9

Virtual work principles

9.1 Introduction

The concept of mechanical work is fundamental to the study of mechanics. The mechanical work done by a force is defined as the scalar product of that force by the displacement through which it acts: work is a scalar quantity, in contrast with forces and displacements, which are vector quantities characterized by magnitudes and directions. Consequently, work quantities are simpler to manipulate than forces and displacements, and this simplification makes work based formulations of mechanics very attractive.

Newtonian mechanics is based on the concepts of forces and moments, which are vector quantities. The equilibrium conditions stated by Newton's law are expressed in their most general form as vector equations, and vector algebra is required for most practical applications. While it is customary to make a distinction between externally applied loads, internal forces and reaction forces, Newton's condition for equilibrium states that *the sum of all forces must vanish,* without making any distinction between them. It follows that all forces explicitly appear in the equilibrium equations of the problem and the solution process involves the determination of all forces, including internal and reaction forces. Newton's approach effectively determines forces and displacements, but it becomes increasingly difficult and tedious for problems of increasing complexity.

Formulations based on the concept of work are a part of what is generally referred to as *analytical mechanics* and provide powerful tools for dealing with complex problems. These methods are very attractive because they deal with quantities that are scalar rather than vector quantities, resulting in simpler analysis processes. Furthermore, specific types of forces, such as reaction forces, can often be eliminated from the solution process if the work they perform vanishes. Analytical mechanics formulations also enable the systematic development of procedures to obtain approximate solutions to very complex problems. In particular, the finite element method, a commonly used tool for structural analysis, has its roots in analytical mechanics.

If analytical mechanics is so powerful and versatile, why has it not completely eclipsed Newton's formulation? An important task in structural analysis is the determination of both magnitude and direction of all forces acting within a structure, which is required to estimate failure conditions. It is logical to use Newton's approach for this task because it is directly expressed in terms of the quantities that must be evaluated. Furthermore, forces are easily visualized as vectors of a specific magnitude and direction that can be directly measured in the laboratory. In contrast, because it can only be measured in an indirect manner, mechanical work is a more abstract concept, which is quite different from the concept of "human work," in the sense of "human labor."

The *Principle of Virtual Work* (PVW) is the most fundamental tool of analytical mechanics, and it will be shown to be entirely equivalent to Newton's law. Both the principle of virtual work and Newton's laws are statements of equilibrium, which must always be satisfied at any point in a structure. In this chapter, simple applications of the principle of virtual work will be presented, focusing on discrete, rather than continuous systems. In many case, both Newtonian and analytical mechanics approaches will be presented in parallel to highlight their respective features.

9.2 Equilibrium and work fundamentals

9.2.1 Static equilibrium conditions

Newton's first law of motion states that *every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.* The expression "state of uniform motion" means that the object moves at a constant velocity; for static problems, however, is is customary to focus on objects at rest. If several forces are applied to the object, the "external force" is, in fact, the resultant, *i.e.*, the vector sum, of all externally applied forces. Finally, the "object" mentioned in the law is to be understood as a "particle." With all these clarifications, Newton's law is then restated as *a particle at rest tends to remain at rest unless the sum of the externally applied force does not vanish.* This also implies that if the sum of the externally applied forces does not vanish, the particle is no longer at rest. A more mathematical statement of Newton's law is: *a particle is at rest if and only if the sum of the externally applied forces vanishes.* The expression "if and only if" is included in the statement because this is both a necessary and sufficient condition: a particle is at rest if the sum of the forces vanishes, and if the sum of the forces vanishes, then the particle is at rest.

In structural mechanics, a particle at rest is said to be in *static equilibrium.* Newton's first law then becomes

> *A particle is in static equilibrium if and only if the sum of the externally applied forces vanishes.*

Newton's first law gives the necessary and the sufficient condition for static equilibrium and can be stated in a mathematical form as: a particle is in static equilibrium if and only if

$$\sum \underline{F} = 0, \tag{9.1}$$

where $\underline{F}$ are the externally applied forces acting on the particle. From a vector algebra standpoint, this equation can be interpreted in various manners: *(1)* the vector sum of all forces acting on the particle must be zero, or *(2)* the vector force polygon must be closed, or *(3)* the components of the vector sum resolved in any coordinate system must vanish, *i.e.*, if $\sum \underline{F} = F_1\bar{\imath}_1 + F_2\bar{\imath}_2 + F_3\bar{\imath}_3$, where $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ is an arbitrary orthonormal basis, $F_1 = F_2 = F_3 = 0$.

Newton's third law is also of fundamental importance to statics; it states: *if particle **A** exerts a force on particle **B**, particle **B** simultaneously exerts on particle **A** a force of identical magnitude and opposite direction.* It is also postulated that *these two forces share a common line of action*. In a more compact manner, Newton's third law states that

> *Two interacting particles exert on each other forces of equal magnitude, opposite directions, and sharing a common line of action.*

Euler's first law

Newton's first and third laws only apply to particles, but they can be extended to a collection of interacting particles. Figure 9.1 shows a system consisting of N particles. Particle i is subjected to an external force, $\underline{F}_i$, and to $N-1$ interaction forces, $\underline{f}_{ij}$, $j = 1, 2, \ldots, N$, $j \neq i$. Newton's first law, eq. (9.1), applied to particle i, then states that

$$\underline{F}_i + \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} = 0. \tag{9.2}$$

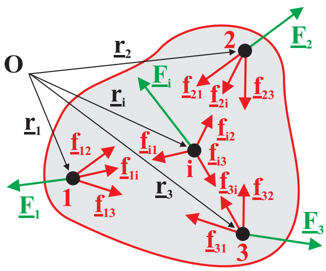

Fig. 9.1. A system of particles.

Note that little has been said about the nature of the *system of particles,* or of the interaction forces. If the system of particles is a rigid body, the interaction forces are those that will ensure that the shape of the body remains unchanged by the externally applied loads. If the system of particles is an elastic body, the interaction forces are the stresses that will result from the deformation of the body. If the system of particles is planetary system, the interaction forces are the gravitational pull that each planet exerts on all others. Although of physically different natures, all interaction forces are assumed to obey Newton's third law, which implies that $\underline{f}_{ij} + \underline{f}_{ji} = 0$.

Since eq. (9.2) applies to each of the N particles of the system, summation of these N equations yields

$$\sum_{i=1}^{N} \underline{F}_i + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} = 0.$$

In the double summation of the second term, interaction forces appear in pairs, $\underline{f}_{ij}$ and $\underline{f}_{ji}$. Newton's third law then implies the vanishing of each pair of interaction forces, $\underline{f}_{ij} + \underline{f}_{ji} = 0$, leading to

$$\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \underline{f}_{ij} = 0. \tag{9.3}$$

Since the second term vanishes, the above equation simplifies to

$$\sum_{i=1}^{N} \underline{F}_i = 0. \tag{9.4}$$

This statement is known as *Euler's first law* for a system of particles; it is very similar to Newton's first law, but now applies to a system of particle.

Note that eq. (9.4) is a necessary condition for the system of particles to be in static equilibrium, but is not a sufficient condition. As implied by Newton's first law, the necessary and sufficient conditions for the static equilibrium of the system are the satisfaction of eq. (9.2) for each of the N particles of the system; in all, this represents N vector equations that must be satisfied. Clearly, eq. (9.4) is a single vector equation, and therefore a much less stringent condition. The main advantage of eq. (9.4), however, is that all interaction forces are eliminated from this statement.

Euler's second law

It is possible to extract additional conditions for the static equilibrium of a system of particles. Let $\underline{r}_i$ be the position vector of particle i with respect to an arbitrary point **O**, see fig. 9.1. Taking a vector product of eq. (9.2) by $\underline{r}_i$, then summing over all particles leads to

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \underline{r}_i \times \underline{f}_{ij} = 0.$$

In the double summation of the second term, interaction forces appear in pairs, $\underline{r}_i \times \underline{f}_{ij}$ and $\underline{r}_j \times \underline{f}_{ji}$. A property of the vector cross product is that $\underline{r}_i \times \underline{f}_{ij} = \underline{r}_\perp \times \underline{f}_{ij}$ and $\underline{r}_j \times \underline{f}_{ji} = \underline{r}_\perp \times \underline{f}_{ji}$, where $\underline{r}_\perp$ is the vector that joins point **O** to the point on the common line of action of the internal force pair that is at the shortest distance from point **O**. For Newton's third law, it then follows that $\underline{r}_i \times \underline{f}_{ij} + \underline{r}_j \times \underline{f}_{ji} = \underline{r}_\perp \times (\underline{f}_{ij} + \underline{f}_{ji}) = 0$, which yields

$$\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \underline{r}_i \times \underline{f}_{ij} = 0. \tag{9.5}$$

With this simplification, the above equation reduces to

$$\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i = \sum_{i=1}^{N} \underline{M}_i = 0, \tag{9.6}$$

where $\underline{M}_i$ is the moment of the external forces applied to particle i. The point about which moments of the externally applied forces is calculated is arbitrary. This statement is known as *Euler's second law* for a system of particles. Euler's first and second laws are both necessary conditions for the system of particles to be in static equilibrium, but are not a sufficient conditions. Taken together, they form two vector equations that clearly fall short of the N vector equations, eq. (9.2) for each of the N particles, required to guarantee static equilibrium of the system.

9.2.2 Concept of mechanical work

Mechanical work is defined as follows: *the work done by a force is the scalar product of the force by the displacement of its point of application.* At first, let the force and displacement vectors be collinear: the force vector is $\underline{F} = F\bar{u}$ and the displacement vector $\underline{d} = d\bar{u}$, where F is the magnitude of the force, d that of the displacement, and $\bar{u}$ is a unit vector along the common direction of the force and displacement. The work, W, done by the force becomes $W = F\bar{u} \cdot d\bar{u} = Fd$. Note that if the force and displacement are both in the same direction, the work is positive, whereas if the force and displacement are in opposite directions, the work is negative.

Next, consider the case where force and displacement vectors are not collinear: the force vector is $\underline{F} = F\bar{u}$ and the displacement vector $\underline{d} = d\bar{v}$, where $\bar{u}$ and $\bar{v}$ are unit vectors along the orientations of the force and displacement vectors, respectively. The work done by the force becomes $W = F\bar{u} \cdot d\bar{v} = Fd\, \bar{u} \cdot \bar{v} = Fd \cos\theta$, where θ is the angle between the unit vectors $\bar{u}$ and $\bar{v}$. If the force acts in the direction perpendicular to the displacement, $\cos\theta = \cos\pi/2 = 0$, and the work done by the force vanishes although both force and displacement vectors do not.

It is often the case that both force and displacement vectors change in time. To deal with this situation, the concept of *incremental work* is introduced as $\mathrm{d}W = \underline{F} \cdot \mathrm{d}\underline{r}$, where $\mathrm{d}\underline{r}$ is the infinitesimal displacement vector. If the point of application of the force moves from $\underline{r}_i$ to $\underline{r}_f$, the total work is then found by integration of the incremental work

$$W = \int_{\underline{r}_i}^{\underline{r}_f} \mathrm{d}W = \int_{\underline{r}_i}^{\underline{r}_f} \underline{F} \cdot \mathrm{d}\underline{r}. \tag{9.7}$$

When the force and incremental displacement are three-dimensional vectors, their scalar product is easily computed by evaluating the components of the two vectors in a common orthonormal basis. The force and infinitesimal displacement vectors are written as $\underline{F} = F_1\bar{e}_1 + F_2\bar{e}_2 + F_3\bar{e}_3$ and $\mathrm{d}\underline{r} = \mathrm{d}r_1\bar{e}_1 + \mathrm{d}r_2\bar{e}_2 + \mathrm{d}r_3\bar{e}_3$, respectively, where $\mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ forms an orthonormal basis, *i.e.*, a set of three mutually orthogonal unit vectors. The incremental work then becomes $\mathrm{d}W = \underline{F} \cdot \mathrm{d}\underline{r} = F_1\mathrm{d}r_1 + F_2\mathrm{d}r_2 + F_3\mathrm{d}r_3$. The force and displacement vectors must both be resolved in a common basis for this formula to apply, although this common basis can be chosen arbitrarily.

Only the component of the force vector acting along the differential displacement vector does work. Let the differential displacement be written as $\mathrm{d}\underline{r} = \mathrm{d}r\, \bar{u}$, where $\bar{u}$ is the unit vector in the direction of the differential displacement. Next, the force vector is written as $\underline{F} = F_{||}\bar{u} + F_{\perp}\bar{v}$, where $\bar{v}$ is a unit vector perpendicular to $\bar{u}$, $F_{||}$ the component of the force acting along the differential displacement vector, and $F_{\perp}$ that acting in the plane perpendicular to $\bar{u}$. The incremental work now becomes

$\mathrm{d}W = (F_{||}\bar{u} + F_{\perp}\bar{v}) \cdot \mathrm{d}r\,\bar{u} = F_{||}\mathrm{d}r$. The component of the force acting along the displacement vector, $F_{||}$, is the sole contributor to the work.

Work is a scalar product, and consequently, superposition holds. Let the applied force be written as $\underline{F} = \underline{F}_1 + \underline{F}_2$; the incremental work now becomes $\mathrm{d}W = \underline{F}\cdot\mathrm{d}\underline{r} = (\underline{F}_1 + \underline{F}_2) \cdot \mathrm{d}\underline{r} = \underline{F}_1 \cdot \mathrm{d}\underline{r} + \underline{F}_2 \cdot \mathrm{d}\underline{r} = \mathrm{d}W_1 + \mathrm{d}W_2$. This implies that the sum of the work done by the two forces, $\underline{F}_1$ and $\underline{F}_2$, denoted $\mathrm{d}W_1$ and $\mathrm{d}W_2$, respectively, equals the work done by the resultant force, $\underline{F}$.

Structural analysis focuses on static problems, as opposed to structural dynamics, which broadens the scope of the investigation to include the dynamic response of structures to time dependent loads. The very definition of work involves a "force that displaces its point of application," which implies a dynamic problem. Why then is work a quantity of interest for the static analysis of structures? The answer to this question is found in the next section, which introduces the concept of *virtual work*, *i.e.*, the work that would be done by a force *if it were to displace its point of application by a fictitious amount*.

9.3 Principle of virtual work

As discussed in the previous section, the static equilibrium condition for a particle, as stated by Newton's first law, is written as a vector equation that imposes the vanishing of the externally applied forces. In the present section, an alternative formulation will be developed, which results in the *Principle of Virtual Work* (PVW). Although expressed in terms of work rather than force vectors, the principle of virtual work will be shown to be entirely equivalent to Newton's first law. First, the principle will be developed for a single particle; next, it will be generalized to enable applications to systems of particles.

The principle of virtual work introduces the fundamental concept of "arbitrary virtual displacements" sometimes called "arbitrary test displacements," or also "arbitrary fictitious displacements," and all of these expressions will be used interchangeably. The word "arbitrary" is easily understood: it simply means that the displacements can be chosen in an arbitrary manner without any restriction imposed on their magnitudes or orientations. More difficult to understand are the words "virtual," "test," or "fictitious." All three imply that these are not real, actual displacements. More importantly, these fictitious displacements *do not affect the forces acting on the particle*. These important concepts will be explained in the following sections.

9.3.1 Principle of virtual work for a single particle

Consider a particle in static equilibrium under a set a externally applied loads, as depicted in fig. 9.2. According to Newton's first law, the sum of the externally applied load must vanish, as expressed by eq. (9.1). Next, consider a fictitious displacement of arbitrary magnitude and orientation, denoted $\underline{s}$ in fig. 9.2. Although the problem appears to be two-dimensional in the figure, both forces and fictitious displacements are three-dimensional quantities.

The virtual work done by the externally applied forces is now evaluated by computing the dot product of the externally applied loads by the fictitious displacement to find

$$W = \left[\sum \underline{F}\right] \cdot \underline{s} = 0. \tag{9.8}$$

Because the particle is in static equilibrium, Newton's first law implies the vanishing of the bracketed term. It follows that the dot product vanishes *for any arbitrary fictitious displacement*.

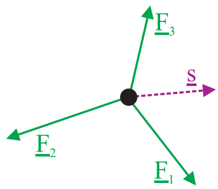

Fig. 9.2. A particle with applied forces subjected to a fictitious test displacement.

This result sheds some light on the special nature of the fictitious, or virtual displacements. If the particle is in static equilibrium in a given configuration, the sum of the forces vanishes, *i.e.*, $\sum \underline{F} = 0$. Assume now that one of the externally applied forces, say $\underline{F}_1$, is the force acting in an elastic spring connected to the particle. If the particle undergoes a *real, but arbitrary displacement*, $\underline{d}$, the force in the spring will change to become $\underline{F}'_1$. All displacement dependent forces applied to the particle will change, and the sum of the externally applied loads becomes $\sum \underline{F}'$. In the new configuration resulting from the application of the real displacement, $\underline{d}$, static equilibrium will not be satisfied, *i.e.*, $\sum \underline{F}' \neq 0$. Indeed, if the particle is in static equilibrium in the configuration resulting from the application of an *arbitrary displacement*, it will be in static equilibrium in *any* configuration, which makes little sense.

In contrast with real displacements, *virtual or fictitious displacements do not affect the loads applied to the particle*. This means that even in the presence of displacement dependent loads such as those arising within an elastic spring, if the particle is in static equilibrium, it remains in static equilibrium when virtual or fictitious displacements are applied. This is the reason why eq. (9.8) remains true for *all arbitrary virtual displacements*. The discussion thus far has thus established that if the particle is in static equilibrium, eq. (9.8) holds for all arbitrary fictitious displacements.

Next, the following question is asked: if eq. (9.8) holds, is the particle in static equilibrium? Consider fig. 9.2, and let the components of the applied forces be $\underline{F}_1 = F_{11}\bar{\imath}_1 + F_{12}\bar{\imath}_2 + F_{13}\bar{\imath}_3$, $\underline{F}_2 = F_{21}\bar{\imath}_1 + F_{22}\bar{\imath}_2 + F_{23}\bar{\imath}_3$, $\underline{F}_3 = F_{31}\bar{\imath}_1 + F_{32}\bar{\imath}_2 + F_{33}\bar{\imath}_3$, while the components of the virtual displacement are $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_2 + s_3\bar{\imath}_3$, where $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ is an orthonormal basis. Equation (9.8) now states $(F_{11} + F_{21} + F_{31})s_1 + (F_{12} + F_{22} + F_{32})s_2 + (F_{13} + F_{23} + F_{33})s_3 = 0$.

At first, assume that the particle is not in static equilibrium, *i.e.*, $\sum \underline{F} \neq 0$. It is always possible to find *a particular virtual displacement* for which eq. (9.8) will be satisfied. Indeed, for a given set of forces, select s_1 and s_2 in an arbitrary manner, then solve eq. (9.8) for s_3 to find $s_3 = -[(F_{11} + F_{21} + F_{31})s_1 + (F_{12} + F_{22} + F_{32})s_2]/(F_{13} + F_{23} + F_{33})$. Consequently, the fact that eq. (9.8) is satisfied *for a particular virtual displacement* does not imply that it is in static equilibrium. In fact, even if it is satisfied *for many virtual displacements*, static equilibrium is still not guaranteed. Indeed, for each new arbitrary choice of s_1 and s_2, it is possible to compute an s_3 for which eq. (9.8) is satisfied.

Different conclusions are reached if eq. (9.8) is satisfied *for all arbitrary virtual displacements*. Indeed, if $(F_{11} + F_{21} + F_{31})s_1 + (F_{12} + F_{22} + F_{32})s_2 + (F_{13} + F_{23} + F_{33})s_3 = 0$ for all arbitrary values of independently chosen s_1, s_2 and s_3, it follows that $F_{11} + F_{21} + F_{31} = 0$, $F_{12} + F_{22} + F_{32} = 0$, and $F_{13} + F_{23} + F_{33} = 0$, is the only solution of eq. (9.8). In turn, this can be written as $(F_{11} + F_{21} + F_{31})\bar{\imath}_1 + (F_{12} + F_{22} + F_{32})\bar{\imath}_2 + (F_{13} + F_{23} + F_{33})\bar{\imath}_3 = 0$, and finally, $\sum \underline{F} = 0$. Thus, if eq. (9.8) is satisfied *for all arbitrary virtual displacements*, then $\sum \underline{F} = 0$, and the particle is in static equilibrium.

In conclusion, if a particle is in static equilibrium, the virtual work done by the externally applied forces vanishes for all arbitrary virtual displacements. Furthermore, it is also true that if the virtual work vanishes for all arbitrary fictitious test displacements, then the sum of the externally applied forces vanishes, and hence, the particle is in static equilibrium. These two facts can be combined into the statement of the principle of virtual work for a particle

Principle 3 (Principle of virtual work for a particle) *A particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes for all arbitrary virtual displacements.*

Since the condition for static equilibrium is nothing but Newton's first law, it follows that the principle of virtual work, which states the condition for static equilibrium, is entirely equivalent to Newton's first law, and either statement provides a fundamental definition of static equilibrium. Simple examples will now be used to illustrate the principle of virtual work.

Example 9.1. Equilibrium of a particle

Consider the particle depicted in fig. 9.3, which is subjected to two vertical forces $\underline{F}_1 = 1\bar{\imath}_1$ and $\underline{F}_2 = -3\bar{\imath}_1$. The following question is asked: is the particle in static equilibrium? Rather than relying on Newton's first law, the principle of virtual work will used to answer the question. Consider the following arbitrary virtual displacement, $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_2$, and its associated virtual work

$$W = (1\bar{\imath}_1 - 3\bar{\imath}_1) \cdot (s_1\bar{\imath}_1 + s_2\bar{\imath}_2) = -2\bar{\imath}_1 \cdot (s_1\bar{\imath}_1 + s_2\bar{\imath}_2) = -2s_1 \neq 0.$$

The fact that $\underline{s}$ is an arbitrary virtual displacement implies that s_1 and s_2 are arbitrary scalars, and hence, $W = -2s_1 \neq 0$. Because the virtual work done by the externally applied forces does not vanish for all virtual displacements, the principle of virtual work, principle 3, implies that the particle is not in static equilibrium.

It is important to understand the implications of the last part of the principle of virtual work, "for all arbitrary virtual displacements." Consider the following arbitrary virtual displacement, $\underline{s} = s_2\bar{\imath}_2$, and its associated virtual work

$$W = (1\bar{\imath}_1 - 3\bar{\imath}_1) \cdot s_2\bar{\imath}_2 = -2\bar{\imath}_1 \cdot s_2\bar{\imath}_2 = 0.$$

This result is due to the fact that the sum of the externally applied loads, $-2\bar{\imath}_1$, is orthogonal to the virtual displacement, $s_2\bar{\imath}_2$, and hence, the virtual work vanishes.

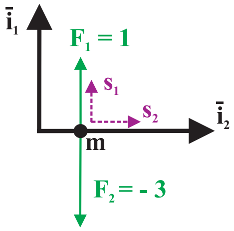

Fig. 9.3. A particle under the action of two forces.

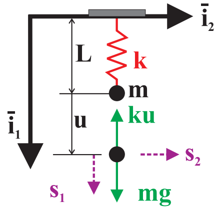

Fig. 9.4. A particle suspended to an elastic spring.

One might be tempted to conclude from the above result that the particle is in static equilibrium because the virtual work vanishes. To satisfy the principle of virtual work, however, the virtual work must vanish *for all arbitrary virtual displacements*.

The above result shows that the virtual work may vanish for "a particular virtual displacement," but this is not a sufficient condition to guarantee static equilibrium. For the two-dimensional problem shown in fig. 9.3, an arbitrary fictitious displacement must span the plane of the problem, *i.e.*, must be of the form $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_2$. For three-dimensional problems, a three-dimensional virtual displacement must be selected, $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_2 + s_3\bar{\imath}_3$, where s_1, s_2, and s_3 are three arbitrary scalars, and $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ a basis that spans the three-dimensional space.

Example 9.2. Equilibrium of a particle connected to an elastic spring
Consider next a particle in static equilibrium under the effect of gravity and the restoring force of an elastic spring of stiffness constant k, as depicted in fig. 9.4. Find the displacement of the particle in its actual static equilibrium configuration.

For this two-dimensional problem, assume that the particle is at position u. An arbitrary fictitious displacement is selected as $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_2$, where s_2 and s_2 are two arbitrary scalars. The virtual work done by the externally applied loads becomes

$$W = (mg\bar{\imath}_1 - ku\bar{\imath}_1) \cdot (s_1\bar{\imath}_1 + s_2\bar{\imath}_2) = [mg - ku]s_1.$$

The principle of virtual work now implies that the particle is in static equilibrium at position u if and only if the virtual work done by the externally applied loads vanishes for all arbitrary virtual displacements, *i.e.*, if and only if $[mg - ku]s_1 = 0$ for all values of s_1. Equation $[mg - ku]s_1 = 0$ possesses two solutions, $[mg - ku] = 0$ or $s_1 = 0$; the second solution, however, is not valid because, as implied by the principle of virtual work, s_1 is arbitrary.

In conclusion, the vanishing of the virtual work for all arbitrary virtual displacements implies that $mg - ku = 0$, and the equilibrium configuration of the system is found as $u = mg/k$. Of course, the same conclusion can be drawn more expeditiously from a direct application of Newton's first law, which requires the sum of the externally applied forces to vanish, *i.e.*, $mg\bar{\imath}_1 - ku\bar{\imath}_1 = 0$, or $(mg - ku)\bar{\imath}_1 = 0$, and finally, $mg - ku = 0$.

This example involves the restoring force of an elastic spring, a displacement dependent force. Indeed, the elastic force in the spring is $-ku\bar{\imath}_1$, and if the particle undergoes a *real downward displacement* of magnitude d, the restoring force becomes $-k(u+d)\bar{\imath}_1$. In contrast, if the particle undergoes a *virtual downward displacement* of magnitude s_1, the restoring force *remains unchanged* as $-ku\bar{\imath}_1$. This difference has profound implications on the computation of work. First, consider the work done by the elastic force, $-ku\bar{\imath}_1 \cdot \mathrm{d}u\,\bar{\imath}_1$, under a *virtual displacement*, s_1,

$$W = \int_u^{u+s_1} -ku\,\mathrm{d}u = -ku\int_u^{u+s_1} \mathrm{d}u = -ku\,[u]_u^{u+s_1} = -kus_1. \tag{9.9}$$

It is possible to remove the elastic force, $-ku$, from the integral because this force remains unchanged by the virtual displacement, and hence, it can be treated as a constant.

In contrast, the work done by the same elastic force under a *real displacement*, d, is

$$W = \int_u^{u+d} -ku\,\mathrm{d}u = \left[-\frac{1}{2}ku^2\right]_u^{u+d} = -kud - \frac{1}{2}kd^2. \tag{9.10}$$

In this case, the real work includes an additional term that is quadratic in d and represents the work done by the change in force that develops due to the stretching of the spring. Even if the magnitude of the real displacement is equal to that of the virtual displacement, *i.e.*, even if $d = s_1$, the two expressions for the work done by the elastic restoring force are not identical.

These observations help explain the terminology used when dealing with the principle of virtual work. The concept of virtual displacement is key to the correct use of the principle of virtual work, which requires the virtual work done by displacement dependent forces to be evaluated according to eq. (9.9) rather than eq. (9.10). Of course, the *real* work done by the elastic force as it undergoes a real displacement is correctly evaluated by eq. (9.10).

Clearly, it is important to keep in mind the crucial difference between "real displacements" and "virtual" or "fictitious displacements." The words "virtual" or "fictitious" are used to emphasize the fact the forces remain unaffected by these displacements. In practice, the term "real displacement" is rarely used; real displacements are simply called displacements. The terms "virtual," "fictitious" or "test displacements" all imply that the forces acting on the system remain unaffected by the application of such displacements. The term "virtual displacement" is the most widely used.

Example 9.3. Equilibrium of a particle sliding on a track

Consider a particle of mass m that can slide on a track, as shown in fig. 9.5. The externally applied horizontal force is resited by friction between the particle and track. Newton's first law expresses the condition for static equilibrium as $mg\bar{\imath}_1 - R\bar{\imath}_1 + P\bar{\imath}_2 - F\bar{\imath}_2 = 0$, where $-R\bar{\imath}_1$ is the reaction force the track exerts on the particle, and $-F\bar{\imath}_2$ the friction force applies to the particle.

Note that the four forces applied to the particle are of different physical natures: $P\bar{\imath}_2$ is an externally applied force, $mg\bar{\imath}_1$ the force of gravity, $-R\bar{\imath}_1$ a reaction force, and $-F\bar{\imath}_2$ a friction force. Yet all forces play an equal role in Newton's law, which states that the sum of all forces must vanish. The law simply states "all forces" without making any distinction among them. Newton's first law is readily solved to find $(mg - R)\bar{\imath}_1 + (P - F)\bar{\imath}_2 = 0$, and finally $R = mg$ and $F = P$, as expected.

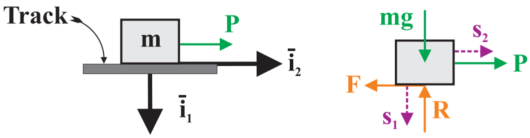

Fig. 9.5. A particle sliding on a track.

Next, the principle of virtual work will be used to solve the same problem. For this two dimensional problem, an arbitrary virtual displacement will be written as $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_2$, and the vanishing of the virtual work it performs implies

$$W = (mg\bar{\imath}_1 - R\bar{\imath}_1 + P\bar{\imath}_2 - F\bar{\imath}_2)\cdot(s_1\bar{\imath}_1 + s_2\bar{\imath}_2) = [mg - R]s_1 + [P - F]s_2 = 0. \quad (9.11)$$

Following a reasoning similar to that developed in the previous example, it is easy to show that the vanishing of the virtual work for all arbitrary scalars s_1 and s_2 implies the vanishing of the two bracketed terms in the above equation: $mg - R = 0$ and $P - F = 0$. This result is identical to that obtained from Newton's first law, as expected, since the principle of virtual work and Newton's first law are identical.

This example illustrates a crucial relationship between Newton's first law and the principle of virtual work. The projection of Newton's law along axes, $\bar{\imath}_1$ and $\bar{\imath}_2$, yields two scalar equilibrium equations, $mg - R = 0$ and $P - F = 0$, respectively. The same two equilibrium equations are obtained by imposing the vanishing of the factors multiplying the arbitrary virtual displacement components, s_1 and s_2, measured along the same axes, $\bar{\imath}_1$ and $\bar{\imath}_2$, respectively.

The principle of virtual work yields scalar equilibrium equations which are the projections of Newton's first law along the directions associated with the virtual displacement components. Because it is based on a scalar quantity, the virtual work, the principle of virtual work yields scalar equations of equilibrium, rather than their vector counterparts inherent to the application of Newton's first law.

9.3.2 Kinematically admissible virtual displacements

Example 9.3 illustrates an important feature of virtual displacements, which are selected to have components in the horizontal direction, $s_2\bar{\imath}_2$, and the vertical direction, $s_1\bar{\imath}_1$. This raises a basic question: how could the particle move in the vertical direction when it is constrained to remain on the track? The answer to the question lies in the nature of the virtual displacements which are not real, but rather are virtual or fictitious displacements. Of course, the particle cannot possibly undergo *real displacements* in the vertical direction because it must remain on the track, but *virtual* or *fictitious displacements* in that same direction are allowed.

In the derivation of the principle of virtual work, it is necessary to use completely arbitrary virtual displacements to prove that the vanishing of the virtual work implies Newton's first law. The completely arbitrary nature of the virtual displacements is key to the successful use of the principle of virtual work. The expression, "arbitrary virtual displacements" means *any virtual displacements, including those that violate the kinematic constraints of the problem.*

In fig. 9.5, the particle is confined to remain on the track; it can move along the track, but not in the direction perpendicular to it. The direction along the track is called the *kinematically admissible direction*, whereas the direction normal to it is called the *kinematically inadmissible direction*, or the *infeasible direction*.

It is sometimes convenient to introduce the concept of *kinematically admissible virtual displacements*. These are virtual displacements that satisfy the kinematic constraints of the problem.

For the problem depicted in fig. 9.5, the kinematic constraint enforces the particle to remain on the track. Arbitrary virtual displacements are written as $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_2$, but since these include a component in the vertical direction, *i.e.*, in a kinematically inadmissible direction, these are not kinematically admissible virtual displacements. On the other hand, virtual displacements of the form $\underline{s} = s_2\bar{\imath}_2$, are kinematically admissible because these are oriented along the track.

At this point, the relationship between kinematic constraints and reaction forces should be clarified. Reaction forces are those forces arising from the enforcement of kinematic constraints. The particle depicted in fig. 9.5 is constrained to move along the track, and this kinematic constraint gives rise to a reaction force. Note that the reaction force acts along the kinematically inadmissible direction, *i.e.*, the direction normal to the track.

Consider now the virtual work done by the reaction force under arbitrary virtual displacements,

$$W = (-R\bar{\imath}_1) \cdot (s_1\bar{\imath}_1 + s_2\bar{\imath}_2) = -Rs_1 \neq 0.$$

Next, consider the virtual work done by the same reaction force under arbitrary *kinematically admissible virtual displacements,*

$$W = (-R\bar{\imath}_1) \cdot (s_2\bar{\imath}_2) = 0.$$

Because the reaction force acts along the infeasible direction, whereas the kinematically admissible virtual displacement is along the admissible direction, these two vectors are normal to each other, and hence, the virtual work done by the reaction force vanishes. In contrast, the work done by the same reaction force under arbitrary virtual displacements does not.

The vanishing of the virtual work done by reaction forces under kinematically admissible virtual displacements has profound implications for applications of the principle of virtual work. The principle is repeated here: "a particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes *for all arbitrary virtual displacements*". Because this principle calls for the use of arbitrary virtual displacements, it is of crucial importance to treat reaction forces as externally applied forces. For instance, in example 9.3, the virtual work done by the reaction force must be included in the statement of the principle, as is done in eq. (9.11), because completely arbitrary virtual displacements are used.

Consider now a modified version of the principle of virtual work: "a particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes *for all arbitrary kinematically admissible virtual displacements*". Rather than considering completely arbitrary virtual displacements, only kinematically admissible virtual displacements are considered now. Because the virtual work done by the constraint forces vanishes for kinematically admissible virtual displacements, constraint forces are automatically eliminated from this statement of the principle of virtual work. This often simplifies the statement of the principle because fewer terms are involved. On the other hand, because the constraint forces are eliminated from the formulation, this modified principle will not yield the equations required to evaluate the reaction forces, which are often quantities of great interest.

As pointed out earlier, Newton's first law requires the sum of all forces to vanish for static equilibrium to be achieved. The "sum of all forces" involves all forces without distinction. While the principle of virtual work is shown to be identical to Newton's first law, this principle creates an important distinction between reaction forces stemming from kinematic constraints, and all other forces. Indeed, reaction forces, also called forces of constraint, can be completely eliminated from the formulation by using kinematically admissible virtual displacements.

All other forces, such as those generated by springs, gravity, friction, temperature, electric or magnetic fields, are of a physical origin. It is easy to recognize such forces because their description involves physical constants that can only be determined by experiment. For instance, the stiffness constant of a spring, the universal constant of gravitation appearing in gravity forces, or the friction coefficient appearing in Coulomb's friction law. All these forces are referred to as *natural forces*, which can be further differentiated into *internal* and *external* forces. *Internal forces* are natural forces arising from and reacted within the structural system under consideration, whereas *external forces* are natural forces that act on the system but stem from outside it; these forces are also called *externally applied loads*.

