{\mathrm{d}\bar{u}_2^2}{\mathrm{d}x_1^2}\right) - p_2 = 0, \tag{12.53}$$

which is the governing equation for the lateral deflection of a beam under load, first developed using the classical differential equation approach, see eq. (5.40). Within the present formalism, the governing differential equation is the *Euler-Lagrange equation* associated with the stationary point of the total potential energy.

The stationarity condition of the total potential energy also yields the boundary conditions of the problem. First, the stationary condition, eq. (12.52) is rewritten as

$$\begin{aligned} \delta\Pi = &\int_0^L \left[\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left(H_{33}^c \, \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2}\right) - p_2\right] \delta\bar{u}_2\mathrm{d}x_1 + \left[M_3\delta\Phi_3\right]_0^L \\ &+ \left[V_2\delta\bar{u}_2\right]_0^L - P_2\delta\bar{u}_2(L) = 0, \end{aligned} \tag{12.54}$$

where the definitions of the bending moment, shear force, and sectional rotation are used.

The last two terms of eq. (12.54) can be written as $[V_2(L) - P_2]\delta\bar{u}_2(L) - V_2(0)\delta\bar{u}_2(0)$. The virtual displacement field must satisfy the geometric boundary condition. Because the beam is clamped at the root, $\delta\bar{u}_2(0) = 0$, and the last two terms of eq. (12.54) reduce to $[V_2(L) - P_2]\delta\bar{u}_2(L)$. This term must vanish for all arbitrary virtual displacements at the tip of the beam, and therefore the bracketed term must vanish, leading to $V_2(L) = P_2$. This equation is the first natural boundary condition at the tip of the beam.

The second bracketed term of eq. (12.54), written as $M_3(L)\delta\Phi_3(L) - M_3(0)\delta\Phi_3(0)$ must also vanish. Because the beam is clamped at the root, $\delta\Phi_3(0) = 0$, and this reduces to $[M_3(L)]\delta\Phi_3(L)$. This term must vanish for all arbitrary virtual rotations at the tip of the beam, and therefore the bracketed term must vanish, leading to $M_3(L) = 0$. This equation is the second natural boundary condition at the tip of the beam.

In summary, the classical Euler-Bernoulli governing differential equation for beam bending problems is derived as the Euler-Lagrange equation associated with the stationarity condition of the total potential energy. Note that both differential equation and natural boundary conditions are recovered. Once again, the equivalence of the various formalisms is demonstrated.

12.3.1 The shear lag problem

The axial and shear flows in thin-walled beams are computed in chapter 8 for both bending and torsional loads. In addition, the cross-sectional warping of both open and closed-section thin-walled beams is examined in section 8.7. In the treatment of warping, it is tacitly assumed that the warping is free of any constraints that might be present at the beam ends, for instance. If the beam is cantilevered at its root, warping must vanish at the root but will develop along the beam's span, causing a redistribution of the stresses in the beam that can become significant under certain conditions.

A good example of this effect is *shear lag*, which is present in thin-walled beams that are fixed in a manner that constrains warping of a cross-section. The result can be a significant redistribution of the axial and shear flows in that cross-section and along the beam's span.

Consider a thin-walled, rectangular box-beam clamped at its root and subjected to a downward tip load, P_0, as shown in fig. 12.6. According to Euler-Bernoulli beam theory, the stress distribution in the upper flange (top panel) of the beam consists of a uniform distribution of tensile axial stress across the width, $2b$, and equal magnitude compressive stresses arise in the lower flange. If concentrated stiffeners are present at the four corners of the section, the top panel will be loaded by shear flows at its edges, as illustrated in fig. 12.7.

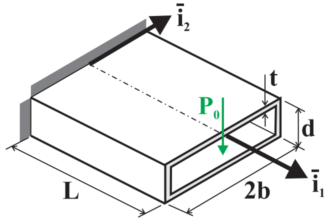

Fig. 12.6. Box beam subjected to tip load.

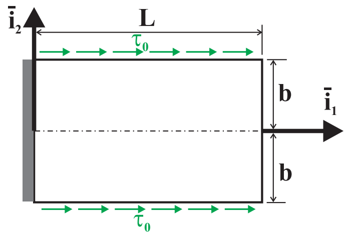

Fig. 12.7. Upper panel of a box beam subjected to edge shear forces.

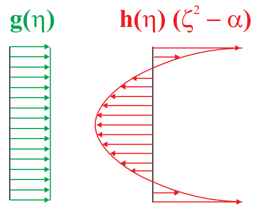

Fig. 12.8. The distribution of axial displacement.

An investigation of the shear lag phenomenon will be performed by considering only the rectangular panel loaded by constant shear stresses, τ_0, at its two edges, as shown in lower portion of fig. 12.7. This configuration is a crude approximation of the top flange of a thin-walled beam with four corner stiffeners. The width of the panel is $2b$, its thickness t, and its length L. At the root of the panel, the axial displacement must vanish, the tip section of the panel is stress free.

The solution procedure begins with kinematic assumptions: the displacement field in the panel is assumed to take the following form

$$u_1(\eta,\zeta) = g(\eta) + h(\eta)(\zeta^2 - \alpha); \quad u_2(\eta,\zeta) = 0, \tag{12.55}$$

where $\eta = x_1/L$ is a non-dimensional variable along the span of the panel and $\zeta = x_2/b$ that across its width. The axial displacement distribution is illustrated in fig. 12.8 and consists of two components. The first term corresponds to a uniform distribution across the width of the panel and gives rise to uniform displacements, strains, and stresses across the width, as predicted by beam theory. The second term describes the variation of the displacement field across the width of the panel. Due to the symmetry of the problem, a symmetric distribution across the width of the panel is selected: a symmetric parabolic distribution defined by parameter α.

This parabolic displacement distribution will result in uneven stress and strain distribution across the width of the panel and will characterize the importance of the shear lag effect. Because a parabolic distribution is arbitrarily selected at the onset of the analysis, an approximate solution is expected.

The strain field is obtained from the assumed displacement field using the strain-displacement relations, see eqs. (1.63) and (1.71), to find

$$\epsilon_1(\eta,\zeta) = \frac{g'}{L} + \frac{h'}{L}(\zeta^2 - \alpha), \quad \epsilon_2(\eta,\zeta) = 0; \quad \gamma_{12}(\eta,\zeta) = \frac{h}{b}2\zeta, \tag{12.56}$$

where the notation $(\cdot)'$ indicates a derivative with respect to η.

The panel is assumed to be made of a linearly elastic material in a plane stress state, and the material stiffness matrix is given by eq. (2.16). Furthermore, the stress in the transverse direction is assumed to be much smaller than the axial stress, $\sigma_2 \ll \sigma_1$, and the constitutive laws reduce to

$$\sigma_1 = E\epsilon_1 = E\left[\frac{g'}{L} + \frac{h'}{L}(\zeta^2 - \alpha)\right], \quad \sigma_2 \approx 0, \quad \tau_{12} = G\gamma_{12} = G\frac{h}{b}2\zeta. \tag{12.57}$$

The strain energy in the panel can now be written in terms of the axial and shear strains as

$$A = \int_0^L \int_{-b}^{+b} \int_{-t/2}^{t/2} \frac{1}{2}(E\epsilon_1^2 + G\gamma_{12}^2)\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = \frac{Lbt}{2} \int_0^1 \int_{-1}^{+1} (E\epsilon_1^2 + G\gamma_{12}^2)\, \mathrm{d}\eta \mathrm{d}\zeta.$$

Introducing the strain distributions, eq. (12.56), and reordering the term leads to

$$A = \frac{Lbt}{2} \int_0^1 \left\{ \frac{Eg'^2}{L^2} \left[\int_{-1}^{+1} \mathrm{d}\zeta \right] + \frac{Eh'^2}{L^2} \left[\int_{-1}^{+1} (\zeta^2 - \alpha)^2\, \mathrm{d}\zeta \right] \right.$$
$$\left. +2\frac{Eg'h'}{L^2} \left[\int_{-1}^{+1} (\zeta^2 - \alpha)\, \mathrm{d}\zeta \right] + \frac{4Gh^2}{b^2} \left[\int_{-1}^{+1} \zeta^2\, \mathrm{d}\zeta \right] \right\} \mathrm{d}\eta.$$

The next step is to evaluate the integrals appearing in the brackets. To simplify the analysis, the free parameter α will be selected to make the third bracketed term vanish: $\int_{-1}^{+1} (\zeta^2 - \alpha)\, \mathrm{d}\zeta = 0$, which leads to $\alpha = 1/3$. With this choice, the third term of the strain energy vanishes, eliminating the coupling between the two deformation modes characterized by functions $g(\eta)$ and $h(\eta)$. Using $\alpha = 1/3$, the strain energy now becomes

$$A = \frac{Ebt}{L} \int_0^1 \left[g'^2 + \frac{4}{45}h'^2 + \frac{4}{3}\left(\frac{L}{b}\right)^2 \left(\frac{G}{E}\right) h^2 \right] \mathrm{d}\eta.$$