Example 9.4. Equilibrium of a particle sliding on a track

Consider once again a particle of mass m resting on a track, as shown in fig. 9.5. For this simple problem, the kinematically admissible direction is along axis $\bar{\imath}_2$, while the infeasible direction is along axis $\bar{\imath}_1$. The free body diagram in the right part of fig. 9.5 shows the forces acting on the particle. The reaction force, $-R\bar{\imath}_1$, acts in the infeasible direction, as expected.

In contrast with example 9.3, which uses completely arbitrary virtual displacements, kinematically admissible virtual displacements will be used here so that $\underline{s} = s_2\bar{\imath}_2$. The vanishing of the virtual work then implies

$$W = (mg\bar{\imath}_1 - R\bar{\imath}_1 + P\bar{\imath}_2 - F\bar{\imath}_2) \cdot s_2\bar{\imath}_2 = [P - F]s_2 = 0.$$

Because s_2 is an arbitrary quantity, the bracketed term must vanish, leading to $F = P$.

First, note that the reaction force, R, is eliminated from the formulation: the statement of the principle of virtual work becomes simply $(P - F)s_2 = 0$ for all values of s_2. The reaction force does not appear in this statement. It is also possible to apply external loads along the infeasible direction: for instance, in this problem, gravity loads act in the infeasible direction and are also eliminated from the formulation. Of course, if gravity acts along the kinematically admissible direction, *i.e.*, along the track, this force will remain in the statement of the principle. In contrast, reaction forces always act along the infeasible direction and hence, are always eliminated from the formulation.

Second, note that less information about the system is obtained. In example 9.3 that uses virtual displacements, two equations are obtained: $F = P$ and $R = mg$. In contrast, the use of kinematically admissible virtual displacements yields a single equation, $F = P$. On the other hand, the solution process is simpler and involves one single equation; however, no information about the reaction force is available.

Finally, it is shown here that the modified version of the principle of virtual work stating "a particle is in static equilibrium if and only if the virtual work done by the externally applied forces vanishes *for all arbitrary kinematically admissible virtual displacements,*" is not entirely correct. The vanishing of the virtual work for all kinematically admissible virtual displacements is a necessary condition, but it is not sufficient, because it does not guarantee equilibrium of the particle in the infeasible direction. Indeed, this latter condition, $R = mg$, is not recovered by the modified principle.

Example 9.5. Equilibrium of a particle on a curved track
Consider a particle of mass m constrained to move on a semi-circular track of radius R under the combined effects of gravity, friction, and a spring force, as depicted in fig. 9.6. Determine the equilibrium position of the particle and the forces acting on it in the equilibrium state.

The spring of stiffness constant k is pinned at point **C** located at coordinates $x_1 = c_1 R$ and $x_2 = c_2 R$ and its un-stretched length is zero. Force N is the reaction force acting on the particle due to its contact with the track and acts in direction $\bar{n}$, which is normal to the track. Force F is the force exerted by the track on the particle and acts in the tangential direction, $\bar{t}$; this force arises from friction between the particle and track.

The position of the particle on the track is conveniently given by angle θ. The unit vector tangent to the circular track is given by $\bar{t} = -\sin\theta\,\bar{\imath}_1 + \cos\theta\,\bar{\imath}_2$, whereas the normal to the track is $\bar{n} = -\cos\theta\,\bar{\imath}_1 - \sin\theta\,\bar{\imath}_2$. For this problem, the kinematically admissible direction is $\bar{t}$, and $\bar{n}$ the infeasible direction. In contrast with the previous example, the admissible direction is not a fixed direction in space, but instead, it depends on the position of the particle on the track, $\bar{t} = \bar{t}(\theta)$. The reaction force of magnitude N acts along the infeasible direction, as expected. The friction force of magnitude F acts in the admissible direction.

The force, $\underline{F}_s$, applied by the elastic spring to the particle is given by the spring stiffness constant times the distance between the particle and point **C** located at $(c_1 R, c_2 R)$ and is oriented in that same direction: $\underline{F}_s = kR[(c_1 - \cos\theta)\bar{\imath}_1 + (c_2 -$

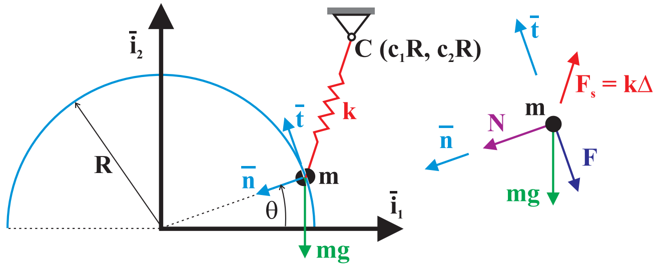

Fig. 9.6. Particle constrained to slide with friction on a circular track.

$\sin\theta)\bar{\imath}_2]$. This can be expressed in terms of admissible and infeasible directions, $\bar{t}$ and $\bar{n}$, respectively, as $\underline{F}_s = kR[(-c_1\sin\theta + c_2\cos\theta)\bar{t} + (1 - c_1\cos\theta - c_2\sin\theta)\bar{n}]$ where use is made of the following relationships: $\bar{\imath}_1 = -\sin\theta\ \bar{t} - \cos\theta\ \bar{n}$ and $\bar{\imath}_2 = \cos\theta\ \bar{t} - \sin\theta\ \bar{n}$.

An arbitrary virtual displacement of the form $\underline{s} = s_t\bar{t} + s_n\bar{n}$ is selected, where s_t and s_n are arbitrary numbers, and the virtual work done by the forces acting on the particle then becomes

$$\begin{aligned} W &= \{kR\left[(-c_1\sin\theta + c_2\cos\theta)\bar{t} + (1 - c_1\cos\theta - c_2\sin\theta)\bar{n}\right] + N\bar{n} - F\bar{t} \\ &\quad + mg(-\cos\theta\bar{t} + \sin\theta\bar{n})\} \cdot (s_t\bar{t} + s_n\bar{n}) \\ &= \left[kR(-c_1\sin\theta + c_2\cos\theta) - F - mg\cos\theta\right] s_t \\ &\quad + \left[kR(1 - c_1\cos\theta - c_2\sin\theta) + N + mg\sin\theta\right] s_n. \end{aligned}$$

Because the virtual work must vanish for arbitrary s_t and s_n, the two bracketed terms must vanish, leading to the two equilibrium equations of the problem,

$$F = \quad kR(\ \ -c_1\sin\theta + c_2\cos\theta) - mg\cos\theta, \tag{9.12a}$$

$$N = -kR(1 - c_1\cos\theta - c_2\sin\theta) - mg\sin\theta. \tag{9.12b}$$

This forms a set of two equations for the three unknowns of the problem: the reaction force, N, the friction force, F, and the equilibrium position of the particle, θ.

One additional equation is required to solve the problem. Coulomb's law of static friction requires the friction force to be smaller than the normal contact force multiplied by the static friction coefficient, μ_s, *i.e.*, $|F| \leq \mu_s|N|$. Substituting the friction and normal forces from eqs. (9.12a) and (9.12b), respectively, leads to

$$\begin{aligned} kR(-c_1\sin\theta + c_2\cos\theta) - mg\cos\theta \\ \leq \pm\mu_s\left[-kR(1 - c_1\cos\theta - c_2\sin\theta) - mg\sin\theta\right]. \end{aligned}$$

This equation can be solved to find two solutions, θ_ℓ and θ_u: the particle is in equilibrium for all configurations, θ, such that $\theta_\ell \leq \theta \leq \theta_u$.

Next, kinematically admissible virtual displacements of the form $\underline{s} = s_t\bar{t}$ will be selected, where s_t is an arbitrary value. The virtual work done by the forces acting on the particle then becomes

$$W = \{kR\,[(-c_1 \sin\theta + c_2 \cos\theta)\bar{t} + (1 - c_1 \cos\theta - c_2 \sin\theta)\bar{n}] + N\bar{n} - F\bar{t} \\ + mg(-\cos\theta\bar{t} + \sin\theta\bar{n})\} \cdot s_t\bar{t} \\ = [kR(-c_1 \sin\theta + c_2 \cos\theta) - F - mg\cos\theta]\, s_t.$$

Because the virtual work must vanish for all arbitrary s_t, the bracketed term must vanish, yielding a single equilibrium equation of the problem, which is the same as eq. (9.12a) above. As expected, the normal reaction force, N, is eliminated from the formulation. The problem still features three unknowns, N, F and θ, and the addition of the static friction law provides a second equation for the problem. Clearly, the principle of virtual work with kinematically admissible virtual displacements does not provide enough equations to solve this problem. This is because the static friction law establishes a relationship between friction and normal forces. By eliminating the normal contact force from the formulation, the use of kinematically admissible virtual displacements yields too little information to solve the problem.

Note that if friction is neglected, the friction force will vanish, $F = 0$, and the single equation stemming from the use of kinematically admissible virtual displacements yields the solution of the problem, $kR(-c_1 \sin\theta + c_2 \cos\theta) - mg\cos\theta = 0$, or $\tan\theta = (c_2 - mg/kR)/c_1$.

In summary, when using kinematically admissible virtual displacements, the principle of virtual work yields a reduced set of equilibrium equations from which the forces of constraints are eliminated. This often greatly simplifies and streamlines the solution process. In some cases, however, too few equations will be obtained, giving the impression that the problem cannot be solved. Arbitrary virtual displacements, *i.e.*, virtual displacements that violate the kinematic constraints must then be used to obtain the missing equations of equilibrium, which correspond to the projection of Newton's first law along the infeasible directions.

9.3.3 Use of infinitesimal displacements as virtual displacements

In the previous sections, three-dimensional virtual displacements are denoted $\underline{s} = s_1\bar{\imath}_1 + s_2\bar{\imath}_1 + s_3\bar{\imath}_3$, where s_1, s_2, and s_3 are arbitrary numbers. In view of the fundamental role they play in energy and variational principles, a special notation is commonly used to denote virtual displacements,

$$\underline{s} = \delta\underline{u}. \tag{9.13}$$

The symbol "δ" is placed in front of the displacement vector, $\underline{u}$, to indicate that it should be understood as a virtual displacement. Similarly, the virtual work done by a force undergoing a virtual displacement will be denoted δW to distinguish it from the real work done by the same force undergoing real displacements. The new notation changes nothing of the special nature of virtual displacements which are fictitious displacements that do not alter the applied forces.

In many applications of the principle of virtual work, it will also be convenient to use virtual displacements of infinitesimal magnitude. Because virtual displacements are of arbitrary magnitude, virtual displacements of infinitesimal magnitude qualify as valid virtual displacements. The infinitesimal magnitude of virtual displacements is a convenience that often simplifies algebraic developments, but is by no means a requirement.

Displacement dependent forces

A key simplification arising from the use of virtual displacements of infinitesimal magnitude is that displacement dependent forces automatically remain unaltered by their application, as illustrated in the following example.

Example 9.6. Equilibrium of a particle connected to an elastic spring
Consider a particle connected to an elastic spring, as illustrated in fig. 9.7. This is the same problem treated in example 9.2.

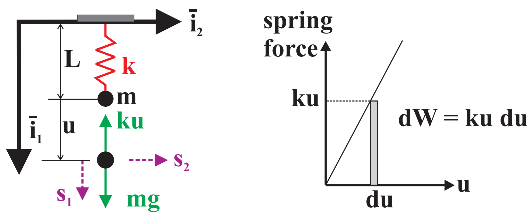

Fig. 9.7. Use of a differential displacement as a virtual displacement.

The principle of virtual work requires that

$$\delta W = (mg\bar{\imath}_1 - ku\bar{\imath}_1) \cdot (\delta u\bar{\imath}_1 + \delta v\bar{\imath}_2) = [mg - ku]\delta u = 0,$$

for all virtual displacements, δu, where the virtual displacements must leave the forces applied to the particle unchanged. Consider now a virtual displacement of infinitesimal magnitude, $\delta u = \mathrm{d}u$. The virtual work done by this virtual displacement of infinitesimal magnitude is still given by eq (9.10) as

$$\int_u^{u+\mathrm{d}u} -ku\,\mathrm{d}u = \left[-\frac{1}{2}ku^2\right]_u^{u+\mathrm{d}u} = -k u\mathrm{d}u - \frac{1}{2}k(\mathrm{d}u)^2 = -ku\,\mathrm{d}u,$$

where the last equality follows from neglecting the higher order differential quantity. The virtual work is now equal to the real work done by an infinitesimal displacement of magnitude $\mathrm{d}u = \delta u$. The right part of fig. 9.7 illustrates the differential work, $\mathrm{d}W$, for a displacement of infinitesimal magnitude.

Rigid bodies

Next, the close relationship between infinitesimal displacements and virtual displacements of infinitesimal magnitude will be explored further in the context of rigid bodies. Consider two arbitrary points, **P** and **Q**, of a rigid body. When the rigid body undergoes arbitrary motions, the velocities of these two points are not independent and must satisfy the following well-known equation from rigid body dynamics, $\underline{v}_P = \underline{v}_Q + \underline{\omega} \times \underline{r}_{QP}$, where $\underline{v}_P$ and $\underline{v}_Q$ are the velocities of points **P** and **Q**, respectively, $\underline{\omega}$ is the angular velocity of the rigid body, and $\underline{r}_{QP}$ the position vector of point **P** with respect to **Q**. This relationship is now written as $\mathrm{d}\underline{u}_P/\mathrm{d}t = \mathrm{d}\underline{u}_Q/\mathrm{d}t + (\underline{\mathrm{d}\psi}/\mathrm{d}t) \times \underline{r}_{QP}$, where $\mathrm{d}\underline{u}_P$ and $\mathrm{d}\underline{u}_Q$ are the infinitesimal displacement vectors of points **P** and **Q**, respectively, and $\underline{\mathrm{d}\psi}$ is the infinitesimal rotation vector for the rigid body. After multiplication by $\mathrm{d}t$, the infinitesimal displacements are found to satisfy the following equation, $\mathrm{d}\underline{u}_P = \mathrm{d}\underline{u}_Q + \underline{\mathrm{d}\psi} \times \underline{r}_{QP}$.

Because virtual displacements can be of infinitesimal magnitude, it is possible to write

$$\delta\underline{u}_P = \delta\underline{u}_Q + \underline{\delta\psi} \times \underline{r}_{QP}. \tag{9.14}$$

where $\delta\underline{u}_P$ and $\delta\underline{u}_Q$ are the virtual displacement vectors of arbitrary points **P** and **Q**, respectively, and $\underline{\delta\psi}$ is the virtual rotation vector for the rigid body. Equation (9.14) describes the field of kinematically admissible virtual displacements for a rigid body. Indeed, these virtual displacements satisfy the kinematic constraints for two points belonging to the same rigid body.

The discussion of the previous paragraph underlines the close relationship between infinitesimal quantities, denoted with symbol "d," and virtual quantities, denoted with symbol "δ." To obtain eq. (9.14) symbol "d" is replaced by "δ" in the last step of the reasoning. While this approach is correct, it must be emphasized that virtual displacements remain fictitious displacements, whereas infinitesimal displacements are real displacements. Furthermore, virtual displacements leave the forces unchanged, whereas no such requirement applies for real infinitesimal displacements. Finally, admissible virtual displacements are allowed to violate the kinematic constraints, whereas real displacement are not.

It is also important to note that eq. (9.14) shows that an infinitesimal rotation, $\underline{\delta\psi}$, is a vector quantity. This is not the case for finite rotations, as discussed in dynamics textbooks [3, 4].

Using virtual displacements of infinitesimal magnitude greatly simplifies the treatment of many problems. In the mathematical treatment of virtual quantities, a branch of mathematics called *calculus of variations,* virtual quantities are systematically assumed to be of infinitesimal magnitude [5, 6].

9.3.4 Principle of virtual work for a system of particles

Consider the system of N particles depicted in fig. 9.8; this problem is treated in section 9.2.1 using the classical Newtonian approach. Particle i is subjected to an external force, $\underline{F}_i$, and to $N-1$ interaction forces, $\underline{f}_{ij}$, $j = 1, 2, \ldots, N$, $j \neq i$. For particle i, the virtual work, denoted δW_i, done by all applied forces when subjected to a virtual displacement, $\delta\underline{u}_i$, is

$$\delta W_i = (\underline{F}_i + \sum_{j=1, j\neq i}^{N} \underline{f}_{ij}) \cdot \delta\underline{u}_i. \tag{9.15}$$

According to the principle of virtual work, this virtual work must vanish for all virtual displacements, $\delta \underline{u}_i$. The principle can be applied to each particle independently, leading to $\delta W_i = 0$, where δW_i is given by eq. (9.15), for $i = 1, 2, \ldots N$.

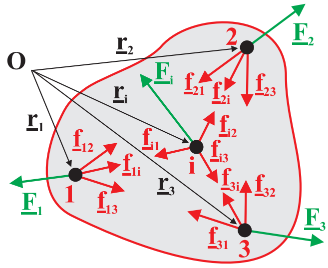

Fig. 9.8. A system of particles.

Because the virtual work must vanish for each particle independently, the sum of the virtual work for all particles must also vanish, leading to the following statement of the principle of virtual work for a system of N particles: a system of particle is in static equilibrium if and only if the virtual work,

$$\delta W = \sum_{i=1}^{N} \left\{ \left[\underline{F}_i + \sum_{j=1, j \neq i}^{N} \underline{f}_{ij} \right] \cdot \delta \underline{u}_i \right\}, \tag{9.16}$$

vanishes for all virtual displacements, $\delta \underline{u}_i$, $i = 1, 2, \ldots, N$. Because the N virtual displacements are all arbitrary and independent, the bracketed term in eq. (9.16) must vanish for $i = 1, 2, \ldots, N$, leading to equilibrium equations that are identical to those obtained from Newton's first law, eq. (9.2).

Because each of the N virtual displacement vectors involves three scalar components, the principle of virtual work yields $3N$ scalar equations for a system of N particles; all must be satisfied for the system to be in static equilibrium. The system is said to present $3N$ *degrees of freedom*. For a two-dimensional, or planar system, the number of scalar equations would reduce to $2N$, *i.e.*, $2N$ degrees of freedom.

The above developments have shown, once again, that the principle of virtual work is entirely equivalent to Newton's first law, and gives the necessary and sufficient conditions for the static equilibrium of the system. Equilibrium is the most fundamental requirement in structural analysis, and must always be satisfied. This means that Newton's first law, or the principle of virtual work since they are both equivalent, always applies. The system of particles considered above is very general; it could represent a rigid body, a flexible body deforming elastically or plastically, a fluid, or a planetary system. Yet, the same equilibrium requirements apply equally to all systems.

Internal and external virtual work

Eq. (9.16) also affords another important interpretation. The forces acting on the system are separated into two groups, the externally applied forces, $\underline{F}_i$, and the internal forces, $\underline{f}_{ij}$. The words "internal" and "external" should be understood with respect to the system of particles. *Internal forces* act and are reacted within the system, whereas *external forces* act on the system but are reacted outside the system. The virtual work done by the external and internal forces, denoted δW_E and δW_I, respectively, are defined as

$$\delta W_E = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{u}_i, \tag{9.17a}$$

$$\delta W_I = \sum_{i=1}^{N} \left[\sum_{j=1, j\neq i}^{N} \underline{f}_{ij} \right] \cdot \delta \underline{u}_i, \tag{9.17b}$$

respectively. With these definitions, eq. (9.16) is becomes

$$\delta W = \delta W_E + \delta W_I = 0, \tag{9.18}$$

for all arbitrary virtual displacements. This leads to the principle of virtual work for a system of particles.

Principle 4 (Principle of virtual work) *A system of particles is in static equilibrium if and only if the sum of the virtual work done by the internal and external forces vanishes for all arbitrary virtual displacements.*

Finally, it is interesting to note that because the virtual displacements are arbitrary, it is possible to choose them to be the actual displacements, and eq. (9.18) then implies

$$W = W_E + W_I = 0, \tag{9.19}$$

where W_E and W_I are the actual work done by the external and internal forces, respectively. Equation (9.19) states that *if a system of particles is in static equilibrium, the sum of the work done by the internal and external forces vanishes.*

Euler's laws

The $3N$ scalar equations implied by the vanishing of the virtual work expressed in eq. (9.16) are often cumbersome to use because they all involve the interaction forces between the particles of the system. To obtain equations that are more convenient to use, a special set of virtual displacements will be selected.

Inspired by eq. (9.14), the virtual displacement of particle i is written as

$$\delta \underline{u}_i = \delta \underline{u}_O + \underline{\delta \psi} \times \underline{r}_i, \tag{9.20}$$

where $\delta \underline{u}_O$ is the virtual displacement of an arbitrary point **O**, see fig. 9.8, $\underline{\delta \psi}$ the virtual rotation vector, and $\underline{r}_i$ the relative position vector of particle i with respect to point **O**. The virtual displacements of all particles are now expressed in terms of a virtual translation of the rigid body, $\delta \underline{u}_O$, and its virtual rotation, $\underline{\delta \psi}$, both chosen to be of infinitesimal magnitude. This corresponds to 6 independent virtual displacement components, far fewer than the original $3N$. The virtual work done by all forces acting on the system under these virtual displacement is

$$\delta W = \sum_{i=1}^{N} \left\{ \left[\underline{F}_i + \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} \right] \cdot (\delta \underline{u}_O + \underline{\delta \psi} \times \underline{r}_i) \right\} = \left(\sum_{i=1}^{N} \underline{F}_i \right) \cdot \delta \underline{u}_O$$
$$+ \left(\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} \right) \cdot \delta \underline{u}_O + \sum_{i=1}^{N} \underline{F}_i \cdot (\underline{\delta \psi} \times \underline{r}_i) + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \underline{f}_{ij} \cdot (\underline{\delta \psi} \times \underline{r}_i).$$

The last two terms of this expression can be simplified by using the triple scalar product identity: $\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{b} \cdot (\underline{c} \times \underline{a})$, which holds for any three vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$. The above equation now becomes

$$\delta W = \delta \underline{u}_O \cdot \left(\sum_{i=1}^{N} \underline{F}_i \right) + \delta \underline{u}_O \cdot \left(\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \underline{f}_{ij} \right) + \underline{\delta \psi} \cdot \left(\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i \right) + \underline{\delta \psi} \cdot \left(\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \underline{r}_i \times \underline{f}_{ij} \right).$$

In view of eqs. (9.3) and (9.5), the terms in the second and last sets of parenthesis now vanish, reducing the expression to

$$\delta W = \delta \underline{u}_O \cdot \left[\sum_{i=1}^{N} \underline{F}_i \right] + \underline{\delta \psi} \cdot \left[\sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i \right].$$

Because the virtual work must vanish for all virtual displacements, $\delta \underline{u}_O$, and virtual rotations, $\underline{\delta \psi}$, the two bracketed terms must also vanish. Clearly, these two equations are identical to Euler's first and second laws obtained directly from Newtonian arguments, see eqs. (9.4) and (9.6).

These two vector equations are necessary but not sufficient conditions to guarantee static equilibrium. Indeed, static equilibrium requires a total of N vector equations to be satisfied; eqs. (9.4) and (9.6) are two linear combinations of those N equations. Only two vector equations are obtained from the principle of virtual work because the virtual displacement field, eq. (9.20), selected for the rigid body involves a single virtual displacement vector, $\delta \underline{u}_O$, and a single virtual rotation vector, $\underline{\delta \psi}$.

9.4 Principle of virtual work applied to mechanical systems

In the previous section, the principle of virtual work is discussed in a rather theoretical setting with applications to single particles and systems of particles. In the present section, the power and efficiency of the same principle will be demonstrated when applied to mechanical systems.

A rigid body is a particular case of the general system of N particles considered in the previous section. The configuration of a rigid body is determined by six parameters: the three components of the position vector of one of its points, and the three rotations that determine its orientation. Equivalently, a "rigid body motion," which is the only motion a body can undergo while remaining rigid, consists of a three-dimensional translation and a three-dimensional rotation. For a rigid body, the virtual displacement field given by eq. (9.20) is kinematically admissible, because it represents the superposition of a translation and a rotation, both in three dimensions.

This kinematically admissible virtual displacement field yields two vector equations, eqs. (9.4) and (9.6), or six scalar equations, which are just enough to determine the equilibrium configuration of the body. The internal forces, $\underline{f}_{ij}$, in the rigid body are the forces of constraint that maintain its shape unchanged and are entirely eliminated from the formulation, as expected, because a kinematically admissible displacement field is used in the application of the principle of virtual work.

When considering two-dimensional or planar mechanisms, the kinematically admissible displacement field defined by eq. (9.20) reduces to

$$\delta \underline{u}_i = \delta \underline{u}_O + \delta\phi \bar{\imath}_3 \times \underline{r}_i. \tag{9.21}$$

The three-dimensional virtual rotation vector, $\underline{\delta\psi}$, now simply becomes $\delta\phi\, \bar{\imath}_3$, where $\delta\phi$ is a virtual rotation of infinitesimal magnitude about axis $\bar{\imath}_3$. The planar mechanism is assumed to be entirely contained in plane $(\bar{\imath}_1, \bar{\imath}_2)$, and hence, rotations are only possible about axis $\bar{\imath}_3$.

Example 9.7. Equilibrium of a lever

Consider the simple lever subjected to two vertical end forces, F_a and F_b, acting at distances a and b, respectively, from the fulcrum, as shown in fig. 9.9.

First, this problem will be solved using the classical equations of statics, considering the free body diagram appearing in the right part of fig. 9.9. The equilibrium of forces in the horizontal and vertical directions yields $H = 0$ and $V = F_a + F_b$, respectively, whereas equilibrium of moments about point **A** leads to $aV \cos\phi = (a + b)F_b \cos\phi$. The Newtonian approach requires the explicit consideration of the horizontal and vertical reaction forces, H and V, respectively, at the lever's fulcrum. Solution of these equations leads to the familiar equilibrium conditions for a lever, $aF_a = bF_b$, $H = 0$, and $V = F_a + F_b$.

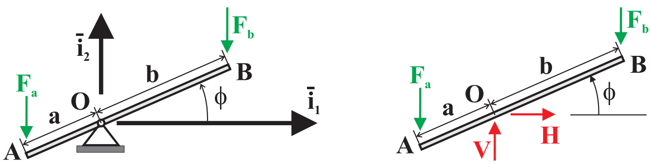

Fig. 9.9. Simple lever acted upon by two vertical end forces.

It is possible to eliminate the reaction forces from the formulation by writing a single moment equilibrium equation about the lever's fulcrum: $aF_a \cos\phi = bF_b \cos\phi$. Because the lines of action of the reaction forces pass through the fulcrum, they are automatically eliminated from the moment equilibrium equation.

Next, this single degree of freedom problem will be solved using the principle of virtual work. Because the lever is fixed at point **O**, a kinematically admissible virtual displacement field simply becomes $\delta \underline{u}_i = \delta\phi \bar{\imath}_3 \times \underline{r}_i$. The translation term, $\delta \underline{u}_O$, appearing in eq. (9.21), is omitted because a translation of point **O** violates the kinematic constraint at this point. The virtual displacement of point **A** now becomes $\delta \underline{u}_A = \delta\phi \bar{\imath}_3 \times \underline{r}_{OA}$, where $\underline{r}_{OA}$ is the position vector of point **A** relative to point **O**. Simple vector algebra then yields $\delta \underline{u}_A = a(\sin\phi \bar{\imath}_1 - \cos\phi \bar{\imath}_2)\delta\phi$. A similar reasoning reveals that $\delta \underline{u}_B = b(-\sin\phi \bar{\imath}_1 + \cos\phi \bar{\imath}_2)\delta\phi$. The virtual work is now

$$\delta W_E = (-F_a \bar{\imath}_2) \cdot \delta \underline{u}_A + (-F_b \bar{\imath}_2) \cdot \delta \underline{u}_B = \delta\phi \left[aF_a \cos\phi - bF_b \cos\phi \right].$$

The reaction forces are eliminated from the formulation because the virtual displacement is kinematically admissible, *i.e.*, it vanishes at point **O**, resulting in the vanishing of the virtual work done by the reaction forces at that point. Because the virtual displacement field is also compatible with the kinematic conditions required for the body to remain rigid (the virtual displacement field consists of a single rotation), all the internal forces that enforce the rigidity of the body are also eliminated.

Because the virtual work must vanish for all arbitrary virtual rotations, $\delta\phi$, the bracketed term in the above equation must vanish, leading to $(aF_a - bF_b)\cos\phi = 0$. The two solutions are $aF_a = bF_b$, the usual lever equilibrium equation, and $\cos\phi = 0$, which corresponds to the lever being in a vertical position. In this latter case, the lever is in equilibrium for any set of applied vertical forces.

As discussed earlier, it is also possible to use a virtual displacement field that violates the kinematic conditions, such as that given by eq. (9.21). The virtual displacement, $\delta \underline{u}_O = \delta u_1 \bar{\imath}_1 + \delta u_2 \bar{\imath}_2$, is a virtual displacement of point **O**, the lever's fulcrum. The virtual displacements of points **A** and **B** now become $\delta \underline{u}_A = \delta \underline{u}_O + a(\sin\phi \bar{\imath}_1 - \cos\phi \bar{\imath}_2)\delta\phi$, and $\delta \underline{u}_B = \delta \underline{u}_O + b(-\sin\phi \bar{\imath}_1 + \cos\phi \bar{\imath}_2)\delta\phi$, respectively. The virtual work done by the externally applied forces is now

$$\begin{aligned}\delta W_E &= (-F_a \bar{\imath}_2) \cdot \delta \underline{u}_A + (-F_b \bar{\imath}_2) \cdot \delta \underline{u}_B + (H \bar{\imath}_1 + V \bar{\imath}_2) \cdot \delta \underline{u}_O \\ &= \delta u_1 [H] + \delta u_2 [V - F_a - F_b] + \delta\phi [aF_a \cos\phi - bF_b \cos\phi].\end{aligned}$$

Because the virtual work done by the reaction forces at the fulcrum does not vanish, is must be included in the formulation. Since the virtual work must vanish for all virtual displacements, δu_1 and δu_2, and rotation, $\delta\phi$, the three bracketed terms must vanish, leading to three equilibrium equations identical to those obtained using the Newtonian approach.

This underlines, once again, the complete equivalence of the principle of virtual work and Newton's first law. The use of a kinematically admissible virtual displacement field automatically eliminates the reactions forces when using the principle of virtual work. Although it is sometimes possible to achieve this elimination by a judicious choice of the point about which moment equilibrium equations are written in Newton's approach, the systematic approach stemming from the use of the principle of virtual work is more efficient and convenient.

Example 9.8. Block and tackle system

Consider the familiar two-pulley block and tackle shown in fig. 9.10. Determine the rope force, F, required to lift a weight, P. The system possesses a single degree of freedom defined by the rotation angle, ϕ, of the upper pulley.

Consider a virtual rotation, $\delta\phi$, of the pulley. The resulting virtual motion of the point of application of force F is $\delta b = R\delta\phi$, where R is the pulley's radius. The resulting motion of the lower block is $\delta a = -R\delta\phi/2$. Because the virtual rotation is kinematically admissible, the only forces that perform work are the externally applied forces, F and P; the reaction forces need not be considered. The principle of virtual work is now simply

$$\delta W = F\delta b + P\delta a = FR\delta\phi - PR\delta\phi/2 = R\left[F - P/2\right]\delta\phi = 0.$$

Because the virtual work must vanish for all $\delta\phi$, the bracketed term must vanish, yielding the equilibrium equation of the system, $F = P/2$.

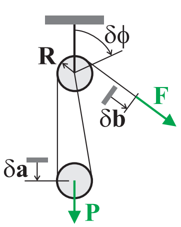

Fig. 9.10. Simple two-pulley block and tackle system.

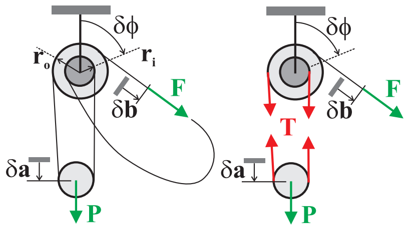

Fig. 9.11. Differential pulley or "chain hoist".

Example 9.9. Differential pulley system

A differential pulley system is depicted in fig. 9.11 and is the basis for the common shop chain hoist. In this device, the two upper pulleys, of radii r_i and r_o, are constrained to rotate together about a common axis. The inextensible chain is not allowed to slip around the upper pulleys. The system features a single degree of freedom, defined by the rotation angle, ϕ, of the upper pulleys assembly.

A virtual rotation, $\delta\phi$, of the upper pulleys causes a virtual displacement of the point of application of force F of $\delta b = r_o\delta\phi$. The virtual motion of the lower pulley is $\delta a = -(r_o\delta\phi - r_i\delta\phi)/2$. This can be explained by noting that for a virtual rotation, $\delta\phi$, of the upper pulleys, the length of chain below these pulleys is decreased by $r_o\delta\phi$ (the length winding on the pulley of radius r_o), but at the same time, is increased by $r_i\delta\phi$ (the length unwinding from the pulley of radius r_i), and the lower pulley rotates to distribute the net shortening between the two segments of chain.

The principle of virtual work now requires

$$\delta W_E = F\delta b + P\delta a = Fr_o\delta\phi - P(r_o - r_i)\delta\phi/2 = \left[Fr_o - P(r_o - r_i)/2\right]\delta\phi = 0,$$

for all virtual rotations, $\delta\phi$, and hence, the bracketed term must vanish, revealing the equilibrium condition of the system $F = (1 - r_i/r_0)P/2$. The mechanical advantage of the differential pulley increases as radii r_i and r_0 approach equal values, at which point force F then vanishes. The load cannot be raised, however, because the amounts of chain winding around the pulley of radius r_0 and unwinding from the pulley of radius r_i become equal.

A kinematically admissible virtual rotation, $\delta\phi$, is used, and because the chain is inextensible, the virtual rotation of the upper pulleys determines the motion of the lower pulley. This is why the system presents a single degree of freedom.

It is also possible to use arbitrary virtual displacements that violate the kinematic constraint imposed by the inextensibility of the chain. Let the virtual displacement of the lower pulley, δa, be independent of the virtual rotation of the upper pulley, $\delta\phi$. In this case, the tension in the chain, the force of constraint that enforces its inextensibility, will perform virtual work. Referring to the right part of fig. 9.11, the virtual work done by the forces acting on the system is

$$\begin{aligned}\delta W_E &= (Fr_o\delta\phi - Tr_o\delta\phi + Tr_i\delta\phi) + (P\delta a - 2T\delta a)\\ &= [Fr_o - Tr_o + Tr_i]\,\delta\phi + [P - 2T]\,\delta a = 0.\end{aligned}$$

The terms inside the two sets of parenthesis represent the virtual work done by the forces acting on the upper and lower pulleys, respectively. Because the virtual rotation, $\delta\phi$, and virtual displacement, δa, are both arbitrary and independent, the two bracketed term must vanish, yielding the equations of equilibrium of the system, $Fr_o = (r_o - r_i)T$, and $P = 2T$. Eliminating the tension in the chain leads to $F = (1 - r_i/r_0)P/2$, as before. Additionally, the tension in the chain, $T = P/2$, is also determined.

Example 9.10. The crank-slider mechanism

Consider the crank-slider mechanism depicted in fig. 9.12. The crank of length R is actuated by a torque, Q, and the link of length L transforms the rotary motion of the crank into a linear motion of the slider. Force F is applied to the slider. The crank angle is denoted ϕ and is measured positive in the counterclockwise direction. Determine the relationship between the torque, Q, and force, F.

For kinematically admissible virtual displacements, the principle of virtual work states that $\delta W_E = -Q\delta\phi - F\delta x = 0$, where x is the distance from the crank axis to the slider. The virtual displacement, δx, and virtual rotation, $\delta\phi$, are *arbitrary but not independent*; consequently, nothing can be concluded from the above statement.

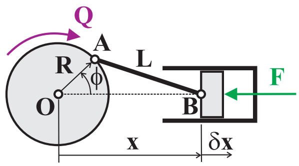

Fig. 9.12. The crank-slider mechanism.

The geometry of the problem links the two variables, x and ϕ, as shown in fig. 9.12. Projections of segments **OA** and **AB** onto the horizontal yield $x = R\cos\phi + \sqrt{L^2 - R^2\sin^2\phi}$. Taking a differential of this equation reveals the relationship between infinitesimal increments in variables x and ϕ as $\mathrm{d}x = -R\sin\phi\,\mathrm{d}\phi - R^2\sin\phi\cos\phi\,\mathrm{d}\phi/\sqrt{L^2 - R^2\sin^2\phi}$. Since virtual displacements are of arbitrary magnitude, it is possible the select virtual displacements of infinitesimal magnitude, *i.e.*, $\delta x = \mathrm{d}x$ and $\delta\phi = \mathrm{d}\phi$. Kinematically admissible infinitesimal virtual displacements must then satisfy the following relationship

$$\delta x = -\left(1 + \frac{R\cos\phi}{\sqrt{L^2 - R^2\sin^2\phi}}\right) R\sin\phi\,\delta\phi.$$

The principle of virtual work now becomes

$$\delta W = -Q\delta\phi - F\delta x = -\left[Q - \left(1 + \frac{R\cos\phi}{\sqrt{L^2 - R^2\sin^2\phi}}\right) FR\sin\phi\right]\delta\phi = 0.$$

Because the virtual rotation, $\delta\phi$, is arbitrary, the bracketed term must vanish, yielding the desired relationship between the torque and force as

$$\frac{Q}{FR} = \left(1 + \frac{\cos\phi}{\sqrt{L^2/R^2 - \sin^2\phi}}\right)\sin\phi.$$

This expression yields the torque developed about the crank axis at a specific angular position, ϕ, of the crank in response to a force, F, applied to the slider.

In the Newtonian formulation, all reaction forces acting on the system must be considered. They include the horizontal and vertical components of the reaction forces at points **O**, **A**, and **B**, and the vertical reaction of the ground on the slider. The use of kinematically admissible virtual displacements with the principle of virtual work automatically eliminates all these forces from the formulation. In this example, only the torque applied to the crank and the force acting on the slider appear in the formulation.

9.4.1 Generalized coordinates and forces

The virtual work done by a force is defined as the scalar product of the force by a virtual displacement. When the principle of virtual work is introduced for a single particle in section 9.3.1, the virtual work is computed according to the definition: $\delta W = \underline{F} \cdot \delta\underline{u} = F_1\delta u_1 + F_2\delta u_2 + F_3\delta u_3$, where the force and virtual displacement vectors are represented by their components in a common orthonormal basis, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$, as $\underline{F} = F_1\bar{\imath}_1 + F_2\bar{\imath}_2 + F_3\bar{\imath}_3$ and $\delta\underline{u} = \delta u_1\bar{\imath}_1 + \delta u_2\bar{\imath}_2 + \delta u_3\bar{\imath}_3$, respectively.

In many cases, however, it is not convenient to work with Cartesian coordinates. Consider, for instance, the crank-slider mechanism shown in fig. 9.12 and treated in example 9.10: the motion of the system is naturally expressed in terms of the crank angle, ϕ, and piston translation, x. Similarly, when dealing with pulleys in examples 9.8 and 9.9, the most natural way to describe the configuration of the system is in terms of the rotation angles of the pulleys.

In general, the configuration of a system will be represented in terms of N variables, called *generalized coordinates* and denoted $q_1, q_2, \ldots q_N$. These variables could be angles, relative motions, Cartesian coordinates, or a mixture thereof, hence the term "generalized coordinates." The displacement vector of any point of the system will be a function of these generalized coordinates, $\underline{u} = \underline{u}(q_1, q_2, \ldots q_N)$. Virtual displacements can now be evaluated using the chain rule for derivatives

$$\delta\underline{u} = \frac{\partial\underline{u}}{\partial q_1}\delta q_1 + \frac{\partial\underline{u}}{\partial q_2}\delta q_2 + \ldots + \frac{\partial\underline{u}}{\partial q_N}\delta q_N.$$

The virtual work done by a force, $\underline{F}$, undergoing this virtual displacement is found as

$$\delta W = \underline{F} \cdot \delta \underline{u} = \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_1} \right) \delta q_1 + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_2} \right) \delta q_2 + \ldots + \left(\underline{F} \cdot \frac{\partial \underline{u}}{\partial q_N} \right) \delta q_N.$$

It is now convenient to define *generalized forces* as follows

$$Q_i = \underline{F} \cdot \frac{\partial \underline{u}}{\partial q_i}, \quad i = 1, 2, \ldots, N, \tag{9.22}$$

and the expression for the virtual work simply becomes

$$\delta W = Q_1 \delta q_1 + Q_2 \delta q_2 + \ldots + Q_N \delta q_N = \sum_{i=1}^{N} Q_i \delta q_i. \tag{9.23}$$

This result helps explain the term "generalized forces" used to denote the quantities defined by eq. (9.22); the virtual work is simply the product of the generalized forces and the corresponding generalized virtual displacements.

The above development is presented for a generic force, $\underline{F}$, which can be an externally applied load or an internal force. To distinguish between the two cases, the following notation is used

$$\delta W_I = \sum_{i=1}^{N} Q_i^I \delta q_i, \tag{9.24a}$$

$$\delta W_E = \sum_{i=1}^{N} Q_i^E \delta q_i, \tag{9.24b}$$

where Q_i^I and Q_i^E are the generalized forces associated with the internal forces and externally applied loads, respectively.

The principle of virtual work, expressed by eq. (9.18), can now be reformulated as

$$\delta W_I + \delta W_E = \sum_{i=1}^{N} Q_i^I \delta q_i + \sum_{i=1}^{N} Q_i^E \delta q_i = \sum_{i=1}^{N} \left[Q_i^I + Q_i^E \right] \delta q_i = 0,$$

for all virtual generalized displacements, δq_i. Clearly, because the virtual generalized displacements, δq_i, are arbitrary, each of the N bracketed terms under the summation sign must vanish, leading to

$$Q_i^I + Q_i^E = 0, \quad i = 1, 2, \ldots, N. \tag{9.25}$$

This equation represents yet another statement of the principle of virtual work.

As discussed in section 9.3.2, the principle of virtual work can be used with either arbitrary or kinematically admissible virtual displacements. Similarly, the present statement of the principle can be used with either arbitrary or kinematically admissible virtual generalized coordinates. When using arbitrary virtual generalized coordinates, the virtual work done by the reaction forces must be included in the evaluation of the virtual work done by the external forces; this implies that the generalized forces associated with the reaction forces must be included in Q_i^E. If the virtual generalized coordinates are kinematically admissible, the reaction forces are eliminated from the formulation.

To illustrate the concepts presented above, consider the pendulum with a torsional spring depicted in fig. 9.13.

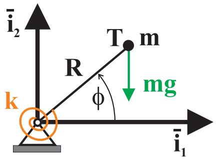

Fig. 9.13. Pendulum with torsional spring.

A rigid arm of length R connects the mass, m, to a pinned support point where a torsional spring of stiffness constant k acts between ground and the rod. The torsional spring is un-stretched when the arm is horizontal. The mass is subjected to gravity loading. The configuration of the system is conveniently represented by the angular position, ϕ, of the arm and is selected to be the single generalized coordinate for this one degree of freedom problem.

Consider first the virtual work done by the gravity load, $\delta W_E = -mg\bar{\imath}_2 \cdot \delta \underline{u}_T$, where $\delta \underline{u}_T$ is the virtual displacement at point **T**. Since $\underline{u}_T = R(\cos\phi\,\bar{\imath}_1 + \sin\phi\,\bar{\imath}_2)$, an infinitesimal virtual displacement of the same quantity is $\delta \underline{u}_T = R(-\sin\phi\,\bar{\imath}_1 + \cos\phi\,\bar{\imath}_2)\delta\phi$. It now follows that $\delta W_E = -mgR\cos\phi\,\delta\phi$, and by defining the generalized force as $Q_\phi^E = -mgR\cos\phi$, the virtual work becomes $\delta W_E = Q_\phi^E \delta\phi$. The same result can be obtained in a more expeditious manner by using eq. (9.22) to find $Q_\phi^E = -mg\bar{\imath}_2 \cdot \partial \underline{u}_T/\partial\phi = -mg\bar{\imath}_2 \cdot R(-\sin\phi\,\bar{\imath}_1 + \cos\phi\,\bar{\imath}_2) = -mgR\cos\phi$.

An even simpler interpretation is as follows. Because the virtual displacement is a rotation, $\delta\phi$, it must be multiplied by a moment to yield a virtual work; hence, the generalized force is simply the moment of the gravity load, $-mgR\cos\phi$.

For this problem, the virtual work done by the internal forces reduces to the virtual work done by the restoring moment of the elastic spring, $\delta W_I = -k\phi\,\delta\phi = Q_\phi^I \delta\phi$, where $Q_\phi^I = -k\phi$ is the generalized internal force of the system. The generalized force is, in this case, a moment, and hence, the expression "generalized force" must be interpreted carefully.

The principle of virtual work, eq. (9.25), yields the equilibrium equation for the system as $Q_\phi^I + Q_\phi^E = -mgR\cos\phi - k\phi = 0$. This is a transcendental equation, but if the angular displacement of the pendulum remains small, $\cos\phi \approx 1$, and the equilibrium configuration becomes $\phi = -mgR/k$.

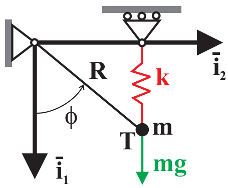

Fig. 9.14. Rotating mass with vertical spring.

Consider next the modified system shown in fig. 9.14 where a rigid arm of length R connects mass m to a pinned support at ground. A linear spring of stiffness constant k supports the mass; this spring remains vertical because its support point is free to move horizontally on rollers. The spring is un-stretched when the arm is horizontal.

As in the previous example, the virtual work done by the gravity load is easily found as $\delta W_E = mg\bar{\imath}_1 \cdot \delta \underline{u}_T$, where $\delta \underline{u}_T$ is the virtual displacement at point **T**. Since $\underline{u}_T = R(\cos\phi\ \bar{\imath}_1 + \sin\phi\ \bar{\imath}_2)$, an infinitesimal virtual displacement of the same quantity is $\delta \underline{u}_T =$

$R(-\sin\phi\ \bar{\imath}_1 + \cos\phi\ \bar{\imath}_2)\delta\phi$. The virtual work done by the gravity load now becomes $\delta W_E = -mgR\sin\phi\ \delta\phi$, and the corresponding generalized force is $Q_\phi^E = -mgR\sin\phi$. Next, the virtual work done by the restoring force in the spring is $\delta W_I = -kR\cos\phi\bar{\imath}_1 \cdot \delta\underline{u}_T$, which yields $Q_\phi^I = kR^2\cos\phi\sin\phi$.

The principle of virtual work as expressed in eq. (9.25) now implies

$$Q_\phi^I + Q_\phi^E = kR^2\cos\phi\sin\phi - mgR\sin\phi = R\sin\phi(kR\cos\phi - mg) = 0.$$

Two solutions are possible. First, $\sin\phi = 0$: this leads to $\phi = 0$ or π, *i.e.*, the arm is in the down or up vertical position, respectively. The second solution is $\cos\phi = mg/(kR)$. For $mg/(kR) > 1$, however, this solution no longer exists, leaving the first solution as the only valid solution of the problem.

Example 9.11. The crank-slider mechanism

Consider the crank-slider mechanism depicted in fig. 9.15. The crank of length R is actuated by a torque Q, and the link of length L transforms the rotary motion of the crank into a linear motion of the slider. A spring of stiffness constant k connects the slider to the ground and is un-stretched when $x = 0$. The configuration of the system is entirely determined by crank angle, ϕ, (measured positive in the counterclockwise direction) which is selected as the generalized coordinate for this problem.

The virtual work done by the externally applied torque, Q, is $\delta W_E = -Q\delta\phi$, and hence, the corresponding generalized force is $Q_\phi^E = -Q$. Similarly, the virtual work done by the internal force in the spring is $\delta W_I = -kx\ \delta x = -kx\ (\partial x/\partial\phi)\ \delta\phi = Q_\phi^I\ \delta\phi$. The position of the slider, x, can be expressed in terms of the generalized coordinate, ϕ. Indeed, projecting segments **OA** and **AB** onto the horizontal yields $x = R\cos\phi + \sqrt{L^2 - R^2\sin^2\phi}$. The generalized force associated with the force in the spring becomes

$$Q_\phi^I = -kx\frac{\partial x}{\partial\phi} = kx\left(1 + \frac{R\cos\phi}{\sqrt{L^2 - R^2\sin^2\phi}}\right)R\sin\phi.$$

The principle of virtual work expressed as eq. (9.25) implies $Q_\phi^I + Q_\phi^E = 0$ leading to the following expression for the applied torque

$$Q = \left(1 + \frac{R\cos\phi}{\sqrt{L^2 - R^2\sin^2\phi}}\right)kxR\sin\phi.$$

Example 9.12. Elastically supported aircraft

For the purpose of dynamic testing, an aircraft is suspended from a hangar's roof by means of three springs of stiffness constants k_L, k_R, and k_T, attached to the aircraft's left wing at point **L**, right wing at point **R**, and tail at point **T**, respectively, as depicted in fig. 9.16. The aircraft's total mass is M and the center of mass is located at point **C**. Determine the equilibrium position of the aircraft under gravity loads.

For simplicity, the aircraft is assumed to be rigid and all spring displacements under load are assumed to remain small (*i.e.*, the airplane remains nearly horizontal).

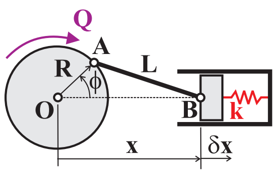

Fig. 9.15. Crank-slider mechanism with a spring.

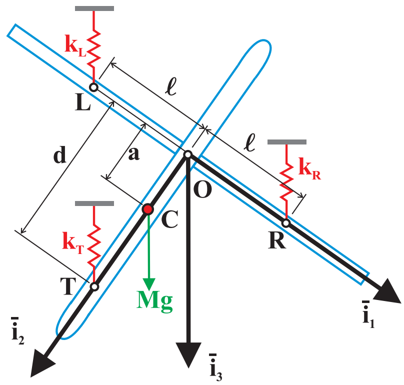

Fig. 9.16. Plan view of the elastically supported aircraft.

The generalized coordinates of the problem will be selected as follows: u is the displacement of point **O** along unit vector $\bar{\imath}_3$, and ϕ_1 and ϕ_2 are the rotations about axes $\bar{\imath}_1$ and $\bar{\imath}_2$, respectively. The displacements of the three suspension points, **L**, **R**, and **T** are now easily expressed as $x_L = u + \ell\phi_2$, $x_R = u - \ell\phi_2$, and $x_T = u + d\phi_1$, respectively. The displacement of the aircraft's center of mass is $x_C = u + a\phi_1$.

The virtual work done by the gravity load is $\delta W_E = Mg\delta x_C = Mg(\delta u + a\delta\phi_1)$, and hence, the corresponding generalized forces are $Q^E_u = Mg$, $Q^E_{\phi_1} = Mga$, and $Q^E_{\phi_2} = 0$. Similarly, the virtual work done by the internal forces in the three springs is $\delta W_I = -k_L x_L \, \delta x_L - k_R x_R \, \delta x_R - k_T x_T \, \delta x_T$, and the corresponding generalized forces become $Q^I_u = k_L x_L - k_R x_R - k_T x_T$, $Q^I_{\phi_1} = -d k_T x_T$, and $Q^I_{\phi_2} = -\ell k_L x_L + \ell k_R x_R$.

The principle of virtual work, eq. (9.25), then yields the three equilibrium equations of the problem as $Q^I_u + Q^E_u = 0$, $Q^I_{\phi_1} + Q^E_{\phi_1} = 0$, and $Q^I_{\phi_2} + Q^E_{\phi_2} = 0$, leading to

$$k_L x_L + k_R x_R + k_T x_T = Mg, \quad d k_T x_T = Mga, \quad \ell k_L x_L - \ell k_R x_R = 0.$$

The solution can be completed using the displacement method (see section 4.3.2). When the displacements of the suspension points are expressed in terms of the generalized coordinates, the three equilibrium equations can be recast as a system of three linear equations that can easily be solved for the generalized coordinates of the system

$$\begin{bmatrix} k_L + k_R + k_T & d k_T & \ell(k_L - k_R) \\ d k_T & d^2 k_T & 0 \\ \ell(k_L - k_R) & 0 & \ell^2(k_L + k_R) \end{bmatrix} \begin{Bmatrix} u \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} Mg \\ Mga \\ 0 \end{Bmatrix}. \tag{9.26}$$

9.4.2 Problems

Problem 9.1. Rotating disk with spring restraint
A mechanism consists of the rotating circular disk pinned at its center as shown in fig. 9.17. A cable is wrapped around the outer edge and a force, P, is applied tangentially. The rotation is resisted by a spring of stiffness constant k attached to a pin on the disk's outer radius and fixed horizontally to a support that can move vertically, leaving the spring horizontal at all times. Use the principle of virtual work to determine the force, P, required to keep the disk in equilibrium as a function of disk angular position, θ.

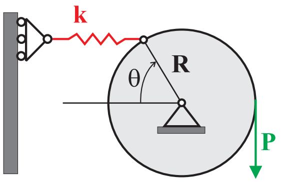

Fig. 9.17. Rotating disk with spring restraint.

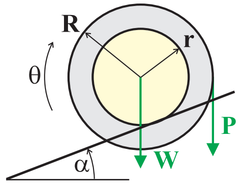

Fig. 9.18. Double-radius wheel on incline.

Problem 9.2. Double-radius wheel on incline
The double radius wheel of weight W shown in fig. 9.18 is of inner radius r and outer radius R. A rope wrapped about the outer radius of the wheel applies a tangential force P. Determine the inclination angle, α, required to maintain the system in static equilibrium.

Problem 9.3. Lever with sliding pivots
Determine the magnitude of force P applied at point **N** required to equilibrate a downward force, F, applied at point **M**, as shown in fig. 9.19. Rod **MN** is of length $a + b$ and is pinned to sleeves which slide along frictionless rods **AO** and **BO**.

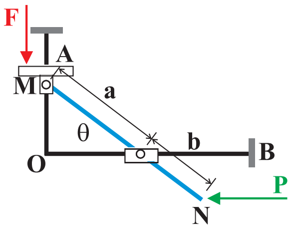

Fig. 9.19. Lever with sliding pivots.

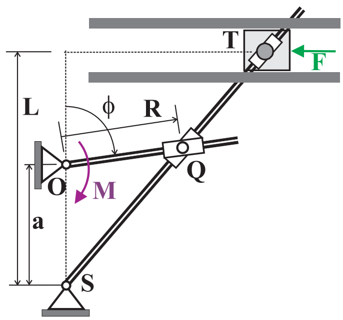

Fig. 9.20. Quick-release mechanism.

Problem 9.4. Quick-release mechanism
In the "quick release mechanism" shown in fig. 9.20, frictionless rod **ST** is a pivoted at point **S** and connected to a sliding hinge and piston at point **T**. At point **Q**, a sliding coupler connects

rods **ST** and **OQ**. Show that the moment, M, necessary to react the applied horizontal force, F, is $M/FL = (R/a)[(R/a) + \cos\phi]/[1 + (R/a)\cos\phi]^2$. Hint: use virtual displacements of infinitesimal magnitude.

Problem 9.5. Lever mechanism

A bar of length $3b$ is pinned at its lower end and supports a normal load, P, applied at its tip, as shown in fig. 9.21. A second bar, of length b, is pinned to the first bar as shown and to a slider that is constrained to move vertically on a frictionless rod. A weight, W, is supported at the slider. Use the principle of virtual work to determine the force, P, required to keep the weight, W, in equilibrium as a function of angle θ.

Problem 9.6. Spring-mass problem with nonlinear geometry

A spring of stiffness constant, k, and un-stretched length, L, is fastened to a support at point **A** and is connected to a weight, W, as shown in fig. 9.22. The weight slides on a frictionless vertical rod and the spring is un-stretched when horizontal. Determine the equilibrium configuration of the system, *i.e.*, position, u, of the weight.

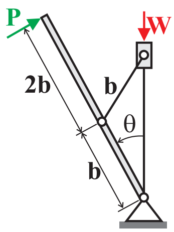

Fig. 9.21. Lever mechanism.

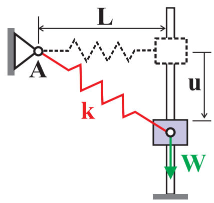

Fig. 9.22. Mechanism with nonlinear geometry.

Problem 9.7. Spring-mass system with nonlinear spring

Consider the spring-mass system depicted in fig. 9.4. Determine the equilibrium position of weight $W = mg$ supported by a vertical spring having a nonlinear stiffness $k = k_0[1 + a(x/L)^2]$, where k_0 is the initial stiffness, *i.e.*, the stiffness for small x, and a is a "hardening coefficient." Solve the problem assuming that $a = 0.5$ and $k_0 = W/8L$.

Problem 9.8. Lever with sliding pivots and spring

Bar **ABC** is of length $b + a$ and constrained to move vertically at point **A** and horizontally at point **B**, while a horizontal force, P, is applied at point **C**, as shown in fig. 9.23. A vertical spring is connected to bar **ABC** at point **A** and is un-stretched when angle $\theta = 0$. Use the principle of virtual work to determine the equilibrium relation(s) of the system.

Problem 9.9. Differential pulley with applied moment

A solid cylinder has two radii, a and b, and its axis is pinned but free to rotate as shown in fig. 9.24. A lever arm of length R is attached to the cylinder and a force, P, acts in the direction normal to the arm. A cable is attached at one end to the smaller radius, a, and at the other to the larger radius, b, and it supports a pulley which carries a vertical weight, W, as shown. Use the principle of virtual work to compute force, P, required to support (or lift) weight, W. Hint: this problem is different from the shop chain hoist discussed in example 9.9: here, as the upper cylinder rotates, cable is taken up on one radius and let out on the other.

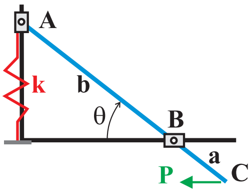

Fig. 9.23. Lever with sliding pivots and spring.

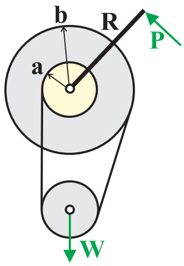

Fig. 9.24. Differential pulley with an applied moment.

Problem 9.10. Linked bars with lateral springs and forces

A mechanical system consists of two articulated bars pinned together at point **B** and to the ground at point **C**, as shown in fig. 9.25. Two springs of stiffness constants k_1 and k_2 support the bars at their mid-span and two forces, P and Q, are applied at points **B** and **A**, respectively. Let q_A and q_B, the downward deflection of points **A** and **B**, be the two generalized coordinates of the system. Use the principle of virtual work to determine the two equilibrium equations of the system. Assume small displacements: $|q_A| \ll L$ and $|q_B| \ll L$.

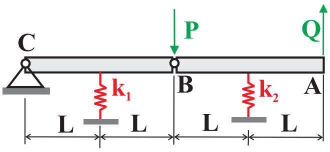

Fig. 9.25. Two articulated bars supported by springs.

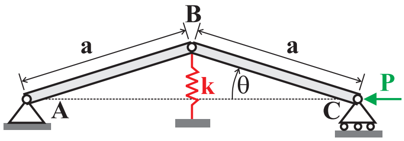

Fig. 9.26. Axially loaded articulated bars with lateral spring restraint.

Problem 9.11. Axially loaded pinned bars with lateral spring restraint

Two rigid bars, **AB** and **BC**, are pinned together at point **B**, as shown in fig. 9.26. The end of the first bar is pinned to the ground at point **A**, whereas the end of the other bar is constrained to slide horizontally at point **C** under the action of load P. A lateral spring of stiffness constant k is attached at point **B**. Angle θ between bar **BC** and the horizontal is the generalized coordinate used to define the system's configuration. Use the principle of virtual work to develop an expression for $P = P(\theta)$. From your analysis, identify the buckling load of the problem.

Problem 9.12. Screw jack scissor lift

Consider the scissor lift (similar to an auto jack) shown in fig. 9.27. The configuration of the system is represented by a single generalized coordinate, θ, the angle between the jack legs and the horizontal. Determine the crank moment, M, required to lift a weight, W. The moment will depend on the configuration of the jack, *i.e.*, on angle θ. The threaded screw has a pitch of N threads per unit length. All bars of the jack are articulated.

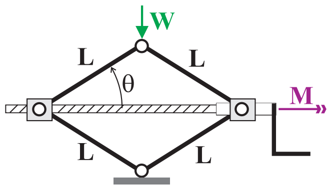

Fig. 9.27. Screw jack type of scissor lift.

9.5 Principle of virtual work applied to truss structures

In the previous section, the principle of virtual work is applied to mechanical systems consisting of rigid bodies and concentrated springs. Attention now turns to applications of the same principle to simple truss structures.

9.5.1 Truss structures

Trusses are a class of structures consisting of slender bars connected together at their ends by what are called *pinned joints*, which transmit forces but no moments. Consequently, the slender bars carry axial forces, in tension or compression, but no bending moments or transverse shear forces.[1] The simplest truss consists of only two members connected at a single joint to which a load may be applied. The resulting structure is isostatic; the addition of a third bar leads to a simple hyperstatic truss, called a "three-bar truss," which is analyzed in section 4.4.

Each bar of a truss acts like a simple rectilinear spring of stiffness constant $k = E\mathcal{A}/L$, where L is the bar's length, $\mathcal{A}$ its cross-sectional area, and E the elastic modulus of the material making up its cross-section. In addition, if all bars lie in a plane, the truss is referred to as a *planar truss*; otherwise, it is called a three-dimensional truss or a *spatial truss*. For both configurations, analysis methods are identical, although the treatment of three-dimensional configurations is usually more cumbersome. This section focuses on planar trusses, but the methods developed here are equally applicable to spatial trusses.

Figure 9.28 illustrates a simple planar truss. A crude sketch of the truss shows the prismatic bars pinned together at their ends. In this illustration, the member widths have been exaggerated: in actual trusses, bars are quite slender. Actual pin joints were commonly employed in early planar truss designs, especially in trusses for railway bridges designed in the 19^{th} century. With the development of higher strength alloys and use of thin-walled tubular sections, bar slenderness, the ratio of their length to diameter, can approach 100. For such truss members, it is practical to design rigid joints using welding or bolting, and although such joints introduce bending moments into the bars, the primary stresses are still almost entirely due to the axial forces, except in the immediate vicinity of the connections.

[1] If the bars are rigidly connected together at the joints so that no relative rotation is possible, bending moments will develop in the bars and bending deflections must be considered. Such structures are generally called *frames* and are more complicated to analyze.

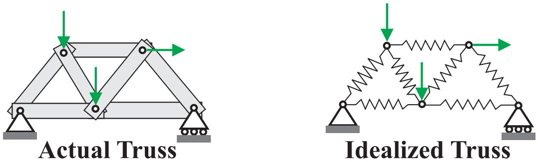

Fig. 9.28. Planar truss and its idealization as an assembly of rectilinear springs.

In simple truss design, each bar is fabricated from a homogeneous material and has a constant cross-section. Equation (4.3) then implies that each member can be represented as a rectilinear spring of stiffness constant $k = E\mathcal{A}/L$. The complete truss can then be viewed as an assembly of springs connected to pinned joints, as illustrated in fig. 9.28.

The spring stiffnesses constants are quite large for a typical bars. For example, the axial stiffness of a bar of length $L = 0.75$ m, sectional area $\mathcal{A} = 75$ mm^2 and modulus $E = 210$ GPa, is $k = 21$ MN/m. When subjected to a 5 kN force, the bar's elongation is $e = F/k = 0.24$ mm. In most practical designs, the maximum deflection of any joint of the truss is very small compared to the bar lengths. If self-weight is a significant component of the overall loading, as is the case for trusses used in civil engineering applications, the gravity force associated with each bar is lumped in two equal forces applied at the bar's two end joints.

Elongation-displacement equations

Consider the generic bar **AB** shown in fig. 9.29. Point **A** is assumed to remain pinned, while point **B** undergoes a displacement $\underline{\Delta} = \Delta_1 \bar{\imath}_1 + \Delta_2 \bar{\imath}_2$. The bar's original length is L, and its elongation e. Elementary geometry then yields $(L+e)^2 = (L_1+\Delta_1)^2 + (L_2+\Delta_2)^2$, where L_1 and L_2 are the projections of the original length along axes $\bar{\imath}_1$ and $\bar{\imath}_2$, respectively.

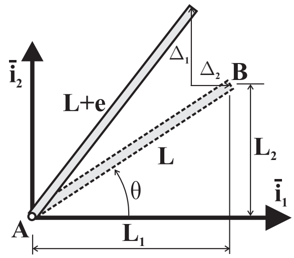

Fig. 9.29. Single bar of a planar truss.

The elongation-displacement relationship is nonlinear, but if the relatively joint displacements, Δ_1 and Δ_2, remain small compared to the bar's length, as is typically the case for engineered structures, this expression can be linearized as follows.

First, a division by the square of the bar's length yields a non-dimensional form of the equation,

$$\left(1 + \frac{e}{L}\right)^2 = \left(\frac{L_1}{L} + \frac{\Delta_1}{L}\right)^2 + \left(\frac{L_2}{L} + \frac{\Delta_2}{L}\right)^2.$$

Expanding all squares then leads to

$$1 + 2\frac{e}{L} + \frac{e^2}{L^2} = \frac{L_1^2}{L^2} + 2\frac{L_1}{L}\frac{\Delta_1}{L} + \frac{\Delta_1^2}{L^2} + \frac{L_2^2}{L^2} + 2\frac{L_2}{L}\frac{\Delta_2}{L} + \frac{\Delta_2^2}{L^2}.$$

It is now assumed that e, Δ_1, and Δ_2 are all small compared to L, and hence, terms e^2/L^2, Δ_1^2/L^2, and Δ_2^2/L^2 become negligible. Finally, noting that $L_1^2/L^2 + L_2^2/L^2 = 1$, the above equation reduces to

$$e \approx \Delta_1 \frac{L_1}{L} + \Delta_2 \frac{L_2}{L} = \Delta_1 \cos\theta + \Delta_2 \sin\theta. \tag{9.27}$$

This equation shows that the bar's elongation is the projection of the relative displacement of its end joint along its direction. Equation (9.27) is an approximate, linearized elongation-displacement relationship, which applies when joint displacements remain small compared to the bar's length.

Internal virtual work for a bar

Figure 9.30 shows a general planar truss member defined by its root and tip joints. Let $\bar{b}$ be the unit vector along the direction of the bar, and $\underline{u}^{\rm t}$ and $\underline{u}^{\rm r}$ the displacements of its tip and root joints, respectively. Let F be the magnitude of the force applied to the bar, and hence, forces $\underline{F}^{\rm r} = -F\bar{b}$ and $\underline{F}^{\rm t} = F\bar{b}$ are applied to the root and tip of the bar, respectively. The virtual work done by these two forces is $\delta W = \underline{F}^{\rm r} \cdot \delta\underline{u}^{\rm r} + \underline{F}^{\rm t} \cdot \delta\underline{u}^{\rm t} = F\bar{b} \cdot (\delta\underline{u}^{\rm t} - \delta\underline{u}^{\rm r})$.

Because the internal and externally applied forces are of opposite sign, the virtual work done by the internal force becomes

$$\delta W_I = -\underline{F}^{\rm r} \cdot \delta\underline{u}^{\rm r} - \underline{F}^{\rm t} \cdot \delta\underline{u}^{\rm t} = -F\bar{b} \cdot (\delta\underline{u}^{\rm t} - \delta\underline{u}^{\rm r}) \tag{9.28}$$

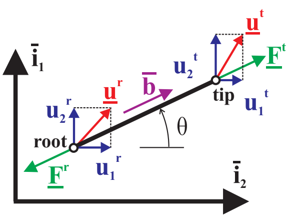

Fig. 9.30. Bar displacements and forces.

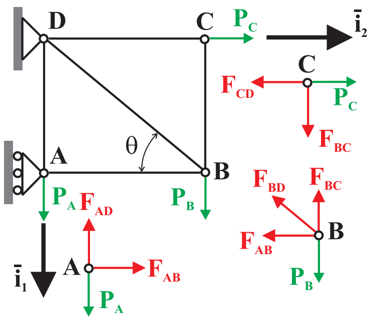

Fig. 9.31. Configuration of the 5-bar truss.

Using eq. (9.27), the elongation, e, of the bar is the projection of the relative displacements of its end points along its direction, and this is expressed by the following dot product, $e = \bar{b} \cdot (\underline{u}^{\rm t} - \underline{u}^{\rm r})$. The virtual elongation then becomes $\delta e = \bar{b} \cdot (\delta\underline{u}^{\rm t} - \delta\underline{u}^{\rm r})$, and the bar's internal virtual work, eq. (9.28), becomes

$$\delta W_I = -F\delta e. \tag{9.29}$$

It is possible to express the virtual elongation in terms of the end nodes virtual displacements as

$$\begin{aligned}\delta e =&(\sin\theta\,\bar{\imath}_1 + \cos\theta\,\bar{\imath}_2)\cdot(\delta u_1^{\rm t}\,\bar{\imath}_1 + \delta u_2^{\rm t}\,\bar{\imath}_2 - \delta u_1^{\rm r}\,\bar{\imath}_1 - \delta u_2^{\rm r}\,\bar{\imath}_2)\\ =&(\delta u_1^{\rm t} - \delta u_1^{\rm r})\sin\theta + (\delta u_2^{\rm t} - \delta u_2^{\rm r})\cos\theta.\end{aligned} \tag{9.30}$$

9.5.2 Solution using Newton's law

Consider the five-bar planar truss depicted in fig. 9.31, subjected to two vertical loads, P_A and P_B, applied at joints **A** and **B**, respectively, and to a horizontal load, P_C, applied at joint **C**. The diagonal bar, **BD**, is inclined at an angle θ with respect to the horizontal. At joint **D**, the two components of displacements are constrained to be zero, whereas at joint **A**, the vertical component of displacement is allowed, but the horizontal is constrained to zero. In the development that follows, the vector notation employed in the previous examples will not be used because it is easier to write the work directly in scalar equations for these two-dimensional problems.