The potential of the applied loads consists of the potential of the shear loads applied along the left and right side edges of the panel,

$$\Phi = -\int_{t/2}^{t/2} \int_0^L \left[\tau_0 u_1(x_1, x_2 = -b) + \tau_0 u_1(x_1, x_2 = +b) \right] \mathrm{d}x_1 \mathrm{d}x_2$$
$$= 2tL \int_0^1 \tau_0 (g + \frac{2h}{3})\, \mathrm{d}\eta,$$

and the total potential energy therefore becomes

$$\Pi = \frac{Ebt}{L} \int_0^1 \left[g'^2 + \frac{4}{45}h'^2 + \frac{4}{3}\left(\frac{L}{b}\right)^2 \left(\frac{G}{E}\right) h^2 - \frac{2\tau_0 L^2}{Eb}(g + \frac{2h}{3}) \right] \mathrm{d}\eta.$$

The principle of minimum total potential energy requires Π to be stationary, $\delta\Pi = 0$. Taking the first variation of Π leads to

$$\int_0^1 \left[g'\delta g' + \frac{4}{45}h'\delta h' + \frac{4}{3}\left(\frac{L}{b}\right)^2 \left(\frac{G}{E}\right) h\delta h - \frac{\tau_0 L^2}{Eb}(\delta g + \frac{2}{3}\delta h) \right] \mathrm{d}\eta = 0.$$

Integration by parts is performed on the first two terms and regrouping yields

$$\int_0^1 \left\{ \delta g \left[-g'' - \frac{\tau_0 L^2}{Eb} \right] + \delta h \left[-\frac{4}{45} h'' + \frac{4}{3} \left(\frac{L}{b} \right)^2 \left(\frac{G}{E} \right) h - \frac{2\tau_0 L^2}{3Eb} \right] \right\} \mathrm{d}\eta$$
$$+ \left[g' \delta g \right]_0^1 + \left[\frac{4}{45} h' \delta h \right]_0^1 = 0.$$

Because the expression must vanish for all arbitrary variations δg and δh, each of the bracketed terms must vanish individually. The first bracketed term leads to the following differential equation for $g(\eta)$: $g'' = -\tau_0 L^2/(Eb)$. The boundary conditions are $g(0) = 0$ at the root of the panel and $g'(1) = 0$ at its tip. The second bracketed term leads to the differential equation for $h(\eta)$: $h'' - \mu^2 h = -15\tau_0 L^2/(2Eb)$. The boundary conditions are $h(0) = 0$ at the root of the panel and $h'(1) = 0$ at its tip. The non-dimensional parameter μ is defined as

$$\mu^2 = 15 \left(\frac{L}{b} \right)^2 \left(\frac{G}{E} \right). \tag{12.58}$$

These second order differential equations can be solved to obtain the axial displacement field as

$$u_1(\eta, \zeta) = \frac{\tau_0}{E} \frac{L^2}{b} \left\{ \left(\eta - \frac{1}{2} \eta^2 \right) + \frac{15}{2\mu^2} \left[1 - \frac{\cosh \mu(1-\eta)}{\cosh \mu} \right] \left(\zeta^2 - \frac{1}{3} \right) \right\}.$$

The first term, $(\eta - \eta^2/2)$, represents the axial displacement, which is constant across the width of the panel as if it were a beam of axial stiffness $S = E2bt$ subjected to a uniform axial load $p_0 = 2\tau_0 t$, see eq. (5.26). The second term represents the displacement variation across the width of the panel and characterizes the shear lag effect that is controlled by the non-dimensional parameter μ.

The stress distribution in the panel can be obtained from the constitutive relationships, eq. (12.57)

$$\begin{aligned} \sigma_1 &= \tau_0 \frac{L}{b} \left[(1-\eta) + \frac{15}{2\mu} \frac{\sinh \mu(1-\eta)}{\cosh \mu} \left(\zeta^2 - \frac{1}{3} \right) \right], \\ \tau_{12} &= \tau_0 \left[1 - \frac{\cosh \mu(1-\eta)}{\cosh \mu} \right] \zeta. \end{aligned} \tag{12.59}$$

The first term describes the constant axial stress distribution across the panel width which is predicted by bean theory. The second term describes the axial stress redistribution due to shear lag.

Consider an aluminum panel with an aspect ratio $L = 8b$. The parameter $\mu = 19.2$ when $E/G = 2(1+\nu) = 2.6$ for a homogeneous, isotropic material with a Poisson's ratio $\nu = 0.3$. On the other hand, $\mu = 6.41$ for a panel with the same aspect ratio but made of a medium modulus graphite fiber reinforced epoxy matrix composite material for which $E = 140\ GPa$, $G = 6\ GPa$ and $\nu = 0.3$. Figure 12.9 shows the non-dimensional axial stress $h\sigma_1/(L\tau_0)$ distribution for both materials at two locations across the panel width: $\zeta = 0$ (corresponding to the panel mid-line) and $\zeta = 1$ (corresponding to its right hand edge).

Figure 12.10 shows the distribution of axial stress across the width at the root of the panel for both materials. In contrast with the uniform distribution predicted by classical beam theory, the present results show a significant over-stress occurring at the root corners of the panel and significant under-stress along its mid-line. The magnitudes of the over- and under-stresses are $5\tanh\mu/\mu$ and $5\tanh\mu/(2\mu)$, respectively, as obtained from eqs.(12.59). For the isotropic panel this translates to 26% and 13% for the over- and under-stress, respectively, and the corresponding numbers are 78% and 39% for the composite panel.

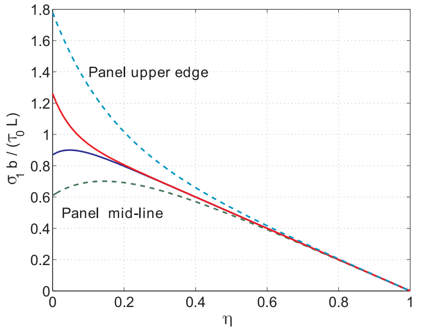

Fig. 12.9. Distribution of axial stress along the panel span. Isotropic material: solid line; anisotropic material: dashed line.

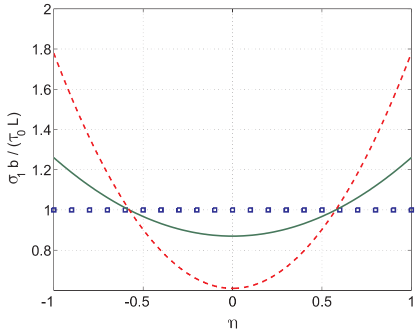

Fig. 12.10. Distribution of axial stress across the panel width at the root of the panel. The uniform distribution predicted by beam theory is indicated by the square boxes. Isotropic material: solid line; anisotropic material: dashed line.

The uniform stress distribution results in optimum structural efficiency because the material is equally stressed at all points across the width of the panel. The shear lag phenomenon considerably redistributes the stresses, creating undesirable overstressed areas and decreasing the structural efficiency. The magnitude of the shear lag effect is controlled by the parameter μ defined in eq. (12.58). The smaller this parameter, the larger the shear lag effect. Clearly, shorter panel (smaller L/b values), made of shear deformable materials (smaller G/E values) will experience the most significant shear lag effects.

A more complicated configuration that includes both panels (sheets) and stringers under similar loading with warping restrained at one end is treated in problem 12.12.

12.3.2 The Saint-Venant torsion problem

Torsion of a beam with an arbitrary cross-section is examined in chapter 7 in section 7.3 using an analysis based on the Newtonian statement of equilibrium. The same problem will formulated here as a variational problem.

For a beam with arbitrary cross-section, Saint-Venant theory assumes that under torsion, each cross-section rotates like a rigid body, and warps out of its own plane.

The resulting assumed displacement field is described by eqs. (7.32a) and (7.32b), which are repeated

$$u_1(x_1,x_2,x_3) = \Psi(x_2,x_3)\,\kappa_1(x_1), \tag{12.60a}$$
$$u_2(x_1,x_2,x_3) = -x_3\Phi_1(x_1), \quad u_3(x_1,x_2,x_3) = x_2\Phi_1(x_1). \tag{12.60b}$$

where $\Phi(x_1)$ is the angle of twist about axis $\bar{\imath}_1$ and $\Psi(x_2,x_3)$ the warping function.