Each member in a truss transmits an axial force that is either tensile or compressive. By convention, tensile forces are defined as positive and compressive forces as negative. The truss can be treated as a system of particles where each particle is a joint of the truss to which two or more members are attached. Newton's law expresses the equilibrium condition at each of the 4 joints, **A**, **B**, **C** and **D** by eq. (9.2). For a planar truss, this yields two scalar equilibrium equations at each joint, a total of 8 equations for the present truss. These equilibrium equations involve the bar forces and reaction forces acting at each joint. The approach is commonly referred to as the *method of joints*.

Considering the free-body diagrams of each joint shown in fig. 9.31, the following 8 equations of equilibrium are obtained

$$P_A - F_{AD} = 0, \quad H_A + F_{AB} = 0 \tag{9.31a}$$

$$P_B - F_{BC} - F_{BD}\sin\theta = 0, \quad -F_{AB} - F_{BD}\cos\theta = 0, \tag{9.31b}$$

$$F_{BC} = 0, \quad P_C - F_{CD} = 0, \tag{9.31c}$$

$$V_D + F_{AD} + F_{BD}\sin\theta = 0, \quad H_D + F_{CD} + F_{BD}\cos\theta = 0. \tag{9.31d}$$

Equations (9.31a) are the vertical and horizontal equilibrium equations at joint **A**; eqs. (9.31b) are the vertical and horizontal equilibrium equations at joint **B**; eqs. (9.31c) are the corresponding equations at joint **C**; and finally, eqs. (9.31d) are the corresponding equations at joint **D**.

These 8 equilibrium equations can used to determine 8 independent forces. If the number of forces (bar and reaction forces) is larger than the number of equations, the problem is hyperstatic. If the number of forces equals the number of equations, the problem is isostatic and a complete solution can be obtained from the equilibrium equations alone. In this case, the problem is isostatic because the 8 equilibrium equations of the problem are sufficient to compute the 5 bar plus the 3 reaction forces.

9.5.3 Solution using kinematically admissible virtual displacements

In section 9.3.1, the principle of virtual work for a single particle is developed by multiplying the particle's equilibrium equation by an arbitrary virtual displacement, to obtain eq. (9.8), which is a statement of the principle of virtual work. A similar approach is followed here to develop the principle of virtual work for the five-bar truss depicted in fig. 9.31. Newton's law is used to obtain the joint equilibrium equations, eqs. (9.31); of these 8 equations, the 5 that correspond to equilibrium in an unconstrained direction are multiplied by virtual displacements to construct the following statement

$$\begin{aligned} &[P_A - F_{AD}]\,\delta u_1^A + [P_B - F_{BC} - F_{BD}\sin\theta]\,\delta u_1^B \\ &+ [-F_{AB} - F_{BD}\cos\theta]\,\delta u_2^B + [F_{BC}]\,\delta u_1^C + [P_C - F_{CD}]\,\delta u_2^C = 0, \end{aligned} \tag{9.32}$$

where δu_1^A is a vertical virtual displacement at joint **A**, δu_1^B and δu_2^B vertical and horizontal virtual displacements at joint **B**, and δu_1^C and δu_2^C the corresponding quantities at joint **C**. These are kinematically admissible virtual displacements because they do not violate any of the geometric boundary conditions of the problem. A horizontal virtual displacement at joint **A**, or any virtual displacements at joint **D** would violate the geometric boundary conditions at those joints and are not considered here. If the truss is in equilibrium, eq. (9.32) vanishes for all kinematically admissible virtual displacements.

The various terms in eq. (9.32) are now regrouped in the following manner

$$\begin{aligned} &P_A\delta u_1^A + P_B\delta u_1^B + P_C\delta u_2^C - F_{AB}\delta u_2^B - F_{AD}\delta u_1^A \\ &-F_{BC}(\delta u_1^B - \delta u_1^C) - F_{BD}(\delta u_1^B\sin\theta + \delta u_2^B\cos\theta) - F_{CD}\delta u_2^C = 0. \end{aligned} \tag{9.33}$$

The first 3 terms of this expression represent the virtual work done by the externally applied forces,

$$\delta W_E = P_A\delta u_1^A + P_B\delta u_1^B + P_C\delta u_2^C, \tag{9.34}$$

where each loading component is multiplied by the virtual displacement in the direction of action of the load. Equation (9.33) now simplifies to

$$\begin{aligned} &\delta W_E - F_{AB}\delta u_2^B - F_{AD}\delta u_1^A - F_{BC}(\delta u_1^B - \delta u_1^C) \\ &-F_{BD}(\delta u_1^B\sin\theta + \delta u_2^B\cos\theta) - F_{CD}\delta u_2^C = 0. \end{aligned}$$

The last 5 terms of this equation represent the virtual work done by the internal forces in the 5 bars. From eq. (9.30), the virtual elongation of bar **AB** is $\delta e_{AB} = \delta u_2^B$, and eq. (9.29) then yields the bar's internal virtual work as $-F_{AB}\delta e_{AB} = -F_{AB}\delta u_2^B$. The virtual elongations in the other four bars are $\delta e_{AD} = \delta u_1^A$, $\delta e_{BC} = \delta u_1^B - \delta u_1^C$, $\delta e_{BD} = \delta u_1^B\sin\theta + \delta u_2^B\cos\theta$, and $\delta e_{CD} = \delta u_2^C$. The virtual work done by all internal forces now becomes

$$\delta W_I = -F_{AB}\delta e_{AB} - F_{AD}\delta e_{AD} - F_{BC}\delta e_{BC} - F_{BD}\delta e_{BD} - F_{CD}\delta e_{CD}, \tag{9.35}$$

and eq. (9.33) reduces to

$$\delta W = \delta W_E + \delta W_I = 0, \tag{9.36}$$

for all kinematically admissible virtual displacements.

The reasoning can be reversed: if eq. (9.36) holds for all kinematically admissible virtual displacements, eq. (9.32) then holds and the equilibrium equations of the problem follow. The results obtained here can be combined into another statement of the principle of virtual work.

Principle 5 (Principle of virtual work) *A structure is in static equilibrium if and only if the sum of the internal and external virtual work vanishes for all kinematically admissible virtual displacements.*

9.5.4 Solution using arbitrary virtual displacements

The development presented above is based on kinematically admissible virtual displacements, but this is not the only possible approach. Newton's law is used to obtain the 8 joint equilibrium equations, eqs. (9.31). Multiplying each of these equilibrium equations by a virtual displacement leads to the following statement

$$\begin{aligned} &[P_A - F_{AD}]\,\delta u_1^A + [H_A + F_{AB}]\,\delta u_2^A + [P_B - F_{BC} - F_{BD}\sin\theta]\,\delta u_1^B \\ &+[-F_{AB} - F_{BD}\cos\theta]\,\delta u_2^B + [F_{BC}]\,\delta u_1^C + [P_C - F_{CD}]\,\delta u_2^C \\ &+[V_D + F_{AD} + F_{BD}\sin\theta]\,\delta u_1^D + [H_D + F_{CD} + F_{BD}\cos\theta]\,\delta u_2^D = 0. \end{aligned} \tag{9.37}$$

If the truss is in equilibrium, eq. (9.37) vanishes for all virtual displacements.

In contrast to eqs. (9.32), these equations include the horizontal and vertical reaction forces acting at joint **D**, denoted V_D and H_D, respectively, and the horizontal reaction force at joint **A**, denoted H_A. In addition, δu_2^A is the horizontal virtual displacement component at point **A**, and δu_1^D and δu_2^D are the vertical and horizontal virtual displacement components at point **D**, respectively. These three virtual displacement components violate the geometric boundary conditions of the problem, *i.e.*, they are not kinematically admissible.

Regrouping terms in eq. (9.37) leads to

$$\begin{aligned} &P_A\delta u_1^A + P_B\delta u_1^B + P_C\delta u_2^C + H_A\delta u_2^A + V_D\delta u_1^D + H_D\delta u_2^D \\ &-F_{AB}(\delta u_2^B - \delta u_2^A) - F_{AD}(\delta u_1^A - \delta u_1^D) - F_{BC}(\delta u_1^B - \delta u_1^C) \\ &-F_{BD}\left[(\delta u_1^B - \delta u_1^D)\sin\theta + (\delta u_2^B - \delta u_2^D)\cos\theta\right] - F_{CD}(\delta u_2^C - \delta u_2^D) = 0. \end{aligned} \tag{9.38}$$

The first 6 terms of this expression represent the virtual work done by the externally applied forces,

$$\delta W_E = P_A\delta u_1^A + P_B\delta u_1^B + P_C\delta u_2^C + H_A\delta u_2^A + V_D\delta u_1^D + H_D\delta u_2^D, \tag{9.39}$$

where each loading component is multiplied by the virtual displacement in the direction of action of the load. Because virtual displacements that violate the geometric boundary conditions are used here, the virtual work done by the reaction forces does not vanish, and the reaction forces must be treated as externally applied forces.

Equation (9.38) now simplifies to

$$\delta W_E - F_{AB}(\delta u_2^B - \delta u_2^A) - F_{AD}(\delta u_1^A - \delta u_1^D) - F_{BC}(\delta u_1^B - \delta u_1^C)$$
$$- F_{BD}\left[(\delta u_1^B - \delta u_1^D)\sin\theta + (\delta u_2^B - \delta u_2^D)\cos\theta\right] - F_{CD}(\delta u_2^C - \delta u_2^D) = 0. \quad (9.40)$$

The last 5 terms of this equation represent the virtual work done by the internal forces in the 5 bars as can be shown by the following reasoning. The virtual elongations in the five bars are given with the help of eq. (9.30) as $\delta e_{AB} = \delta u_2^B - \delta u_2^A$, $\delta e_{AD} = \delta u_1^A - \delta u_1^D$, $\delta e_{BC} = \delta u_1^B - \delta u_1^C$, $\delta e_{BD} = (\delta u_1^B - \delta u_1^D)\sin\theta + (\delta u_2^B - \delta u_2^D)\cos\theta$, and $\delta e_{CD} = \delta_2^C - \delta u_2^D$, which are updated to reflect the presence of virtual displacements that violate the geometric constraints. Using these, the last 5 terms in eq. (9.40) become $-F_{AB}\delta e_{AB} - F_{AD}\delta e_{AD} - F_{BC}\delta e_{BC} - F_{BD}\delta e_{BD} - F_{CD}\delta e_{CD} = \delta W_I$ so that eq. (9.38) reduces to eq. (9.36).

The reasoning can be reversed: if eq. (9.36) holds for all virtual displacements, eq. (9.37) then holds and the equilibrium equations of the problem follow.

These results can be combined into the following statement of the principle of virtual work.

Principle 6 (Principle of virtual work) *A structure is in static equilibrium if and only if the sum of the internal and external virtual work vanishes for all virtual displacements.*

Although the above two principles are established for the simple truss structure depicted in fig. 9.31, it will be shown later that they are applicable to general structures. These two principles are nearly identical. When the principle of virtual work is used with kinematically admissible virtual displacements as principle 5, the virtual work done by the externally applied forces does not include the reaction forces; the virtual work they perform automatically vanishes because the corresponding virtual displacements are zero, see eq. (9.34). For the 5 bar truss, application of this principle yields the five equilibrium equations in eqs. (9.31) that do not involve the reaction forces.

When the principle of virtual work is used with arbitrary virtual displacements as principle 6, the virtual work done by the reactions forces must be included in the statement of the external virtual work, see eq. (9.39). For the 5 bar truss, application of this principle yields all 8 of the equilibrium equations, eqs. (9.31), two at each of the four joints in the truss.

These principles are derived from Newton's law to which they are entirely equivalent. When arbitrary virtual displacements are used, all equilibrium equations of the problem are recovered. If the virtual displacements are limited to those that are kinematically admissible, a subset of the equilibrium equations is recovered.

Example 9.13. Three-bar truss using principle of virtual work

The three-bar planar truss depicted in fig. 9.32 is a simple hyperstatic truss with a single free joint. It is subjected to a vertical load P at joint **O**, where the three bars are pinned together. The three bars are identified by the joints at which they are pinned to the ground, denoted as **A**, **B**, and **C**. $\mathcal{A}_A$, $\mathcal{A}_B$ and $\mathcal{A}_C$ are the cross-sectional areas of bars **A**, **B**, and **C**, respectively, and E_A, E_B and E_C denote their respective Young's moduli. The axial stiffnesses of the three bars are $k_A = (E\mathcal{A})_A/L_A = (E\mathcal{A})_A \cos\theta/L$, $k_B = (E\mathcal{A})_B/L$, and $k_C = (E\mathcal{A})_C \cos\theta/L$, respectively.

This problem is hyperstatic of order 1, and it is solved using the displacement and force methods in examples 4.4 on page 147 and example 4.6 on page 152. In these two earlier examples, the stiffnesses of bars **A** and **C** are assumed to be identical to simplify the problem. This assumption is not made in the present example.

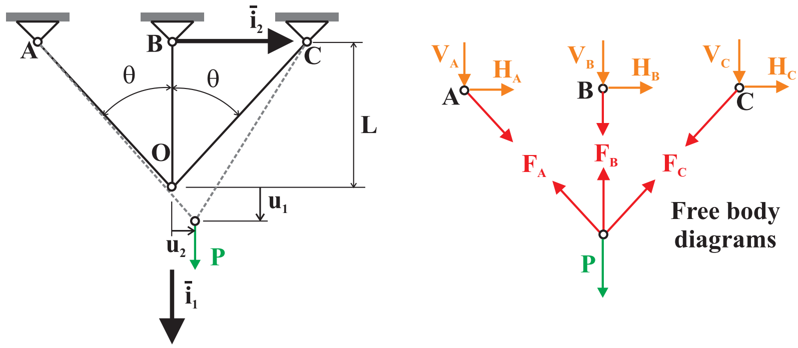

Fig. 9.32. Three-bar truss configuration with free-body diagram.

The virtual displacement vector for point **O** is $\delta\underline{u} = \delta u_1 \bar{\imath}_1 + \delta u_2 \bar{\imath}_2$. Equation (9.30) then gives the bar virtual elongations as $\delta e_A = \delta u_1 \cos\theta + \delta u_2 \sin\theta$, $\delta e_B = \delta u_1$, and $\delta e_C = \delta u_1 \cos\theta - \delta u_2 \sin\theta$, for bars **A**, **B** and **C**, respectively. The principle of virtual work now states that

$$\begin{aligned}\delta W &= \delta W_E + \delta W_I \\ &= P\delta u_1 - F_A(\delta u_1 \cos\theta + \delta u_2 \sin\theta) - F_B \delta u_1 - F_C(\delta u_1 \cos\theta - \delta u_2 \sin\theta) \\ &= -\left[F_A \cos\theta + F_B + F_C \cos\theta - P\right] \delta u_1 - \sin\theta \left[F_A - F_C\right] \delta u_2 = 0,\end{aligned}$$

for all arbitrary virtual displacement components, δu_1 and δu_2. Because these components are arbitrary, the two bracketed terms must vanish, leading to the two equilibrium equations of the problem, $F_A \cos\theta + F_B + F_C \cos\theta = P$ and $F_A = F_C$. Because these two equilibrium equations are not sufficient to evaluate the three bar forces, the problem is hyperstatic of order 1.

The solution of the problem can be completed using the displacement method, see section 4.3.2. The elongation-displacement equations are obtained by expressing the bar elongations in terms of the end-joint displacements using eq. (9.27). This yields $e_A = u_1 \cos\theta + u_2 \sin\theta$, $e_B = u_1$, and $e_C = u_1 \cos\theta - u_2 \sin\theta$, where e_A, e_B, and e_C are the elongations of bars **A**, **B** and **C**, respectively, and $\underline{u} = u_1 \bar{\imath}_1 + u_2 \bar{\imath}_2$ is the displacement vector of point **O**. The constitutive laws are $F_A = e_A (E\mathcal{A})_A \cos\theta/L$, $F_B = e_B (E\mathcal{A})_B/L$, and $F_C = e_C (E\mathcal{A})_C \cos\theta/L$, for bars **A**, **B** and **C**, respectively. Introducing the constitutive laws into the equilibrium equations, and the elongation-displacement equations into the resulting relationships yield two equations for the two displacement components. It is convenient to write these in matrix form as

$$\cos^2\theta \begin{bmatrix} 1+(\bar{k}_A+\bar{k}_C)\cos\theta & (\bar{k}_A-\bar{k}_C)\sin\theta \\ (\bar{k}_A-\bar{k}_C)\sin\theta & (\bar{k}_A+\bar{k}_C)\sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \bar{P}L \\ 0 \end{Bmatrix},$$

where $\bar{k}_A = (E\mathcal{A})_A/(E\mathcal{A})_B$ and $\bar{k}_C = (E\mathcal{A})_C/(E\mathcal{A})_B$ are non-dimensional bar stiffness ratios, and $\bar{P} = P/(E\mathcal{A})_B$ is the non-dimensional applied load. With simple matrix manipulations, this set of equations yields the non-dimensional displacement components of point **O** as

$$\frac{u_1}{\bar{P}L} = \frac{\bar{k}_A+\bar{k}_C}{\bar{k}}, \quad \frac{u_2}{\bar{P}L} = -\frac{\bar{k}_A-\bar{k}_C}{\bar{k}}\frac{\cos\theta}{\sin\theta}, \tag{9.41}$$

where $\bar{k} = \bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta$. With the help of the elongation-displacement equations, the bar non-dimensional elongations are found to be

$$\frac{e_A}{\bar{P}L} = \frac{2\bar{k}_C\cos\theta}{\bar{k}}, \quad \frac{e_B}{\bar{P}L} = \frac{\bar{k}_A+\bar{k}_C}{\bar{k}}, \quad \frac{e_C}{\bar{P}L} = \frac{2\bar{k}_A\cos\theta}{\bar{k}}.$$

Finally, the non-dimensional bar forces are obtained from the constitutive laws as

$$\frac{F_A}{P} = \frac{F_C}{P} = \frac{2\cos\theta^2\bar{k}_A\bar{k}_C}{\bar{k}}, \quad \frac{F_B}{P} = \frac{\bar{k}_A+\bar{k}_C}{\bar{k}}. \tag{9.42}$$

Except for the fact that equilibrium equations are obtained from the principle of virtual work rather than from Newton's first law, the solution process presented here is identical to that of the displacement method presented in section 4.3.2.

In this example, the principle of virtual work is used in conjunction with kinematically admissible virtual displacements. Figure 9.32 also shows free body diagrams of the four nodes of the truss, and these involve the reaction forces at joints **A**, **B**, and **C**. These reaction forces do not appear in the above developments because the virtual displacements at joints **A**, **B**, and **C** are selected to be kinematically admissible, *i.e.*, all three are assumed to vanish. If arbitrary virtual displacements are selected, the virtual work done by the reaction forces no longer vanishes. The external virtual work is now

$$\delta W_E = V_A\delta u_1^A + H_A\delta u_2^A + V_B\delta u_1^B + H_B\delta u_2^B + V_C\delta u_1^C + H_C\delta u_2^C + P\delta u_1^O,$$

where δu_1^A and δu_2^A are the vertical and horizontal components of the virtual displacement at joint **A**, and similar notations are used for the corresponding virtual displacements at joints **B**, **C**, and **O**.

Next, the internal virtual work is evaluated as

$$\begin{aligned} \delta W_I = & -F_A(\cos\theta\bar{\imath}_1 + \sin\theta\bar{\imath}_2)\cdot\left[(\delta u_1^O - \delta u_1^A)\bar{\imath}_1 + (\delta u_2^O - \delta u_2^A)\bar{\imath}_2\right] \\ & -F_B\bar{\imath}_1\cdot\left[(\delta u_1^O - \delta u_1^B)\bar{\imath}_1 + (\delta u_2^O - \delta u_2^B)\bar{\imath}_2\right] \\ & -F_C(\cos\theta\bar{\imath}_1 - \sin\theta\bar{\imath}_2)\cdot\left[(\delta u_1^O - \delta u_1^C)\bar{\imath}_1 + (\delta u_2^O - \delta u_2^C)\bar{\imath}_2\right], \end{aligned}$$

where the expressions for the root and tip virtual displacements of the bars reflect the virtual displacements at joints **A**, **B**, and **C**, which violate the geometric boundary conditions.

Invoking the principle of virtual work, principle 6, then yields

$$
\begin{aligned}
&[V_A + F_A \cos\theta]\,\delta u_1^A + [H_A + F_A \sin\theta]\,\delta u_2^A + [V_B + F_B]\,\delta u_1^B + [H_B]\,\delta u_2^B \\
&+ [V_C + F_C \cos\theta]\,\delta u_1^C + [H_C - F_C \sin\theta]\,\delta u_2^C \\
&+ [P - F_A \cos\theta - F_B - F_C \cos\theta]\,\delta u_1^O + [F_A \sin\theta - F_C \sin\theta]\,\delta u_2^O = 0.
\end{aligned}
$$

In this expression, all virtual displacement components are arbitrary, and this implies that all bracketed terms must vanish. The last two terms yield the vertical and horizontal equilibrium equations at joint **O**, which are the only two equations obtained when kinematically admissible virtual displacements are used. The first six bracketed terms yield the six equilibrium equations involving the reaction forces at joints **A**, **B**, and **C**.

The complete solution process mirrors that used earlier. First, the displacements of joint **O** can be obtained from eq. (9.41). Next, the forces in bars **A**, **B**, and **C** follow from eq. (9.42). Finally, the three bar forces can be introduced into the six equilibrium equations at joints **A**, **B**, and **C** to obtain the six components of the reaction forces at the corresponding points.

9.6 Principle of complementary virtual work

The basic equations of linear elasticity are derived in chapter 1. As shown in fig. 9.33, these equations are divided into three groups: the equilibrium equations, the strain-displacement relationships, and the constitutive laws. Given the proper boundary conditions, these three groups of equations are sufficient to obtain solutions of elasticity problems.

In addition, the strain compatibility equations impose constraints on the body's strain field. Because they can be derived from the strain-displacement relationships, the compatibility equations do not form an independent set of equations and are not required to solve elasticity problems. Their importance, however, arises in situations where the displacement field is to be evaluated from the strain field. Because the six strain components are expressed in terms of three displacement components only, the problem is over-determined. The compatibility equations ensure that the strain field can be derived from a compatible displacement field, *i.e.*, a displacement field that creates no gaps or overlaps in the solid. Consequently, the compatibility equations can become a critical element of any solution procedure.

The principle of virtual work derived in the first part of this chapter is shown to be entirely equivalent to the equilibrium equations, which are themselves a mathematical restatement of Newton's laws. The equivalence of these three statements is indicated in fig. 9.33 by the double headed arrows joining the corresponding boxes.

Although expressed within markedly different formalisms, the principle of virtual work and Newton's laws are two entirely equivalent statements. Because New-

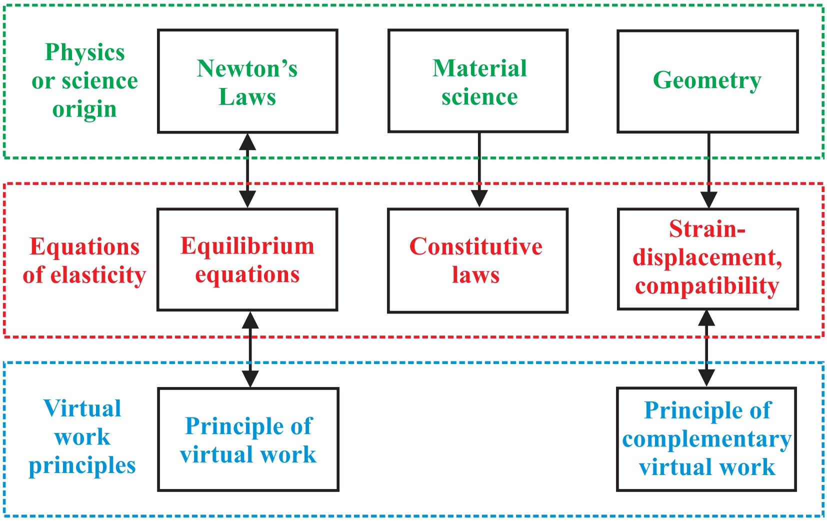

Fig. 9.33. Relationship between the equations of elasticity and virtual work principles.

ton's laws are the foundation of mechanics and elasticity, the principle of virtual work is an equivalent, alternative foundation of mechanics and elasticity.

As discussed in chapter 1, the solution of any elasticity problem requires the three groups of basic equations appearing in the middle row of fig. 9.33. Consequently, because it is equivalent to only the equilibrium equations, the principle of virtual work alone does not provide enough information to solve elasticity problems. To obtain complete solutions to elasticity problems, it must be complemented with strain-displacement relationships and constitutive laws.

In the next part of this chapter, a second virtual work principle will be developed: the *principle of complementary virtual work*. As indicated in fig. 9.33, this virtual work principle is entirely equivalent to the compatibility equations of the problem. The principle of complementary virtual work alone does not provide enough information to solve elasticity problems. It must be augmented with equilibrium equations and constitutive laws to derive complete solutions to elasticity problems.

Clearly, the strain-displacement relationships and compatibility equations are central to the understanding of the principle of complementary virtual work. Rather than consider the more abstract general case, these concepts will be examined in the next section for simple truss structures and in subsequent sections for more complex beam structures.

9.6.1 Compatibility equations for a planar truss

Before deriving the principle of complementary virtual work, the kinematics and compatibility equations for planar trusses will be investigated.

Compatibility conditions

Consider the two-bar planar truss depicted in fig. 9.34, consisting of two bars, denoted bars **A** and **C**, joined together at point **O**, connected to the ground at points **A** and **C**, respectively, and of lengths L_A and L_C, respectively. Let the two bars undergo arbitrary elongations of magnitudes e_A and e_C, respectively; the loads that create these elongations are irrelevant to the present discussion and are not shown in the figure.

The configuration of the truss that is compatible with these elongations is easily found with the help of a purely geometric reasoning. Draw two circles of radii $L_A + e_A$ and $L_C + e_C$, centered at points **A** and **C**, respectively; the intersection of these two circles, denoted point **O'**, is the connection point of the two bars in their deformed configuration. Given any two arbitrary elongations, the truss's configuration is easily found, assuming, of course, that the two circles intersect.

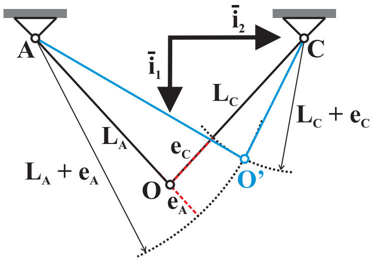

Fig. 9.34. Two-bar truss in the original and deformed configurations.

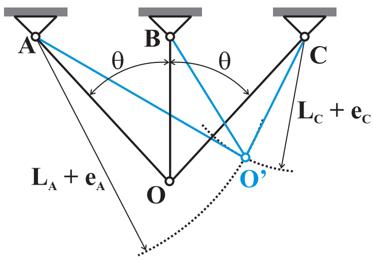

Fig. 9.35. Three-bar truss in the original and deformed configurations.

Consider next the three-bar, planar truss shown in fig. 9.35 which is similar to the two-bar truss considered in fig. 9.34 but with the addition of the middle bar, denoted bar **B**, of length L_B. Let bars **A** and **C** undergo arbitrary elongations of magnitudes e_A and e_C, respectively. Using the geometric construction described in the previous paragraph, the configurations of bars **A** and **C** are readily found, and point **O'** is obtained. The configuration of bar **B** is now uniquely defined, because it must join points **B** and **O'**. If the deformed length of bar **B** is denoted L'_B, its elongation is then $e_B = L'_B - L_B$. Clearly, the elongations of the three bars are no longer independent because, given e_A and e_C, e_B can be obtained from simple geometric considerations.

Instead of using this purely a geometric reasoning, the same conclusions can be reached by using the elongation-displacement equations of the problem. Consider first the two-bar truss depicted in fig. 9.34. Let $\underline{u} = u_1\bar{\imath}_1 + u_2\bar{\imath}_2$ be the displacement vector of point **O**. The elongations of bars **A** and **C** simply correspond to the projections of this displacement vector along the directions of the bars, see eq. (9.27), to find

$$e_A = u_1 \cos\theta + u_2 \sin\theta, \quad e_C = u_1 \cos\theta - u_2 \sin\theta. \tag{9.43}$$

These equations relate the elongations, e_A and e_C, of the two bars to the two components of displacement of point **O**, u_1 and u_2, and can be inverted to find the displacement components as a function of the elongations: $u_1 = (e_A + e_C)/(2\cos\theta)$ and $u_1 = (e_A - e_C)/(2\sin\theta)$. The final configuration of the system is uniquely defined if the two displacement components, u_1 and u_2, are given. Alternatively, if the two elongations are given, the two displacement components can be evaluated using eq. (9.43), and the final system configuration is obtained.

Consider now the three-bar truss shown in fig. 9.35. The elongations of bars **A**, **B** and **C**, correspond to the projections of the displacement vector of point **O** along the directions of the bars, see eq. (9.27), to find

$$e_A = u_1 \cos\theta + u_2 \sin\theta, \quad e_B = u_1, \quad e_C = u_1 \cos\theta - u_2 \sin\theta. \tag{9.44}$$

The present situation is quite different from that examined in the previous paragraph. Whereas the elongations of the three bars can readily be expressed in terms of the two displacement components, it is not possible to express the two displacement components in terms of the three elongations. This stems from the fact that given the three elongations, eqs. (9.44) form an over-determined set three equations for the two unknown displacement components. It is possible, however, to eliminate the two displacement components from eqs. (9.44) to find the compatibility equation of the problem

$$e_A - 2e_B \cos\theta + e_C = 0. \tag{9.45}$$

With this approach to the problem, it is easy to predict the number of compatibility equations: because the three bar elongations are expressed in terms of two displacement components, a single compatibility condition exists.

This simple example underlines a fundamental difference between the two- and three-bar trusses shown in figs. 9.34 and 9.35, respectively. For the two-bar truss, the deformed configuration of the system can be determined through purely geometric constructions, given the elongations of each bar. For the three-bar truss, the bar elongations are not independent of each other and must satisfy a compatibility condition. If this condition is satisfied, the deformed configuration of the system can be determined through purely geometric constructions.

Another fundamental difference between these two trusses exists: the two-bar truss is isostatic, whereas its three-bar counterpart is hyperstatic. In fact, the number of compatibility equations is equal to the order of redundancy of the hyperstatic problem, as defined in section 4.3. For isostatic problems, the order of redundancy is zero and therefore, the number of compatibility equations is zero.

To better understand the relationship between the number of compatibility equations and the order of a hyperstatic system, consider the following reasoning. The two-bar truss involves two force components (the forces in the two bars) that are linked by two equilibrium equations, the horizontal and vertical equilibrium equations for the forces acting at joint **O**. These two forces can be determined solely from the equilibrium equations, and hence, the system is isostatic. The same truss involves two elongations (the elongations in the two bars) that are related to the two displacement components of joint **O**. The two displacements are uniquely defined by the two elongations, leaving no compatibility conditions.

In contrast, the three-bar truss involves three force components (the forces in the three bars) that are linked by two equilibrium equations, the horizontal and vertical equilibrium equations for the forces acting at joint **O**. These three forces cannot be determined from the two equilibrium equations, and hence, the system is hyperstatic of degree 1. The same truss involves three elongations (the elongations in the three bars) that are related to the two displacement components of joint **O**. The three elongations must satisfy one condition to be compatible with the two displacement components.

In a general planar truss, the number of force components (one per bar) equals the number of elongations (one per bar). Each node of the truss introduces two independent displacement components and two independent equilibrium equations along two orthogonal directions. Starting from an isostatic configuration, the addition of one bar connecting existing joints results in a hyperstatic system of order 1 and creates one compatibility equation. Each additional bar connecting existing joints increase the order by one and creates a new compatibility equation. Therefore, the number of compatibility equations will always equal the order of the hyperstatic problem.

9.6.2 Principle of complementary virtual work for trusses

As illustrated in fig. 9.33, the principle of complementary virtual work focuses on a single group of equations, the strain-displacement relationships, instead of Newton's law, which the principle of virtual work focuses on. The strain-displacement equations simply provide a definition of the strains, and they are based on purely geometric arguments, as discussed in section 1.4.

Three-bar truss under applied load

Consider the three-bar truss depicted in fig. 9.36. It is assumed to undergo compatible deformations so that the three bar elongations satisfy the elongation-displacement relationships, eqs. (9.44). The following statement is now constructed

$$\begin{aligned} \delta W' = &- [e_A - u_1 \cos\theta - u_2 \sin\theta]\, \delta F_A - [e_B - u_1]\, \delta F_B \\ &- [e_C - u_1 \cos\theta + u_2 \sin\theta]\, \delta F_C = 0, \end{aligned} \tag{9.46}$$

where δF_A, δF_B, and δF_C are three arbitrary quantities, called *virtual forces*, and $\delta W'$ is the *complementary virtual work*.

The bracketed terms are the three elongation-displacement relationships of the system. Since the truss undergoes compatible deformations, the elongation-displacement equations are satisfied, and the bracketed terms in the statement vanish. Hence, the above statement is true for any arbitrary virtual forces, δF_A, δF_B, and δF_C. Simple algebraic manipulations then lead to

$$\begin{aligned} \delta W' = &- e_A \delta F_A - e_B \delta F_B - e_C \delta F_C \\ &+ u_1 \left(\delta F_A \cos\theta + \delta F_B + \delta F_C \cos\theta\right) + u_2 \sin\theta \left(\delta F_A - \delta F_C\right) = 0, \end{aligned} \tag{9.47}$$

for all virtual forces.

Figure 9.36 depicts a free body diagram of joint **O**, and the equilibrium equations are found as $F_A \cos\theta + F_B + F_C \cos\theta = P$ and $F_A - F_C = 0$. A set of forces that satisfies these equilibrium equations is said to be *statically admissible*.

Figure 9.36 also shows a set of virtual forces acting at joint **O**. These virtual forces are said to be *statically admissible virtual forces* if they satisfy the following equilibrium equations at the joint,

$$\begin{aligned} \delta F_A \cos\theta + \delta F_B + \delta F_C \cos\theta &= 0, \\ \delta F_A - \delta F_C &= 0. \end{aligned} \tag{9.48}$$

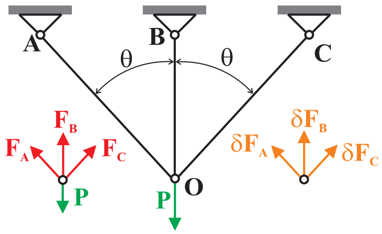

Fig. 9.36. Three-bar truss with applied load.

These equations do not include the externally applied loads because $\delta P = 0$ for a specified or given value of P. The geometry of the system is given, hence $\delta\theta = 0$.

Statement (9.47) is true for all arbitrary virtual forces. If the virtual forces, however, are required to be statically admissible, that is if eqs. (9.48) are satisfied, a much simpler statement results

$$\delta W' = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = 0, \tag{9.49}$$

for all statically admissible virtual forces. The internal virtual work done by a force, F, in a bar undergoing an elongation, e, is defined by eq. (9.29) as $\delta W_I = -F \delta e$. In statement (9.49), the three terms represent the *internal complementary virtual work* of the truss,

$$\delta W_I' = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = -\sum_{i=1}^{N_b} e_i \delta F_i, \tag{9.50}$$

where the last equality gives the general expression for internal complementary virtual work in a truss consisting of N_b bars. Statement (9.49) is now recast in a compact form as

$$\delta W' = \delta W_I' = 0, \tag{9.51}$$

for all statically admissible virtual forces.