Under the assumption of uniform torsion, The strain field is then computed using the strain-displacement relations, and is given by eqs. (7.33a) to (7.33c). The only nonzero strains are the two shear strains components, γ_{12} and γ_{13}, given by eqs. (7.33c). The only non-vanishing stress components are τ_{12} and τ_{13}, given by eqs. (7.34c).

For uniform torsion, the total potential energy can be evaluated for a slice of the beam of unit length only. Indeed, for uniform torsion, the deformation of all slices are identical. The strain energy per unit length of the beam is

$$A = \frac{1}{2}\int_{\mathcal{A}} (\tau_{12}\gamma_{12} + \tau_{13}\gamma_{13})\ \mathrm{d}\mathcal{A}.$$

The potential of the externally applied load is the negative of the work done by the torque, M_1, acting along the unit length segment, $\Phi = -M_1\kappa_1 \cdot 1$. Using eqs. (7.34c) and (7.33c) for the stress and strains fields, respectively, the total potential energy becomes

$$\Pi = \frac{1}{2}\int_{\mathcal{A}} \left[\left(\frac{\partial\Psi}{\partial x_2} - x_3\right)^2 + \left(\frac{\partial\Psi}{\partial x_3} + x_2\right)^2\right] G\kappa_1^2\ \mathrm{d}\mathcal{A} - M_1\kappa_1.$$

The principle of minimum total potential energy requires the total potential energy to be stationary with respect to admissible displacement fields, leading to

$$\delta\Pi = G\kappa_1^2\int_{\mathcal{A}} \left[\left(\frac{\partial\Psi}{\partial x_2} - x_3\right)\frac{\partial\delta\Psi}{\partial x_2} + \left(\frac{\partial\Psi}{\partial x_3} + x_2\right)\frac{\partial\delta\Psi}{\partial x_3}\right]\ \mathrm{d}\mathcal{A} - 0 = 0. \tag{12.61}$$

To eliminate the derivatives of variation in the warping function, Green's theorem [7] will now be used, leading to

$$\begin{aligned}\delta\Pi = -G\kappa_1^2 \iint_{\mathcal{A}} &\left[\frac{\partial^2\Psi}{\partial x_2^2} + \frac{\partial^2\Psi}{\partial x_3^2}\right]\delta\Psi\ \mathrm{d}\mathcal{A}\\ &+ G\kappa_1^2\oint_{\mathcal{C}}\left[\left(\frac{\partial\Psi}{\partial x_2} - x_3\right)n_2 + \left(\frac{\partial\Psi}{\partial x_3} + x_2\right)n_3\right]\delta\Psi \mathrm{d}s = 0,\end{aligned}$$

where $n_2 = \mathrm{d}x_3/\mathrm{d}s$ and $n_3 = -\mathrm{d}x_2/\mathrm{d}s$ are the direction cosines of the unit normal to curve $\mathcal{C}$. Because the variation in the warping function are arbitrary, the bracketed terms in the expression above must vanish. The first bracketed term yields the governing partial differential equation of the problem, and the second, the boundary conditions along curve $\mathcal{C}$,

$$\frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2} = 0 \quad \text{over } \mathcal{A} \tag{12.62a}$$

$$\left(\frac{\partial \Psi}{\partial x_2} - x_3\right) \frac{\mathrm{d}x_3}{\mathrm{d}s} - \left(\frac{\partial \Psi}{\partial x_3} + x_2\right) \frac{\mathrm{d}x_2}{\mathrm{d}s} = 0 \quad \text{along } \mathcal{C}. \tag{12.62b}$$

Equations (12.62) are identical to eqs. (7.40) obtained using the classical Newtonian approach. The boundary condition of the problem, eq. (12.62b), corresponds to the vanishing of the shear stress components acting in the direction normal to curve $\mathcal{C}$, $\tau_n = 0$. In the variational approach, this natural boundary condition results from the application of Green's theorem.

12.3.3 The Saint-Venant torsion problem using the Prandtl stress function

The Saint-Venant uniform torsion problem is formulated in the previous section as a variational problem in terms of the warping function, Ψ, which defines the deformation of the beam. To avoid the complicated boundary condition expressed by eq. (12.62b), the problem is reformulated using Prandtl's stress function as developed in section 7.3.2. In this case, however, the principle of minimum total potential energy is no longer appropriate, and the principle of minimum complementary energy must be employed instead.

In Prandtl's stress function formulation, the non-vanishing shear stress components are expressed in terms of the stress function by eqs. (7.41), $\tau_{12} = \partial \phi / \partial x_3$ and $\tau_{13} = \partial \phi / \partial x_2$, where $\phi(x_2, x_3)$ is Prandtl's stress function. The complementary strain energy stored in a unit slice of the beam now becomes

$$A' = \frac{1}{2} \iint_{\mathcal{A}} \frac{1}{G} \left(\tau_{12}^2 + \tau_{13}^2\right) \mathrm{d}\mathcal{A} = \frac{1}{2} \iint_{\mathcal{A}} \frac{1}{G} \left[\left(\frac{\partial \phi}{\partial x_2}\right)^2 + \left(\frac{\partial \phi}{\partial x_3}\right)^2\right] \mathrm{d}\mathcal{A}.$$

The potential of the prescribed displacements, Φ', see section (12.2.6), requires more careful consideration. In this case, the beam of unit length is fixed at one end, and a rotation, Φ_1, is prescribed at the other. The associated potential is $\Phi' = -M_1 \kappa_1 \cdot 1 = -\iint_{\mathcal{A}} 2\phi\kappa_1 \mathrm{d}\mathcal{A}$ where use is made of eq. (7.48) for a cross-section bounded by a single curve, $\mathcal{C}$. The total complementary potential energy is then[1]

$$\Pi' = A' + \Phi' = \frac{1}{2} \iint_{\mathcal{A}} \frac{1}{G} \left[\left(\frac{\partial \phi}{\partial x_2}\right)^2 + \left(\frac{\partial \phi}{\partial x_3}\right)^2\right] \mathrm{d}\mathcal{A} - \iint_{\mathcal{A}} 2\phi\kappa_1 \mathrm{d}\mathcal{A}. \tag{12.63}$$

The principle of stationary complementary energy, principle 18, requires the total complementary energy to be stationary value with respect to arbitrary choices of statically admissible stress fields, leading to

$$\delta \Pi' = \iint_{\mathcal{A}} \frac{1}{G} \left[\frac{\partial \phi}{\partial x_2} \frac{\partial \delta\phi}{\partial x_2} + \frac{\partial \phi}{\partial x_3} \frac{\partial \delta\phi}{\partial x_3} - 2G\kappa_1 \delta\phi\right] \mathrm{d}\mathcal{A} = 0,$$

[1] The notation is treacherous: Φ_1 is a rotation about axis $\bar{\imath}_1$; Φ' is the potential of prescribed displacements; and ϕ is Prandtl's stress function.

To eliminate the derivatives of variations of the stress function, Green's theorem is applied to the first two terms in the integrand to find

$$\iint_{\mathcal{A}} \frac{1}{G}\left[-\frac{\partial^2\phi}{\partial x_2^2} - \frac{\partial^2\phi}{\partial x_3^2} - 2G\kappa_1\right]\delta\phi \mathrm{d}\mathcal{A} + \frac{1}{G}\oint_{\mathcal{C}}\left[\frac{\partial\phi}{\partial x_2}n_2 + \frac{\partial\phi}{\partial x_3}n_3\right]\delta\phi\,\mathrm{d}s = 0.$$

Equation (7.44) implies that the stress function, ϕ, must remain constant along curve $\mathcal{C}$. Hence, $\delta\phi = 0$ along the same curve and the second integral vanishes. The first integral must vanish for all arbitrary variation, $\delta\phi$. Consequently, the first bracketed term must vanish, and the Euler-Lagrange equation for this variational problem is

$$\frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = -2G\kappa_1, \tag{12.64}$$

This result is identical to that obtained with the classical approach, see eq. (7.45). Here again, the governing differential of the problem is recovered as the Euler-Lagrange equation of a variational problem

Example 12.1. Torsion of rectangular section - a crude solution
Consider a bar with a rectangular cross-section of width $2a$ and depth $2b$ as depicted in fig. 12.11. Determine the stress function and the sectional torsional stiffness using the principle of complementary strain energy.

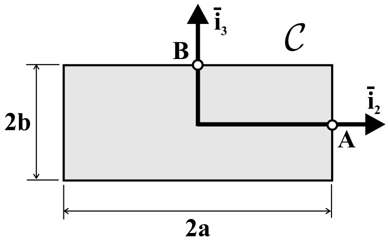

Fig. 12.11. Bar with a rectangular cross-section.

The following expression will be assumed for the stress function

$$\phi(\eta,\zeta) = c_0(\eta^2 - 1)(\zeta^2 - 1), \tag{12.65}$$

where c_0 is an unknown constant, $\eta = x_2/a$ is the non-dimensional coordinate along axis $\bar{\imath}_2$, and $\zeta = x_3/b$ is that along axis $\bar{\imath}_3$. This choice of the stress function satisfies the boundary conditions of the problem along $\mathcal{C}$: $\phi(\eta = \pm 1, \zeta) = 0$ and $\phi(\eta, \zeta = \pm 1) = 0$.

The principle of minimum complementary energy will be used to determine the constant, c_0, that minimize the total complementary energy. The total complementary energy is given by eq. (12.63), and introducing the approximation of the stress function given by eq. (12.65) yields

$$\Pi' = \int_{\mathcal{A}} \frac{c_0^2}{2G} \left[\frac{4\eta^2}{a^2}(\zeta^2-1)^2 + \frac{4\zeta^2}{b^2}(\eta^2-1)^2 \right] \mathrm{d}\mathcal{A} \\ - 2c_0\kappa_1 \int_{\mathcal{A}} (\eta^2-1)(\zeta^2-1)\,\mathrm{d}\mathcal{A}. \tag{12.66}$$

After integration over the cross-section, this becomes

$$\Pi' = \frac{c_0^2}{2G}\left[\frac{8}{3a}\frac{16b}{15} + \frac{16a}{15}\frac{8}{3b}\right] - 2c_0\kappa_1 \frac{4a}{3}\frac{4b}{3}.$$

The total complementary energy must assumes a stationary value with respect to arbitrary statically admissible stress fields, which implies

$$\delta\Pi' = \left[\frac{2c_0}{2G}\left(\frac{8}{3a}\frac{16b}{15} + \frac{16a}{15}\frac{8}{3b}\right) - 2\kappa_1\frac{16ab}{9}\right]\delta c_0 = 0.$$

This expression must vanish for arbitrary δc_0, and hence, the bracketed term must vanish. Solving the resulting equation for constant c_0 yields the stress function as

$$\phi(\eta,\zeta) = \frac{5}{4}\frac{a^2b^2}{a^2+b^2}\,G\kappa_1\,(\eta^2-1)(\zeta^2-1). \tag{12.67}$$

For this section bounded by a single curve, the externally applied torque is given by eq. (7.48),

$$M_1 = 2\int_{\mathcal{A}} \phi\,\mathrm{d}\mathcal{A} = \frac{5}{2}\frac{a^2b^2}{a^2+b^2}\,G\kappa_1 \int_{\mathcal{A}} (\eta^2-1)(\zeta^2-1)\,\mathrm{d}\mathcal{A} = \frac{40}{9}\frac{a^3b^3}{a^2+b^2}G\kappa_1.$$

The sectional torsional stiffness, H_{11}, then follows as

$$H_{11} = \frac{40}{9}\frac{a^3b^3}{a^2+b^2}G. \tag{12.68}$$

The stress field is easily found from the derivatives of the stress function. This solution should be compared to the development in section 7.3.3 based on solutions to the governing partial differential equations of linear elasticity.

Example 12.2. Torsion of rectangular section, a refined solution

Consider once again a bar with a rectangular cross-section of width $2a$ and depth $2b$ analyzed in example 12.1 and shown in fig. 12.11. Study the behavior of this section in uniform torsion using the following approximation for the stress function,

$$\phi(\eta,\zeta) = (\zeta^2-1)g(\eta), \tag{12.69}$$

where $g(\eta)$ is an unknown function, $\eta = x_2/a$ the non-dimensional coordinate along axis $\bar{\imath}_2$, and $\zeta = x_3/b$ that along axis $\bar{\imath}_3$. This choice for the stress function explicitly satisfies the boundary conditions along two edges of the section, $\phi(\eta,\zeta=\pm1)=0$, but to satisfy the boundary conditions along the other two edges, the following conditions are required, $g(\eta=\pm1)=0$.

The total complementary energy for the problem is given by eq. (12.63), and introducing the approximation of the stress function given by eq. (12.69) yields

$$\Pi' = \int_{\mathcal{A}} \left[(\zeta^2 - 1)^2 \frac{g'^2}{a^2} + \frac{4\zeta^2}{b^2} g^2 \right] \, \mathrm{d}\mathcal{A} - 2\kappa_1 \int_{\mathcal{A}} (\zeta^2 - 1) g(\eta) \, \mathrm{d}\mathcal{A},$$

where the notation $(.)'$ indicates a derivative with respect to η. The integration over variable ζ can be performed to yield

$$\Pi' = \frac{ab}{2G} \int_{-1}^{+1} \left[\frac{16}{15a^2} g'^2 + \frac{8}{3b^2} g^2 \right] \, \mathrm{d}\eta - 2\kappa_1 ab \int_{-1}^{+1} \left(-\frac{4}{3} \right) g \, \mathrm{d}\eta.$$

When the total complementary energy is a minimum, Π' assumes a stationary value with respect to statically admissible stresses which implies

$$\delta\Pi' = \frac{ab}{2G} \int_{-1}^{+1} \left[\frac{16}{15a^2} 2g'\delta g' + \frac{8}{3b^2} 2g\delta g + 4G\kappa_1 \frac{4}{3} \delta g \right] \, \mathrm{d}\eta = 0.$$

Performing an integration by parts of the first term then leads to

$$\int_{-1}^{+1} \left[-\frac{16}{15a^2} g'' + \frac{8}{3b^2} g + G\kappa_1 \frac{8}{3} \right] \delta g \, \mathrm{d}\eta + \left[\frac{16}{15a^2} g' \delta g \right]_{-1}^{+1} = 0.$$

The variation, δg, is arbitrary, and hence, the first bracketed term must vanish. The Euler-Lagrange equation of this stationarity problem now becomes $g'' - \mu^2 g = \mu^2 b^2 G\kappa_1$, where $\mu = \sqrt{5/2}\ a/b$. Since the boundary conditions of the problem require $g(\eta = \pm 1) = 0$, the variations at the end points vanish, $\delta g(\eta = \pm 1) = 0$, and the second bracketed terms vanishes.

The general solution of the differential equation is $g(\eta) = C_1 \sinh \mu\eta + C_2 \cosh \mu\eta - b^2 G\kappa_1$, where C_1 and C_2 are two integration constants that must be evaluated with the help of the boundary conditions, $g(\eta = \pm 1) = 0$. The stress function then becomes

$$\phi(\eta, \zeta) = \left(\frac{\cosh \mu\eta}{\cosh \mu} - 1 \right) (\zeta^2 - 1)\, b^2 G\kappa_1. \tag{12.70}$$

For this section bounded by a single curve, the externally applied torque is given by eq. (7.48), leading to the following expression for the torsional stiffness

$$H_{11} = \frac{16ab^3}{3} \left(1 - \frac{\tanh \mu}{\mu} \right) G. \tag{12.71}$$

The stress field can be found from the derivatives of the stress function. This solution should be compared to the development in section 7.3.3 based on solutions to the governing partial differential equations on linear elasticity.

12.3.4 The non-uniform torsion problem

The equations governing the non-uniform torsion problem for a beam with a thin-walled cross-section are derived in section 8.9 using classical arguments. The derivation is complex, relying on a number of equilibrium arguments. It is possible to obtain these governing equations based solely on variational and energy arguments. The development starts with the following expression for the assumed displacement field in a general thin-walled beam,

$$\begin{gathered} u_1(x_1, s) = \Psi(s)\, \kappa_1(x_1), \\ u_2(x_1, s) = -(x_3 - x_{3k})\, \Phi_1(x_1), \quad u_3(x_1, s) = (x_2 - x_{2k})\, \Phi_1(x_1). \end{gathered} \tag{12.72}$$

This displacement field is similar to that of the Saint-Venant solution described in section 7.3.2. The axial displacement component is proportional to the twist rate and distributed over the cross-section according to the warping function, $\Psi(s)$, assumed to be that found in section 8.7. The in-plane displacement components describe a rigid body rotation of the section about the center of twist (or shear center) which is computed using the procedure described in section 8.7.

The strain field is now computed from this assumed displacement field to find

$$\epsilon_1 = \Psi(s)\, \frac{\mathrm{d}\kappa_1}{\mathrm{d}x_1}, \quad \gamma_s = \left(\frac{\mathrm{d}\Psi}{\mathrm{d}s} + r_k \right) \kappa_1,$$

where r_k is defined by eq. (8.11). The remaining strain components all vanish, $\epsilon_2 = \epsilon_3 = \gamma_{23} = 0$, as should be expected since the in-plane displacement components describe a rigid body rotation.