The reasoning developed in the previous paragraphs can be reversed. Equation (9.51) is equivalent to statement (9.49). If this statement holds for all statically admissible virtual forces, eq. (9.47) must also hold under the same conditions. Simple algebraic manipulations then lead to eq. (9.46) which implies the elongation-displacement relationships of the problem, because the virtual forces are arbitrary. Although the developments presented above apply to a three-bar truss, all the steps of the reasoning would still hold for trusses of arbitrary configuration.

Three-bar truss under prescribed displacement

The three-bar truss depicted in fig. 9.37 is similar to the truss discussed in the previous section, except that instead of being subjected to a concentrated load at joint **O**, the downward vertical displacement of joint **B** is now prescribed to be of magnitude Δ.

Prescribed displacements form an important class of problem parameters. Imagine that a displacement-controlled actuator, such as a screw-jack or a displacement-controlled servo-hydraulic actuator, is oriented downward at point **B**. The natural length of vertical bar **BO** is L, and displacement Δ is prescribed. The actuator will provide whatever force is required to obtain the specified displacement. The force required to obtain the specified displacement, often called the *"driving force,"* D, is as yet unknown.

A prescribed displacement of this type fundamentally affects the statement of the principle of complementary virtual work because it directly impacts the compatibility equations. Indeed, the elongation-displacement relationship for bar **B** now becomes $e_B = u_1 - \Delta$, instead of $e_B = u_1$.

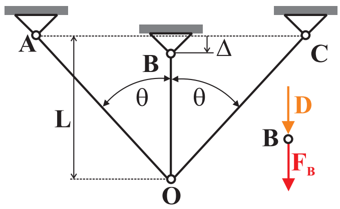

Fig. 9.37. Three-bar truss with prescribed displacement.

The following statement of complementary virtual work is constructed from the compatibility equations expressed in homogeneous form

$$\begin{aligned}\delta W' = &- \left[e_A - u_1 \cos\theta - u_2 \sin\theta\right] \delta F_A - \left[e_B - u_1 + \Delta\right] \delta F_B \\ &- \left[e_C - u_1 \cos\theta + u_2 \sin\theta\right] \delta F_C = 0,\end{aligned} \tag{9.52}$$

where δF_A, δF_B, and δF_C are, here again, arbitrary virtual forces. This statement should be compared to eq. (9.46), written for the same truss in the absence of a prescribed displacement.

Because the truss undergoes compatible deformations, the elongation-displacement equations are satisfied, and the bracketed terms in the statement vanish. Simple algebraic manipulations then lead to

$$\begin{aligned}\delta W' = &-e_A \delta F_A - e_B \delta F_B - e_C \delta F_C - \Delta\, \delta F_B \\ &+ u_1 \left(\cos\theta \delta F_A + \delta F_B + \cos\theta \delta F_C\right) + u_2 \sin\theta \left(\delta F_A - \delta F_C\right) = 0,\end{aligned} \tag{9.53}$$

for all arbitrary virtual forces.

Consider now a set of statically admissible virtual forces that satisfy the following equilibrium equations,

$$\delta F_A \cos\theta + \delta F_B + \delta F_C \cos\theta = 0, \;\; \delta F_A - \delta F_C = 0, \;\; \delta F_B + \delta D = 0, \tag{9.54}$$

where the first two equations correspond to the equilibrium conditions of joint **O** and the third to that at joint **B**. The virtual driving force, δD, does not vanish because this force is unknown.

Statement (9.53) is true for all arbitrary virtual forces. If the virtual forces are required to be statically admissible, however, a simpler statement results,

$$\delta W' = \Delta\, \delta D - e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = 0. \tag{9.55}$$

As observed earlier, the last three terms of this expression represent the internal complementary virtual work, $\delta W_I'$, as defined in eq. (9.50). The product of a force, D, by the displacement of its point of application in the direction of the force, Δ, is defined in section 9.2.2 as the work done by the force, $W = D\Delta$. The virtual work done by the same force is introduced in section 9.3.4 as the product of the real force by a virtual displacement, $\delta W = D\ \delta\Delta$. In statement 9.55, the first term is the product of the true displacement by a virtual force, and is called the *external complementary virtual work*,

$$\delta W_E' = \Delta\ \delta D. \tag{9.56}$$

Statement (9.55) is now recast in a compact form as

$$\delta W' = \delta W_E' + \delta W_I' = 0, \tag{9.57}$$

for all statically admissible virtual forces. In the absence of prescribed displacements, the external complementary virtual work vanishes and the simpler statement of eq. (9.51) remains. As before, the reasoning can be reversed: statement 9.52 can be recovered from statement 9.57. Consequently, the elongation-displacement are obtained and the truss undergoes compatible deformations.

In summary, the elongation-displacement relationships of the problem are satisfied if and only if the sum of the external and internal complementary virtual work vanishes for all statically admissible virtual forces. This leads to the principle of complementary virtual work.

Principle 7 (Principle of complementary virtual work) *A truss undergoes compatible deformations if and only if the sum of the internal and external complementary virtual work vanishes for all statically admissible virtual forces.*

In this principle, the virtual forces must be statically admissible. If the complementary virtual work is required to vanish for all arbitrary virtual forces, *i.e.*, for all independently chosen arbitrary δF_A, δF_B, δF_C, and δD, eq. (9.55) implies $e_A = e_B = e_C = \Delta = 0$, which in turn, implies that the truss cannot deform. This is clearly not correct.

Because the virtual forces are statically admissible, they must satisfy eq. (9.54), which forms a set of three equations for the four statically admissible virtual forces. This means that it is possible to express three of the virtual force components in terms of the fourth: $\delta F_B = -2\delta F_A \cos\theta$, $\delta F_C = \delta F_A$, and $\delta D = 2\delta F_A \cos\theta$, for example. The principle of complementary virtual work now becomes

$$\begin{aligned}\delta W' &= \Delta(2\delta F_A \cos\theta) - e_A\delta F_A - e_B(-2\delta F_A\cos\theta) - e_C\delta F_A\\ &= [2\Delta\cos\theta - e_A + 2e_B\cos\theta - e_C]\,\delta F_A = 0.\end{aligned}$$

Since the remaining virtual force component, δF_A, is now entirely arbitrary, the bracketed term must vanish, yielding the compatibility equation of the problem: $e_A - 2(e_B + \Delta)\cos\theta + e_C = 0$.

Clearly, when the concept of statically admissible virtual forces is properly interpreted, the principle of complementary virtual work yields the correct compatibility equation of the problem.

9.6.3 Complementary virtual work

Complementary virtual work is defined as the work done by virtual forces acting through real displacements. In this sense, it is complementary to the virtual work done by real forces acting through virtual displacements. In both cases, the real quantities are assumed to remain fixed during the application of the virtual quantities. The meaning of the term "complementary" will be illustrated in the following discussion.

Consider a uniform bar fixed at one end and subjected to an axial load, F, at its tip and let the resulting tip deflection or elongation be denoted u. The material the bar is made of is not necessarily linearly elastic, and hence, the load-displacement curve is not a straight line, as illustrated in fig. 9.38.

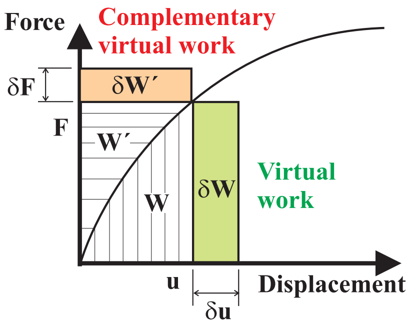

Fig. 9.38. Work and complementary work and their virtual counterparts.

As the displacement increases from 0 to u, the work done by the tip force is $W = \int_0^u F \,\mathrm{d}u$, see eq. (9.7). This integral corresponds to the area under the curve shown in fig. 9.38. If the material is linearly elastic, say $F = ku$, where k is the axial stiffness of the bar, the work simply becomes $W = \int_0^u ku \,\mathrm{d}u = ku^2/2 = Fu/2$.

The complementary work is defined as $W' = \int_0^F u \,\mathrm{d}F$, and corresponds to the area to the left of the load-displacement curve, as indicated in fig. 9.38. Note that for a nonlinearly elastic material, the work and complementary work are not equal. For a linearly elastic material, however, $W' = \int_0^F F/k \,\mathrm{d}F = F^2/(2k) = Fu/2 = W$. For either linearly or nonlinearly materials, $W + W' = Fu$, which explains why the complementary work is called "complementary."

The virtual work and complementary virtual work are also shown in fig. 9.38. The virtual work is the shaded area to the right of that representing the work itself. While computing the work done by the applied force, the force is a function of the displacement, $F = F(u)$, but when computing the virtual work, the force is held constant. As discussed in example 9.2, the term "virtual" used to qualify the virtual displacements indicates that the forces remain unchanged by these virtual displacements.

The complementary virtual work is the shaded area above that representing the complementary work itself. While computing the complementary work done by the applied force, the displacement is a function of the force, $u = u(F)$, but when computing the complementary virtual work, the displacement is held constant. Here again, the term "virtual forces" emphasizes the fact that the displacements remain unchanged by these virtual forces.

9.6.4 Applications to trusses

By now, the similarities and differences between the principle of virtual work and the principle of complementary virtual work are clear.

- The principle of virtual work focuses of the equilibrium equations of the system, whereas the principle of complementary virtual work focuses on the strain-displacement equations, see fig. 9.33.
- The principle of virtual work identifies the equilibrium state of the system from among all kinematically compatible configurations; the principle of complementary virtual work identifies the kinematically compatible configuration from among all statically admissible states.
- The concepts of virtual displacements and virtual forces are subjected to similar restrictions: the former leave the real system forces unchanged, whereas the latter leave real system displacements unchanged.
- The work and its complementary counterpart complement each other as illustrated in fig. 9.38.

To generalize these results, consider a planar truss consisting of a number of bars connected at N nodes. When using arbitrary virtual displacements, the principle of virtual work will provide $2N$ equilibrium equations corresponding to 2 equilibrium equations at each of the N nodes. On the other hand, the principle of complementary virtual work yields the compatibility equations of the problem: n equations will be produced for a hyperstatic truss of order n. If the truss is isostatic, however, no compatibility equations exist, and the principle of complementary virtual work yields no information about the system.

The principle of virtual work provides a statement of equilibrium. When supplemented with the strain-displacement relationships and constitutive laws, it enables the systematic development of the displacement method of solution first presented in section 4.3.2. This approach is easily implemented using computers that can solve the resulting large sets of linear equations.

The principle of complementary virtual work, on the other hand, provides a statement of compatibility. When supplemented with the constitutive laws and equilibrium equations, it enables the development of the force method first presented in section 4.3.3. More often than not, the hyperstatic order is much less than the number of nodes, $n \ll N$, and hence, the principle of complementary virtual work generates only a few ($n \ll N$) equations, which would seem to lead to a simpler solution process. The principle of complementary virtual work, however, suffers from a major drawback: the virtual forces must be statically admissible, *i.e.*, they must form a set of self-equilibrating virtual forces. This implies that the equilibrium equations must be derived at each joint of the truss before the principle can be applied. Thus, while the principle of complementary virtual work generates fewer equations, it requires much more extensive work for the generation of these equations. This problem hinders the systematic application of this principle, and helps explain why the principle of virtual work is used much more widely than its complementary counterpart.

Example 9.14. Three-bar truss under vertical load at joint
The three-bar truss problem depicted in fig. 9.36 is treated earlier in example 9.13 using the principle of virtual work. In this example, the principle of complementary virtual work will be used to solve the same problem, which involves the three forces, F_A, F_B, and F_C, acting in bars **A**, **B** and **C**, respectively. The equations of equilibrium of the problem are found earlier to be $F_A \cos\theta + F_B + F_C \cos\theta = P$ and $F_A = F_C$, and the statically admissible virtual forces satisfy eqs. (9.48).

The principle of complementary virtual work now can be written as

$$\delta W' = -e_A \delta F_A - e_B \delta F_B - e_C \delta F_C = 0, \tag{9.58}$$

for all statically admissible virtual forces. In this example, e_A, e_B, and e_C are the elongations of the three bars are the true displacements components of joint **O** along the directions of bars **A**, **B** and **C** respectively. It is easy to verify from kinematics that $e_A = u_1 \cos\theta + u_2 \sin\theta$, $e_B = u_1$, and $e_C = u_1 \cos\theta - u_2 \sin\theta$, where $\underline{u} = u_1 \bar{\imath}_1 + u_2 \bar{\imath}_2$ is the true displacement vector of joint **O**. The reaction forces at the attachment points **A**, **B**, and **C** do not appear in the above statements because the true displacements at those points are zero, and therefore the complementary virtual work they perform also vanishes.

Because the virtual forces must be statically admissible, the three virtual forces components are linked by the two equilibrium conditions, eqs. (9.48), and hence, it is possible to express two of them in terms of the third: $\delta F_A = \delta F_C$ and $\delta F_B = -2\cos\theta \delta F_C$. The complementary virtual work then becomes

$$\delta W' = -e_A \delta F_C + 2 e_B \cos\theta \delta F_C - e_C \delta F_C = -\left[e_A - 2 e_B \cos\theta + e_C \right] \delta F_C = 0.$$

The virtual force δF_C is arbitrary, and therefore the bracketed term must vanish, yielding the elongation compatibility equation for the problem

$$e_A - 2 e_B \cos\theta + e_C = 0. \tag{9.59}$$

Using the force method, this equation, when combined with the two equilibrium equations, can be used to solve for the 3 unknown bar forces. The constitutive laws are $F_A = e_A \cos\theta (E\mathcal{A})_A / L$, $F_B = e_B (E\mathcal{A})_B / L$, and $F_C = e_C \cos\theta (E\mathcal{A})_C / L$, for bars **A**, **B** and **C**, respectively. Expressing the elongations in terms of forces in the compatibility equation, eq. (9.59), yields

$$\frac{F_A}{\bar{k}_A} - 2 F_B \cos^2\theta + \frac{F_C}{\bar{k}_C} = 0,$$

where $\bar{k}_A = (E\mathcal{A})_A / (E\mathcal{A})_B$ and $\bar{k}_C = (E\mathcal{A})_C / (E\mathcal{A})_B$ are non-dimensional stiffness ratios.

The equilibrium equations of the problem are $F_A \cos\theta + F_B + F_C \cos\theta = P$ and $F_A = F_C$, and these two equilibrium equations, together with the compatibility equation expressed in terms of forces, form a set of three equations for the three force components, which are found to be

$$\frac{F_A}{P} = \frac{F_C}{P} = \frac{2\bar{k}_A\bar{k}_C\cos^2\theta}{\bar{k}}, \quad \frac{F_B}{P} = \frac{\bar{k}_A + \bar{k}_C}{\bar{k}},$$

where $\bar{k} = \bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta$ is the non-dimensional stiffness of the truss. This result is identical to that found with the principle of virtual work, see eq. (9.42).

Bar elongations are then evaluated with the help of the constitutive laws, and finally, the displacement can be obtained from the elongation-displacement relationships.

As expected, all results are identical to those found using the principle of virtual work. Except for the fact that the compatibility equations are obtained from the principle of complementary virtual work rather than geometric arguments, the solution process presented here mirrors that of the force method developed in section 4.3.3.

Example 9.15. Three-bar truss under prescribed displacement
Next, the same three-bar truss problem will be treated again, but the structure is now subjected to a prescribed displacement, Δ, at point **B**, as depicted in fig. 9.37. This problem involves involves the three forces, F_A, F_B, and F_C, acting in bars **A**, **B**, and **C**, respectively, and the driving force, D, applied at point **B** to achieve the specified displacement, Δ. The equilibrium equations of the problem are found earlier as $F_A\cos\theta + F_B + F_C\cos\theta = 0$, $F_A - F_C = 0$, and $F_B + D = 0$, and the statically admissible virtual forces must satisfy eqs. (9.54).

The principle of complementary virtual work is

$$\delta W' = \Delta\,\delta D - e_A\delta F_A - e_B\delta F_B - e_C\delta F_C = 0, \tag{9.60}$$

for all statically admissible virtual forces. The elongations of the three bars, e_A, e_B, and e_C, are associated with the true displacements of point **O** along the directions of bars **A**, **B** and **C** respectively.

Because the virtual forces must be statically admissible, the three virtual forces components are linked by the two equilibrium conditions, eqs. (9.54), and hence, it is possible to express three of them in terms of the fourth: $\delta F_A = \delta F_C$, $\delta F_B = -2\cos\theta\;\delta F_C$, and $\delta D = 2\cos\theta\;\delta F_C$. The complementary virtual work then becomes

$$\delta W' = [2\Delta\cos\theta - e_A + 2e_B\cos\theta - e_C]\,\delta F_C = 0.$$

The virtual force δF_C is arbitrary and therefore, the bracketed term must vanish. This yields the elongation compatibility equation for the problem

$$e_A - 2(e_B + \Delta)\cos\theta + e_C = 0. \tag{9.61}$$

Using the force method, this equation, when combined with the 3 equilibrium equations, can be used to solve for the 4 unknown bar forces. The constitutive laws are $F_A = e_A\cos\theta(E\mathcal{A})_A/L$, $F_B = e_B(E\mathcal{A})_B/L$, and $F_C = e_C\cos\theta(E\mathcal{A})_C/L$, for bars **A**, **B**, and **C**, respectively. Expressing the elongations in terms of forces in the compatibility equation, eq. (9.59), yields

$$\frac{F_A}{2\bar{k}_A\cos^2\theta} - F_B + \frac{F_C}{2\bar{k}_C\cos^2\theta} = \frac{\Delta}{L}(E\mathcal{A})_B,$$

where $\bar{k}_A = (E\mathcal{A})_A/(E\mathcal{A})_B$ and $\bar{k}_C = (E\mathcal{A})_C/(E\mathcal{A})_B$ are non-dimensional stiffness ratios for bars **A** and **C**, respectively.

The equilibrium equations of the problem are $F_A \cos\theta + F_B + F_C \cos\theta = 0$, $F_A - F_C = 0$, and $F_B + D = 0$. These three equilibrium equations, together with the above compatibility equation expressed in terms of forces, form a set of four equations for the four force components. The non-dimensional forces in bars **A** and **C** are found as

$$\frac{F_A}{(E\mathcal{A})_B} = \frac{F_C}{(E\mathcal{A})_B} = \frac{2\bar{k}_A\bar{k}_C\cos^2\theta}{\bar{k}}\frac{\Delta}{L},$$

whereas the non-dimensional driving force, D, and the force in bar **B** are

$$\frac{D}{(E\mathcal{A})_B} = -\frac{F_B}{(E\mathcal{A})_B} = \left(1 - \frac{\bar{k}_A + \bar{k}_C}{\bar{k}}\right)\frac{\Delta}{L},$$

where $\bar{k} = \bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta$ is the non-dimensional stiffness of the truss.

Bar elongations are evaluated with the help of the constitutive laws, and finally, the displacement can be obtained from the elongation-displacement relationships.

9.6.5 Problems

Problem 9.13. Three springs in series

(1) Use the principle of complementary virtual work to determine the forces in each of the three springs connected in series depicted in fig. 9.39. *(2)* Find the solution of the same problem using the force method developed in section 4.3.3. *(3)* Compare the two solution approaches.

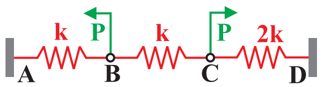

Fig. 9.39. Three-spring hyperstatic system.

9.6.6 Unit load method for trusses

In section 9.6, the principle of complementary virtual work is shown to yield the compatibility equations of a system. It is also the basis for a general approach, called the *unit load method,* to determine deflections at specific points of structures. This simple and elegant method provides the displacement or rotation at any point of a structure. It will be presented here in the context of truss structures.

Consider the two-bar truss subjected to a load P, as depicted in fig. 9.40. Based on Newton's law or on the principle of virtual work, the equilibrium equations of the problem can be derived, and with the help of the free body diagram shown in the figure can be used to find the actual forces in the bars, F_A and F_B.

Next, the principle of complementary virtual work will be use to determine the displacement of point **O** in the direction of the applied load. To accomplish this, imagine that the displacement of point **O** is prescribed to be of magnitude Δ in the vertical direction, as indicated in the right part of fig. 9.40. Because the displacement at point **O** is prescribed to be Δ, the external complementary work is given by eq. (9.56) as $\delta W'_E = \Delta\delta D$, where δD is the virtual driving force that is applied to

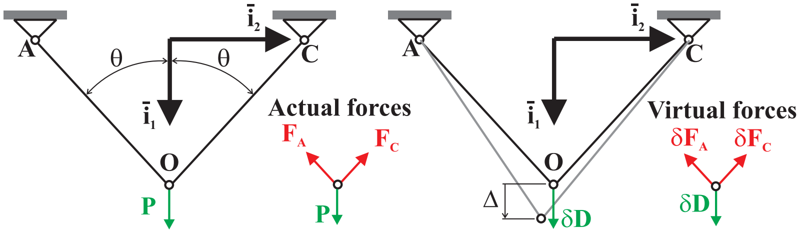

Fig. 9.40. The unit load method for a two-bar truss.

achieve the desired displacement, Δ. The principle of complementary virtual work, eq. (9.57), now implies $\delta W_E' + \delta W_I' = 0$ and can be recast as

$$\Delta \delta D = -\delta W_I', \tag{9.62}$$

for all statically admissible virtual forces.

The internal complementary virtual work is given by eq. (9.50) as $\delta W_I' = -e_A \delta F_A - e_C \delta F_C$, where e_A and e_C are actual elongations of bars **A** and **C**, respectively. The principle of complementary virtual work, eq. (9.62), becomes $\Delta \delta D = e_A \delta F_A + e_C \delta F_C$. For a more general truss consisting of N_b bars, this statement can be written as

$$\Delta \delta D = \sum_{i=1}^{N_b} e_i \delta F_i, \tag{9.63}$$

for all statically admissible virtual forces.

Let δD, δF_A and δF_C be a set of statically admissible virtual forces; the free body diagram in the right part of fig. 9.40 leads to $\delta F_A - \delta F_C = 0$ and $\delta D - (\delta F_A + \delta F_C) \cos\theta = 0$. Because the two equilibrium equations of the system link the three virtual forces, δD, δF_A and δF_C, it is possible to select one arbitrarily, and the other two are then obtained from the equilibrium equations. In the unit load method, *the virtual driving force is select to be a unit load*, $\delta D = 1$, from which it follows that $\delta F_A = \delta F_C = \Delta\delta D/(2\cos\theta) = 1/(2\cos\theta)$.

To simplify the notation, let $\delta D = 1$ be a unit virtual driving force and let $\delta F_A = \hat{F}_A$ and $\delta F_C = \hat{F}_C$ denote the corresponding statically admissible virtual forces. Equation (9.63) now becomes

$$\Delta = \sum_{i=1}^{N_b} \hat{F}_i \, e_i. \tag{9.64}$$

This equation yields the desired displacement at a point of the truss.

Two distinct sets of forces are involved in the unit load approach: F_i and $\hat{F}_i$. Forces F_i are the actual forces that develop in the bars under the externally applied loads. Because these forces are the actual forces acting in the system, they must satisfy all equilibrium conditions, and the associated elongations must be compatible. Forces $\hat{F}_i$ and the unit driving force form a set of statically admissible forces, as required by the principle of complementary virtual work. Because these forces are statically admissible, they must satisfy the equilibrium equations, but the associated elongations are not required to be compatible.

Equation (9.64) seems to be incorrect because the left-hand side has units of displacement, whereas the right-hand side has units of force times displacement. This is because the virtual driving force is selected to be of unit magnitude. Equation (9.64) should be written as $1 \cdot \Delta = \sum_{i=1}^{N_b} \hat{F}_i \, e_i$, where the term "1" has units of force, which reconciles the units on the two sides of the equation.

If the bars are made of a linearly elastic material, the actual, compatible elongations of the bars are obtained as $e_i = F_i L_i / (E_i \mathcal{A}_i)$. Equation (9.64) now becomes

$$\Delta = \sum_{i=1}^{N_b} \frac{\hat{F}_i F_i L_i}{E_i \mathcal{A}_i}. \tag{9.65}$$

The unit load method is based on the principle of complementary virtual work, which expresses compatibility conditions. Material constitutive laws are not considered in the derivation of this principle, which therefore remains true for all constitutive laws. Consequently, the unit load method for trusses as expressed by eq. (9.64) applies for all material behavior, whereas the use of eq. (9.65) is limited to linearly elastic materials.

The unit load method applied to truss structures can be summarized in the following three steps.

1. Determine the forces, F_i, $i = 1, 2, \ldots, N_b$, acting in the bars under the effect of the externally applied loads. From these determine e_i, $i = 1, 2, \ldots, N_b$ using the constitutive law for the material.
2. Determine the bar forces, $\hat{F}_i$, $i = 1, 2, \ldots, N_b$. These form a set of statically admissible forces in equilibrium with a unit load applied at the point and in the direction of the desired displacement component. This is called the *unit load system*.
3. Use eq. (9.64) to evaluate the displacement component at the point and in the direction of application of the unit load. If the truss is made of a linearly elastic material, use eq. (9.65).

The first two steps of the procedure require the evaluation of two sets of bar forces generated by two distinct loading conditions: first, the externally applied loads, and second, a unit load applied at a joint. If the truss is isostatic, bar forces can be evaluated based on the equilibrium equations at each joint of the truss. If the truss is hyperstatic, the unit load method is still applicable, although the evaluation of bars forces for these two loading conditions becomes more cumbersome; such cases will be treated in in section 9.8.1.

The unit load method can also be used to determine rotation at a point of the structure. A slight modification of the procedure presented above is then required: instead of applying a unit load, a unit moment is applied. For prescribed rotations, the complementary external virtual work is $\delta W_E' = \Phi \delta M$, where Φ is the prescribed rotation and δM the virtual driving moment. The principle of complementary virtual work, eq. (9.57), now implies $\delta W'_E + \delta W'_I = 0$, or

$$\Phi \delta M = -\delta W'_I, \tag{9.66}$$

Instead of using a unit force, $\delta D = 1$, a unit moment is used, $\delta M = 1$, and the rest of the procedure is identical to that described above. For such cases, the method becomes the "unit moment method." Equation (9.64) now becomes $\Phi = \sum_{i=1}^{N_b} \hat{F}_i \, e_i$, where $\hat{F}_i$ are the forces acting in the bars under the action of this unit moment.

As pointed out earlier, the unit load method is not restricted to linearly elastic materials. If a nonlinear material is employed in the truss, eq. (9.64) still applies. For an isostatic truss, once the bar forces associated with the externally applied loads have been determined from the joint equilibrium equations, bar extensions can be evaluated from the nonlinear material constitutive law. In fact, it is also possible to consider bar extensions that are not due to mechanical loads. These include manufacturing imperfections and thermal deformations.

In the examples presented below, a tabular presentation will be used to keep track of the contributions of individual bars. This approach is convenient and minimizes computational errors.

Example 9.16. Joint deflection in a simple 2-bar truss

A simple two-bar truss shown in fig. 9.41 will be used to illustrate the unit load method. The truss member stiffnesses are $k_A = (E\mathcal{A}/L)_A$ and $k_C = (E\mathcal{A}/L)_C$.

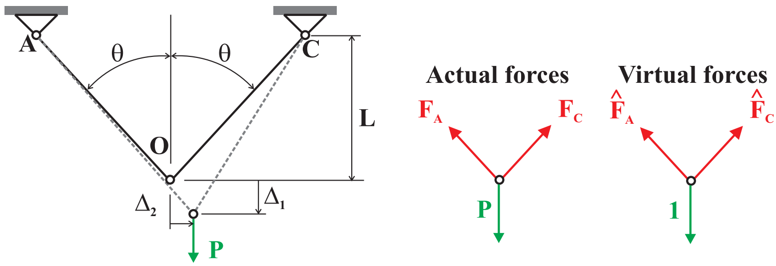

Fig. 9.41. Two-bar truss with unsymmetric properties and vertical load at joint.

Step 1 starts with the determination of the bar forces and extensions due to the externally applied loads. The equilibrium equations for the joint **O** yield $F_A = F_C = P/(2\cos\theta)$. The elongations associated with these bar forces are $e_A = F_A L_A/(E\mathcal{A})_A$ and $e_C = F_C L_C/(E\mathcal{A})_C$, and hence,

$$e_A = \frac{PL}{(E\mathcal{A})_A} \frac{1}{2\cos^2\theta}, \quad e_C = \frac{PL}{(E\mathcal{A})_C} \frac{1}{2\cos^2\theta}.$$

In step 2, a unit load is applied at the point and in the direction of the desired deflection component. Next, the bar forces arising from the application of this unit load are evaluated. This is illustrated in the second free body diagram in the right part of fig. 9.41. Because the unit virtual load acts at the same point and in the same direction as the applied load, the same equilibrium equation yields the desired forces as

$$\hat{F}_A = \frac{1}{2\cos\theta}, \quad \hat{F}_C = \frac{1}{2\cos\theta}.$$

The last step of the procedure uses eq. (9.65) to find the vertical displacement of joint **O** as

$$\begin{aligned}\Delta_1 = &\sum_{i=1}^{N_b} \hat{F}_i e_i = \frac{1}{2\cos\theta}\frac{PL}{2\cos^2\theta(E\mathcal{A})_A} + \frac{1}{2\cos\theta}\frac{P}{2\cos^2\theta(E\mathcal{A})_C} \\ =&\frac{PL}{4\cos^3\theta}\frac{(E\mathcal{A})_A + (E\mathcal{A})_C}{(E\mathcal{A})_A(E\mathcal{A})_C}.\end{aligned} \qquad (9.67)$$

Next, the horizontal deflection component of joint **O**, denoted Δ_2, will be evaluated. The first step of the procedure is identical to that developed above. In step 2, a unit load is applied in the horizontal direction at joint **O**, because the desired displacement component is in that direction. Based on equilibrium equations, the virtual forces associated with this horizontal unit load are $\hat{F}_A = 1/2 \ \sin\theta$ and $\hat{F}_C = -1/2 \ \sin\theta$. Equation (9.64) then yields the desired displacement components as

$$\begin{aligned}\Delta_2 = &\sum_{i=1}^{N_b} \hat{F}_i e_i = \frac{1}{2\sin\theta}\frac{PL}{2\cos^2\theta(E\mathcal{A})_A} - \frac{1}{2\sin\theta}\frac{P}{2\cos^2\theta(E\mathcal{A})_C} \\ =&\frac{PL}{4\sin\theta\cos^2\theta}\frac{(E\mathcal{A})_A - (E\mathcal{A})_C}{(E\mathcal{A})_A(E\mathcal{A})_C}.\end{aligned} \qquad (9.68)$$

Example 9.17. Joint deflection in a 2-bay planar truss

A more complicated planar truss will be analyzed next. Consider the two-bay cantilevered truss subjected to externally applied vertical loads depicted in fig. 9.42. Determine the vertical deflection of joint **F**. The following data are provided: $L = 30$ in., and for all bars, $E = 30 \times 10^6$ psi and $\mathcal{A} = 0.1$ in^2. The load applied at joints **B** and **C** are each 1,000 lbs.

The truss material is linearly elastic and therefore eq. (9.65) can be used. This is most conveniently carried out in a tabular form in table 9.1. The first column in the table lists the bars, and the second column lists the bar flexibility factors, $L_i/(E_i\mathcal{A}_i)$.

The first step of the unit load method calls for the determination of the bar forces under the externally applied loads; the results of this computation are listed in the third column of table 9.1. In the second step, a unit load is applied in the vertical direction at joint **F**. The bar forces generated by the unit load are listed in the fourth column of table 9.1. The last column of the table lists the products $\hat{F}_i F_i L_i/(E_i\mathcal{A}_i)$. In view of eq. (9.65), the sum of the numbers in that column yields the vertical displacement component at joint **F**, $\Delta = 0.185$ in.

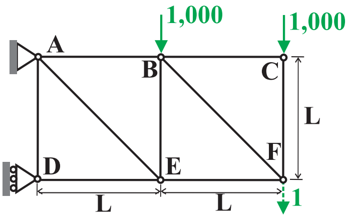

Fig. 9.42. Tip deflection of two-bay planar truss.

Table 9.1. Calculation of vertical deflection at joint **F** for the 2-bay planar truss.

Bar	$10^6 \times L_i/(E_i\mathcal{A}_i)$	F_i	$\hat{F}_i$	$\hat{F}_i F_i L_i/(E_i\mathcal{A}_i)$
AB	10	1,000	1	0.01
BC	10	0	0	0
DE	10	-3,000	-2	0.06
EF	10	-1,000	-1	0.01
AD	10	0	0	0
BE	10	-2,000	-1	0.02
CF	10	-1,000	0	0
AE	$10\sqrt{2}$	$2,000\sqrt{2}$	$\sqrt{2}$	0.056
BF	$10\sqrt{2}$	$1,000\sqrt{2}$	$\sqrt{2}$	0.028

Example 9.18. Rotation of a bar in a 2-bay planar truss
The unit load method also allows the determination of the rotation of an individual bar. To illustrate the process, the rotation of bar **CF** will be computed for the 2-bay planar truss depicted in fig. 9.43. The physical properties of the truss are identical to those used in example 9.17.

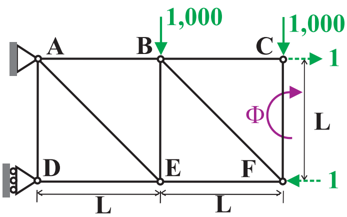

Fig. 9.43. Rotation of a bar in a two-bay planar truss.

As shown in fig. 9.43, a moment $\delta M = 1 \cdot L$ is applied to bar **CF**; this is provided by two unit load acting in opposite directions at joints **C** and **F**. Rather than a unit virtual moment, a virtual moment of magnitude $1 \cdot L$ is applied; this does not matter because the magnitude of the virtual moment is arbitrary.

The first step of the method calls for the evaluation of the actual forces in all bars when the truss is subjected to the externally applied loads. These results are listed in the third column of table 9.2 and are identical to the corresponding results listed in the third column of table 9.1, because the same external loads are applied to the truss. The fourth column of table 9.2 lists the bar virtual forces generated by the unit load system applied to bar **CF**. Equation (9.65) now yields

$$1 \cdot L \cdot \Phi = \sum_{i=1}^{N_b} \frac{\hat{F}_i F_i L_i}{E_i \mathcal{A}_i}, \quad \text{or} \quad \Phi = \frac{1}{L} \sum_{i=1}^{N_b} \frac{\hat{F}_i F_i L_i}{E_i \mathcal{A}_i}.$$

The last column of the table lists the individual contributions of each bar, and summing up all contributions yields $\Phi = 0.05/L = 0.00166$ rad $= 0.96°$.

Table 9.2. Calculation of rotation of bar **CF** in the 2-bay planar truss.

Bar	$10^6 \times L_i/(E_i\mathcal{A}_i)$	F_i	$\hat{F}_i$	$\hat{F}_i F_i L_i/(E_i\mathcal{A}_i)$
AB	10	1,000	1	0.01
BC	10	0	1	0
DE	10	-3,000	-1	0.03
EF	10	-1,000	-1	0.01
AD	10	0	0	0
BE	10	-2,000	0	0
CF	10	-1,000	0	0
AE	$10\sqrt{2}$	$2,000\sqrt{2}$	0	0
BF	$10\sqrt{2}$	$1,000\sqrt{2}$	0	0

9.6.7 Problems

Problem 9.14. Deflection of a simple square truss

Consider the square planar truss shown in fig. 9.44 and assume that all bars are of cross-sectional area, $\mathcal{A}$, and modulus, E. Joints **D**, **E**, and **F** are pinned to the ground. This problem presents symmetries that may be helpful in simplifying the force calculations. *(1)* Find the vertical deflection at joint **A**. *(2)* Find the increase in horizontal distance between joints **B** and **C**.

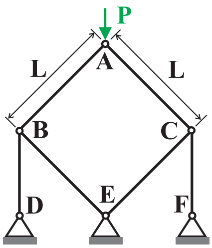

Fig. 9.44. Simple square planar truss with vertical load at joint **A**.

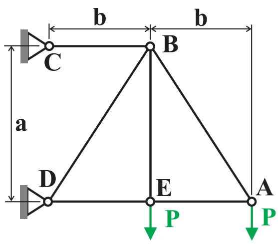

Fig. 9.45. Planar truss with nonlinear material properties.

Problem 9.15. Deflection of a truss with nonlinear material properties
The truss shown in fig. 9.45 is made of a material presenting a nonlinearly elastic behavior described by the following constitutive equation: $\sigma_1 = E_0 \epsilon_1^n$, where $E_0 = 500,000$ psi and $n = 1/3$. The truss dimensions are $a = 40$ in and $b = 30$ in, while the bar cross-sectional areas are $\mathcal{A}_{AB} = 2$, $\mathcal{A}_{BC} = 1$, $\mathcal{A}_{BD} = 2$, $\mathcal{A}_{BE} = 0.5$, $\mathcal{A}_{AE} = \mathcal{A}_{DE} = 1.5$ in^2, and the load $P = 10,000$ lbs. *(1)* Find the vertical and horizontal deflections of joint **A**.

Problem 9.16. Deflection of a planar truss
All bars of the planar truss shown in fig. 9.46 are of cross-sectional area, $\mathcal{A}$, and modulus, E. Joints **C** and **D** are pinned to the vertical wall. *(1)* Use the unit load method to find the vertical and the horizontal deflections of joint **A**, and *(2)* the rotation of bar **AB**.

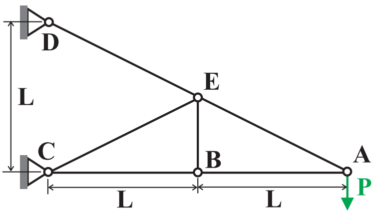

Fig. 9.46. Planar truss with tip load.

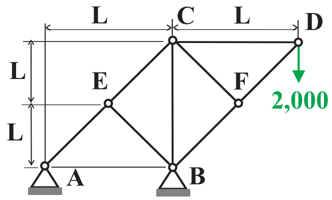

Fig. 9.47. Planar truss with tip load.

Problem 9.17. Deflection of a planar truss
Determine the vertical deflection of joint **D** in the planar truss shown in fig. 9.47. The truss is supported at point **A** and **B**, but no bar joins these two points. All bar are of cross-sectional area, $\mathcal{A}$, and modulus, E.

9.7 Internal virtual work in beams and solids

The previous sections of this chapter focus on simple mechanical problems such as particles, systems of particles, and trusses. In each case, expressions for work, virtual work, and complementary virtual work are developed. A formal proof of the principle of virtual work for three-dimensional solids will be presented in chapter 12, but by induction, it is assumed here that the principle developed for particles, systems of particles, and trusses, also holds for beams and three-dimensional solids. The key to this generalization is to develop an expression for the internal virtual work that is adapted to the structure under investigation.

Similarly, the principle of complementary virtual work is derived for trusses, but it will be shown in chapter 12 that it remains true for three-dimensional solids. By induction, the principle of complementary virtual work for trusses, stated as principle 7 and expressed by eq. (9.57), is generalized to state that "a structure undergoes compatible deformations if and only if the sum of the internal and external complementary virtual work vanishes for all statically admissible virtual stresses." Here again, the key to this generalization is to develop an expression for the complementary internal virtual work that is adapted to the structure under investigation.

Expressions for the virtual work and complementary virtual work in beams and three-dimensional solids will be developed in this section.

9.7.1 Beam bending

Euler-Bernoulli beam bending will be investigated first. The associated kinematic assumptions and their implications are presented in section 5.2. For simplicity, it is assumed that plane $(\bar{\imath}_1, \bar{\imath}_2)$ is a plane of symmetry of the problem. A bending moment, $M_3(x_1)$, acts, resulting in a rotation of the section, denoted $\Phi_3(x_1)$, and a transverse displacement, $\bar{u}_2(x_1)$.

Consider an infinitesimal slice of a beam depicted in fig. 9.48. Under the action of the bending moment, the two neighboring sections rotate by angles Φ_3 and $\Phi_3 + \mathrm{d}\Phi_3$, at span-wise locations x_1 and $x_1 + \mathrm{d}x_1$, respectively. The differential rotation of the two cross-sections generates the curvature of the differential element, $\kappa_3 = \Phi_3' = \bar{u}_2''$, see eq. (5.6).

The work done by a moment is the product of the moment by the rotation of its point of application. The work done by the moment acting on the left-hand side of the differential element of the beam is $-M_3\Phi_3$, where the minus sign is due to the fact that the moment and rotation are counted positive about opposite axes. The work done by the moment acting on the other side of the element is $M_3(\Phi_3 + \mathrm{d}\Phi_3)$. Finally, the net work done by the two moments, $\mathrm{d}W$, is found by summing up the two contributions to find $\mathrm{d}W = M_3 \mathrm{d}\Phi_3 = M_3(\mathrm{d}\Phi_3/\mathrm{d}x_1)\mathrm{d}x_1$. The total internal work done by the moment distribution acting in the beam of length L is then found by integration,

$$W_I = -\int_0^L M_3 \frac{\mathrm{d}\Phi_3}{\mathrm{d}x_1}\,\mathrm{d}x_1 = -\int_0^L M_3 \kappa_3\,\mathrm{d}x_1. \tag{9.69}$$

The minus sign is due to the fact that the internal work is that done by the internal moment, which is opposite in sign to the moment applied externally to the cross-section.

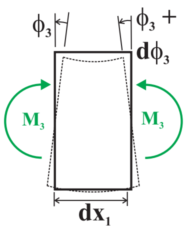

Fig. 9.48. Bending deformation of an infinitesimal segment of a beam.

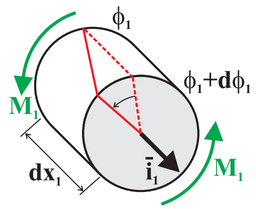

Fig. 9.49. Torsional deformation of an infinitesimal segment of a beam.

The virtual internal work and its complementary counterpart are then found by considering the virtual work done by moments undergoing virtual curvatures, $\delta\kappa_3$, and the virtual work done by virtual moments, δM_3, undergoing actual curvatures, respectively. The internal virtual work and its complementary counterpart for Euler-Bernoulli beam bending can thus be written as

$$\delta W_I = -\int_0^L M_3 \delta\kappa_3 \, \mathrm{d}x_1, \tag{9.70a}$$

$$\delta W_I' = -\int_0^L \kappa_3 \delta M_3 \, \mathrm{d}x_1. \tag{9.70b}$$

Of course, similar expressions can be derived for the internal virtual work and its complementary counterpart for bending moments, M_2, and curvatures, κ_2, acting in the orthogonal plane.

9.7.2 Beam twisting

Next, the problem of torsion of a circular bar will be investigated. The associated kinematic assumptions and their implications are presented in section 7.1.1. A torque, $M_1(x_1)$, is acting, resulting in a rotation of the section, denoted $\Phi_1(x_1)$. Consider the infinitesimal slice of a beam depicted in fig. 9.49. Under the action of the torque, the two neighboring sections rotate by angles Φ_1 and $\Phi_1 + \mathrm{d}\Phi_1$, at span-wise locations x_1 and $x_1 + \mathrm{d}x_1$, respectively. The differential rotation of the two cross-sections generates the twist rate of the differential element, $\kappa_1 = \Phi_1'$, see eq. (7.7).

The work done by the torque acting on the left-hand side of the differential element of the beam is $-M_1\Phi_1$, where the minus sign is due to the fact that the torque and rotation are counted positive about opposite axes. The work done by the torque acting on the other side of the element is $M_1(\Phi_1 + \mathrm{d}\Phi_1)$. Finally, the net work done by the two torques, $\mathrm{d}W$, is found by summing up the two contributions to find $\mathrm{d}W = M_1 \mathrm{d}\Phi_1 = M_1(\mathrm{d}\Phi_1/\mathrm{d}x_1)\mathrm{d}x_1$. The total internal work done by the torque distribution acting in the beam of length L is then found by integration,

$$W_I = -\int_0^L M_1 \frac{\mathrm{d}\Phi_1}{\mathrm{d}x_1} \, \mathrm{d}x_1 = -\int_0^L M_1 \kappa_1 \, \mathrm{d}x_1. \tag{9.71}$$

The minus sign is due to the fact that the internal work is that done by the internal torque, which is opposite in sign to the torque applied externally to the cross-section.

The virtual internal work and its complementary counterpart are then found by considering the virtual work done by torques undergoing virtual twist rates, $\delta\kappa_1$, and the virtual work done by virtual torques, δM_1, undergoing actual twist rates, respectively. The internal virtual work and its complementary counterpart for torsion of circular cylinders now become

$$\delta W_I = -\int_0^L M_1 \delta\kappa_1 \, \mathrm{d}x_1, \tag{9.72a}$$

$$\delta W_I' = -\int_0^L \kappa_1 \delta M_1 \, \mathrm{d}x_1. \tag{9.72b}$$

A similar development will reveal that identical expressions hold for the torsion of bars with cross-sections of arbitrary shape. This development is based on the kinematic assumptions of Saint-Venant's theory of uniform torsion, as presented in section 7.3.2.

9.7.3 Three-dimensional solid

The more general case of a three-dimensional solid will now be addressed. The strain-displacement equations for a three-dimensional solid are presented in section 1.4.1. At a specific point of the solid, three direct and three shear stress components are acting. To simplify the presentation, the work done by each of these six stress components will be computed separately, and because work is an additive quantity, the total work will be found by summing up the contributions of each stress component.

Axial stresses

Consider the infinitesimal differential element of a solid depicted in fig. 9.50. Under the effect of the axial stress component, σ_1, the displacement components of two neighboring edges become u_1 and $u_1 + \mathrm{d}u_1$, at locations x_1 and $x_1 + \mathrm{d}x_1$, respectively. The differential displacement of the two edges generates the axial strain of the differential element, $\epsilon_1 = \partial u_1/\partial x_1$, see eq. (1.63).

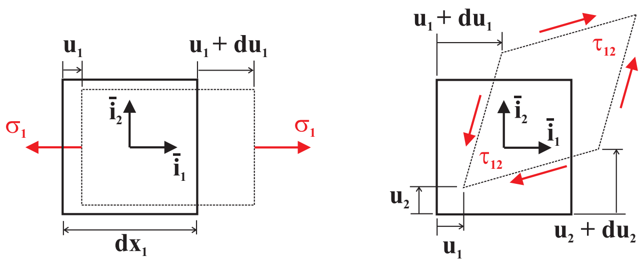

Fig. 9.50. Deformation of a differential element of a solid.

The work done by the force, $\sigma_1 \mathrm{d}x_2 \mathrm{d}x_3$, acting on the left-hand side of the differential element is $-(\sigma_1 \mathrm{d}x_2 \mathrm{d}x_3)u_1$ where the minus sign is due to the fact that the force and displacement are counted positive in opposite directions. The work done by the force acting on the other side of the element is $(\sigma_1 \mathrm{d}x_2 \mathrm{d}x_3)(u_1 + \mathrm{d}u_1)$. Finally, the net work done by the two forces, $\mathrm{d}W$, is found by summing up the two contributions to find $\mathrm{d}W = (\sigma_1 \mathrm{d}x_2 \mathrm{d}x_3)\mathrm{d}u_1 = (\sigma_1 \mathrm{d}x_2 \mathrm{d}x_3)(\partial u_1/\partial x_1)\mathrm{d}x_1$. The total internal work done by the axial stress distribution acting in the solid of volume $\mathcal{V}$ is then found by integration,

$$W_I = -\int_{\mathcal{V}} \sigma_1 \frac{\partial u_1}{\partial x_1} \,\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = -\int_{\mathcal{V}} \sigma_1 \epsilon_1 \,\mathrm{d}\mathcal{V}. \tag{9.73}$$

Again, the minus sign is due to the fact that the internal work is that done by the internal axial stresses, which are opposite in sign to the axial stresses applied to the differential element. The differential element of volume is written as $\mathrm{d}\mathcal{V} = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$. Following a similar reasoning, the internal work associated with the other two axial stress components, σ_2 and σ_3, can also be found.

Shear stresses

The work done by the three shear stress components requires further attention. Due to the principle of reciprocity of shear stresses, eq. (1.5), shear stress components will act on the right and left edges of the differential element but also on the top and bottom edges of the same element, as illustrated in the right part of fig. 9.50. The work done by the force, $\tau_{12}\mathrm{d}x_1\mathrm{d}x_3$, acting on the bottom edge of the differential element of the solid is $-(\tau_{12}\mathrm{d}x_1\mathrm{d}x_3)u_1$, where the minus sign is due to the fact that the force and displacement are counted positive in opposite directions. The work done by the force acting on the top edge of the differential element is $(\tau_{12}\mathrm{d}x_1\mathrm{d}x_3)(u_1 + \mathrm{d}u_1)$. Finally, the net work done by these two forces, $\mathrm{d}W$, is found by summing up the two contributions to find $\mathrm{d}W = (\tau_{12}\mathrm{d}x_1\mathrm{d}x_3)\mathrm{d}u_1 = (\tau_{12}\mathrm{d}x_1\mathrm{d}x_3)(\partial u_1/\partial x_2)\mathrm{d}x_2$.

Next, the work done by the force, $\tau_{12}\mathrm{d}x_2\mathrm{d}x_3$, acting on the left edge of the differential element of the solid is $-(\tau_{12}\mathrm{d}x_2\mathrm{d}x_3)u_2$, where the minus sign is due to the fact that the force and displacement are counted positive in opposite directions. The work done by the force acting on the right edge of the differential element is $(\tau_{12}\mathrm{d}x_2\mathrm{d}x_3)(u_2 + \mathrm{d}u_2)$. Finally, the net work done by these two forces, $\mathrm{d}W$, is found by summing up the two contributions to find $\mathrm{d}W = (\tau_{12}\mathrm{d}x_2\mathrm{d}x_3)\mathrm{d}u_2 = (\tau_{12}\mathrm{d}x_2\mathrm{d}x_3)(\partial u_2/\partial x_1)\mathrm{d}x_1$. The total internal work done by the shear stress distribution acting in the solid of volume $\mathcal{V}$ is then found by integration,

$$W_I = -\int_{\mathcal{V}} \tau_{12} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = -\int_{\mathcal{V}} \tau_{12}\gamma_{12} \, \mathrm{d}\mathcal{V}. \tag{9.74}$$

Again, the minus sign is due to the fact that the internal work is that done by the internal shear stresses, which are opposite in sign to the shear stresses applied to the differential element. Following a similar reasoning, the internal work associated with the other two shear stress components, τ_{23} and τ_{13}, can also be found.

The total work done by all six stress components is found by summing up all contributions to find

$$W_I = -\int_V (\sigma_1\epsilon_1 + \sigma_2\epsilon_2 + \sigma_3\epsilon_3 + \tau_{23}\gamma_{23} + \tau_{13}\gamma_{13} + \tau_{12}\gamma_{12}) \, \mathrm{d}\mathcal{V}. \tag{9.75}$$

To simplify the notation, the strain and stress arrays defined in eqs. (2.11a) and (2.11b), respectively, will be used here. The internal work then becomes

$$W_I = -\int_{\mathcal{V}} \underline{\sigma}^T \underline{\epsilon} \, \mathrm{d}\mathcal{V}. \tag{9.76}$$

The virtual internal work and its complementary counterpart are then found by considering the virtual work done by stress components undergoing a virtual strains,

$\delta\underline{\epsilon}$, and the virtual work done by virtual stresses, $\delta\underline{\sigma}$, undergoing actual strains, respectively. The internal virtual work and its complementary counterpart for three-dimensional solids now become

$$\delta W_I = -\int_{\mathcal{V}} \underline{\sigma}^T \delta\underline{\epsilon} \, \mathrm{d}\mathcal{V}, \tag{9.77a}$$

$$\delta W_I' = -\int_{\mathcal{V}} \underline{\epsilon}^T \delta\underline{\sigma} \, \mathrm{d}\mathcal{V}. \tag{9.77b}$$

9.7.4 Euler-Bernoulli beam

The Euler-Bernoulli beam will be reexamined, but rather than using the procedure described in section 9.7.1, the beam will be viewed as a three-dimensional solid, and the results of section 9.7.3 will be used. Equations (5.5) give the complete strain field in Euler-Bernoulli beams: all strain components vanish, except for the axial strain, given by eq. (5.7). Using eq. (9.75), the total work done by the sole non-vanishing strain component becomes

$$W_I = -\int_V \sigma_1 \epsilon_1 \, \mathrm{d}\mathcal{V} = -\int_0^L \int_{\mathcal{A}} \sigma_1 (\bar{\epsilon}_1 + x_3 \kappa_2 - x_2 \kappa_3) \, \mathrm{d}\mathcal{A}\mathrm{d}x_1$$
$$= -\int_0^L \left\{ \left[\int_{\mathcal{A}} \sigma_1 \mathrm{d}\mathcal{A} \right] \bar{\epsilon}_1 + \left[\int_{\mathcal{A}} \sigma_1 x_3 \mathrm{d}\mathcal{A} \right] \kappa_2 + \left[-\int_{\mathcal{A}} \sigma_1 x_2 \mathrm{d}\mathcal{A} \right] \kappa_3 \right\} \mathrm{d}x_1.$$

The integration over the beam's volume is separated into an integration along the beam's length, L, followed by an integration over its cross-section, $\mathcal{A}$. The first bracketed term is the axial force, N_1, defined by eq. (5.8), whereas the next two bracketed terms are the two bending moments, M_2 and M_3, both defined by eqs. (5.10). The internal work done by the axial stress component in an Euler-Bernoulli beam now reduces to

$$W_I = -\int_0^L \left(N_1 \bar{\epsilon}_1 + M_2 \kappa_2 + M_3 \kappa_3 \right) \, \mathrm{d}x_1. \tag{9.78}$$

Clearly, the internal work presented in equation (9.69) is a particular case of this more general result.

The virtual internal work and its complementary counterpart are then found by considering the virtual work done by stress resultants undergoing virtual deformations, $\delta\bar{\epsilon}_1$, $\delta\kappa_2$, and $\delta\kappa_3$, and the virtual work done by virtual stress resultants, δN_1, δM_2, and δM_3, undergoing actual deformations, respectively. The internal virtual work and its complementary counterpart for Euler-Bernoulli beam thus become

$$\delta W_I = -\int_0^L \left(N_1 \delta\bar{\epsilon}_1 + M_2 \delta\kappa_2 + M_3 \delta\kappa_3 \right) \, \mathrm{d}x_1, \tag{9.79a}$$

$$\delta W_I' = -\int_0^L \left(\bar{\epsilon}_1 \delta N_1 + \kappa_2 \delta M_2 + \kappa_3 \delta M_3 \right) \, \mathrm{d}x_1. \tag{9.79b}$$

9.7.5 Problems

Problem 9.18. Virtual work in circular tubes
(1) Starting from eq. (9.75) and using the kinematic assumption for the torsion of circular bars presented in section 7.1.1, develop a general expression for the internal work in circular bars. *(2)* Develop a general expression of the internal virtual work and complementary internal virtual work in circular bars.

Problem 9.19. Virtual work in torsion of bars
(1) Starting from eq. (9.75) and using the kinematic assumption for Saint-Venant's torsion theory of bars presented in section 7.3.2, develop a general expression for the internal work for the torsion of bars. *(2)* Develop a general expression of the internal virtual work and complementary internal virtual work for the torsion of bars.

9.7.6 Unit load method for beams

In section 9.6.6, the unit load method is presented for application to truss structures. Because the unit load method is a direct consequence of the principle of complementary virtual work, its application is easily generalized to Euler-Bernoulli beam structures. If the displacement at a point of the beam is prescribed to be of magnitude Δ, the principle of complementary virtual work, eq. (9.57), requires

$$\Delta \delta D + \delta W_I' = 0,$$

for all statically admissible virtual forces, where δD is the virtual driving force. The complementary internal virtual work in the beam, $\delta W_I'$, is given by eq. (9.79b), and the above equation can be written as

$$\Delta \delta D = \int_0^L \left(\bar{\epsilon}_1 \delta N_1 + \kappa_2 \delta M_2 + \kappa_3 \delta M_3 \right) \, \mathrm{d}x_1, \tag{9.80}$$

where δD, δN_1, δM_2, and δM_3, are statically admissible virtual forces and moments, whereas $\bar{\epsilon}_1$, κ_2, and κ_3, are the actual deformations of the beam.

Following the procedure developed for truss structures, see section 9.6.6, the virtual driving force is selected to be a unit force, $\delta D = 1$, and $\delta N_1 = \hat{N}_1$, $\delta M_2 = \hat{M}_2$ and $\delta M_3 = \hat{M}_3$ are the resulting statically admissible axial forces and bending moments. Equation (9.80) now becomes

$$\Delta = \int_0^L \left(\hat{N}_1 \bar{\epsilon}_1 + \hat{M}_2 \kappa_2 + \hat{M}_3 \kappa_3 \right) \, \mathrm{d}x_1, \tag{9.81}$$

Next, the beam is assumed to be made of a linearly elastic material. If the origin of the axis system is selected to be at the centroid of the cross-section, the sectional constitutive laws are given by eq. (6.13), and these can be used to eliminate the sectional strain and curvatures in eq. (9.81) to yield

$$\Delta = \int_0^L \left[\frac{\hat{N}_1 N_1}{S} + \frac{\hat{M}_2 (H_{33}^c M_2 + H_{23}^c M_3)}{\Delta_H} + \frac{\hat{M}_3 (H_{23}^c M_2 + H_{22}^c M_3)}{\Delta_H} \right] \mathrm{d}x_1, \tag{9.82}$$

If the axes are selected to be the principal centroidal axes of bending, this result further simplifies to

$$\Delta = \int_0^L \left[\frac{\hat{N}_1 N_1}{S} + \frac{\hat{M}_2 M_2}{H_{22}^c} + \frac{\hat{M}_3 M_3}{H_{33}^c} \right] \mathrm{d}x_1, \tag{9.83}$$

The unit load method applied to Euler-Bernoulli beams can be summarized in the following three steps.

1. Find the actual force and moment distributions acting in the beam under the action of the externally applied loads.
2. Apply a unit load at the point and in the direction of the desired displacement component. Evaluate the statically admissible force and moment distributions acting in the beam that are in equilibrium with this unit load; this is called the *unit load system*.
3. Use eq. (9.81) to evaluate the displacement component at the point and in the direction of application of the unit load. If the truss is made of a linearly elastic material and the origin of the axes is at the section's centroid, use eq. (9.82). If principal axes of bending are used, use eq. (9.83).

The first two steps of the procedure require the evaluation of two sets of force and moment distributions generated by two distinct loading conditions: first, the externally applied loads, and second, a unit load. If the beam is isostatic, these distributions can be evaluated based on the equilibrium equations. If the beam is hyperstatic, the unit load method is still applicable, although the evaluation of the force and moment distributions for these two loading conditions becomes more cumbersome.

The unit load method can also be used to determine rotation at a point of the beam. A slight modification of the procedure presented above is then required: instead of applying a unit load, a unit moment is applied. For prescribed rotations, the complementary virtual work is $\delta W_E' = \phi \delta M$, where ϕ is the prescribed rotation and δM the virtual driving moment. Instead of using a unit force, $\delta D = 1$, a unit moment is used, $\delta M = 1$. For such cases, the method becomes the "unit moment method."

The unit load method described above also applies to torsion problems. In this case, the relevant complementary internal virtual work expression is given by eq. (9.72b). If the beam is subjected to both bending moments and torques, the relevant complementary internal virtual work expression is the sum of those for bending and torsion, *i.e.*, the sum of eqs. (9.79b) and (9.72b).

Example 9.19. Deflection of a tip-loaded cantilevered beam

Consider the cantilevered beam of length L subjected to a concentrated load, P, as depicted in fig. 9.51. In this example, the load is applied at the beam's tip, $\alpha = 1$. Find the tip deflection of the beam. This problem is treated using the classical, differential equation approach in example 5.8 on page 201.

The first step of the unit load method calls for the evaluation of the bending moment distribution under the externally applied loads. Simple equilibrium arguments yield $M_3(x_1) = P(x_1 - L)$. Since the tip deflection is desired, a vertical unit load

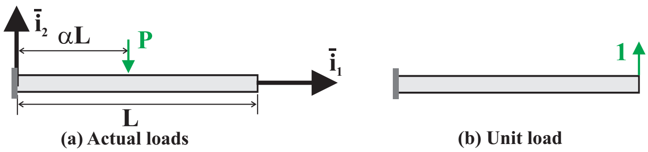

Fig. 9.51. Cantilevered beam under tip load.

is applied at the tip of the beam, as illustrated in the right part of fig. 9.51. Here again, equilibrium arguments yield the corresponding bending moment distribution as $\hat{M}_3(x_1) = -1(x_1 - L)$. The tip deflection, Δ, now follows from eq. (9.83) as

$$\Delta = \int_0^L \frac{\hat{M}_3 M_3}{H_{33}^c} \, \mathrm{d}x_1 = \int_0^L \frac{[P(x_1 - L)][-(x_1 - L)]}{H_{33}^c} \, \mathrm{d}x_1 = -\frac{PL^3}{3H_{33}^c}.$$

This result is in agreement with that found using the classical differential equation approach, see eq. (5.55). The minus sign in the present solution indicates that the tip displacement is *in the direction opposite to that of the unit load*, *i.e.*, downward, as expected. Indeed, in the derivation of the unit load method, the external complementary virtual work is expressed as $\delta W_E' = \Delta \delta D$, where Δ is the desired displacement component and δD the virtual driving force. For his expression to be correct, both displacement and driving force must be counted as positive *along the same direction*. Hence, a positive displacement Δ is along the direction of the unit driving force.

The unit load method yields the desired result without requiring the solution of the governing differential equation, thereby considerably easing the solution process. Of course, if the transverse displacement distribution at all points along the beam is desired, the solution of the governing differential equation would be more expeditious, see example 5.8.

Example 9.20. Tip deflection of a cantilever beam with concentrated load
Consider now the cantilevered beam of length L subjected to a concentrated load, P, applied at a distance αL from the beam's root, as depicted in fig. 9.51. Find the tip deflection of the beam.

First, the bending moment distribution under the externally applied loads is obtained from equilibrium arguments as

$$M_3(\eta) = PL \begin{cases} -(\alpha - \eta), & 0 \leq \eta \leq \alpha, \\ 0, & \alpha < \eta \leq 1, \end{cases}$$

where $\eta = x_1/L$ is the non-dimensional variable along the beam's span.

Since the tip deflection is desired, a vertical unit load is applied at the tip of the beam, as illustrated in the right part of fig. 9.51. Here again, equilibrium arguments yield the corresponding bending moment distribution as $\hat{M}_3(\eta) = L(1 - \eta)$.

The tip deflection, Δ, now follows from eq. (9.83) as

$$\Delta = \int_0^\alpha \frac{[-PL(\alpha-\eta)][L(1-\eta)]}{H_{33}^c} L\mathrm{d}\eta + \int_\alpha^1 \frac{[0][L(1-\eta)]}{H_{33}^c} L\mathrm{d}\eta$$
$$= -\frac{PL^3}{H_{33}^c}\int_0^\alpha (\alpha-\eta)(1-\eta)\,\mathrm{d}\eta = -\frac{PL^3}{6H_{33}^c}\alpha^2(3-\alpha).$$

Here again, this result is in agreement with that found using the classical differential approach, see eq. (5.55).

Example 9.21. Displacement field of a uniformly loaded cantilever beam
The unit load method can also be applied to situations involving distributed loading. Consider the case of a cantilever with a uniform load, p_0, as shown in fig. 9.52. Determine the tip displacement and the entire displacement field for the cantilevered beam.

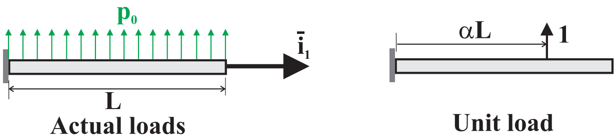

Fig. 9.52. Tip deflection of a uniformly loaded cantilever.

The bending moment distribution caused by the uniform loading is found from equilibrium arguments as $M_3 = p_0L^2(1-\eta)^2/2$, where $\eta = x_1/L$ is the non-dimensional variable along the beam's span.

First, the beam's tip deflection will be computed, and hence, a unit load is applied at its tip; the associated bending moment distribution is $\hat{M}_3 = L(1-\eta)$. The tip deflection, Δ, now follows from eq. (9.83) as

$$\Delta = \int_0^1 \frac{[p_0L^2(1-\eta)^2/2][L(1-\eta)]}{H_{33}^c} L\mathrm{d}\eta = \frac{p_0L^4}{2H_{33}^c}\int_0^1 (1-\eta)^3\,\mathrm{d}\eta = \frac{p_0L^4}{8H_{33}^c},$$

The unit load method can also be used to compute the entire displacement field. For this purpose, a unit load is applied at location αL along the beam's span, as illustrated in fig. 9.52. The associated bending moment distribution is $\hat{M}_3 = L(\alpha-\eta)$ for $0 \leq \eta \leq \alpha$ and $\hat{M}_3 = 0$ for $\alpha \leq \eta < 1$. The displacement field, $\Delta(\eta)$, now follows from eq. (9.83) as

$$\Delta(\alpha) = \int_0^\alpha \frac{[p_0L^2(1-\eta)^2/2][L(\alpha-\eta)]}{H_{33}^c} L\mathrm{d}\eta + \int_\alpha^1 \frac{[p_0L^2(1-\eta)^2/2][0]}{H_{33}^c} L\mathrm{d}\eta$$
$$= \frac{p_0L^4}{2H_{33}^c}\int_0^\alpha (1-\eta)^2(\alpha-\eta)\,\mathrm{d}\eta = \frac{p_0L^4}{24H_{33}^c}\alpha^2(6-4\alpha+\alpha^2).$$

As α varies along the beam's span, the entire displacement field is recovered. The present result matches that obtained with the classical differential equation approach, see eq. (5.54).

Example 9.22. Tip deflection of a simply supported beam
Consider a simply supported beam of length L with an overhang of length $L/2$. The first portion of the beam is subjected to a uniform loading, p_0. Determine the deflection and rotation at point **T**, as indicated in fig. 9.53.

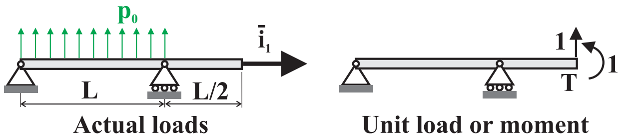

Fig. 9.53. Deflection at tip of a uniformly loaded simply-supported beam with overhang.

The bending moment distribution associated with the externally applied load found from statics as $M_3 = p_0L^2(\eta^2 - \eta)/2$ for $0 \le \eta \le 1$ and $M_3 = 0$ for $1 \le \eta \le 3/2$, where $\eta = x_1/L$ is the non-dimensional variable along the beam's span.

To determine the deflection at point **T**, a vertical unit load is applied at that point, and the associated bending moment distribution is $\hat{M}_3 = L\eta/2$, for $0 \le \eta \le 1$; and $\hat{M}_3 = L(3/2 - \eta)$, for $1 \le \eta \le 3/2$. Equation (9.83) then yields the deflection, Δ, at point **T**, as

$$\Delta = \int_0^1 \frac{[p_0L^2(\eta^2-\eta)/2][L\eta/2]}{H^c_{33}}\, L\mathrm{d}\eta + \int_1^{3/2} \frac{[0][L(3/2-\eta)]}{H^c_{33}}\, L\mathrm{d}\eta$$
$$= \frac{p_0L^4}{4H^c_{33}} \int_0^1 (\eta^3 - \eta^2)\,\mathrm{d}\eta = -\frac{p_0L^4}{48H^c_{33}}.$$

The negative sign means that the tip deflection is downward, *i.e.*, in the opposite direction of the unit load. Because the bending moment distribution associated with the externally applied loads vanishes in the overhang portion of the beam, the second integral in the above equation vanishes; consequently, it is not required to compute the bending moment distribution associated with the unit load over that portion of the beam, a further simplification of the procedure.

To determine the rotation at point **T**, a unit moment is applied at that point, and the associated bending moment distribution is $\hat{M}_3 = \eta$, for $0 \le \eta \le 1$; the rest of the bending moment distribution need not be computed. Equation (9.83) then yields the rotation, Φ, at point **T**, as

$$\Phi = \int_0^1 \frac{[p_0L^2(\eta^2-\eta)/2][\eta]}{H^c_{33}}\, L\mathrm{d}\eta = \frac{p_0L^3}{2H^c_{33}} \int_0^1 (\eta^3 - \eta^2)\,\mathrm{d}\eta = -\frac{p_0L^3}{24H^c_{33}}.$$

This final result is non-dimensional, as should be expected for a rotation, which is measured in radians.

Example 9.23. Bent beam assembly under tip load
Consider the three-dimensional, bent beam assembly depicted in fig. 9.54. The beam's cross-section is assumed to be circular, and hence, $H_{11} = GI_{11}$ and

$H_{22}^c = H_{33}^c = EI_{11}/2$, where I_{11} is the section's second area moment. The beam consists of three segments **AB**, **BC**, and **CD** connected at right angles to each other. Load P is applied at point **A**, along the direction of segment **CD**. Find the deflection, Δ, of point **A**, in the direction of the applied load. To simplify the computation, a different coordinate system is assigned to each beam segment, as depicted in the right part of fig. 9.54.

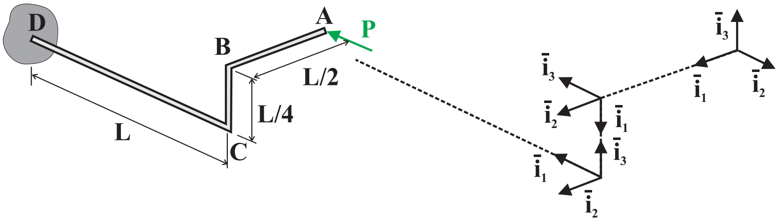

Fig. 9.54. Bent beam assembly under tip load.