The strain energy of the beam under non-uniform torsion reduces to the following expression

$$A = \frac{1}{2} \int_0^L \int_{\mathcal{A}} (\tau_s \gamma_s + \sigma_1 \epsilon_1)\, \mathrm{d}\mathcal{A}\, \mathrm{d}x_1.$$

Assuming the beam to be made of homogeneous, linearly elastic material, the strain energy can be written in terms of deformation as

$$A = \frac{1}{2} \int_0^L \int_{\mathcal{A}} \left[G \left(\frac{\mathrm{d}\Psi}{\mathrm{d}s} + r_k \right)^2 \kappa_1^2 + E\Psi^2 \left(\frac{\mathrm{d}\kappa_1}{\mathrm{d}x_1} \right)^2 \right] \mathrm{d}\mathcal{A}\, \mathrm{d}x_1.$$

Recognizing that the twist rate, κ_1, and its spatial derivative are functions only of the span-wise variable, x_1, this expression can be recast as

$$A = \frac{1}{2} \int_0^L \left\{ \left[\int_{\mathcal{C}} G \left(\frac{\mathrm{d}\Psi}{\mathrm{d}s} + r_k \right)^2 t \mathrm{d}s \right] \kappa_1^2 + \left[\int_{\mathcal{C}} E\Psi^2\, t \mathrm{d}s \right] \left(\frac{\mathrm{d}\kappa_1}{\mathrm{d}x_1} \right)^2 \right\} \mathrm{d}x_1.$$

For uniform torsion problems, the twist rate is a constant along the span of the beam, and the second term in the integral vanishes. The first term then represents the strain energy associated with this uniform torsion problem, $A = \frac{1}{2} \int_0^L H_{11} \kappa_1^2 \, \mathrm{d}x_1$, where H_{11} is the torsional stiffness of the section. For closed section, the torsional stiffness is given by eq. (8.67), whereas for open sections, expressions are given in section 7.5.

The second term in the integral represents the strain energy associated the axial strain component that arises in non-uniform torsion problems. The bracketed term is the *warping stiffness* that is identified in the classical approach, see eq. (8.103). Hence, the strain energy becomes

$$A = \frac{1}{2} \int_0^L \left[H_{11} \left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right)^2 + H_w \left(\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2} \right)^2 \right] \mathrm{d}x_1.$$

Consider a beam clamped at the root and subjected to a distributed torque, $q_1(x_1)$, and a concentrated torque, Q_1, at the tip. The total potential energy of the system is then

$$\Pi = \frac{1}{2} \int_0^L \left[H_{11} \left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right)^2 + H_w \left(\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2} \right)^2 \right] \mathrm{d}x_1 - \int_0^L q_1\Phi_1 \,\mathrm{d}x_1 - Q_1\Phi_1(L).$$

The principle of minimum total potential energy requires this expression to be a minimum with respect to all the possible choices of the twist distribution, $\Phi_1(x_1)$, that are compatible with the geometric boundary conditions. This occurs when $\delta\Pi = 0$ and can be expressed as

$$\int_0^L \left[H_{11} \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \left(\frac{\mathrm{d}\delta\Phi_1}{\mathrm{d}x_1} \right) + H_w \frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2} \left(\frac{\mathrm{d}^2\delta\Phi_1}{\mathrm{d}x_1^2} \right) - q_1\delta\Phi_1 \right] \mathrm{d}x_1 - Q_1\delta\Phi_1(L) = 0$$

where the order of the variational and differential operators are interchanged. The first term is integrated by parts once and the second term twice to yield

$$\int_0^L \left[-\frac{\mathrm{d}}{\mathrm{d}x_1} \left(H_{11} \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right) + \frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \left(H_w \frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2} \right) - q_1 \right] \delta\Phi_1 \,\mathrm{d}x_1 + \left[H_{11} \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \delta\Phi_1 \right]_0^L$$
$$+ \left[H_w \frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2} \left(\frac{\mathrm{d}\delta\Phi_1}{\mathrm{d}x_1} \right) \right]_0^L - \left[\frac{\mathrm{d}}{\mathrm{d}x_1} \left(H_w \frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2} \right) \delta\Phi_1 \right]_0^L - Q_1\delta\Phi_1(L) = 0.$$

Because the integral must vanish for all arbitrary variations $\delta\Phi_1$, the first bracketed term must vanish, leading to the governing differential equation of the problem

$$\frac{\mathrm{d}}{\mathrm{d}x_1} \left(H_{11} \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right) - \frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \left(H_w \frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2} \right) = -q_1. \tag{12.73}$$

This equation is identical to eq. (8.105) obtained using the classical approach. At the root of the beam, the geometric boundary conditions imply $\Phi_1(0) = 0$ and $\mathrm{d}\Phi_1(0)/\mathrm{d}x_1 = 0$, where the second condition stems from the required vanishing of the axial displacement component, see eq. (12.72). Because the variations $\delta\Phi_1(L)$ and $\delta(\mathrm{d}\Phi_1(L)/\mathrm{d}x_1)$ are arbitrary, the boundary conditions at the tip of the beam are the natural conditions obtained from the boundary terms as

$$\left[H_{11}\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} - \frac{\mathrm{d}}{\mathrm{d}x_1}\left(H_w\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2}\right)\right]_{x_1=L} = Q_1, \quad \text{and} \quad H_w\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x_1^2}\bigg|_{x_1=L} = 0. \tag{12.74}$$

The first condition requires the torque in the beam to be equal to the externally applied tip torque, Q_1. The second condition implies the vanishing of the tip axial stresses.

Clearly, the classical and energy approaches lead to the same governing equation and boundary conditions for non-uniform torsion. It should be noted, however, that the energy approach is much more convenient because the governing equation and boundary conditions are both obtained from the expression of the total potential energy by means of a purely algebraic procedure.

12.3.5 The non-uniform torsion problem (closed sections)

The variational and energy approach to the non-uniform torsion problem developed in the previous section is based on the displacement field given by eq. (12.72). When using an energy approach, different approximations to a problem can be easily obtained by starting from different displacement fields.

In this section, the following displacement field will be investigated

$$\begin{gathered} u_1(x_1,s) = \Psi(s)\,\alpha(x_1), \\ u_2(x_1,s) = -(x_3 - x_{3k})\,\Phi_1(x_1), \quad u_3(x_1,s) = (x_2 - x_{2k})\,\Phi_1(x_1), \end{gathered} \tag{12.75}$$

where $\alpha(x_1)$ is an unknown function that characterizes the amplitude of the axial displacement, and Φ_1 is the rigid body rotation of the cross-section. This contrasts with the displacement field of the previous approach, eq. (12.72), where the amplitude of the axial displacement is taken to be proportional to the twist rate. Here again, the warping function, $\Psi(s)$, is assumed to be that found in section 8.7, and the in-plane displacement components describe a rigid body rotation of the section about the shear center.

The strain field is now computed from this assumed displacement field as

$$\epsilon_1 = \Psi(s)\,\frac{\mathrm{d}\alpha}{\mathrm{d}x_1}, \quad \gamma_s = \frac{\mathrm{d}\Psi}{\mathrm{d}s}\alpha + r_k\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}, \tag{12.76}$$

where r_k is defined by eq. (8.11). The remaining strain components all vanish, $\epsilon_2 = \epsilon_3 = \gamma_{23} = 0$, as should be expected because the in-plane displacement components describe a rigid body rotation.

Assuming the beam has a closed, single-cell, thin-walled cross-section made of homogeneous, linearly elastic material, the strain energy becomes

$$A = \frac{1}{2}\int_0^L\int_{\mathcal{C}}\left[E\Psi^2\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x_1}\right)^2 + G\left(\frac{\mathrm{d}\Psi}{\mathrm{d}s}\alpha + r_k\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}\right)^2\right]t\,\mathrm{d}s\,\mathrm{d}x_1.$$

Expanding this expression leads to a number of sectional integrals. The first two are

$$H_p = \int_{\mathcal{C}} G r_k^2 \, t\mathrm{d}s; \quad \text{and} \quad H_w = \int_{\mathcal{C}} E\Psi^2 \, t\mathrm{d}s,$$

where the second integral defines the torsional warping of the section. Two other integrals are encountered,

$$\int_{\mathcal{C}} G \frac{\mathrm{d}\Psi}{\mathrm{d}s} r_k \, t\mathrm{d}s = \int_{\mathcal{C}} G \left[\frac{H_{11}}{2\mathcal{A}Gt} - r_k \right] r_k \, t\mathrm{d}s = H_{11} - H_p,$$

and

$$\int_{\mathcal{C}} G \left(\frac{\mathrm{d}\Psi}{\mathrm{d}s} \right)^2 \, t\mathrm{d}s = \int_{\mathcal{C}} G \left[\frac{H_{11}}{2\mathcal{A}Gt} - r_k \right]^2 \, t\mathrm{d}s = H_{11} + H_p - 2H_{11} = H_p - H_{11}.$$