Given the geometry of the system, segment **AB** is under bending in plane $(\bar{\imath}_1, \bar{\imath}_2)$, segment **BC** is under torsion and bending in plane $(\bar{\imath}_1, \bar{\imath}_3)$, and finally, segment **CD** is under bending in both planes $(\bar{\imath}_1, \bar{\imath}_2)$ and $(\bar{\imath}_1, \bar{\imath}_3)$. The bending and twisting moment distributions in each beam segment are readily found from equilibrium consideration for both the externally applied load and unit load.

For this problem, eq. (9.83) becomes

$$\begin{aligned}\Delta = &\int_0^{L/2} \frac{M_3 \hat{M}_3}{H_{33}^c}\, \mathrm{d}x_1 \\ &+ \int_0^{L/4} \frac{M_2 \hat{M}_2}{H_{22}^c}\, \mathrm{d}x_1 + \int_0^{L/4} \frac{M_1 \hat{M}_1}{H_{11}}\, \mathrm{d}x_1 \\ &+ \int_0^{L} \frac{M_2 \hat{M}_2}{H_{22}^c}\, \mathrm{d}x_1 + \int_0^{L} \frac{M_3 \hat{M}_3}{H_{33}^c}\, \mathrm{d}x_1.\end{aligned}$$

The first line of this expression represents the contribution from the bending of segment **AB**, the second line provides the bending and torsion contributions for segment **BC**, and the third line gives the two bending contributions for segment **CD**. Introducing the bending and twisting moment distributions then leads to

$$\begin{aligned}\Delta = &\int_0^{L/2} \frac{P x_1^2}{H_{33}^c}\, \mathrm{d}x_1 + \int_0^{L/4} \frac{P x_1^2}{H_{22}^c}\, \mathrm{d}x_1 + \int_0^{L/4} \frac{P (L/2)^2}{H_{11}}\, \mathrm{d}x_1 \\ &+ \int_0^{L} \frac{P (L/4)^2}{H_{22}^c}\, \mathrm{d}x_1 + \int_0^{L} \frac{P (L/2)^2}{H_{33}^c}\, \mathrm{d}x_1.\end{aligned}$$

Performing the integrals the yields the desired deflection as

$$\Delta = \frac{23}{64} \frac{PL^3}{H_{22}^c} + \frac{1}{16} \frac{PL^3}{H_{11}} = \frac{23}{64} \frac{PL^3}{H_{22}^c} \left[1 + \frac{2}{23} \frac{E}{G}\right] = \frac{23}{64} \frac{PL^3}{H_{22}^c} \left[1 + \frac{4(1+\nu)}{23}\right].$$

If the beam assembly is made of a material that obeys Hooke's law, eq. (2.8) implies $E = 2G(1+\nu)$. In the last bracketed expression, the first term represents the contribution due to bending of the various beam segments, whereas the second term represents that of twisting of segment **BC**. The twisting of the middle segment of the assembly accounts for about 20% of the deflection at point **A**, assuming $\nu = 0.3$.

Example 9.24. Bending of a cantilever with a "Z" cross-section
Consider the cantilevered beam with a thin-walled "Z" shaped cross-section subjected to a uniform load, p_0, as shown in fig. 9.55. Find the beam's tip deflection along axes $\bar{\imath}_2$ and $\bar{\imath}_3$ using the unit load method. This problem is treated in example 6.6 on page 249 using the classical differential equation approach.

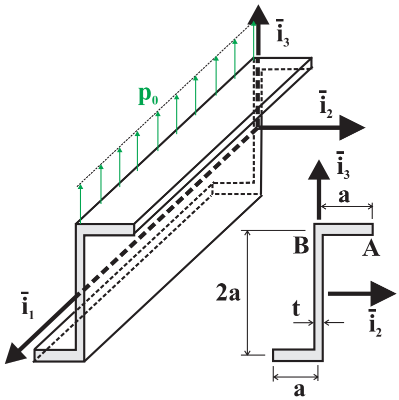

Fig. 9.55. Cantilevered Z-section beam under a uniform load.

The first step of the process is to compute the bending moment distribution due to the externally applied load, p_0: $M_2 = -p_0L^2(1-\eta)^2/2$ and $M_3 = 0$. To compute the tip deflection along axis $\bar{\imath}_3$, the unit load must be applied at the tip along the same axis. The resulting bending moment distribution is found as $\hat{M}_2 = -L(1-\eta)$ and $\hat{M}_3 = 0$. Substituting these bending moment distributions into eq. (9.82) then yields

$$\Delta_3 = \int_0^L \frac{H_{33}^c}{\Delta_H}\hat{M}_2 M_2 \,\mathrm{d}x_1 = \frac{6}{7Ea^3t}\int_0^1 [-L(1-\eta)][-p_0L^2(1-\eta)^2/2]\,L\mathrm{d}\eta$$
$$= \frac{3p_0L^4}{7Ea^3t}\int_0^1 (1-\eta)^3\mathrm{d}\eta = \frac{3}{28}\frac{p_0L^4}{Ea^3t},$$

where $\eta = x_1/L$ is the non-dimensional variable along the beam's span, and the sectional bending stiffnesses, $H_{33}^c = 2Ea^3t/3$ and $\Delta_H = 7(Ea^3t)^2/9$, are evaluated in example 6.6. This result is in agreement with that found in example 6.6, see eq. (6.58).

To compute the tip deflection along axis $\bar{\imath}_2$, a tip unit load must be applied along that direction. The associated bending moment distribution is $\hat{M}_2 = 0$ and $\hat{M}_3 =$

$L(1-\eta)$. The bending moment distribution due to the externally applied load is, of course, unchanged, and eq. (9.82) then leads to

$$\Delta_2 = \int_0^L \frac{H_{23}^c}{\Delta_H} \hat{M}_3 M_2 \, \mathrm{d}x_1 = \frac{9}{7Ea^3t} \int_0^1 [L(1-\eta)][-p_0 L^2 (1-\eta)^2/2] \, L\mathrm{d}\eta$$
$$= -\frac{9p_0L^4}{14Ea^3t} \int_0^1 (1-\eta)^3 \mathrm{d}\eta = -\frac{9}{56} \frac{p_0 L^4}{Ea^3t},$$

where the sectional bending stiffness, $H_{22}^c = 8Ea^3t/3$, is evaluated in example 6.6. Here again, this result matches that found in example 6.6, see eq. (6.57).

Evaluation of the beam's tip deflection is far easier when using the unit load method as compared to the classical approach that required the solution of coupled differential equations. If the complete displacement field of the beam is desired, unit loads should be applied at location αL along unit vectors $\bar{\imath}_2$ and $\bar{\imath}_3$.

Example 9.25. Torsion of a thin-walled tube with a closed section

The torsion of a thin-walled tube with a closed cross-section of arbitrary shape is investigated in section 8.5.2. The Bredt-Batho formula, $M_1 = 2\mathcal{A}f$, relates the applied torque, M_1, to the constant shear flow, f, in the thin wall, where $\mathcal{A}$ is the area enclosed by curve $\mathcal{C}$, which defines the shape of the cross-section, as depicted in fig. 9.56. To find the torsional stiffness of the structure, the first law of thermodynamics is invoked in section 8.5.2: the work done by the applied torque must equal the strain energy stored in the structure. In this example, the torsional stiffness of the structure is calculated using the unit load method. The structure is assumed to be fixed at one end, and a torque, M_1, is applied at the other, as shown in fig. 9.56.

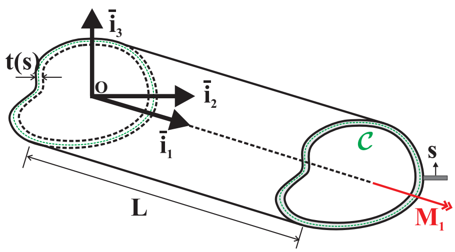

Fig. 9.56. Twisting of a thin walled tube of arbitrary cross-sectional shape.

The principle of complementary virtual work for a prescribed rotation, Φ, can be stated as

$$\Phi \delta M = -\delta W_I' = \int_{\mathcal{V}} \underline{\epsilon}^T \delta \underline{\sigma} \, \mathrm{d}\mathcal{V} = \int_0^L \int_{\mathcal{C}} \gamma_s \delta\tau_s \, t\mathrm{d}s\mathrm{d}x_1.$$

In this expression, the complementary external virtual work, $\Phi\delta M$, is the product of the prescribed tip rotation, Φ, by a virtual tip torque, δM. The complementary internal virtual work is taken to be that of a general three-dimensional solid, given in eq. (9.77b). Finally, as discussed in section 8.5.2, the only vanishing stress component in a thin-walled tube undergoing uniform torsion is the tangential shear strain component, γ_s.

Next, the virtual driving moment is select to be of unit magnitude, $\delta M = 1$, and the corresponding statically admissible virtual stress field is denoted $\delta\tau_s = \hat{\tau}_s$. The Bredt-Batho formula then yields $\hat{\tau}_s = \hat{M}_1/(2\mathcal{A}t) = 1/(2\mathcal{A}t)$ and $\gamma_s = \tau_s/G = M_1/(2G\mathcal{A}t)$. The tip twist now becomes

$$\Phi = \int_0^L \int_{\mathcal{C}} \frac{M_1}{2G\mathcal{A}t}\frac{1}{2\mathcal{A}t}\, t\mathrm{d}s\mathrm{d}x_1 = \frac{M_1}{4\mathcal{A}^2}\int_0^L \left[\int_{\mathcal{C}} \frac{\mathrm{d}s}{Gt}\right]\mathrm{d}x_1 = \frac{M_1 L}{4\mathcal{A}^2}\int_{\mathcal{C}} \frac{\mathrm{d}s}{Gt}.$$

Because this is a uniform torsion problem, the twist rate is simply $\kappa_1 = \Phi/L$ or

$$\kappa_1 = \frac{\Phi}{L} = \frac{M_1}{4\mathcal{A}^2}\int_{\mathcal{C}} \frac{\mathrm{d}s}{Gt}.$$

The torsional stiffness, H_{11}, is the constant of proportionality between the torque and the twist rate, $H_{11} = M_1/\kappa_1$, which leads to $H_{11} = 4\mathcal{A}^2/\left[\int_{\mathcal{C}} \mathrm{d}s/(Gt)\right]$. This result is identical to that developed in section 8.5.2, see eq. (8.67). Here again, the principle of complementary virtual work provides an elegant solution of the problem.

9.7.7 Problems

Problem 9.20. Cantilevered beam subjected to two concentrated loads
Consider the cantilevered beam subjected to two concentrated loads of equal magnitude and opposite direction applied at points **M** and **T**, as shown in fig. 9.57. *(1)* Compute the beam's transverse deflection at point **M**. *(2)* Compute the beam's transverse deflection at point **T**.

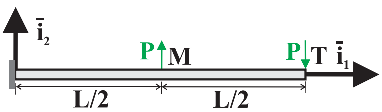

Fig. 9.57. Cantilevered beam subjected to two concentrated loads.

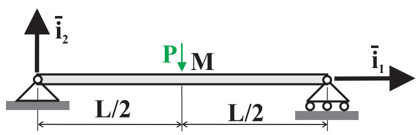

Fig. 9.58. Simply supported beam subjected to concentrated load.

Problem 9.21. Simply supported beam subjected to concentrated load
Consider the simply supported beam subjected to a mid-span concentrated load applied at point **M**, as shown in fig. 9.58. *(1)* Compute the beam's transverse deflection at point **M**. *(2)* Determine the beam's transverse displacement field, $\bar{u}_2(x_1)$.

Problem 9.22. Cantilevered beam subjected to triangular loading
Consider the cantilevered beam subjected to a distributed triangular loading of magnitude p_0 at the root and vanishing at the tip, as shown in fig. 9.59. *(1)* Compute the beam's transverse deflection at point **T**. *(2)* Determine the beam's transverse displacement field, $\bar{u}_2(x_1)$.

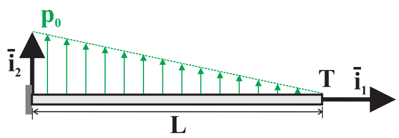

Fig. 9.59. Cantilevered beam subjected to triangular loading.

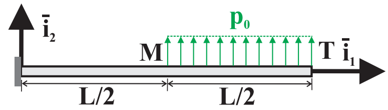

Fig. 9.60. Cantilevered beam under uniform loading.

Problem 9.23. Cantilevered beam subjected to uniform loading
Consider the cantilevered beam subjected to a uniform loading of magnitude p_0 extending from the beam's mid-span to its tip, as shown in fig. 9.60. *(1)* Compute the beam's transverse deflection at point **M**. *(2)* Compute the beam's transverse deflection at point **T**. *(2)* Determine the beam's transverse displacement field, $\bar{u}_2(x_1)$.

Problem 9.24. Pivoted beam supported by three-bar truss
A root pivoted beam carries a concentrated mid-span load, P, and is supported by a three-bar truss, as shown in fig. 9.61. *(1)* Combine the unit load method for beams and trusses to determine the midpoint deflection for the beam. *(2)* Determine the vertical deflection of point **A**.

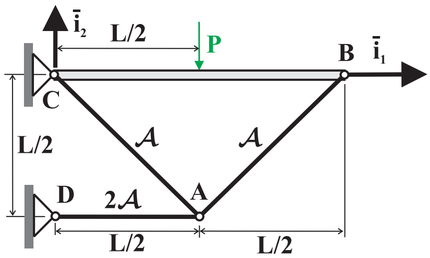

Fig. 9.61. Pivoted beam supported by three-bar truss.

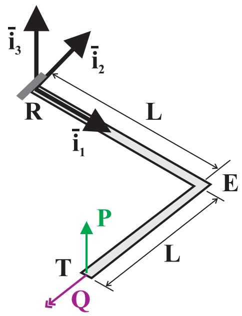

Fig. 9.62. Beam under combined bending and torsion.

Problem 9.25. Beam under combined bending and torsion
The beam shown in fig. 9.62 has a circular cross-section with bending stiffnesses, $H_{22} = H_{33} = EI_{11}/2$, and torsional stiffness, $H_{11} = GI_{11}$. A torque, Q, and a vertical force, P, are applied at the beam's tip as indicated in the figure. Consider both bending and torsional deformation. *(1)* Determine the beam's tip deflection along axis $\bar{\imath}_3$. *(2)* Determine the beam's tip twist about axis $\bar{\imath}_2$.

Problem 9.26. Cantilevered beam under combined loads
A cantilevered beam of length $3L$ is subjected to a uniformly distributed loading, p_0, over its central portion and concentrated transverse loads of magnitude P acting in opposite directions at points **B** and **C**, as depicted in fig. 9.63. *(1)* Find the beam's deflection at point **A**. *(2)* Determine it rotation at point **C**.

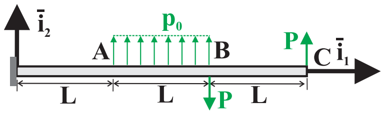

Fig. 9.63. Cantilevered beam under combined loads.

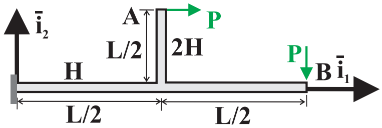

Fig. 9.64. Cantilevered beam under eccentric loading.

Problem 9.27. Cantilevered beam under eccentric loading
The cantilevered beam shown in fig. 9.64 is of length L and bending stiffness H, and carries a tip transverse load, P. A vertical beam of length $L/2$ and bending stiffness $2H$ is connected at its mid-span and carries an axial load, P. *(1)* Determine the rotation at point **A**. *(1)* Determine the transverse deflection at point **B**.

9.8 Application of the unit load method to hyperstatic problems

In the previous section, the unit load method is developed for trusses, beams, and solids. As explained in section 9.6.6, the approach calls for the determination of two sets of statically admissible forces corresponding to two distinct loading cases: the first associated with the externally applied loads, the second with the unit load. In all examples treated in the previous section, the structures are isostatic, and consequently, the two sets of statically admissible forces can always be determined solely from the equilibrium equations. The unit load method applies equally to iso- and hyperstatic system; in the latter case, however, the evaluation of the two sets of statically admissible forces is more arduous, because equilibrium equations are not sufficient for this task.

In chapter 4, two approaches are presented for the analysis of hyperstatic structures: the displacement or stiffness method and the force or flexibility method, see sections 4.3.2 and 4.3.3, respectively. The force method is particularly well-suited for dealing with hyperstatic problems because it focuses on the determination internal forces, moments and reactions. A key step of the procedure is the development of the compatibility equations that must complement the equilibrium equations to enable the solution of the problem. Because the principal of complementary virtual work is equivalent to the compatibility equations of the system, it seems logical to combine the force method with this principle.

The force method is intuitively described as the "method of cuts." For each cut made to the system, the order of the hyperstatic system is decreased by one because one internal force or moment then vanishes. For a hyperstatic system of n^{th} order, n cuts are required to transform the original hyperstatic system into an isostatic system. Statically admissible forces in this isostatic system are then obtained solely from the equilibrium equations, and relative displacements at the cuts are evaluated. At each cut, sets of self-equilibrated forces are added, and their magnitudes are determined by enforcing the vanishing of the relative displacement at the cut.

The approach involves two crucial steps. First, determine the relative displacements at the cuts under the externally applied loads alone, and second, evaluate the internal forces applied at the cuts that are required to eliminate the relative displacements at the cuts. The principle of complementary virtual work is a powerful tool to solve both problems.

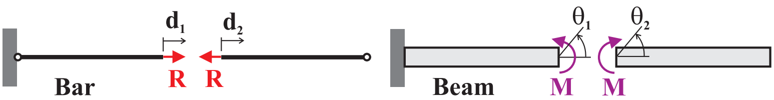

Fig. 9.65. Relative displacements and rotations.

The left part of fig. 9.65 depicts a single bar of a truss. The bar is cut to release the axial force, and a set of self-equilibrating forces of magnitude, R, is applied at the cut. Let d_1 and d_2 be the displacements at the two sides of the cut. If the displacements, d_1 and d_2, are prescribed, the associated complementary external virtual work is $\delta W'_E = d_1 \delta R - d_2 \delta R = (d_1 - d_2)\delta R$. The relative displacement at the cut is $\varDelta = d_1 - d_2$, measured positive when the two segments of the cut bar overlap. The principle of complementary virtual work, eq. (9.57), stated as $\delta W'_E + \delta W'_I = 0$, now implies

$$\varDelta \delta R = -\delta W'_I. \tag{9.84}$$

This result is very similar to eq. (9.62), but $\varDelta$ now represents the relative displacement at a cut and δR the set of self-equilibrating virtual forces applied at the cut.

The right part of fig. 9.65 depicts a cantilevered beam with a cut to release the bending moment[2] and a set of self-equilibrating moments of magnitude, M, applied at the cut. Let θ_1 and θ_2 be the rotations at the two sides of the cut. If the rotations, θ_1 and θ_2, are prescribed, the associated complementary external virtual work is $\delta W'_E = \theta_1 \delta M - \theta_2 \delta M = (\theta_1 - \theta_2)\delta M$. The relative rotation at the cut is $\varPhi = \theta_1 - \theta_2$. The principle of complementary virtual work, eq. (9.57), stated as $\delta W'_E + \delta W'_I = 0$, now implies

$$\varPhi \delta M = -\delta W'_I. \tag{9.85}$$

This result is very similar to eq. (9.66), but $\varPhi$ now represents the relative rotation at a cut and δM the set of self-equilibrating virtual moments applied at the cut.

9.8.1 Force method for trusses

In this section, the force method will be combined with the unit load method to find internal forces in hyperstatic trusses. The basic steps of the force method are presented in section 4.3.3, and the same procedure will be followed here. The approach will be described using the three-bar hyperstatic truss depicted in fig. 9.66 as an example. The truss carries a load, P, at joint **O**.

This hyperstatic system is of order 1, and hence, a single cut is required to transform it into an isostatic system. The middle bar is cut, and fig. 9.66 shows the resulting isostatic truss. The actual system is viewed as the superposition of two problems.

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[2] The cut must release only the moment and not the shear. It can be imagined as a hinge.

First, the isostatic system obtained by cutting one member, subjected to the externally applied load, and second, the internal force system in which an internal force of unknown magnitude, R, is applied at the cut.

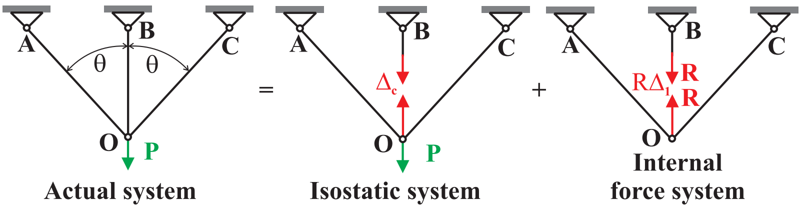

Fig. 9.66. Force method for the three-bar truss.

For the isostatic system, the unit load method developed in section 9.6.6 is directly applicable to compute the relative displacement, Δ_c, at the cut by using eq. (9.84), and results in

$$\Delta_c = \sum_{i=1}^{N_b} \frac{\hat{F}_i F_i L_i}{(E\mathcal{A})_i}, \tag{9.86}$$

where F_i are the bar forces in the isostatic truss subjected to the externally applied loads, and $\hat{F}_i$ the statically admissible virtual forces corresponding to the self-equilibrating unit load system applied at the cut. The bar forces corresponding to the externally applied load are $F_A = F_C = P/(2\cos\theta)$ and $F_B = 0$. The bar forces corresponding to a self-equilibrating unit load system applied at the cut are $\hat{F}_A = \hat{F}_C = -1/(2\cos\theta)$ and $\hat{F}_B = 1$. Equation (9.86) then yields the relative displacement at the cut as

$$\begin{aligned}\Delta_c &= \frac{-1}{2\cos\theta}\,\frac{P}{2\cos\theta}\,\frac{L}{(E\mathcal{A})_A\cos\theta} + \frac{-1}{2\cos\theta}\,\frac{P}{2\cos\theta}\,\frac{L}{(E\mathcal{A})_C\cos\theta}\\ &= -\left(\frac{1}{(E\mathcal{A})_A} + \frac{1}{(E\mathcal{A})_C}\right)\frac{PL}{4\cos^3\theta}.\end{aligned}$$

The minus sign reflects the fact that the externally applied load opens the cut.

Next, the internal force system illustrated fig. 9.66 is investigated. The relative displacement at the cut, Δ_1, due to a unit internal force in bar **B** is computed using the unit load once again. Equation (9.84) now yields

$$\Delta_1 = \sum_{i=1}^{N_b} \frac{\hat{F}_i^2 L_i}{(E\mathcal{A})_i}, \tag{9.87}$$

In this case, the bar forces due to a self-equilibrating unit load system applied at the cut, $\hat{F}_i$, represent both the loads due to the external loading and the unit load system. This set of forces is the same as that computed in the previous step. For the three-bar truss,

$$\Delta_1 = \frac{1}{(2\cos\theta)^2} \frac{L}{(E\mathcal{A})_A \cos\theta} + \frac{L}{(E\mathcal{A}_B)} + \frac{1}{(2\cos\theta)^2} \frac{L}{(E\mathcal{A})_C \cos\theta}$$
$$= \frac{L}{(E\mathcal{A})_B 4\cos^3\theta} \frac{\bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta}{\bar{k}_A\bar{k}_C}.$$

where $\bar{k}_A = (E\mathcal{A})_A/(E\mathcal{A})_B$ and $\bar{k}_C = (E\mathcal{A})_C/(E\mathcal{A})_B$ are the non-dimensional stiffnesses of bars **A** and **C**, respectively,

In the last step of this process, the results of the two loading cases are superposed. The sum of the relative displacements at the cut for the isostatic and internal force systems must vanish, because it is artificially introduced. This implies the following compatibility condition at the cut

$$\Delta_c + R\Delta_1 = 0, \tag{9.88}$$

where R is the internal force in bar **B**. Equation (9.88) is solved for the unknown force in the cut bar,

$$R = -\frac{\Delta_c}{\Delta_1}. \tag{9.89}$$

For the three-bar truss example, this yields

$$R = -\frac{\Delta_c}{\Delta_1} = \frac{\bar{k}_A + \bar{k}_C}{\bar{k}_A + \bar{k}_C + 4\bar{k}_A\bar{k}_C\cos^3\theta} P.$$

Bar forces are then found by superposition

$$F_i + R\hat{F}_i, \quad i = 1, 2, \ldots N_b. \tag{9.90}$$

Summary of the force method for hyperstatic trusses of order 1

The procedure described in the previous section, which combines the force and unit load methods, can be summarized by the following steps.

1. Transform the original, hyperstatic truss into an isostatic truss by cutting one bar or one support of the system. The cut must transform the original system into an isostatic system, not a mechanism. This can be achieved in different ways, although specific choices might be more or less cumbersome from an algebraic standpoint.
2. Determine the bar forces, F_i, in the isostatic system subjected to the externally applied loads.
3. Determine the bar forces, $\hat{F}_i$, in the isostatic system loaded by a pair of unit forces at the cut.
4. Determine the relative displacement at the cut, Δ_c, due to the externally applied loads using eq. (9.86). Determine the relative displacement at the cut, Δ_1, due to the pair of unit forces applied at the cut using eq. (9.87).
5. Impose the compatibility condition given by eq. (9.88), and find the internal force in the cut bar, eq. (9.89).

6. Find the forces in all bars by superposition using eq. (9.89).

Once the procedure is completed, displacements at selected points of the truss can be evaluated using the unit load method, and this will be illustrated in the examples below. Because this method uses the principle of superposition, it is only valid for structures made of linearly elastic materials undergoing small displacement.

Example 9.26. Six-bar hyperstatic truss
Consider the six-bar, hyperstatic truss shown in fig. 9.67. All bars have identical Young's modulus, E, and cross-sectional area, $\mathcal{A}$. Determine the forces in the bars of the truss. The combination of the force and unit load methods will be used to solve this problem. Because the two diagonal bars of the square bay are present, the truss is hyperstatic of order 1.

First, an isostatic truss is created by cutting one of the two diagonal members, as indicated in fig. 9.67; this truss is subjected to the externally applied loads. The internal force system consists of the isostatic truss loaded by unit forces at the cut.

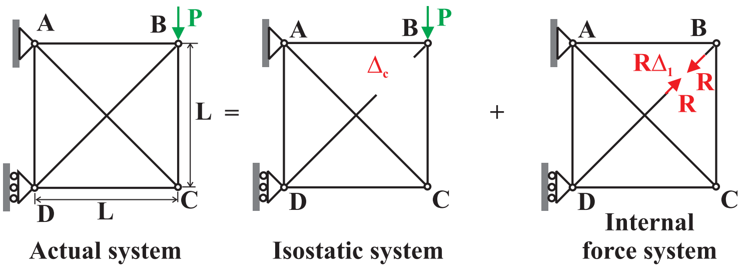

Fig. 9.67. Six-bar hyperstatic truss.

Table 9.3 presents the following information: the second column lists the bar flexibility factors, $L_i/(E\mathcal{A})$, the third lists the bar forces, F_i, in the isostatic system subjected to the externally applied loads, and the fourth lists the bar forces, $\hat{F}_i$, in the isostatic system subjected to a set of unit loads applied at the cut. The relative displacement at the cut, Δ_c, due to the externally applied loads is evaluated using eq. (9.86), and the intermediate results involved in the evaluation of this relative displacement are listed in the fifth column of the table. The relative displacement at the cut, Δ_1, due to a set of unit forces applied at the cut is computed using eq. (9.87), and intermediate results are presented in the sixth column. The set of unit forces applied at the cut is assumed to create a tensile force in the cut bar to comply with the customary sign convention of positive tensile bar forces.

The internal force in the cut bar is now evaluated using eq. (9.89). The two relative displacements at the cut, Δ_c and Δ_1, are found by adding the entries in columns 5 and 6 of table 9.3, respectively, to find

$$R = -\frac{\Delta_c}{\Delta_1} = -\frac{2(1+1/\sqrt{2})PL/(E\mathcal{A})}{(2+2\sqrt{2})L/(E\mathcal{A})} = -\frac{P}{\sqrt{2}}.$$

The bar forces are then found by superposition, see eq. (9.89), and are listed in the last column of the table.

Table 9.3. Calculation of Δ_c and Δ_1 for the six-bar hyperstatic truss.

Bar	$L_i/(E\mathcal{A})$	F_i	$\hat{F}_i$	$F_i\hat{F}_iL_i/(E\mathcal{A})$	$\hat{F}_i^2L_i/(E\mathcal{A})$	$F_i + R\hat{F}_i$
AB	1	0	$-1/\sqrt{2}$	0	$1/2$	$P/2$
BC	1	$-P$	$-1/\sqrt{2}$	$P/\sqrt{2}$	$1/2$	$-P/2$
CD	1	$-P$	$-1/\sqrt{2}$	$P/\sqrt{2}$	$1/2$	$-P/2$
DA	1	0	$-1/\sqrt{2}$	0	$1/2$	$P/2$
AC	$\sqrt{2}$	$P\sqrt{2}$	1	$2P$	$\sqrt{2}$	$P/\sqrt{2}$
BD	$\sqrt{2}$	0	1	0	$\sqrt{2}$	$-P/\sqrt{2}$

Example 9.27. Truss with redundant support

The two-bay truss depicted in fig. 9.68 is isostatic, but the presence of the tip support makes the complete system hyperstatic. All bars are of identical Young's modulus, E, and cross-sectional area, $\mathcal{A}$. Determine the forces in the bars of the truss.

Instead of cutting one of the bars, the tip support at point **F** will be removed to render the truss isostatic. The isostatic truss subjected to the externally applied loads is shown in fig. 9.68, and the same isostatic truss loaded by a set of internal forces of magnitude R at the support is also depicted.

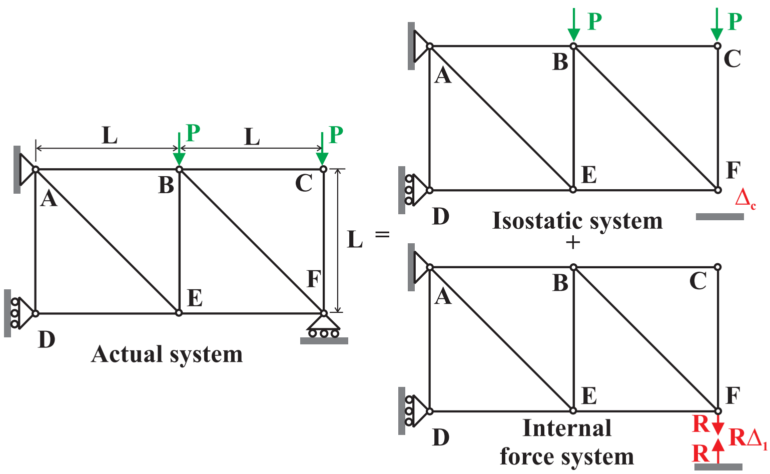

Fig. 9.68. Calculation of member forces in a 2-bay truss with a redundant support.

Table 9.4 presents the results of the analysis. The second column lists the bar flexibility factors, $L_i/(E\mathcal{A})$. The next two columns list the bar forces in the isostatic truss subjected to two loading cases: F_i for the isostatic truss subjected to the externally applied loads, and $\hat{F}_i$ for the same truss subjected to a set of unit loads at the cut. Columns 5 and 6 list the contributions of each bar to the relative displacements at the cut, Δ_c and Δ_1, according to eqs. (9.86) and (9.87), respectively. The reaction force at the support then follows from eq. (9.89) as $R = -\Delta_c/\Delta_1 = (10 + 6\sqrt{2})P/(7 + 4\sqrt{2})$. The bar forces are found by superposition and are listed in the last column of table 9.4.

Table 9.4. Calculation of Δ_c and Δ_1 for the two-bay truss. $\alpha = 7 + 4\sqrt{2}$.

Bar	$L_i/(E\mathcal{A})$	F_i	$\hat{F}_i$	$F_i\hat{F}_iL_i/(E\mathcal{A})$	$\hat{F}^2L_i/(E\mathcal{A})$	$F_i + R\hat{F}_i$
AB	1	P	- 1	$-P$	1	$-(3+2\sqrt{2})P/\alpha$
BC	1	0	0	0	0	0
DE	1	$-3P$	2	$-6P$	4	$-P/\alpha$
EF	1	$-P$	1	$-P$	1	$(3+2\sqrt{2})P/\alpha$
AD	1	0	0	0	0	0
BE	1	$-2P$	1	$-2P$	1	$-(4+2\sqrt{2})P/\alpha$
CF	1	$-P$	0	0	0	$-P$
AE	$\sqrt{2}$	$2\sqrt{2}P$	$-\sqrt{2}$	$-4\sqrt{2}P$	$2\sqrt{2}$	$(4+4\sqrt{2})P/\alpha$
BF	$\sqrt{2}$	$\sqrt{2}P$	$-\sqrt{2}$	$-2\sqrt{2}P$	$2\sqrt{2}$	$-(4+3\sqrt{2})P/\alpha$

Example 9.28. Deflection of a hyperstatic truss

The previous examples focus on the determination of bar and reaction forces in hyperstatic trusses. In some cases, it is also necessary to compute displacements at specific points of hyperstatic trusses, as is done for isostatic trusses and beams in sections 9.6.6 and 9.7.6, respectively. Here again, the unit load method will be used for this task. In this case, the first step of the method will be to compute the bar forces in the hyperstatic system subjected to the externally applied loads.

Consider the two-bay truss depicted in fig. 9.69. The bar forces in this hyperstatic truss are computed in example 9.27. In this example, the vertical displacement at joint **E** will be evaluated using the unit load method. This approach requires the computation of two sets of bar forces: the bar forces, F_i, associated with the externally applied loads, and the bar forces, $\hat{F}_i$, generated by a unit load applied at joint **E**, as illustrated in fig. 9.69. The first set of forces, F_i, are listed in the last column of table 9.4 and repeated, for convenience, in the third column of table 9.5.

To complete the problem, it is necessary to evaluate a set of statically admissible bar forces that are in equilibrium with a unit load applied at joint **E**. At first it appears that a procedure similar to that developed in the previous example will yield the desired bar forces. Indeed, it is possible to compute the forces in all the bars of the hyperstatic truss when subjected to a unit load at joint **E**. But while this is feasible and will lead to the desired result, it is a cumbersome approach that can be easily bypassed using the following reasoning. The unit load method is a direct application of the principle of complementary virtual work for which the bar forces, $\hat{F}_i$, are required to be statically admissible, but are not necessarily those acting in the truss as it undergoes compatible deformations. In particular, the principle of complementary

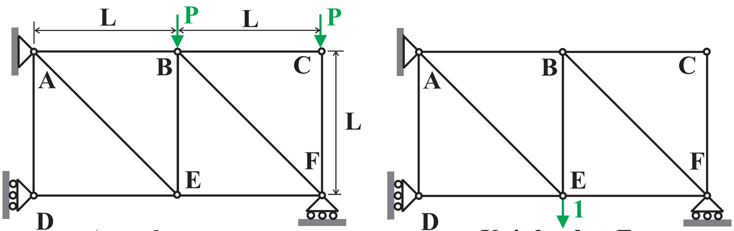

Fig. 9.69. Deflection of a joint in a two-bay truss with a redundant support.

virtual work holds "for all statically admissible virtual forces." Rather than selecting those statically admissible virtual forces acting in the hyperstatic truss as it undergoes compatible deformations, it is much simpler to select a set of statically admissible virtual forces corresponding to an arbitrary choice of the redundant force in the truss.