The derivative of the warping function is expressed in terms of eq. (8.94), and the torsional stiffness of the closed section is given by eq. (8.67). Using these, the strain energy of the beam now becomes

$$A = \frac{1}{2} \int_0^L \left[H_w \left(\frac{\mathrm{d}\alpha}{\mathrm{d}x_1} \right)^2 + (H_p - H_{11})\alpha^2 + H_p \left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right)^2 - 2\,(H_p - H_{11})\alpha \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right] \mathrm{d}x_1.$$

Consider a cantilevered beam subjected to a distributed torque, $q_1(x_1)$, and a tip torque, Q. The total potential energy of the beam is then

$$\Pi = \frac{1}{2} \int_0^L \left[H_w \left(\frac{\mathrm{d}\alpha}{\mathrm{d}x_1} \right)^2 + (H_p - H_{11})\alpha^2 + H_p \left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right)^2 - 2\,(H_p - H_{11})\alpha \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right] \mathrm{d}x_1 - \int_0^L q_1 \Phi_1 \, \mathrm{d}x_1 - Q\Phi_1(L).$$

Invoking the principle of stationary total potential energy, the total potential energy must be a stationary quantity,

$$\begin{aligned} \delta\Pi = & \int_0^L \delta\alpha \left[-\frac{\mathrm{d}}{\mathrm{d}x_1} \left(H_w \frac{\mathrm{d}\alpha}{\mathrm{d}x_1} \right) + (H_p - H_{11})\alpha - (H_p - H_{11}) \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right] \mathrm{d}x_1 \\ & - Q\Phi_1(L) + \left[H_w \frac{\mathrm{d}\alpha}{\mathrm{d}x_1} \delta\alpha \right]_0^L \\ & + \int_0^L \delta\Phi_1 \left[-\frac{\mathrm{d}}{\mathrm{d}x_1} \left(H_p \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \right) + \frac{\mathrm{d}}{\mathrm{d}x_1} (H_p - H_{11})\alpha - q_1 \right] \mathrm{d}x_1 \\ & + \left[H_p \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \delta\Phi_1 \right]_0^L - \left[(H_p - H_{11})\alpha\delta\Phi_1 \right]_0^L = 0. \end{aligned}$$

Because this expression must vanish for all arbitrary variations, $\delta\alpha$ and $\delta\Phi_1$, the bracketed terms must be zero, leading to the two differential equations of the problem

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(H_w\frac{\mathrm{d}\alpha}{\mathrm{d}x_1}\right)+(H_p-H_{11})\left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}-\alpha\right)=0, \tag{12.77a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left[H_p\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}-(H_p-H_{11})\alpha\right]=-q_1. \tag{12.77b}$$

At the root of the beam, the geometric boundary conditions, $\Phi_1 = 0$, and $\alpha = 0$, must be imposed; the second condition stems from the required vanishing of the axial displacement component, see eq. (12.75). At the tip of the beam, variations $\delta\alpha(L)$ and $\delta\Phi_1(L)$ are arbitrary, leading to the natural boundary conditions, $H_w\ \mathrm{d}\alpha/\mathrm{d}x_1 = 0$, and $H_p\ \mathrm{d}\Phi_1/\mathrm{d}x_1 - (H_p - H_{11})\alpha = Q$. The first condition implies the vanishing of the axial stresses at the tip of the beam, the second the equilibrium of the moment in the beam with the externally applied torque.

Consider next the case of a cantilevered beam under a tip torque alone, *i.e.*, $q_1 = 0$. Integration of the second equation of the problem, eq. (12.77b), yields

$$H_p\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}-(H_p-H_{11})\alpha=Q, \tag{12.78}$$

where the second boundary condition at the beam's tip is used to evaluate the integration constant. This result is used to substitute for $\mathrm{d}\Phi_1/\mathrm{d}x_1$ in terms of α in the first governing equation to find

$$H_w\frac{\mathrm{d}^2\alpha}{\mathrm{d}x_1^2}-H_{11}\left(1-\frac{H_{11}}{H_p}\right)\alpha=-\left(1-\frac{H_{11}}{H_p}\right)Q.$$

This second order, ordinary differential equation is readily solved to find

$$\alpha=\frac{Q}{H_{11}}\left[1-\frac{\cosh\bar{k}(1-\eta)}{\cosh\bar{k}}\right], \tag{12.79}$$

where $\eta = x_1/L$ is a non-dimensional variable along the span of the beam, and coefficient $\bar{k}$ is defined as

$$\bar{k}^2=\frac{H_{11}L^2}{H_w}\left(1-\frac{H_{11}}{H_p}\right). \tag{12.80}$$

Finally, eq. (12.79) is introduced into eq. (12.78) to determine the beam's twist,

$$\Phi_1=\frac{QL}{H_{11}}\left[\eta-\left(1-\frac{H_{11}}{H_p}\right)\frac{\sinh\bar{k}-\sinh\bar{k}(1-\eta)}{\bar{k}\cosh\bar{k}}\right].$$

Of course, the stresses in the beam can now be obtained from the strain field, eq. (12.76), as

$$\sigma_1(\eta,s)=E\frac{QL}{H_{11}}\,\frac{\Psi(s)}{L^2}\,\frac{\bar{k}\sinh\bar{k}(1-\eta)}{\cosh\bar{k}}.$$

for the axial stress and

$$\tau_s(\eta,s)=\frac{Q}{2\mathcal{A}t}\left[1-\left(1-\frac{2\mathcal{A}tG}{I_p}r_k\right)\frac{\cosh\bar{k}(1-\eta)}{\cosh\bar{k}}\right],$$

for the shear flow.

12.3.6 The non-uniform torsion problem (open sections)

The previous section is focused on beams with closed, single-cell, thin-walled cross-sections made of homogeneous, linearly elastic material. If the section is an open section, the same developments will apply except for the fact that the warping function now satisfies eq. 8.85, rather than eq. 8.94. Consequently, the strain field now simplifies to

$$\epsilon_1 = \Psi(s)\,\frac{\mathrm{d}\alpha}{\mathrm{d}x_1}, \quad \gamma_s = \left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} - \alpha\right) r_k, \tag{12.81}$$

and the strain energy in the beam becomes

$$A = \frac{1}{2}\int_0^L \left[H_w\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x_1}\right)^2 + H_p\left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} - \alpha\right)^2\right]\,\mathrm{d}x_1.$$

This approximation does not take into account the linear through-the-thickness shear strain distribution that develops under torsion, see section 7.5. Indeed, the shear strain defined by eq. (12.81) is *uniform through-the-thickness*. To account for this effect, the strain energy expression is corrected as follows

$$A = \frac{1}{2}\int_0^L \left[H_w\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x_1}\right)^2 + H_p\left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} - \alpha\right)^2 + H_{11}\left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}\right)^2\right]\,\mathrm{d}x_1,$$

where H_{11} is the classical torsional stiffness for the thin-walled, open sections, see eq. (7.64).

Consider a cantilevered beam subjected to a distributed torque $q_1(x_1)$ and a tip torque Q. The total potential energy of the beam can be written

$$\begin{aligned}\Pi =& \frac{1}{2}\int_0^L \left[H_w\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x_1}\right)^2 + H_p\left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} - \alpha\right)^2 + H_{11}\left(\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1}\right)^2\right]\,\mathrm{d}x_1 \\ &- \int_0^L q_1\Phi_1\,\mathrm{d}x_1 - Q\Phi_1(L).\end{aligned}$$

Invoking the principle of minimum total potential energy and following a procedure similar to that presented in the previous section yields the solution of the problem as

$$\alpha = \frac{Q}{H_{11}}\left[1 - \frac{\cosh k(1-\eta)}{\cosh k}\right], \tag{12.82}$$

$$\Phi_1 = \frac{QL}{H_{11}}\left[\eta - \left(1 + \frac{H_{11}}{H_p}\right)\frac{\sinh\bar{k} - \sinh\bar{k}(1-\eta)}{\bar{k}\cosh\bar{k}}\right].$$

where $\eta = x_1/L$ is the non-dimensional variable along the beam's span, and coefficient $\bar{k}$ is defined as

$$\bar{k}^2 = \frac{H_{11}H_pL^2}{H_w(H_p + H_{11})} = \frac{H_{11}L^2}{H_w}\,\frac{1}{1 + H_{11}/H_p}. \tag{12.83}$$

For open section, $H_{11} \ll H_p$, and coefficient $\bar{k}$ becomes nearly equal to its counterpart in the classical formulation of the non-uniform torsion problem, see eq. (8.107). Of course, the stress field can be recovered using the strain field given by eq. (12.81).