For the problem at hand, a particularly simple set of statically admissible bar forces, $\hat{F}_i$, is the set associated with a vanishing reaction at the support, point **F**, as listed in the fourth column of table 9.5. While this set of statically admissible forces is not equal to the set acting in the truss as it undergoes compatible deformations, it nonetheless is statically admissible, and hence is a valid set of forces for application of the unit load method.

Table 9.5. Calculation of vertical deflection at joint **E**. $\alpha = 7 + 4\sqrt{2}$

Bar	$L_i/(E\mathcal{A})$	F_i	$\hat{F}_i$	$F_i\hat{F}_i L_i/(E\mathcal{A})$
AB	1	$-(3+2\sqrt{2})P/\alpha$	0	0
BC	1	0	0	0
DE	1	$-P/\alpha$	-1	P/α
EF	1	$(3+2\sqrt{2})P/\alpha$	0	0
AD	1	0	0	0
BE	1	$-(4+2\sqrt{2})P/\alpha$	0	0
CF	1	$-P$	0	0
AE	$\sqrt{2}$	$(4+4\sqrt{2})P/\alpha$	$\sqrt{2}$	$8(1+\sqrt{2})P/\alpha$
BF	$\sqrt{2}$	$-(4+3\sqrt{2})P/\alpha$	0	0

The last column of table 9.5 lists the partial results necessary for the application of eq. (9.65): summing up the entries in the last column yields the vertical displacement at point **E** as

$$\Delta_E = \sum_{i=1}^{N_b} \frac{F_i \hat{F}_i L_i}{(E\mathcal{A})_i} = \frac{(9+8\sqrt{2})}{(7+4\sqrt{2})} \frac{PL}{E\mathcal{A}}.$$

It will be left to the reader to verify that an identical answer will be obtained by using other sets of statically admissible virtual forces that are in equilibrium with the applied unit load. Various sets of statically admissible forces are readily obtained by setting selected bar or reaction forces to zero and computing forces in the remaining bars based on equilibrium. This procedure automatically produces statically admissible virtual forces.

By simplifying the evaluation of the statically admissible bar forces in equilibrium with the unit load, the overall amount of effort required to compute the displacement at a point of the truss is dramatically reduced, as demonstrated in this example.

9.8.2 Force method for beams

In this section, the combined use of the force and unit load methods will be developed for the analysis of hyperstatic beams. The procedure closely follows that developed in section 9.8.1 for truss structures. As compared to trusses, relatively few beam structures involve internally redundant configurations. Examples of beam structures featuring internal redundancy include closed circular beams or rings and beam grillage. In most cases, however, beam structures become hyperstatic due to the presence of multiple supports.

Figure 9.70 depicts a cantilevered beam with an additional mid-span support. Without this additional support, the structure is isostatic and it is possible to compute the root reactions and the bending moment distribution from equilibrium considerations alone. When the mid-span support is added, an additional reaction, R, arises and equilibrium equations are no longer sufficient to determine the reaction forces and moment.

The essence of the force method described in section 4.3.3 is to transform the original, hyperstatic problem into an isostatic system. When the redundancy of the beam structure is due to multiple supports, this is achieved by eliminating, or cutting, the appropriate number of supports to render the beam isostatic. Reaction forces and moments, as well as shear force and bending moment diagrams are then readily obtained from statics.

For the simple example depicted in fig. 9.70, the mid-span support is eliminated, leaving an isostatic, cantilevered beam. The unit load method will be used to compute the deflection, Δ_c, at the location of the support that is eliminated. Equation (9.83) will be used for this purpose and yields

$$\Delta_c = \int_0^L \frac{M_3 \hat{M}_3}{H_{33}^c} \, \mathrm{d}x_1, \tag{9.91}$$

where $M_3(x_1)$ is the bending moment distribution in the isostatic beam subjected to the externally applied loads, and $\hat{M}_3(x_1)$ the statically admissible bending moment distribution in the isostatic beam subjected to a set of self-equilibrating unit forces applied at the support.

Next, the unit load method is used to compute the relative deflection at the support due to a set of self-equilibrating, unit loads applied at that location, as illustrated in fig. 9.70. Equation (9.84) then yields the desired relative displacement, denoted Δ_1, as

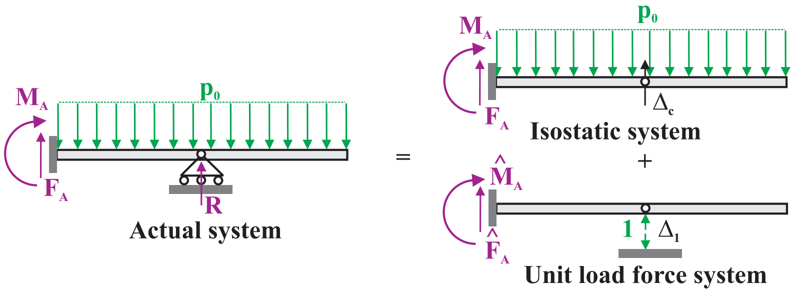

Fig. 9.70. Cantilever with a mid-span support. The isostatic system is obtained by eliminating the mid-span support.

$$\Delta_1 = \int_0^L \frac{\hat{M}_3^2}{H_{33}^c} \, dx_1. \tag{9.92}$$

where $\hat{M}_3(x_1)$ is the statically admissible bending moment distribution in the isostatic beam subjected to a set of self-equilibrating unit forces applied at the support. This moment distribution is identical to that used in eq. (9.91).

The displacement compatibility equation at the support is now expressed as

$$\Delta_c + R\,\Delta_1 = 0, \tag{9.93}$$

where $R\Delta_1$ is the deflection at the cut when the isostatic beam is subjected to a set of self-equilibrating loads of magnitude R applied at the cut. Equation (9.93) provides an additional relationship to evaluate the unknown reaction force at the support as

$$R = -\frac{\Delta_c}{\Delta_1}. \tag{9.94}$$

Once the redundant reaction force, R, is computed, the other reaction forces and bending moments can be obtained from the principle of superposition as $F_A + R\hat{F}_A$ and $M_A + R\hat{M}_A$. Finally, superposition also yields the beam's bending moment distribution as $M_3(x_1) + R\hat{M}_3(x_1)$.

The essence of the force method is to transform the original, hyperstatic system into an isostatic system by cutting or eliminating one support. Usually, this can be done in several different ways, by cutting or eliminating any one of the beam's support. The only requirement is that after the cut, the structure must be isostatic and free of any mechanism.

For instance, fig. 9.71 depicts the cantilevered beam with a mid-span support treated in the previous paragraphs. To transform the system into an isostatic structure, a cut will be made at the root to allow rotation of the beam at this point. This is equivalent to transforming the root clamp into a simple support, as illustrated in fig. 9.71.

When dealing with trusses, this first step of the force method is adequately described as "cutting one of the bars." When dealing with supports, however, the expression "cutting one of the supports" is confusing. The expression "eliminating one support" more accurately describes the removal of the mid-span support illustrated in fig. 9.70. The expression "releasing one constraint" is more appropriate when describing the replacement of the root clamp by a simple support.

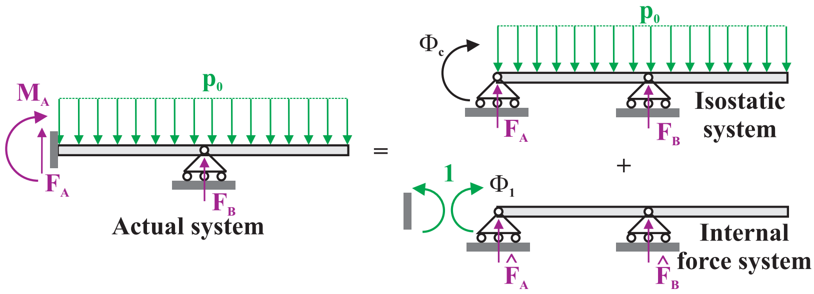

Fig. 9.71. Cantilever with a mid-span support. The isostatic system is obtained by eliminating the mid-span support.

After releasing the root rotation constraint, the unit load method is used to compute the relative root rotation, Φ_c, in the isostatic structure using eq. (9.85). Next, a set of self-equilibrating moments are applied at the root of the beam, as illustrated in fig. 9.71, and the unit load method is used once again to compute the associated root rotation, Φ_1. Finally, the root rotation compatibility condition is expressed as $\Phi_c + M_A \Phi_1 = 0$, where Φ_c is the root rotation when the isostatic beam is subjected to the externally applied loads, and $M_A \Phi_1$ the rotation of the isostatic beam at the same location where a set of self-equilibrating moments of magnitude M_A is applied at the beam's simply supported root. The compatibility condition implies the vanishing of the root rotation when the system is subjected to the combined loading, and provides an additional relationship to evaluate the unknown root reaction moment as $M_A = -\Phi_c/\Phi_1$. The other reactions forces and the beam's bending moment distribution are then obtained by superposition.

This discussion illustrates two ways of transforming the original hyperstatic beam problem into an isostatic system. Both approaches yield identical results. The choice between the two is entirely a matter of convenience: the approach that will minimize the burden of the solution process is the preferred course of action.

Example 9.29. Cantilevered beam with tip support

Consider a cantilevered beam of length L subjected to a uniform loading distribution, p_0, as illustrated in fig. 9.72. Find the bending moment distribution in the beam.

To transform this hyperstatic system into an isostatic problem, the tip support is eliminated, *i.e.*, the tip constraint is released. In the first step of the process, the beam's tip deflection is computed using the unit load method. The bending moment distribution in the isostatic beam generated by the externally applied load is $M_3(\eta) = -p_0 L^2 (1-\eta)2/2$, where $\eta = x_1/L$ is the non-dimensional variable along the beam's span. The statically admissible bending moment distribution associated with a unit load applied at the beam's tip is $\hat{M}_3(\eta) = L(1-\eta)$. The tip deflection of the isostatic beam is then

$$\Delta_c = \int_0^L \frac{M_3(x_1)\hat{M}_3(x_1)}{H_{33}^c}\,\mathrm{d}x_1 = -\frac{p_0 L^4}{2H_{33}^c}\int_0^1 (1-\eta)^3\,\mathrm{d}\eta = -\frac{p_0 L^4}{8H_{33}^c}.$$

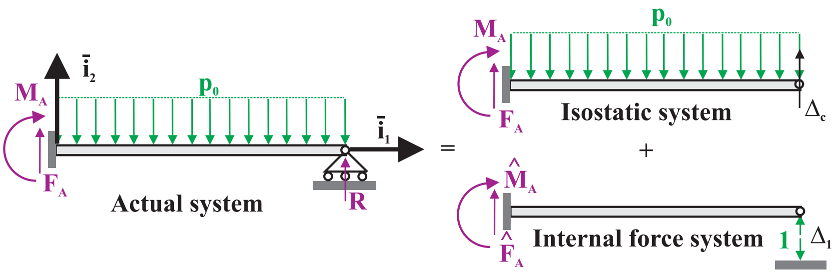

Fig. 9.72. Cantilever with a redundant support.

Next, the tip deflection of the isostatic beam subjected to a set of self-equilibrating tip unit loads is evaluated using the unit load method to find

$$\Delta_1 = \int_0^L \frac{\hat{M}_3^2(x_1)}{H_{33}^c}\,\mathrm{d}x_1 = \frac{L^3}{H_{33}^c}\int_0^1 (1-\eta)^2\,\mathrm{d}\eta = \frac{L^3}{3H_{33}^c}.$$

The compatibility condition, eq. (9.93), allows determination of the reaction force at the tip support as

$$R = -\frac{\Delta_c}{\Delta_1} = \frac{p_0 L^4}{8H_{33}^c}\frac{3H_{33}^c}{L^3} = \frac{3p_0 L}{8}.$$

The solution of the original, hyperstatic problem is now found by superposition. In particular, the bending moment distribution is

$$M_3 + R\hat{M}_3 = -\frac{p_0 L^2}{2}(1-\eta)^2 + \frac{3p_0 L^2}{8}(1-\eta) = \frac{p_0 L^2}{8}\left[3(1-\eta) - 4(1-\eta)^2\right].$$

Example 9.30. Tip rotation of a cantilevered beam with tip support
Consider a cantilevered beam of length L subjected to a uniform load, p_0, as illustrated in fig. 9.72. Determine the beam's tip rotation, Φ, using the unit load method.

The unit load method is equally applicable to iso- and hyperstatic problems; hence, the desired tip rotation is given as

$$\Phi = \int_0^L \frac{M_3(x_1)\hat{M}_3(x_1)}{H_{33}^c}\,\mathrm{d}x_1,$$

where $M_3(x_1)$ is the bending moment distribution in the hyperstatic beam subjected to the externally applied loads, and $\hat{M}_3(x_1)$ is any statically admissible bending moment distribution in equilibrium with a unit moment applied at the beam's tip.

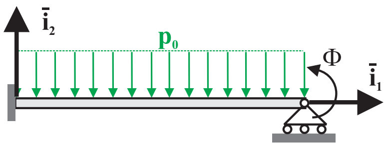

Fig. 9.73. Tip rotation for cantilever with a redundant tip support.

The bending moment distribution in the hyperstatic beam is evaluated in example 9.29 as $M_3 = p_0 L^2 \left[3(1-\eta) - 4(1-\eta)^2\right]/8$. Because the unit load method is a direct application of the principle of complementary virtual work, the bending moment distribution, $\hat{M}_3(x_1)$, can be selected as any statically admissible virtual bending moment distribution in equilibrium with the unit tip moment. Instead of trying the determine the statically admissible bending moment distribution acting in the beam undergoing compatible deformations, it is simpler to determine a statically admissible distribution acting in the beam undergoing incompatible deformations. This is acceptable because the principle of complementary virtual work calls for "any statically admissible bending moment distribution."

A simple, acceptable bending moment distribution is found by setting the tip reaction force to zero and evaluating the statically admissible bending moment distribution associated with a tip unit moment to find $\hat{M}_3(\eta) = 1$. By setting the tip reaction force to zero, the beam becomes isostatic, and the statically admissible bending moment distribution associated with a tip unit moment is then easily evaluated based on equilibrium considerations. Another suitable bending moment distribution is found by setting the root reaction moment to zero and evaluating the statically admissible bending moment distribution associated with a tip unit moment to find $\hat{M}_3(\eta) = \eta$. In this case, an isostatic problem with simple supports is obtained by releasing the beam's root rotation.

With the first bending moment distribution, the tip rotation becomes

$$\Phi = \int_0^L \frac{M_3(x_1)\hat{M}_3(x_1)}{H_{33}^c} \,\mathrm{d}x_1 = \frac{p_0 L^3}{8H_{33}^c} \int_0^1 \left[3(1-\eta) - 4(1-\eta)^2\right] 1 \,\mathrm{d}\eta = \frac{p_0 L^3}{48 H_{33}^c}.$$

Using the second bending moment distribution, the resulting tip rotation is

$$\Phi = \frac{p_0 L^3}{8H_{33}^c} \int_0^1 \left[3(1-\eta) - 4(1-\eta)^2\right] \eta \,\mathrm{d}\eta = \frac{p_0 L^3}{48 H_{33}^c}.$$

As expected, the results obtained with both bending moment distributions are identical. This surprising conclusion stems from the principle of complementary virtual work, which holds for "any statically admissible bending moment distribution." In fact, for hyperstatic problems of order 1, an infinite number of statically admissible bending moment distributions can be generated, each corresponding to an arbitrary choice of any one of the reaction forces of moments. All will generate the same tip rotation. The analyst should select the approach that simplifies computations as much as possible.

9.8.3 Combined truss and beam problems

The approach developed in the previous sections combines the force method with the unit load method, which itself is a direct application of the principle of complementary virtual work. In the previous examples, truss and beam structures are treated separately using expressions for the complementary internal virtual work given by eqs. (9.50) and (9.69), respectively. Because the complementary internal virtual work is an additive quantity, the complementary internal virtual work of a structure composed of a combination of beams and trusses is found by adding the contributions of each of the beams and bars. The rest of the unit load method remains unchanged.

Example 9.31. Beam with hyperstatic truss bracing
Consider the simply supported beam of bending stiffness H_{33}^c subjected to a uniform loading, as depicted in fig. 9.74. At mid-span, a vertical strut **CD** of infinite stiffness and height $h = L$ is pinned to the beam. Cable **ACB** braces the beam through the strut to provide additional support. Each of the two cable segments will be treated as bars of axial stiffness $E\mathcal{A}$. Determine the bending moment distribution in the beam and the forces in all bars; also find the beam's mid-span vertical deflection.

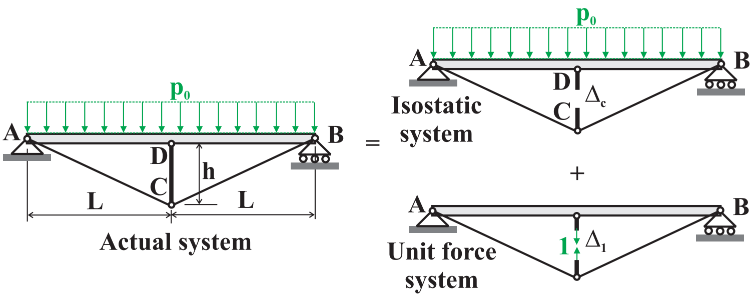

Fig. 9.74. Calculation of bending moment distribution in beam and forces in bars in a hyperstatic beam-truss problem.

First, the original, hyperstatic system is transformed into an isostatic system by cutting the vertical strut **CD**. This is not the only way to proceed: an isostatic system is also obtained if *(1)* cable **AC** is cut, *(2)* cable **CB** is cut, or *(3)* the beam's mid-span rotation is released by introducing a hinge at **D**.

For the isostatic system subjected to the externally applied loads, the beam's bending moment distribution is $M_3(\eta) = p_0 L^2(\eta - \eta^2/2)$, where $\eta = x_1/L$ is a non-dimensional variable along the beam's span, and the bar forces are $F_{AC} = F_{BC} = F_{CD} = 0$. The moment distribution is symmetric with respect to the beam's mid-span.

The deflection, Δ_c, at the cut in the strut is

$$\Delta_c = \frac{2}{H_{33}^c} \int_0^L M_3(x_1)\hat{M}_3(x_1)\,\mathrm{d}x_1 + \sum_{i=1}^{N_b} \frac{F_i \hat{F}_i L_i}{(E\mathcal{A})_i}.$$

The right-hand side represents the negative complementary internal virtual work in the structure and comprises two terms: the integral is the contribution of the flexible beam, while the sum represents the contributions from the individual bars. In view of the symmetry of the beam's bending moment distribution, the integral is computed from 0 to L and the result is then doubled. Introducing the bending moment distribution and bar forces then leads to

$$\Delta_c = \frac{2}{H_{33}^c} \int_0^L p_0 L^2 \left(\eta - \frac{\eta^2}{2}\right) \frac{L\eta}{2}\, L\mathrm{d}\eta = \frac{5}{24} \frac{p_0 L^4}{H_{33}^c}.$$

Note that in the isostatic system, the bar forces all vanish and hence, do not contribute to the displacement at the cut.

Analysis of the isostatic structure subjected to a system of self-equilibrating unit loads applied at the cut yields the following bending moment distribution in the beam $\hat{M}_3(\eta) = L\eta/2$, for $0 \leq \eta \leq 1$ and bar forces $\hat{F}_{AC} = \hat{F}_{BC} = -\sqrt{2}/2$, $\hat{F}_{CD} = 1$. Here again, the moment distribution is symmetric with respect to the beam's mid-span. The deflection at the cut, Δ_1, due to the unit load system is

$$\Delta_1 = \frac{2}{H_{33}^c} \int_0^L \hat{M}_3^2(x_1)\, \mathrm{d}x_1 + \sum_{i=1}^{N_b} \frac{\hat{F}_i^2 L_i}{(E\mathcal{A})_i}.$$

Introducing the beam's bending moment distribution and bar forces results in

$$\Delta_1 = \frac{2}{H_{33}^c} \int_0^1 \frac{L^2\eta^2}{4}\, L\mathrm{d}\eta + \frac{L}{E\mathcal{A}} \left[\left(-\frac{\sqrt{2}}{2}\right)^2 \sqrt{2} + 1 + \left(-\frac{\sqrt{2}}{2}\right)^2 \sqrt{2}\right]$$
$$= \frac{L^3}{6H_{33}^c} + (1+\sqrt{2})\frac{L}{E\mathcal{A}}.$$

The compatibility condition at the cut, $\Delta_c + R\Delta_1 = 0$, then gives the force, R, in the strut

$$R = -\frac{\Delta_c}{\Delta_1} = -\frac{5p_0 L}{4} \frac{1}{1 + 6(1+\sqrt{2})H_{33}^c/(E\mathcal{A}\, L^2)}. \tag{9.95}$$

Finally, superposition yields the bar forces $F_{AC} = F_{BC} = -\sqrt{2}R/2$, $F_{CD} = R$, and beam bending moment distribution as $M_3(\eta) = p_0 L^2(\eta - \eta^2/2) + RL\eta/2$.

Next, the beam's mid-span deflection, Δ_D, is determined using the unit load method as follows

$$\Delta_D = \frac{2}{H_{33}^c} \int_0^L \hat{M}_3(x_1) M_3(x_1)\, \mathrm{d}x_1 + \sum_{i=1}^{N_b} \frac{\hat{F}_i F_i L_i}{(E\mathcal{A})_i}.$$

In this expression, the bending moment distribution, $M_3(x_1)$, and bar forces, F_i, are those acting in the hyperstatic structure subjected to the externally applied loads, which are computed in the first part of this example. The virtual bending moment distribution, $\hat{M}_3(x_1)$, and bar forces, $\hat{F}_i$, are any statically admissible internal forces in equilibrium with a unit load applied at point **D**. For this problem, it is convenient to compute a set of statically admissible forces acting in the isostatic system obtained by setting the strut force to zero. This leads the following statically admissible beam bending moment distribution $\hat{M}_3(\eta) = L\eta/2$ for $0 \leq \eta \leq 1$ and bar forces $F_{CD} = F_{AC} = F_{BC} = 0$. Using these results, the mid-span deflection becomes

$$\Delta_D = \frac{2}{H_{33}^c} \int_0^1 \left[p_0 L^2 \left(\eta - \frac{\eta^2}{2} \right) + \frac{RL\eta}{2} \right] \left[\frac{L\eta}{2} \right] L\mathrm{d}\eta = \frac{5}{24} \frac{p_0 L^4}{H_{33}^c} + \frac{1}{6} \frac{RL^3}{H_{33}^c}$$

where R is given by eq. (9.95). Substituting for R and simplification then yields

$$\Delta_D = \frac{5}{24} \frac{p_0 L^4}{H_{33}^c} \left[1 - \frac{1}{1 + 6(1 + \sqrt{2}) H_{33}^c / (E\mathcal{A}\, L^2)} \right].$$

It is interesting to verify limiting cases for this mid-span deflection. First, if the stiffness of the cable becomes negligible compared to that of the beam, *i.e.*, if $E\mathcal{A}\, L^2/H_{33}^c \rightarrow 0$, $\Delta_D \approx 5p_0L^4/(24H_{33}^c)$, as expected for a uniformly loaded, simply supported beam. Second, if the stiffness of the cable becomes very large compared to that of the beam, *i.e.*, if $E\mathcal{A}\, L^2/H_{33}^c \rightarrow \infty$, then $\Delta_D \approx 0$, as expected because the truss essentially provides a pinned support at the beam's mid-span. For intermediate cable stiffness values, the bracketed term is always smaller than unity and $\Delta_D < 5p_0L^4/(24H_{33}^c)$. The cables stiffen the structure and reduce the beam's mid-span deflection.

9.8.4 Multiple redundancies

All hyperstatic problems treated in the previous sections are of order 1, *i.e.*, a single cut is sufficient to transform the hyperstatic system into an isostatic system. Many practical hyperstatic structures are of higher order. If a hyperstatic structure is of order N, then N cuts will be required to create an isostatic problem. The combination of the force and unit load methods still leads to an efficient but possible tedious solution process: N compatibility equations are generated that can be solved for the N unknown internal forces. The process will be illustrated in the following example.

Example 9.32. Cantilevered beam with redundant supports
Consider the cantilevered beam of length L subjected to a uniform load distribution, as depicted in fig. 9.75. The beam also features mid-span and tip supports, and this is therefore a hyperstatic system of order 2.

In the first step of the force method, the structure is transformed into an isostatic system by eliminating the two supports, as illustrated in fig. 9.75. The bending moment distribution in the isostatic structure subjected to the externally applied loads is found from statics as $M_3(\eta) = -p_0L^2(1-\eta)^2/2$.

Next, the unit load method is used to compute the beam's deflections at the support locations, denoted Δ_{c1} and Δ_{c2}, for the mid-span and tip support locations,

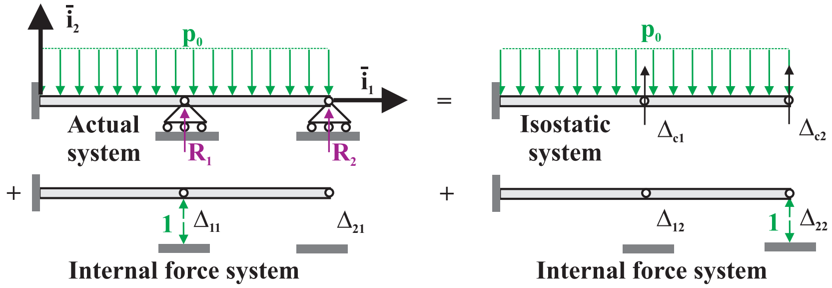

Fig. 9.75. Cantilevered beam with two support.

respectively. To accomplish this task, the statically admissible bending moment distribution in equilibrium with a set of self-equilibrating unit loads applied at the location of the mid-span support is found to be $\hat{M}_3^{[1]} = L(1/2 - \eta)$ for $0 \leq \eta \leq 1/2$ and $\hat{M}_3^{[1]} = 0$ for $1/2 \leq \eta \leq 1$. The corresponding bending moment distribution associated with a set of self-equilibrating unit loads applied at the location of the tip support is $\hat{M}_3^{[2]} = L(1 - \eta)$. The desired deflection at the mid-span support then follows as

$$\Delta_{c1} = \int_0^L \frac{M_3 \hat{M}_3^{[1]}}{H_{33}^c} \, dx_1 = -\frac{p_0 L^4}{2H_{33}^c} \int_0^{1/2} (1-\eta)^2 (1/2 - \eta) \, d\eta = -\frac{17 p_0 L^4}{384 H_{33}^c},$$

and the deflection at the tip support is found in a similar manner as

$$\Delta_{c2} = \int_0^L \frac{M_3 \hat{M}_3^{[2]}}{H_{33}^c} \, dx_1 = -\frac{p_0 L^4}{2H_{33}^c} \int_0^{1} (1-\eta)^2 (1 - \eta) \, d\eta = -\frac{p_0 L^4}{8 H_{33}^c}.$$

Next, a set of self-equilibrating unit loads are applied at the location of the mid-span support, and the resulting deflections at the location of the mid-span and tip supports, denoted $\Delta_1^{[1]}$ and $\Delta_2^{[1]}$, respectively, are found

$$\Delta_1^{[1]} = \int_0^L \frac{\hat{M}_3^{[1]} \hat{M}_3^{[1]}}{H_{33}^c} \, dx_1 = \frac{L^3}{H_{33}^c} \int_0^{1/2} (1/2-\eta)(1/2 - \eta) \, d\eta = \frac{L^3}{24 H_{33}^c},$$

$$\Delta_2^{[1]} = \int_0^L \frac{\hat{M}_3^{[1]} \hat{M}_3^{[2]}}{H_{33}^c} \, dx_1 = \frac{L^3}{H_{33}^c} \int_0^{1/2} (1/2-\eta)(1 - \eta) \, d\eta = \frac{5L^3}{48 H_{33}^c}.$$

Similarly, a set of self-equilibrating unit loads are applied at the location of the tip support, and the resulting deflections at the location of the mid-span and tip supports, denoted $\Delta_1^{[2]}$ and $\Delta_2^{[2]}$, respectively, are found

$$\Delta_1^{[2]} = \int_0^L \frac{\hat{M}_3^{[2]} \hat{M}_3^{[1]}}{H_{33}^c} \, dx_1 = \frac{L^3}{H_{33}^c} \int_0^{1/2} (1-\eta)(1/2 - \eta) \, d\eta = \frac{5L^3}{48 H_{33}^c}.$$

$$\Delta_2^{[2]} = \int_0^L \frac{\hat{M}_3^{[2]}\hat{M}_3^{[2]}}{H_{33}^c}\, dx_1 = \frac{L^3}{H_{33}^c}\int_0^1 (1-\eta)^2\, d\eta = \frac{L^3}{3H_{33}^c}.$$

The two compatibility conditions impose the vanishing of the displacement at the mid-span support, $\Delta_{c1} + R_1\Delta_1^{[1]} + R_2\Delta_1^{[2]} = 0$, and at the tip support, $\Delta_{c2} + R_1\Delta_2^{[1]} + R_2\Delta_2^{[2]} = 0$, where R_1 and R_2 are the unknown reaction forces at the mid-span and tip supports, respectively. Introducing the various displacement components in these compatibility conditions yields a set of algebraic equations for the two unknown reaction forces

$$\begin{bmatrix} 1/24 & 5/48 \\ 5/48 & 1/3 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = p_0 L \begin{Bmatrix} 17/384 \\ 1/8 \end{Bmatrix}.$$

Solution of this set of two equations in two unknowns yields $R_1 = 4p_0L/7$ and $R_2 = 11p_0L/56$. The bending moment distribution in the hyperstatic beam then follows from the principle of superposition as $p_0L^2(1-\eta)^2/2 + R_1\hat{M}_3^{[1]} + R_2\hat{M}_3^{[2]}$.

This example demonstrates the use of the force method for hyperstatic systems of higher order. As the order increases, the solution process becomes increasingly cumbersome. The solution of a hyperstatic system of order N will call for the solution of a system of N compatibility equations written in terms of N unknown reaction components. While this example presents the approach for a beam with multiple redundant supports, it can also be used for hyperstatic trusses, beams, combined beam and truss, or three-dimensional structures.

9.8.5 Problems

Problem 9.28. Redundant planar frame with tip load

Consider the cantilevered beam consisting of two segments of length L connected at a 90 degree angle, as shown in fig. 9.76. A simple support is located at point **B**, and a horizontal load, P, is applied at point **A**. *(1)* Find the magnitude and location of the maximum bending moment in the bent beam. *(2)* Find the horizontal deflection at point **A**.

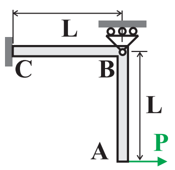

Fig. 9.76. Planar right angle frame with tip load.

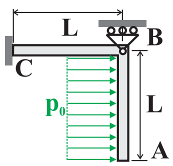

Fig. 9.77. Planar right angle frame with distributed load.

Problem 9.29. Redundant planar frame with tip load

Consider the cantilevered beam consisting of two segments of length L connected at a 90 degree angle, as shown in fig. 9.76. A simple support is located at point **B**, and a horizontal load, P, is applied at point **A**. *(1)* Find the magnitude and location of the maximum bending moment in the bent beam. *(2)* Find the rotation at point **A**.

Problem 9.30. Redundant planar frame with distributed loading
Consider the cantilevered beam consisting of two segments of length L connected at a 90 degree angle, as shown in fig. 9.77. A simple support is located at point **B**, and a distributed horizontal load, p_0, is acting along segment **BA**. *(1)* Find the magnitude and location of the maximum bending moment in the bent beam. *(2)* Find the horizontal tip deflection at point **A**.

Problem 9.31. Cantilevered beam with truss bracing
A cantilevered beam of length, L, and bending stiffness, H, carries a tip load, P, as shown in fig. 9.78. At mid-span, the bean is braced by a bar, **BM**, of stiffness $S = E\mathcal{A}$, oriented at an angle $\phi = 60$ degrees. *(1)* Find the magnitude and location of the maximum bending moment in the beam. The effect of the axial load in portion **RM** of the beam is negligible because the beam's axial stiffness of very large. *(2)* Find the transverse deflection at point **T**.

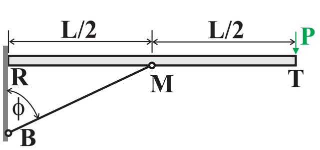

Fig. 9.78. Cantilevered beam with supporting truss.

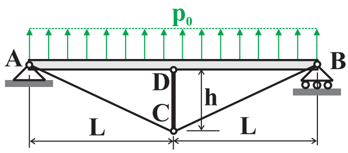

Fig. 9.79. Simply supported beam with supporting truss.

Problem 9.32. Simply supported beam with truss bracing
The structure depicted in fig. 9.79 consists of a simply supported beam, **AB**, supported at its mid-point by cable **ACB** and a rigid vertical strut, **CD**, of length $h = L$ connecting points **D** and **C**. Cable **ACB** can be modeled as two bars with sectional stiffnesses, $E\mathcal{A}$, and the strut can be modeled as a bar of infinite stiffness. Ignore the axial force developed in the beam itself because the beam's axial stiffness is much larger than that of the cable. *(1)* Find the bending moment distribution in the beam and the forces in the cable segments and vertical strut. Hint: It will be convenient to cut the vertical strut. *(2)* Find the mid-span deflection of the beam.

Problem 9.33. Curved cantilevered beam with mid-support
The cantilevered beam shown in fig. 9.80 is straight from point **A** to point **B**. From point **B** to point **C**, the beam has the shape of a quarter circle of radius R. A horizontal load of magnitude P is applied at point **C**. *(1)* Determine the tip displacement of the beam at point **C**. Assume the the beam only undergoes bending deformations. For the beam's curved portion, express the bending moment as a function of $\theta \in [0, \pi/2]$ and use $\mathrm{d}x_1 = R\mathrm{d}\theta$.

Problem 9.34. Redundant truss
A vertical load, P_1, is applied to the six-bar hyperstatic planar truss depicted in fig. 9.81. Bars **AD**, **BD** and **CD** are of equal length and joint **D** is at the center of the triangle. *(1)* Determine all bar forces and the displacement at point **C** when load P_1 is acting alone. *(2)* If loads P_1 and P_2 are applied simultaneously, find the value of the horizontal load, P_2, for which the displacement at joint **B** vanishes. *(3)* Determine all corresponding bar forces. Use the following data: $S = E\mathcal{A} = 1 \times 10^6$ psi for all bars; $L = 48$ inches and $P_1 = 4,000$ lbs.

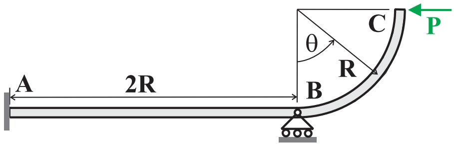

Fig. 9.80. Curved cantilevered beam with tip load.

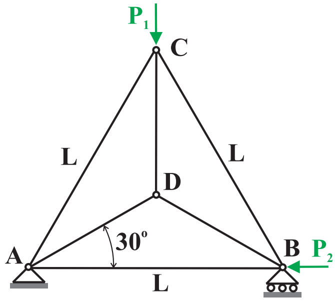

Fig. 9.81. Triangular truss with internal redundancy.