12.3.7 Problems

Problem 12.1. Cantilevered beam with elastic foundation

A cantilevered beam of length L is subjected to a tip load P_2, a tip bending moment Q_3, a transverse distributed load $p_2(x_1)$, and a distributed bending moment $q_3(x_1)$, as shown in fig. 12.12. The cantilevered beam is supported by an elastic foundation of stiffness k, not shown on the figure, for clarity. The total potential energy of the system is

$$\Pi = \int_0^L \left[\frac{1}{2} H_{33} \left(\frac{\mathrm{d}^2 u_2}{\mathrm{d}x_1^2} \right)^2 + \frac{1}{2} k\, u_2^2 \right] \mathrm{d}x_1 - \int_0^L \left(p_2\, u_2 + q_3 \frac{\mathrm{d}u_2}{\mathrm{d}x_1} \right) \mathrm{d}x_1 - P_2\, u_2(L) - Q_3 \left. \frac{\mathrm{d}u_2}{\mathrm{d}x_1} \right|_L .$$

(1) Find the governing differential equations and boundary conditions for this problem using the principle of minimum total potential energy. *(2)* Derive the same equations and boundary conditions based on simple free body diagrams for a differential element of the beam.

Problem 12.2. Comparing solutions for torsion of a rectangular section

In examples 12.1 and 12.2, two solutions are developed for the uniform torsion of the rectangular section depicted in fig 12.11, leading to the stress functions given by eqs. (12.67) and (12.70), respectively. *(1)* On one graph, plot the non-dimensional torsional stiffnesses, $H_{11}/(4abG)$, predicted by the two solutions as a function of $a/b \in [1, 12]$. *(2)* On one graph, plot the non-dimensional shear stress at point **B**, $8ab^2\tau^B/M_1$, predicted by the two solutions as a function of $a/b \in [1, 12]$. *(3)* On one graph, plot the non-dimensional shear stress at point **A**, $8ab^2\tau^A/M_1$, predicted by the two solutions as a function of $a/b \in [1, 12]$.

Problem 12.3. Cantilevered beam with various loading

The uniform cantilevered beam of span L depicted in fig. 12.13 has a bending stiffness H_{33} and is supported by an elastic foundation of stiffness k over its first half. A concentrated spring of stiffness k_1 supports the beam at its free end. A mid-span concentrated load P is applied together with a uniform distributed load p_0 that acts over the second half of the beam span. Write the principle of minimum total potential energy for this system.

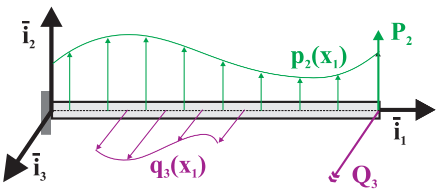

Fig. 12.12. Cantilevered beam with concentrated an distributed moments.

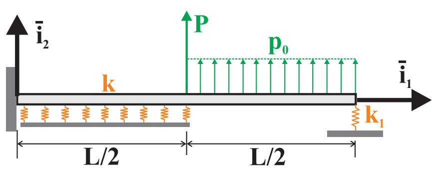

Fig. 12.13. Simply supported beam with partial elastic foundation.

Problem 12.4. Simply supported beam with concentrated load
Consider the uniform simply supported beam of span L subjected to a concentrated load acting at a distance αL from the left support, as shown in fig. 5.23. Write the principle of minimum total potential energy for the system. From this principle, derive the governing differential equations of the problem and the associated boundary conditions.

Problem 12.5. Cantilevered beam with tip spring
The uniform cantilevered beam of span L shown in fig. 12.14 features a tip spring of stiffness k and a tip concentrated load P. Write the principle of minimum total potential energy for the system. From this principle, derive the governing differential equations of the problem and the associated boundary conditions. Explain the physical meaning of the boundary conditions at $x_1 = L$ using a free body diagram.

Problem 12.6. Simply supported beam with end torsional springs
Consider a simply supported, uniform beam of length L with two end point torsional springs of stiffness k_1 and a mid-span spring of stiffness k_2. The beam, shown in fig. 12.15, is subjected to a uniform transverse loading $p_2(x_1) = p_0$. Write the principle of minimum total potential energy for the system. From this principle, derive the governing differential equations of the problem and the associated boundary conditions. Explain the physical meaning of the boundary conditions at $x_1 = L/2$ using a free body diagram.

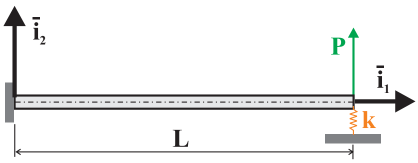

Fig. 12.14. Cantilevered beam with tip concentrated load and elastic spring.

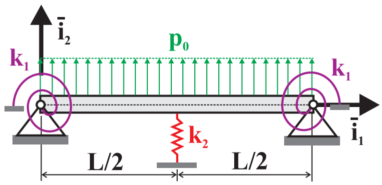

Fig. 12.15. Simply supported beam with mid-span and end point springs.

Problem 12.7. Torsion of a beam with circular cross-section
Consider a uniform beam of length, L, with a circular cross-section of area, $\mathcal{A}$, and modulus, G. It is fixed at one end and loaded with both a distributed twisting moment, $q_1(x_1)$, and a concentrated moment, Q_1, at the other end. (1) Use the principle of minimum total potential energy to develop the governing differential equation. (2) Determine the possible boundary conditions that could be applied at each end of a general beam. (3) Indicate which of these boundary conditions apply for the present problem.

Problem 12.8. Torsion of a beam with concentrated torque applied at a mid-point
A uniform beam of length, L, with a circular cross-section of area, $\mathcal{A}$, and modulus, G, is clamped at both ends and subjected to a concentrated torque, Q_0, applied at $x_1 = a$. (1) Use the principle of minimum total potential energy to develop the governing differential equation. (2) Determine the possible boundary conditions that could be applied at each end of a general beam and at the point of application of the torque, Q_0. (3) Indicate which of these boundary conditions apply for the present problem.

Problem 12.9. Torsion stress in a thin-walled box beam

Consider the thin-walled beam with a rectangular cross-section depicted in fig. 12.16. The beam is clamped at the root and subjected to a tip torque Q_1. *(1)* Solve this problem using the following assumed displacement field $u_1(x_1, x_2, x_3) = 0$; $u_2(x_1, x_2, x_3) = -x_3\ \Phi_1(x_1)$; $u_3(x_1, x_2, x_3) = x_2\ \Phi_1(x_1)$. This leads to the classical theory for torsion: the solution is denoted $\Phi_1^{\rm SV}$. *(2)* Develop a beam theory for torsion based on the following assumed displacement field $u_1(x_1, x_2, x_3) = \Psi(s)\ \alpha(x_1)$; $u_2(x_1, x_2, x_3) = -x_3\ \Phi_1(x_1)$; $u_3(x_1, x_2, x_3) = x_2\ \Phi_1(x_1)$, where $\Psi(s)$ is the warping function obtained from thin-walled beam theory, and $\alpha(x_1)$ an unknown function. The warping function is given as

$$\Psi_1(s) = \frac{a-b}{a+b}\, bs; \quad \Psi_2(s) = -\frac{a-b}{a+b}\, as; \quad \Psi_3(s) = \frac{a-b}{a+b}\, bs; \quad \Psi_4(s) = -\frac{a-b}{a+b}\, as.$$

This leads to a theory for torsion that takes into account the effects of nonuniform torsion: the solution is denoted $\Phi_1^{\rm NU}$. *(3)* Plot the twist distributions $\Phi_1^{\rm SV}$ and $\Phi_1^{\rm NU}$ along the span of the beam on the same graph. *(4)* Sketch the shear stress flow distribution $f^{\rm SV}$ and $f^{\rm NU}$ over the cross-section of the beam at $x_1 = 0$ and L. *(5)* Plot the shear flow distribution along the span of the beam at points **A** and **B**, denoted $f_{\rm A}^{\rm SV}$ and $f_{\rm B}^{\rm SV}$, respectively, for the classical theory, and $f_{\rm A}^{\rm NU}$, and $f_{\rm B}^{\rm NU}$, respectively, for the non-uniform torsion theory. *(6)* Plot the axial stress flow distribution $n^{\rm SV}$ and $n^{\rm NU}$ over the cross-section of the beam at $x_1 = 0$ and $L/2$. *(7)* Predict the failure loads $Q_{\rm fail}^{\rm SV}$ and $Q_{\rm fail}^{\rm NU}$ according Von Mises strength criterion. Compute $Q_{\rm fail}^{\rm SV}/Q_{\rm fail}^{\rm NU}$.

Use the following parameters: $a = 4b$; $L = 6a$; $E/G = 20$; $n = Et\epsilon_{11}$; $f = Gt\gamma$.

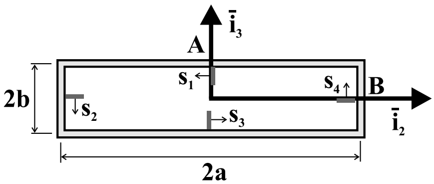

Fig. 12.16. Rectangular cross-section of a thin-walled beam.

Fig. 12.17. Clamped beam with a rigid tip plate.

Problem 12.10. Axial stress in a thin-walled box beam

Repeat the previous problem. However, the beam is now clamped at the root and a rigid plate prevents any warping deformation at the tip, see fig. 12.17

Problem 12.11. Beam with bending, axial and shear deformations

NOTE: this problem requires material on shear deformation in beams from chapter 15. Consider a uniform cantilevered beam of span L subjected to distributed axial and transverse loads $p_1(x_1)$ and $p_2(x_1)$, respectively, to tip concentrated axial and transverse loads P_1 and P_2, respectively, and to distributed and concentrated bending moments $q_3(x_1)$ and Q_3, respectively. The beam possesses a strain energy density function $a = 1/2\ (S\epsilon_1^2 + K_{22}\gamma_{12}^2 + H_{33}\kappa_3^2)$, where the axial strain $\epsilon_1 = {\rm d}u_1/{\rm d}x_1$, the transverse shearing strain $\gamma_{12} = {\rm d}u_2/{\rm d}x_1 - \Phi_3$, and the curvature $\kappa_3 = {\rm d}\Phi_3/{\rm d}x_1$. respectively. The equilibrium equations of the problem are ${\rm d}F_1/{\rm d}x_1 + p_1 = 0$; ${\rm d}F_2/{\rm d}x_1 + p_2 = 0$; and ${\rm d}M_3/{\rm d}x_1 + F_2 + q_3 = 0$, where F_1 is the axial force, F_2 the transverse shearing force, and M_3 the bending moment. *(1)* Derive the principle of virtual work for this problem. *(2)* Derive the principle of minimum total potential energy. *(3)* Derive Hu-Washizu's principle. *(4)* Derive Hellinger-Reissner's principle.

Consider now a cantilever beam subjected to a single transverse tip load $P_2 = P$. *(5)* Solve the problem using the principle of minimum total potential energy with the following assumed modes $u_2 = \eta u_T$ and $\Phi_3 = \eta \Phi_T$, where $\eta = x_1/L$, and u_T and Φ_T are unknown coefficients. *(6)* Solve the problem using Hellinger-Reissner's principle with the following assumed modes $u_2 = \eta u_T$, $\Phi_3 = \eta \Phi_T$, $F_2 = F_0$, and $M_3 = M_R + \eta M_T$, where u_T, Φ_T, F_0, M_R, and M_T are unknown coefficients. *(7)* On a single graph, plot the non-dimensional displacement fields $(H_{33}u_2)/(PL^3)$ and $(H_{33}\Phi_3)/(PL^2)$ versus η for the exact solution and each approximate solution. Choose $s^2 = 2.0\ 10^{-3}$, where $s^2 = H_{33}/K_{22}l^2$. *(8)* On a single graph, plot the non-dimensional internal force fields M_3/PL and F_2/P versus η for the exact solution and each approximate solution. *(9)* On a single graph, plot the non-dimensional strain fields $(H_{33}\kappa_3)/(PL)$ and $(K_{22}\gamma_{12})/(P)$ versus η for the exact solution and each approximate solution. *(10)* For each approximate solution check how well, or how poorly, the equilibrium equations, the strain displacement equations, and the constitutive laws are satisfied. *(11)* Plot the approximate non-dimensional tip displacements normalized by the exact non-dimensional tip displacement versus $1/s^2$. Use a log scale to plot $1/s^2$. Comment on your results.

Problem 12.12. Axial stress in a reinforced panel

The thin panel depicted in fig. 12.18 is reinforced by stiffeners of axial stiffness S. The panel has a length L and the distance between two stiffeners is $2b$. At one end of the panel, loads P are applied to each stiffener, and the panel is clamped along its other end. The concentrated loads applied to the stiffener diffuses in to the panel and the purpose of the analysis is to determine how fast this diffusion process takes place. It can be assumed that at a large distance from the point of application of the concentrated loads, the axial strain and axial stress distributions become uniform across the width of the panel, *i.e.*, $\epsilon_1(x_1/L \rightarrow \infty, x_2) = \epsilon_f$ and $\sigma_1(x_1/L \rightarrow \infty, x_2) = \sigma_f$.

The number of stiffeners is assumed to be large so that a typical cell can be studied. In that typical cell, a panel of width $2b$ and length $L/b = \infty$ is attached to stiffeners of stiffness $S/2$ along each edge, and the stiffeners are subjected to concentrated loads $P/2$. At $x_1 = \infty$ the stress in the panel is σ_f and the loads in the stiffeners are $P_f/2$.

To analyze this problem, use an energy approach with the following assumed displacement field: $u_1(\eta, \zeta) = g(\eta) + h(\eta)(\zeta^2 - \alpha)$, $u_2(\eta, \zeta) = 0$, where $\eta = x_1/L$ and $\zeta = x_2/b$. The following geometric boundary conditions can be selected: $g(x_1 = 0) = 0$, implying the vanishing of the average axial displacement at $x_1 = 0$, and $h(x_1 = \infty) = 0$, implying a uniform axial displacement at $x_1 = \infty$.

(1) Express σ_f and $P_f/2$ as a function of the applied load P. *(2)* Write the total strain energy in the structure. Calculations will greatly simplify if you select the coefficient α so that the coupling term between g and h vanishes. Note that the strain energy of a stiffener is $1/2 \int_0^\infty S\ (\mathrm{d}u_1/\mathrm{d}x_1)^2_{x_2=h}\ \mathrm{d}x_1$. *(3)* Write the total potential of the externally applied loads. Do not forget to include the work done by the loads applied at $x_1 = \infty$. *(4)* Solve the problem for the unknown displacement functions g and h. *(5)* Determine the non-dimensional axial stress distribution in the panel, σ_1/σ_f, and the non-dimensional load in the stiffener, $F_s/(P_f/2)$. *(6)* Plot $\sigma_1/\sigma_f(\eta, \zeta = 1)$ and $\sigma_1/\sigma_f(\eta, \zeta = 0)$ versus η, for $E/G = 2.6$ and $E/G = 28$, on the same graph. *(7)* $F_s/(P_f/2)$ versus η, for $E/G = 2.6$ and $E/G = 28$, on the same graph. *(8)* Determine the non-dimensional shear stress distribution in the panel, $h\tau_{12}/P$. *(9)* Plot $h\tau_{12}/P(\eta, \zeta = 1)$ versus η, for $E/G = 2.6$ and $E/G = 28$, on the same graph. *(10)* On the same graph, plot the distribution of non-dimensional axial stress across the

width of the panel for η = 0, 2, 4 and 6, for $E/G = 2.6$; same question for $E/G = 28$. *(11)* On the same graph, plot the distribution of non-dimensional shear stress across the width of the panel for η = 0, 2, 4 and 6, for $E/G = 2.6$; same question for $E/G = 28$. *(12)* Find the diffusion length d for the panel, *i.e.*, the distance it takes for the maximum stress to decrease to within 1% of σ_f. *(13)* Plot the diffusion length as a function of $k \in [0, 1]$. Interpret your results.

Use the following data: The panel and stiffener are made of a homogeneous, linearly elastic material; $\sigma_1 = E\,\epsilon_1$, $\tau_{12} = G\,\gamma_{12}$; $F_s(x_1) = S\,\epsilon_1(x_1, x_2 = \pm b)$ for the right and left stiffeners, respectively. The stiffener axial stiffness is $S = E\mathcal{A}$, where $\mathcal{A}$ is its cross-sectional area; $k = \mathcal{A}/(2bt) = 0.15$ is the ratio of the stiffener area to the panel area.

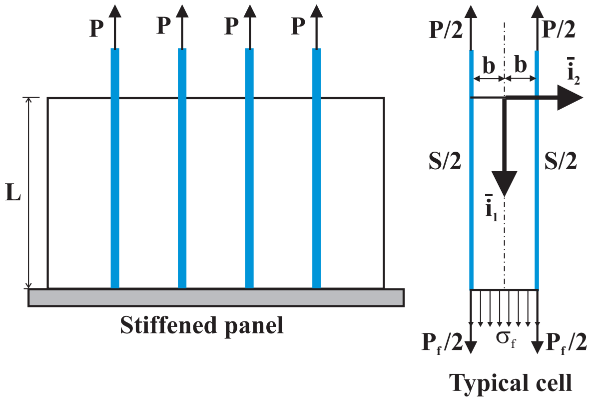

Fig. 12.18. Configuration of the stiffened panel.